I.2 Metric and normed linear spaces

Throughout this work, we will be dealing with sets of functions or operators or other objects and we will often need a way of measuring the distance between the objects in the sets. It is reasonable to define a notion of distance that has the most important properties of ordinary distance in $\mathbb{R}^3$.

Definition A **metric space** is a set M and a real-valued function $d(\cdot, \cdot)$ on $M \times M$ which satisfies:

(i) $d(x, y) \geq 0$
(ii) $d(x, y) = 0$ if and only if $x = y$
(iii) $d(x, y) = d(y, x)$
(iv) $d(x, z) \leq d(x, y) + d(y, z)$ [triangle inequality]

The function d is called a **metric** on M.

We often call the elements of a metric space points. Notice that a metric space is a set M together with a metric function d; in general, a given set X can be made into a metric space in different ways by employing different metric functions. When it is not clear from the context which metric we are talking about, we will denote the metric space by $\langle M, d\rangle$, so that the metric is explicitly displayed.

Example 1 Let $M = \mathbb{R}^n$ with the distance between two points $x = \langle x_1, \ldots, x_n\rangle$ and $y = \langle y_1, \ldots, y_n\rangle$ given by

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$$

Example 2 Let M be the unit circle in $\mathbb{R}^2$, that is, the set of all pairs of real numbers $\langle \alpha, \beta\rangle$ with $\alpha^2 + \beta^2 = 1$, and let

$$d_1(\langle \alpha, \beta\rangle, \langle \alpha', \beta'\rangle) = \sqrt{(\alpha - \alpha')^2 + (\beta - \beta')^2}$$

Another possible metric is $d_2[p, p'] =$ arc length between the points p, p' (see Figure I.1).

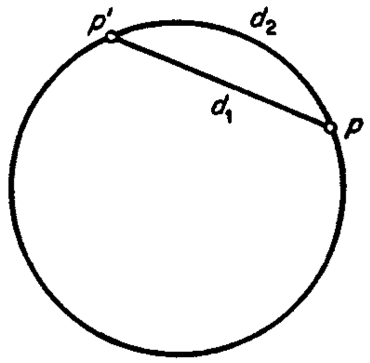

FIGURE I.1 The metrics d_1 and d_2.

Example 3 Let $M = C[0, 1]$, the continuous real-valued functions on $[0, 1]$ with either of the metrics

$$d_1(f, g) = \max_{x \in [0,1]} |f(x) - g(x)| \qquad d_2(f, g) = \int_0^1 |f(x) - g(x)|\, dx$$

Now that we have a notion of distance, we can say what we mean by convergence.

Definition A sequence of elements $\{x_n\}_{n=1}^{\infty}$ of a metric space $\langle M, d\rangle$ is said to **converge** to an element $x \in M$, if $d(x, x_n) \to 0$ as $n \to \infty$. We will often denote this by $x_n \xrightarrow{d} x$ or $\lim_{n\to\infty} x_n = x$. If x_n does not converge to x, we will write $x_n \overset{d}{\not\longrightarrow} x$.

In Example 2, $d_1(p, p') \le d_2(p, p') \le \pi d_1(p, p')$ which we will write $d_1 \le d_2 \le \pi d_1$. Thus $p_n \xrightarrow{d_1} p$ if and only if $p_n \xrightarrow{d_2} p$. But in Example 3, the metrics induce distinct notions of convergence. Since $d_2 \le d_1$, $f_n \xrightarrow{d_1} f$ implies $f_n \xrightarrow{d_2} f$, but the converse is false. A counterexample is given by the functions g_n defined in Figure I.2, which converge to the zero function in the metric d_2

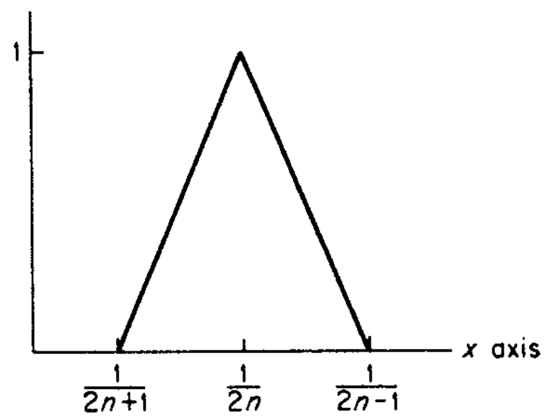

FIGURE I.2 The graph of $g_n(x)$.

but which do not converge in the metric d_1. This may be seen by introducing the important notion of Cauchy sequence.

Definition A sequence of elements $\{x_n\}$ of a metric space $\langle M, d\rangle$ is called a **Cauchy sequence** if $(\forall \varepsilon > 0)(\exists N)$ $n, m \ge N$ implies $d(x_n, x_m) < \varepsilon$.

Proposition Any convergent sequence is Cauchy.

Proof Given $x_n \to x$ and ε, find N so $n \ge N$ implies $d(x_n, x) < \varepsilon/2$. Then $n, m \ge N$ implies $d(x_n, x_m) \le d(x_n, x) + d(x, x_m) < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon$. ▮

We now return to the functions in Figure I.2. It is easy to see that if $n \neq m$, $d_1(g_n, g_m) = 1$. Thus g_n is not a Cauchy sequence in $\langle C[0, 1], d_1\rangle$ and therefore not a convergent sequence. Thus, the sequence $\{g_n\}$ converges in $\langle C[0, 1], d_2\rangle$ but not in $\langle C[0, 1], d_1\rangle$.

Although every convergent sequence is a Cauchy sequence, the following example shows that the converse need not be true. Let $\mathbb{Q}$ be the rational numbers with the usual metric (that is, $d(x, y) = |x - y|$) and let x^* be any irrational number (that is, $x^* \in \mathbb{R}\backslash\mathbb{Q}$). Find a sequence of rationals x_n with $x_n \to x^*$ in $\mathbb{R}$. Then x_n is a Cauchy sequence of numbers in $\mathbb{Q}$, but it cannot converge in $\mathbb{Q}$ to some $y \in \mathbb{Q}$ (for, if $x_n \to y$ in $\mathbb{Q}$, then $x_n \to y$ in $\mathbb{R}$, so we would have $y = x^*$).

Definition A metric space in which all Cauchy sequences converge is called **complete.**

For example, $\mathbb{R}$ is complete, but $\mathbb{Q}$ is not. It can be shown (Sections I.3 and I.5) that $\langle C[0, 1], d_1\rangle$ is complete but $\langle C[0, 1], d_2\rangle$ is not. The example of $\mathbb{Q}$ and $\mathbb{R}$ suggests what we need to do to an incomplete space X to make it complete. We need to enlarge X by adding "all possible limits of Cauchy sequences." The original space X should be dense in the larger space $\tilde{X}$ where:

Definition A set B in a metric space M is called **dense** if every $m \in M$ is a limit of elements in B.

Of course, if the incomplete space is not already contained in a larger complete space (like $\mathbb{Q}$ is contained in $\mathbb{R}$) it is not clear what "all possible limits" means. That this "completion" can be done is the content of a theorem that we shall shortly state; but first some definitions:

Definition A function f from a metric space $\langle X, d\rangle$ to a metric space $\langle Y, \rho\rangle$ is called **continuous at x** if $f(x_n) \xrightarrow{\langle Y, \rho\rangle} f(x)$ whenever $x_n \xrightarrow{\langle X, d\rangle} x$.

We have already had an example of a sequence of elements in $C[0, 1]$ with $f_n \xrightarrow{d_2} 0$ but $f_n \stackrel{d_1}{\not\to} 0$. Thus the identity function from $\langle C[0, 1], d_2\rangle$ to $\langle C[0, 1], d_1\rangle$ is *not* continuous but the identity from $\langle C[0, 1], d_1\rangle$ to $\langle C[0, 1], d_2\rangle$ is continuous.

Definition A bijection h from $\langle X, d\rangle$ to $\langle Y, \rho\rangle$ which preserves the metric, that is,

$$\rho(h(x), h(y)) = d(x, y)$$

is called an **isometry**. It is automatically continuous. $\langle X, d\rangle$ and $\langle Y, \rho\rangle$ are said to be **isometric** if such an isometry exists.

Isometric spaces are essentially identical as metric spaces; a theorem concerning only the metric structure of $\langle X, d\rangle$ will hold in all spaces isometric to it.

We now state precisely in which sense an incomplete space can be fattened out to be complete:

Theorem I.3 If $\langle M, d\rangle$ is an incomplete metric space, it is possible to find a complete metric space $\tilde{M}$ so that M is isometric to a dense subset of $\tilde{M}$.

Sketch of proof Consider the Cauchy sequences $\{x_n\}$ of elements of M. Call two sequences, $\{x_n\}$, $\{y_m\}$, equivalent if $\lim_{n\to\infty} d(x_n, y_n) = 0$. Let $\tilde{M}$ be the family of equivalence classes of Cauchy sequences under this equivalence relation. One can show that for any two Cauchy sequences $\lim_{n\to\infty} d(x_n, y_n)$ exists and depends only on the equivalence classes of $\{x_n\}$ and $\{y_n\}$. This limit defines a metric on $\tilde{M}$ and $\tilde{M}$ is complete. Finally, map M into $\tilde{M}$ by taking x into the constant sequence in which each x_n equals x. M is dense in $\tilde{M}$ and the map is isometric. ∎

To complete our discussion of metric spaces, we want to introduce the notions of open and closed sets. The reader should keep the example of open and closed sets on the real line in mind.

Definition Let $\langle X, d\rangle$ be a metric space:

(a) The set $\{x \mid x \in X, d(x, y) < r\}$ is called the **open ball**, $B(y; r)$, of radius r about the point y.

(b) A set $O \subset X$ is called **open** if $(\forall y \in O)(\exists r > 0)\ B(y; r) \subset O$.

(c) A set $N \subset X$ is called a **neighborhood** of $y \in N$ if $B(y; r) \subset N$ for some $r > 0$.

(d) Let $E \subset X$. A point x is called a **limit point** of E, if $(\forall r > 0)$ $B(x; r) \cap (E\backslash\{x\}) \neq \varnothing$, that is, x is a limit point of E if E contains points *other than* x arbitrarily near x.

(e) A set $F \subset X$ is called **closed** if F contains all its limit points.

(f) If $G \subset X$, $x \in G$ is called an **interior point** of G, if G is a neighborhood of x.

The reader can prove for himself the following collection of elementary statements:

Theorem I.4 Let $\langle X, d\rangle$ be a metric space:

(a) A set, O, is open if and only if $X\backslash O$ is closed.
(b) $x_m \xrightarrow{d} x$ if and only if for each neighborhood N of x, there exists an M so that $m \geq M$ implies $x_m \in N$.
(c) The set of interior points of a set is open.
(d) The union of a set E with its limit points is a closed set (denoted by $\bar{E}$ and called the **closure** of E).
(e) A set is open if and only if it is a neighborhood of each of its points.

One of the main uses of open sets is to check for convergence using Theorem I.4.b and in particular to check for continuity via the following criteria, the proof of which we leave as an exercise:

Theorem I.5 A function $f(\cdot)$ from a metric space X to another space Y is continuous if and only if for all open sets $O \subset Y$, $f^{-1}[O]$ is open.

Finally, we warn the reader that often in incomplete metric spaces, closed sets may not appear to be closed at first glance. For example, $[\frac{1}{2}, 1)$ is closed in $(0, 1)$ (with the usual metric).

We complete this section with a discussion of two of the central concepts of functional analysis: normed linear spaces and bounded linear transformations.

Definition A **normed linear space** is a vector space, V, over $\mathbb{R}$ (or $\mathbb{C}$) and a function, $\|\cdot\|$ from V to $\mathbb{R}$ which satisfies:

(i) $\|v\| \geq 0$ for all v in V
(ii) $\|v\| = 0$ if and only if $v = 0$
(iii) $\|\alpha v\| = |\alpha|\ \|v\|$ for all v in V and α in $\mathbb{R}$ (or $\mathbb{C}$)
(iv) $\|v + w\| \leq \|v\| + \|w\|$ for all v and w in V

Definition A **bonnded·linear transformation** (or bounded operator) from a normed linear space $\langle V_1, \|\ \|_1\rangle$ to a normed linear space $\langle V_2, \|\ \|_2\rangle$ is a function, T, from V_1 to V_2 which satisfies:

(i) $T(\alpha v + \beta w) = \alpha T(v) + \beta T(w) \quad (\forall v, w \in V)(\forall \alpha, \beta \in \mathbb{R}$ or $\mathbb{C})$
(ii) For some $C \geq 0$, $\|Tv\|_2 \leq C\|v\|_1$

The smallest such C is called the **norm of** T, written $\|T\|$ or $\|T\|_{1,2}$. Thus

$$\|T\| = \sup_{\|v\|_1 = 1} \|Tv\|_2$$

Since we will study these concepts in detail later, we will not give many examples now but merely note that $\mathbb{R}^n$ with the norm

$$\|\langle x_1, \ldots, x_n \rangle\| = \sqrt{|x_1|^2 + \cdots + |x_n|^2}$$

and $C[0, 1]$ with either the norm

$$\|f\|_\infty = \sup_{x \in [0,1]} |f(x)| \qquad \text{or} \qquad \|f\|_1 = \int_0^1 |f(x)|\, dx$$

are normed linear spaces. Observe also that any normed linear space $\langle V, \|\cdot\| \rangle$ is a metric space when given the distance function $d(v, w) = \|v - w\|$. There is thus a notion of continuity of functions, and for linear functions this is precisely captured by bounded linear transformations. The proof of this fact is left to the reader.

Theorem I.6 Let T be a linear transformation between two normed linear spaces. The following are equivalent:

(a) T is continuous at one point.
(b) T is continuous at all points.
(c) T is bounded.

Definition We say $\langle V, \|\cdot\| \rangle$ is **complete** if it is complete as a metric space in the induced metric.

If $\langle X, \|\cdot\| \rangle$ is a normed linear space, then X has a completion as a metric space by Theorem I.3. Using the fact that X is dense in $\tilde{X}$, it is easy to see that $\tilde{X}$ can be made into a normed linear space in exactly one natural way. All these concepts are well illustrated by the following important theorem and its proof:

Theorem I.7 (the B.L.T. theorem) Suppose T is a bounded linear transformation from a normed linear space $\langle V_1, \|\cdot\|_1 \rangle$ to a complete normed linear space $\langle V_2, \|\cdot\|_2 \rangle$. Then T can be uniquely extended to a bounded linear transformation (with the same bound), $\tilde{T}$, from the completion of V_1 to $\langle V_2, \|\cdot\|_2 \rangle$.

Proof Let $\tilde{V}_1$ be the completion of V_1. For each x in $\tilde{V}_1$, there is a sequence of elements $\{x_n\}$ in V_1 with $x_n \to x$ as $n \to \infty$. Since x_n converges, it is Cauchy, so given ε, we can find N so that $n, m > N$ implies $\|x_n - x_m\|_1 \leq \varepsilon / \|T\|$. Then $\|Tx_n - Tx_m\|_2 = \|T(x_n - x_m)\|_2 \leq \|T\| \, \|x_n - x_m\|_1 \leq \varepsilon$ which proves that Tx_n is a Cauchy sequence in V_2. Since V_2 is complete, $Tx_n \to y$ for some y. Set $\tilde{T}x = y$. We must first show that this definition is independent of the sequence $x_n \to x$ chosen. If $x_n \to x$ and $x_n' \to x$, then the sequence $x_1, x_1', x_2, x_2', \ldots \to x$ so $Tx_1, Tx_1', \ldots \to \hat{y}$ for some $\hat{y}$ by the above argument. Thus $\lim Tx_n' = \hat{y} = \lim Tx_n$. Moreover, we can show $\tilde{T}$ so defined is bounded because

$$\begin{aligned} \|\tilde{T}x\|_2 &= \lim_{n\to\infty} \|Tx_n\|_2 && \text{(see Problem 8)} \\ &\leq \overline{\lim_{n\to\infty}} \, C\|x_n\|_1 && \text{(see Appendix to I.2)} \\ &= C\|x\|_1 \end{aligned}$$

Thus $\tilde{T}$ is bounded. The proofs of linearity and uniqueness are left to the reader. ∎

Example 4 (the bounded operators) In Section I.3 we defined the concept of a bounded linear transformation or bounded operator from one normed linear space, X, to another Y; we will denote the set of all bounded linear operators from X to Y by $\mathscr{L}(X, Y)$. We can introduce a norm on $\mathscr{L}(X, Y)$ by defining

$$\|A\| = \sup_{x \in X,\, x \neq 0} \frac{\|Ax\|_Y}{\|x\|_X}$$

This norm is often called the **operator norm.**

Theorem III.2 If Y is complete, $\mathscr{L}(X, Y)$ is a Banach space.

Proof Since any finite linear combination of bounded operators is again a bounded operator, $\mathscr{L}(X, Y)$ is a vector space. It is easy to see that $\|\cdot\|$ is a norm; for example, the triangle inequality is proven by the computation

$$\begin{aligned}\|A + B\| &= \sup_{x\neq 0} \frac{\|(A + B)x\|}{\|x\|} \leq \sup_{x\neq 0} \frac{\|Ax\| + \|Bx\|}{\|x\|} \\ &\leq \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|} + \sup_{x\neq 0} \frac{\|Bx\|}{\|x\|} \\ &= \|A\| + \|B\|\end{aligned}$$

To show that $\mathscr{L}(X, Y)$ is complete, we must prove that if $\{A_n\}_{n=1}^{\infty}$ is a Cauchy sequence in the operator norm, then there is a bounded linear operator A so that $\|A_n - A\| \to 0$. Let $\{A_n\}_{n=1}^{\infty}$ be Cauchy in the operator norm; we construct A as follows. For each $x \in X$, $\{A_n x\}_{n=1}^{\infty}$ is a Cauchy sequence in Y. Since Y is complete, $A_n x$ converges to an element $y \in Y$. Define $Ax = y$. It is easy to check that A is a linear operator. From the triangle inequality it follows that

$$|\,\|A_n\| - \|A_m\|\,| \leq \|A_n - A_m\|$$

so $\{\|A_n\|\}_{n=1}^{\infty}$ is a Cauchy sequence of real numbers converging to some real number C. Thus,

$$\begin{aligned}\|Ax\|_Y &= \lim_{n\to\infty} \|A_n x\|_Y \leq \lim_{n\to\infty} \|A_n\|\,\|x\|_X \\ &= C\|x\|_X\end{aligned}$$

so A is a bounded linear operator. We must still show that $A_n \to A$ in the operator norm. Since $\|(A - A_n)x\| = \lim_{m\to\infty} \|(A_m - A_n)x\|$, we have

$$\frac{\|(A - A_n)x\|}{\|x\|} \leq \lim_{n\to\infty} \|A_m - A_n\|$$

which implies

$$\|A - A_n\| = \sup_{x\neq 0} \frac{\|(A - A_n)x\|}{\|x\|} \leq \lim_{m\to\infty} \|A_m - A_n\|$$

which is arbitrarily small for n large enough. The triangle inequality shows that the norm of A is actually equal to C. ∎