Mathematical Methods of Physics

Lecture Notes

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1 Metric spaces

Reference: [RS80, Chapters I.2, I.5].

Throughout these notes, we shall deal with sets of functions, operators, and other mathematical objects. We shall often need a way of measuring the distance between two objects in such a set. This motivates a notion of distance having the essential properties of the ordinary Euclidean distance.

Definition 1.1 (Metric space)**.** A *metric space* is a set M together with a function

$$d\colon M \times M \to \mathbb{R}$$

such that, for all $x, y, z \in M$,

(i) $d(x, y) \geq 0$;

(ii) $d(x, y) = 0$ if and only if $x = y$;

(iii) $d(x, y) = d(y, x)$;

(iv) $d(x, z) \leq d(x, y) + d(y, z)$ *(triangle inequality).*

The function d is called a *metric* on M.

The elements of a metric space are often called *points*. Notice that a metric space consists of both the set and the metric. A given set X may be made into a metric space in different ways by choosing different metrics. When the metric is not clear from the context, we denote the metric space by (M, d).

Example 1.2 (Euclidean space)**.** Let $M = \mathbb{R}^n$. For

$$x = (x_1, \ldots, x_n), \qquad y = (y_1, \ldots, y_n),$$

the Euclidean distance is

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}.$$

Example 1.3 (Two metrics on the unit circle)**.** Let

$$M = \{(\alpha, \beta) \in \mathbb{R}^2 : \alpha^2 + \beta^2 = 1\}.$$

One metric is the Euclidean chord distance

$$d_2((\alpha, \beta), (\alpha', \beta')) = \sqrt{(\alpha - \alpha')^2 + (\beta - \beta')^2}.$$

Another possible metric is the length $d_a(p, p')$ of the shorter circular arc joining p and p'.

Example 1.4 (Two metrics on a function space)**.** Let $M = C([0, 1])$, the space of continuous complex-valued functions on $[0, 1]$. Two metrics on M are

$$d_\infty(f, g) = \max_{t \in [0,1]} |f(t) - g(t)|, \qquad d_1(f, g) = \int_0^1 |f(t) - g(t)| \, \mathrm{d}t.$$

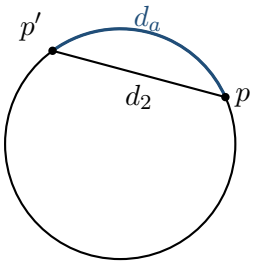


Figure 1.1: The chord metric d_2 and the arc-length metric d_a.

1.1 Convergence, Cauchy sequences, and completeness

Definition 1.5 (Convergence)**.** A sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space (M, d) is said to *converge* to $x \in M$ if

$$d(x_n, x) \to 0 \qquad (n \to \infty).$$

We write $x_n \xrightarrow{d} x$, or simply $x_n \to x$ when the metric is understood.

In Example 1.3,

$$d_2(p, p') \le d_a(p, p') \le \frac{\pi}{2}\, d_2(p, p'). \qquad (🏠)$$

Consequently, a sequence converges with respect to d_2 if and only if it converges with respect to d_a. The constant $\pi/2$ is optimal: equality in the upper bound is attained when p and p' are antipodal.

The two metrics in Example 1.4, on the other hand, induce different notions of convergence. Since

$$d_1(f, g) \le d_\infty(f, g),$$

uniform convergence implies convergence in the integral metric, but the converse is false.

Consider the continuous functions g_n whose graphs are triangles of height 1, supported on

$$\left[\frac{1}{2} - \frac{1}{2n}, \frac{1}{2} + \frac{1}{2n}\right],$$

with their maximum at $x = 1/2$; see Figure 1.2. Then $d_1(g_n, 0) \to 0$, whereas $d_\infty(g_n, 0) = 1$ for every n.

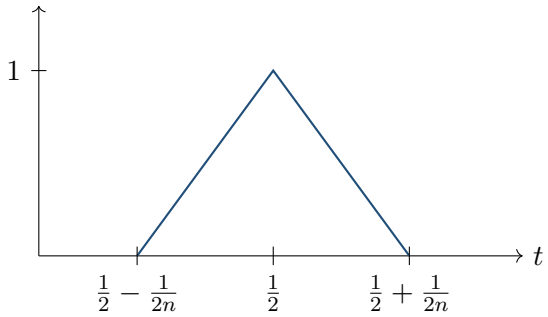


Figure 1.2: The graph of g_n.

Definition 1.6 (Cauchy sequence)**.** A sequence (x_n) in a metric space (M, d) is a *Cauchy sequence* if, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$n, m \ge N \quad \to \quad d(x_n, x_m) < \varepsilon.$$

Proposition 1.7 (Convergent implies Cauchy)**.** Every convergent sequence is a Cauchy sequence.

Proof. Suppose that $x_n \to x$. Given $\varepsilon > 0$, choose N such that $d(x_n, x) < \varepsilon/2$ for all $n \geq N$. If $n, m \geq N$, then

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

Returning to the functions in Figure 1.2, one has

$$d_\infty(g_n, g_m) = 1 - \frac{m}{n} \qquad (n \geq m).$$

Hence (g_n) is not a Cauchy sequence for d_∞ and therefore cannot converge in that metric, in agreement with Proposition 1.7.

1.2 Continuity and isometries

Definition 1.8 (Continuity)**.** Let (X, d) and (Y, ρ) be metric spaces. A function $f\colon X \to Y$ is *continuous at* $x \in X$ if

$$x_n \xrightarrow{d} x \quad \to \quad f(x_n) \xrightarrow{\rho} f(x).$$

It is *continuous* if it is continuous at every point of X.

The sequence (g_n) above converges to zero for the metric d_1 but not for d_∞. Thus the identity map

$$\mathrm{id}\colon (C([0,1]), d_1) \to (C([0,1]), d_\infty)$$

is not continuous, whereas the identity map in the opposite direction is continuous.

Definition 1.9 (Isometry)**.** A map $h\colon (X, d_X) \to (Y, d_Y)$ is an *isometry* if

$$d_Y\big(h(x), h(y)\big) = d_X(x, y) \qquad \text{for all } x, y \in X.$$

The metric spaces (X, d_X) and (Y, d_Y) are said to be *isometric* if there exists a bijective isometry $h : X \to Y$.

Every isometry is injective and continuous. Isometric spaces have identical metric structure: any theorem formulated solely in terms of that structure transfers from one to the other.

1.3 Open and closed sets

Definition 1.10. Let (X, d) be a metric space.

(a) The set

$$B(y, r) = \{x \in X : d(x, y) < r\}$$

is the *open ball* of radius $r > 0$ centred at y.

(b) A set $O \subseteq X$ is *open* if, for every $y \in O$, there is an $r > 0$ such that $B(y, r) \subseteq O$.

(c) A set $N \subseteq X$ is a *neighbourhood* of $y \in X$ if N is open and $y \in N$.

(d) A point $x \in X$ is a *limit point* of $E \subseteq X$ if

$$B(x,r) \cap (E \setminus \{x\}) \neq \varnothing \qquad \text{for every } r > 0.$$

(e) A set $F \subseteq X$ is *closed* if it contains all its limit points.

(f) A point $x \in G \subseteq X$ is an *interior point* of G if there is an $r > 0$ such that $B(x,r) \subseteq G$.

(g) A set $B \subset X$ is *dense* in X if every point of X is the limit of a sequence of elements of B.

Example 1.11. The geometry of the open balls depends on the metric used. Figure 1.3 illustrates the unit circles $\partial B_p(0,1) = \{x \in \mathbb{R}^2 : d_p(x,0) = 1\}$ in $(\mathbb{R}^2, d_p)$ with the metrics

$$d_p(x,y) = \left(|x_1 - y_1|^p + |x_2 - y_2|^p\right)^{1/p} \quad (p = 1, 2), \qquad d_\infty(x,y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}.$$

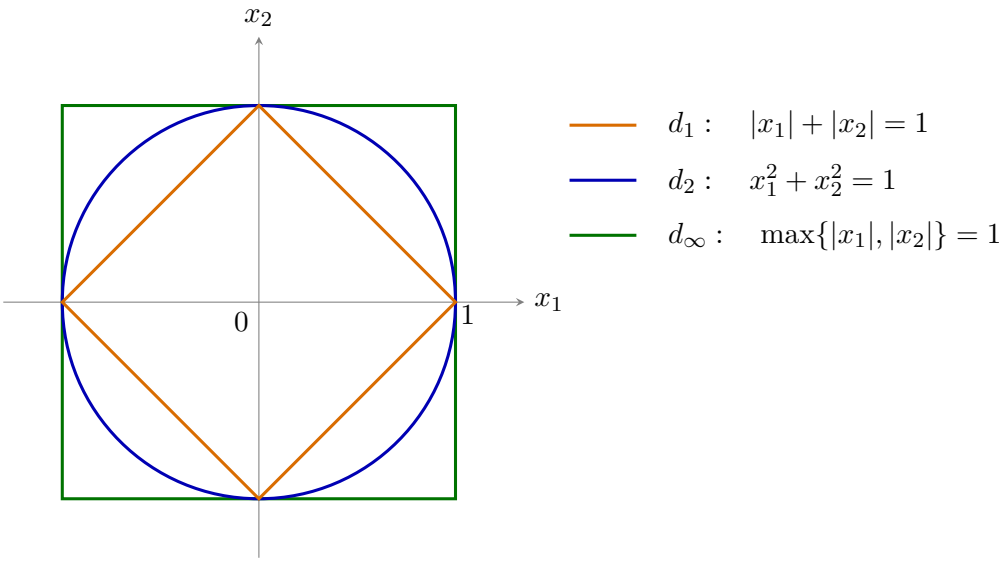


Figure 1.3: Unit circles $\partial B(0,1)$ in $\mathbb{R}^2$.

Theorem 1.12 (Elementary properties)**.** Let (X,d) be a metric space.

(a) A set O is open if and only if $X \setminus O$ is closed.

(b) $x_n \to x$ if and only if every neighbourhood of x contains all but finitely many terms of the sequence.

(c) The set of interior points of any set is open.

(d) The union of E with all its limit points is a closed set, denoted by $\overline{E}$ and called the *closure* of E.

(e) A set is open if and only if it is a neighbourhood of each of its points.

Theorem 1.13 (Topological criterion for continuity)**.** A function $f\colon X \to Y$ between metric spaces is continuous if and only if $f^{-1}(O)$ is open in X for every open set $O \subseteq Y$.

Closedness is relative to the ambient metric space. For instance, $[1/2, 1)$ is closed as a subset of $(0,1)$ with its usual metric, although it is not closed as a subset of $\mathbb{R}$.

1.4 Completeness and completion

Although every convergent sequence is Cauchy, the converse is not always true. Let $\mathbb{Q}$ have its usual metric $d(x,y) = |x-y|$, choose $x_* \in \mathbb{R} \setminus \mathbb{Q}$, and let (x_n) be a sequence of rational numbers converging to x_* in $\mathbb{R}$. Then (x_n) is Cauchy in $\mathbb{Q}$, but it cannot converge in $\mathbb{Q}$.

Definition 1.14 (Complete metric space)**.** A metric space in which every Cauchy sequence converges is called *complete*.

For example, $\mathbb{R}$ is complete, whereas $\mathbb{Q}$ is not. Moreover, $(C([0,1]), d_\infty)$ is complete, while $(C([0,1]), d_1)$ is not; see Theorem 1.18 and Example 1.15.

Example 1.15. The metric space $(C[0,1], d_1)$ is not complete.

Indeed, consider the continuous functions f_n whose graphs are given in Figure 1.4.

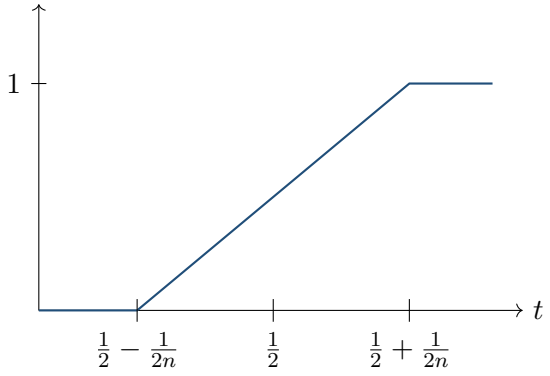


Figure 1.4: The graph of f_n.

Then, one has

$$d_1(f_n, f_m) = \left| \frac{1}{4m} - \frac{1}{4n} \right|,$$

so that (f_n) is a Cauchy sequence. However, the (pointwise) limit function is the function $u(t) = \mathbf{1}_{(1/2,1]}(t) + \frac{1}{2}\delta_{t,1/2}$, and for any continuous function $f \in C[0,1]$ one gets

$$\int_0^1 |f(t) - u(t)|\,\mathrm{d}t \le \int_0^1 |f(t) - f_n(t)|\,\mathrm{d}t + \int_0^1 |f_n(t) - u(t)|\,\mathrm{d}t.$$

Since, by the continuity of f,

$$\int_0^1 |f(t) - u(t)|\,\mathrm{d}t = c_f > 0,$$

while

$$\int_0^1 |f_n(t) - u(t)|\,\mathrm{d}t \to 0,$$

one gets that

$$d_1(f_n, f) = \int_0^1 |f(t) - f_n(t)|\,\mathrm{d}t$$

cannot converge to 0 as $n \to \infty$.

Problem 1.16. Evaluate $d_\infty(f_n, f_m)$ for the above sequence, and show that, at variance with d_1, (f_n) is not a Cauchy sequence for d_∞.

The example of $\mathbb{Q} \subset \mathbb{R}$ suggests that an incomplete space may be completed by adding all possible limits of its Cauchy sequences.

Theorem 1.17 (Completion theorem)**.** Let (M, d) be a metric space. There exists a complete metric space $(\widetilde{M}, \widetilde{d})$ and an isometry

$$\iota \colon M \to \widetilde{M}$$

such that $\iota(M)$ is dense in $\widetilde{M}$.

Sketch of proof. Consider all Cauchy sequences (x_n) in M. Declare two such sequences (x_n) and (y_n) equivalent when

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

Let $\widetilde{M}$ be the set of equivalence classes. For any two Cauchy sequences, the limit

$$\lim_{n\to\infty} d(x_n, y_n)$$

exists and depends only on their equivalence classes; it therefore defines a metric $\widetilde{d}$ on $\widetilde{M}$. One then proves that $\widetilde{M}$ is complete. Finally, map $x \in M$ to the class of the constant sequence $(x, x, \dots)$. This map is an isometry and its image is dense in $\widetilde{M}$. □

Theorem 1.18 (Completeness of $(C[0,1], d_\infty)$)**.**

The metric space $(C[0,1], d_\infty)$ is complete.

Proof. Let (f_n) be a Cauchy sequence in the uniform metric.

- (Pointwise limit) Given $\varepsilon > 0$, find N such that $d_\infty(f_n, f_m) < \varepsilon$ for all $n, m \geq N$. Therefore, for every fixed $t \in [0,1]$,

$$|f_n(t) - f_m(t)| \leq d_\infty(f_n, f_m) < \varepsilon, \qquad n, m \geq N. \tag{1.1}$$

  Thus $(f_n(t))_{n\geq 1}$ is Cauchy in $\mathbb{C}$. By completeness of $\mathbb{C}$, it has a limit, which we denote by $f(t)$. This defines a function $f \colon [0,1] \to \mathbb{C}$; its continuity is not yet known.

- (Uniform convergence)

  Fix $n \geq N$ and $t \in [0,1]$, and let $m \to \infty$ in the estimate (1.1):

$$|f_n(t) - f(t)| = \lim_{m\to\infty} |f_n(t) - f_m(t)| \leq \varepsilon.$$

  The bound is independent of t, so

$$\sup_{t\in[0,1]} |f_n(t) - f(t)| \leq \varepsilon \qquad (n \geq N).$$

  Hence $f_n \to f$ uniformly. Thus, if we can show that $f \in C[0,1]$, we can conclude that $d_\infty(f_n, f) \to 0$, so $f_n \to f$ in $(C[0,1], d_\infty)$.

- (Continuity of the limit) To prove that "a uniform limit of continuous functions is continuous", we use an $\varepsilon/3$ *argument.*

  Fix $x \in [0,1]$ and $\varepsilon > 0$. We want to find a $\delta > 0$ so that $|y - x| < \delta$ implies $|f(y) - f(x)| < \varepsilon$.

  Pick n so that

$$\sup_{t\in[0,1]} |f_n(t) - f(t)| < \frac{\varepsilon}{3}.$$

Now, since f_n is continuous, pick $\delta > 0$ so that $|y - x| < \delta$ implies

$$|f_n(y) - f_n(x)| < \frac{\varepsilon}{3}.$$

Then, $|y - x| < \delta$ implies

$$\begin{aligned} |f(y) - f(x)| &\leq |f(y) - f_n(y)| + |f_n(y) - f_n(x)| + |f_n(x) - f(x)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Thus f is continuous.

□

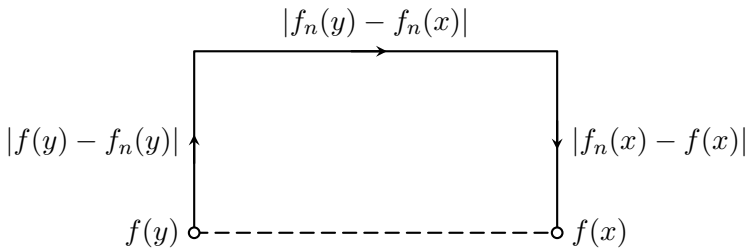


Figure 1.5: The $\varepsilon/3$ argument.

What is the essence of the $\varepsilon/3$ argument? We had a family of convergent sequences $f_n(x) \to f(x)$ for each x and had *uniform control on the rate of convergence*, that is control independent of the object x that parametrised the family. The central idea is to prove continuity of the limit f by comparing it with one fixed continuous approximant f_n. We write

$$f(y) - f(x) = (f(y) - f_n(y)) + (f_n(y) - f_n(x)) + (f_n(x) - f(x)).$$

There are three errors to control: approximation at y, variation along f_n, and approximation at x. Assigning less than $\varepsilon/3$ to each ensures that their sum is less than ε.

What we did is pictorially indicated in Figure 1.5: one could also call the $\varepsilon/3$ argument the "up, over, and around" proof.

2 Normed linear spaces and Banach spaces

Reference: [RS80, Chapters I.2, III.1].

We now turn to two central concepts of functional analysis: normed linear spaces and bounded linear operators.

Definition 2.1 (Normed linear space)**.** A *normed linear space* is a vector space V over $\mathbb{R}$ or $\mathbb{C}$ together with a function $\|\cdot\| : V \to \mathbb{R}$ such that, for all $v, w \in V$ and all scalars α,

(i) $\|v\| \geq 0$;

(ii) $\|v\| = 0$ if and only if $v = 0$;

(iii) $\|\alpha v\| = |\alpha| \, \|v\|$;

(iv) $\|v + w\| \leq \|v\| + \|w\|$.

The Euclidean spaces $\mathbb{R}^n$ and $\mathbb{C}^n$, equipped with

$$\|(x_1, \ldots, x_n)\|_2 = \sqrt{|x_1|^2 + \cdots + |x_n|^2},$$

are normed linear spaces. The space $C([0,1])$ may be equipped with either

$$\|f\|_\infty = \max_{t \in [0,1]} |f(t)| \qquad \text{or} \qquad \|f\|_1 = \int_0^1 |f(t)| \, \mathrm{d}t.$$

Every normed linear space $(V, \|\cdot\|)$ is a metric space (🏠) with the *induced metric*

$$d(v, w) = \|v - w\| .$$

In the induced metric, the norm $\|\cdot\| : V \to \mathbb{R}$ is a continuous function, that is $\lim_n \|v_n\| = \|\lim_n v_n\|$, as a consequence of the following inequality.

Lemma 2.2 (Reverse triangle inequality)**.** For all $v, w \in V$,

$$\Big| \|v\| - \|w\| \Big| \leq \|v - w\| . \tag{2.1}$$

Proof. By the triangle inequality,

$$\|v\| = \|v - w + w\| \leq \|v - w\| + \|w\|,$$

so that

$$\|v\| - \|w\| \leq \|v - w\|.$$

By swapping v and w one gets (2.1). □

Problem 2.3. Prove that the norm $\|\cdot\| : V \to \mathbb{R}$ is a continuous function.

Definition 2.4 (Banach space)**.** A normed linear space is a *Banach space* if it is complete with respect to the metric induced by its norm.

If $(X, \|\cdot\|)$ is a normed linear space, then X has a completion as a metric space by Theorem 1.17. Using the fact that X is dense in $\widetilde{X}$, one can prove that $\widetilde{X}$ can be endowed, in a unique natural way, with a vector-space structure and a norm extending those of X, by

$$\|\widetilde{x}\|_{\widetilde{X}} = \widetilde{d}(\widetilde{x}, 0) = \lim_n d(x_n, 0) = \lim_n \|x_n\|,$$

with $(x_n) \in \tilde{x}$ being a representative Cauchy sequence of $\widetilde{x}$; see the proof of the completion theorem 1.17.

2.1 Bounded operators

Definition 2.5 (Bounded operator)**.** Let $(V_1, \|\cdot\|_1)$ and $(V_2, \|\cdot\|_2)$ be normed linear spaces. A map $T\colon V_1 \to V_2$ is a *bounded operator* if

(i) T is linear: $T(\alpha v + \beta w) = \alpha T(v) + \beta T(w)$ for all $v, w \in V_1$ and all scalars α, β;

(ii) there exists $C \geq 0$ such that

$$\|Tv\|_2 \leq C \|v\|_1 \qquad \text{for all } v \in V_1.$$

The smallest admissible constant C is the *operator norm* $\|T\|$ of T. One has (🏠)

$$\|T\| = \sup_{\|v\|_1=1} \|Tv\|_2 = \sup_{\|v\|_1\leq 1} \|Tv\|_2 = \sup_{v\neq 0} \frac{\|Tv\|_2}{\|v\|_1}. \tag{2.2}$$

Problem 2.6. Prove that

$$\min\Big\{C \geq 0 \,:\, \|Tv\|_2 \leq C \|v\|_1, \forall v \in V_1\Big\} = \sup_{\|v\|_1=1} \|Tv\|_2 = \sup_{v\neq 0} \frac{\|Tv\|_2}{\|v\|_1}. \tag{2.3}$$

For operators, continuity with respect to the induced metric is equivalent to boundedness.

Theorem 2.7 (Continuity of bounded operators)**.** Let A be a linear map between normed linear spaces. The following statements are equivalent:

(a) A is bounded;

(b) A is continuous at every point;

(c) A is continuous at one point.

Proof.

- $a) \Rightarrow b)$.

  By assumption $\|A\| < \infty$, thus given a point $x \in V$ and a sequence $(x_n)_{n\in\mathbb{N}} \subset V$ with $x_n \to x$, that is $d(x_n, x) = \|x_n - x\| \to 0$, as $n \to \infty$, one has

  $$d(Ax_n, Ax) = \|Ax_n - Ax\| = \|A(x_n - x)\| \leqslant \|A\| \|x_n - x\| \to 0,$$

  as $n \to \infty$.

- $b) \Rightarrow c)$ is trivial

- $c) \Rightarrow a)$.

  Assume that the operator A is continuous at $x_0 \in V$. Fix $\varepsilon = 1$. There is a $\delta > 0$ such that

  $$\forall x \in V, \|x - x_0\| < \delta : \ \|Ax - Ax_0\| < 1. \tag{2.4}$$

  Our claim is that

  $$\exists C > 0 \text{ s.t. } \forall y \in V : \ \|Ay\| \leqslant C \|y\|. \tag{2.5}$$

  The inequality (2.5) is true for $y = 0$. Assume $y \neq 0$ and define

  $$x = x_0 + \frac{\delta}{2\|y\|} y.$$

We have

$$\|x - x_0\| = \left\| \frac{\delta}{2} \frac{y}{\|y\|} \right\| = \frac{\delta}{2} < \delta.$$

Therefore, by (2.4),

$$\|Ax - Ax_0\| < 1,$$

that is

$$\|A(x - x_0)\| = \frac{\delta}{2\|y\|} \|Ay\| < 1$$

whence

$$\|Ay\| \leq \frac{2}{\delta} \|y\|,$$

and we are done.

□

Theorem 2.8 (BLT theorem)**.** Let X be a normed linear space, Y a Banach space, and $Z \subset X$ a dense subspace of X. Let $T\colon Z \to Y$ be a bounded operator, i.e., $T \in \mathcal{B}(Z, Y)$.

Then T has a unique bounded linear extension

$$\widetilde{T}\colon X \to Y$$

to X, and $\|\widetilde{T}\| = \|T\|$.

Proof. For $x \in X$, choose a sequence (x_n) in Z with $x_n \to x$. Since (x_n) is Cauchy and

$$\|Tx_n - Tx_m\| \leq \|T\| \, \|x_n - x_m\|,$$

the sequence (Tx_n) is Cauchy in Y. Completeness of Y gives a limit, and we define

$$\widetilde{T}x = \lim_{n\to\infty} Tx_n.$$

This definition does not depend on the approximating sequence. Indeed, if $x'_n \to x$, then

$$\|Tx_n - Tx'_n\| \leq \|T\| \, \|x_n - x'_n\| \to 0$$

as $n \to \infty$. Thus, $\lim_n Tx'_n = \lim_n Tx_n = \widetilde{T}x$.

Linearity follows by passing to limits, and $\widetilde{T}x = Tx$ for all $x \in Z$, so that $\widetilde{T}$ is an extension of T.

Moreover,

$$\|\widetilde{T}x\| = \lim_{n\to\infty} \|Tx_n\| \leq \lim_{n\to\infty} \|T\| \, \|x_n\| = \|T\| \, \|x\|.$$

Thus $\widetilde{T}$ is bounded and $\|\widetilde{T}\| \leq \|T\|$. Since it extends T, the reverse inequality is immediate. Finally, any continuous extension must agree with T on the dense subspace Z, which proves uniqueness (🏠). □

The theorem is often used to extend a bounded operator on a normed linear space Z to its Banach-space completion $\widetilde{Z}$.

Problem 2.9. Prove uniqueness of $\widetilde{T}$.

2.2 The Banach space of bounded operators

Let X and Y be normed linear spaces. We denote by $\mathcal{B}(X,Y)$ the space of all bounded linear operators from X to Y, equipped with the operator norm

$$\|A\| = \sup_{x\in X,\, x\neq 0} \frac{\|Ax\|_Y}{\|x\|_X}. \tag{2.6}$$

Theorem 2.10 (Space of bounded operators)**.** The space $\mathcal{B}(X,Y)$ of bounded operators, equipped with the operator norm (2.6) is a normed linear space.

Proof.

- $\mathcal{B}(X,Y)$ is a vector space.

  Let $A, B \in \mathcal{B}(X,Y)$, and α, β be scalars. Then, for every $x \in X$,

  $$\|(\alpha A + \beta B)(x)\|_Y = \|\alpha Ax + \beta Bx\|_Y \leq (|\alpha|\,\|A\| + |\beta|\,\|B\|)\,\|x\|_X\,.$$

  Thus $\alpha A + \beta B \in \mathcal{B}(X,Y)$.

- The operator norm (2.6) is a norm.

  Let $A, B \in \mathcal{B}(X,Y)$ be bounded operators. Then

  (i) $\|A\| \geq 0$;

  (ii)
  $$\|A\| = \sup_{x\neq 0} \frac{\|Ax\|_Y}{\|x\|_X} = 0 \quad \text{iff} \quad \|Ax\|_Y = 0 \text{ for all } x \in X \quad \text{iff} \quad A = 0;$$

  (iii) for every scalar α,
  $$\|\alpha A\| = \sup_{\|x\|_X=1} \|\alpha Ax\|_Y = |\alpha| \sup_{\|x\|_X=1} \|Ax\|_Y = |\alpha|\|A\|;$$

  (iv)
  $$\|A+B\| = \sup_{\|x\|_X=1} \|(A+B)x\|_Y \leq \sup_{\|x\|_X=1} (\|Ax\|_Y + \|Bx\|_Y) \leq \|A\| + \|B\|\,.$$

□

Theorem 2.11 (Completeness of the space of bounded operators)**.** If Y is a Banach space, then $\mathcal{B}(X,Y)$ is a Banach space.

Proof. Let $(A_n)_{n\geq 1}$ be a Cauchy sequence in the operator norm. Fix $\varepsilon > 0$. Then there is a $N > 0$ such that for all $n, m \geq N$ one has $\|A_n - A_m\| < \varepsilon$.

Thus, for every $x \in X$,

$$\|A_n x - A_m x\|_Y \leq \|A_n - A_m\|\ \|x\|_X < \varepsilon\,\|x\|_X\,, \tag{2.7}$$

so $(A_n x)$ is Cauchy in Y. Since Y is complete, we may define

$$Ax = \lim_{n\to\infty} A_n x.$$

Passing to the limit in the linearity identities for A_n shows that A is linear.

It remains to prove that i) $A \in \mathcal{B}(X,Y)$ and ii) convergence in norm.

i) For $n \geq N$ and for all $x \in X$ one has

$$\|A_n x - Ax\| = \left\| A_n x - \lim_m A_m x \right\| = \lim_m \|A_n x - A_m x\| \leq \varepsilon \|x\| \tag{2.8}$$

where we used the continuity of norm and (2.7). Thus, $A_N - A$ belongs to $\mathcal{B}(X,Y)$. Since $A_N \in \mathcal{B}(X,Y)$ and $\mathcal{B}(X,Y)$ is a vector space, then $A = A_N - (A_N - A) \in \mathcal{B}(X,Y)$.

ii) Convergence in operator norm follows from (2.8), which reads

$$\|A_n - A\| \leq \varepsilon, \quad \text{for } n \geq N, \tag{2.9}$$

that is $\|A_n - A\| \to 0$, as $n \to \infty$.

□

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