The Two-Layer Ising Film Correlation Length and Critical Curve by CLE and CVM Methods (Physica A 219 (1995) 447-466) ⋆

L. Angelini [1], D. Caroppo [2], M. Pellicoro [3], M. Villani [4]

Dipartimento di Fisica dell'Università di Bari, 70126 Bari, Italy
and
Istituto Nazionale di Fisica Nucleare, Sezione di Bari, 70126 Bari, Italy

Abstract

On the basis of an improved elaboration of the method already proposed by the present authors, the correlation length along a vertical section of a two-layer ferromagnetic Ising film is calculated for all $T \geq T_c$. As a by-product the critical curve is determined and a very interesting factorization property, related to scaling limits, is pointed out. A comparison is made with previous results from other methods.

1 Introduction

This paper deals with a two-layer ferromagnetic film, made of two isotropic planes of Ising spins coupled by nearest neighbour interactions also along the vertical direction. The Hamiltonian of the model is given by

$$\begin{aligned} H = -J_1 \sum_{i,j} (s_{i,j}s_{i+1,j} + s_{i,j}s_{i,j+1} + u_{i,j}u_{i+1,j} + u_{i,j}u_{i,j+1}) \\ -J_2 \sum_{i,j} s_{i,j}u_{i,j} \qquad\qquad (J_i > 0, \quad s_{i,j}, u_{i,j} = \pm 1) \end{aligned} \tag{1}$$

⋆ PACS numbers: 05.50.+q, 02.50.+s
[1] E-mail: angelini@bari.infn.it
[2] E-mail: caroppo@bari.infn.it
[3] E-mail: pellicoro@bari.infn.it
[4] E-mail: villani@bari.infn.it

J_1 is the horizontal intralayer coupling, while J_2 is the inter-layer vertical coupling.

This quite interesting model is not exactly soluble. However it has been studied by several methods, such as series analysis of high-temperature expansions [1–3], perturbation theory and scaling laws [4], Monte Carlo simulations [5,6], constrained variational approach [7], mean field and interfacial approximations [8].

In this paper we report calculations concerning the correlation length of the model (1), in the whole region $T \geq T_{\rm c}$. The results are obtained through an improved elaboration of the approach already proposed by the present authors [7], which has been also applied successfully in an another context [9]. As a matter of fact, besides the specific properties of the above model, we are also interested in a further test of the powerfulness of our method.

The preliminary steps of the approach which we will follow for the two layer Ising film, can be described as follows. These are given for completeness, by adopting a general language. We consider a vertical section Σ_v (Figure 1a) of the film and introduce the reduced probability distribution (RPD) $P_1 = P_1(\sigma,\tau)$ of a spin configuration $(\sigma,\tau) = (s_1, s_2, \ldots, s_M; u_1, u_2, \ldots, u_M)$ on Σ_v, obtained by averaging over all the configurations of the other sections. We will be interested in the limit $M \to \infty$. We also consider the RPD $P_2 = P_2(\sigma,\tau;\sigma',\tau')$ on two adjacent sections Σ_v and Σ'_v (Figure 1b), again by averaging over the configurations of the other sections. In the same way we introduce the RPD's $P_3, P_4, \ldots,$ for strips made of three adjacent sections $\Sigma_v, \Sigma'_v, \Sigma''_v$, four adjacent sections, and so on.

We now consider the situation where at the left of Σ_v and at the right of Σ_v, or Σ'_v, or $\Sigma''_v, \ldots$, there are N vertical sections, with N very large. We are interested in the limit $N \to +\infty$. In such a situation the averaging process gives a unique function $\Phi(\sigma,\tau)$, independent of N, such that

$$
\begin{aligned}
P_1(\sigma,\tau) &= \frac{P_1^0(\sigma,\tau)\ \Phi^2(\sigma,\tau)}{Z_1}, \\
P_2(\sigma,\tau;\sigma',\tau') &= \frac{\Phi(\sigma,\tau)\ P_2^0(\sigma,\tau;\sigma',\tau')\ \Phi(\sigma',\tau')}{Z_2}, \\
P_3(\sigma,\tau;\sigma',\tau';\sigma'',\tau'') &= \frac{\Phi(\sigma,\tau)\ P_3^0(\sigma,\tau;\sigma',\tau';\sigma'',\tau'')\ \Phi(\sigma'',\tau'')}{Z_3}, \\
\ldots &= \ldots\ldots
\end{aligned}
\tag{2}
$$

Each P_n^0 describes an Ising film, infinite in one direction (when $M \to \infty$), but of finite width n, with free boundary conditions. The Z_n's are constants fixed by $\mathbf{Tr} P_n = 1$ $(n = 1, 2, \ldots)$.

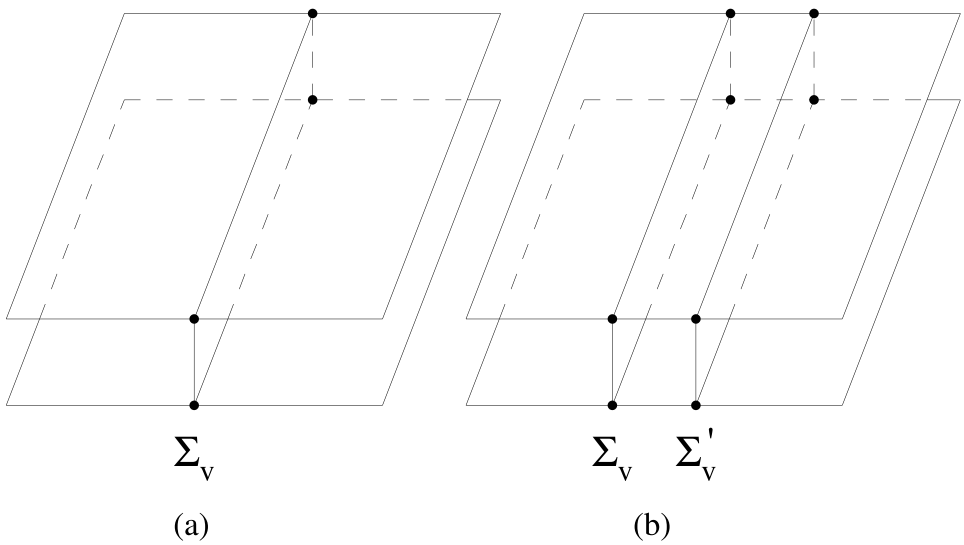

Fig. 1. A vertical section Σ_v (a) and two adjacent vertical sections Σ_v and Σ'_v (b) of the film.

The P_n's provide exact informations on the model. The ratios

$$\frac{Z_2}{Z_1} = \frac{Z_3}{Z_2} = \ldots = \frac{Z_{n+1}}{Z_n} \tag{3}$$

are related to the free energy. However it is very difficult (perhaps impossible) to obtain them. In principle $\Phi(\sigma, \tau)$ is determined by the marginality or consistency condition

$$\sum_{(\sigma',\tau')} P_2(\sigma, \tau; \sigma', \tau') = P_1(\sigma, \tau), \tag{4}$$

which implies (and is also implied by) the other marginality conditions

$$\sum_{(\sigma'',\tau'')} P_3(\sigma, \tau; \sigma', \tau'; \sigma'', \tau'') = P_2(\sigma, \tau; \sigma', \tau'), \tag{5}$$

$$\ldots = \ldots$$

The aim of the presentation outlined above is to provide a framework for a reliable approximate study of our model. In order to have a problem that can be dealt with, we introduce approximate descriptions $\widetilde{\Phi}(\sigma, \tau)$ of $\Phi(\sigma, \tau)$, given in terms of a finite number of parameters. On the basis of (2), we are led to consider the probability distributions $\widetilde{P}_n$, with

$$\begin{aligned}
\widetilde{P}_1(\sigma,\tau) &= \frac{P_1^0(\sigma,\tau)\ \widetilde{\Phi}^2(\sigma,\tau)}{\widetilde{Z}_1}, \\
\widetilde{P}_2(\sigma,\tau;\sigma',\tau') &= \frac{\widetilde{\Phi}(\sigma,\tau)\ P_2^0(\sigma,\tau;\sigma',\tau')\ \widetilde{\Phi}(\sigma',\tau')}{\widetilde{Z}_2}, \\
\ldots &= \ldots
\end{aligned} \tag{6}$$

Then we are faced with the choice of the marginality conditions to be imposed on the distributions $\widetilde{P}_n$, in place of the full conditions (4), (5).

Our approach, as proposed in the previous works [7,9,10] is characterized by the basic requirement of the correlation length equality (CLE) for the distributions $\widetilde{P}_n$. The correlation length along the direction of the vertical sections is the fundamental quantity which is calculated. The CLE and the related constrained variational method (CVM) are discussed in section 2, in the framework of a very simple ansatz $\widetilde{\Phi}_{\mathrm{I}}(\sigma,\tau)$, from which some preliminary results are deduced. In section 3 we introduce an alternative, and in some respect improved ansatz $\widetilde{\Phi}_{\mathrm{II}}(\sigma,\tau)$, fixing the attention to a greater extent on the critical region. Through $\widetilde{\Phi}_{\mathrm{II}}(\sigma,\tau)$ it is also possible in part to avoid the partial exploitation of the variational method and to consider the double correlation length equality (DCLE), involving the CLE for three distributions. Very accurate results for the correlation length in the whole range $T \geq T_{\mathrm{c}}$ and for the critical curve are obtained. A very interesting factorization property is pointed out. In the last section 4 we conclude with some comments about the model and the approach which we have followed.

2 The CLE condition, the constrained variational method and preliminary results by a simple ansatz

By construction $\Phi(\sigma,\tau)$ is strictly positive, so it can be written in terms of an effective "boundary" hamiltonian $h(\sigma,\tau)$ [7],

$$\Phi(\sigma,\tau) = \exp[h(\sigma,\tau)], \tag{7}$$

with

$$h(\sigma,\tau) = A_1 \sum_i (s_i s_{i+1} + u_i u_{i+1}) + A_2 \sum_i s_i u_i + A_3 \sum_i (s_i u_{i+1} + u_i s_{i+1})$$

$$+ A_4 \sum_i s_i u_i s_{i+1} u_{i+1} + \ldots \tag{8}$$

There is a very large number (infinite when $M \to +\infty$) of coupling parameters A_i, related to all the possible clusters of spins on Σ_v. In principle the A_i's are determined by (4) or (5), or equivalently, by the requirement that all the correlation functions on Σ_v be the same for P_n and P_{n+1} ($n = 1, 2, \ldots$). A compact form of the set of equations for $\{A_i\}$, is given by

$$\frac{\partial}{\partial A_i} \frac{Z_{n+1}}{Z_n} = 0 \quad (i = 1, 2, \ldots). \tag{9}$$

The link between consistency conditions and the stationariness of the free energy, has been already discussed in the context of the cluster variation method [11].

One formal structure for an approximate description $\widetilde{\Phi}(\sigma, \tau)$ of $\Phi(\sigma, \tau)$ can be obtained by truncating the expansion (8). Within this procedure, the simplest function, suggested by our Hamiltonian (1), is given by

$$\widetilde{\Phi}_{\mathrm{I}}(\sigma, \tau) = \exp \left[A \sum_{i=1}^{M} (s_i s_{i+1} + u_i u_{i+1}) + B \sum_{i=1}^{M} s_i u_i \right], \tag{10}$$

where we take

$$A, B > 0. \tag{11}$$

The effect of averaging is described through effective horizontal and vertical couplings. An alternative procedure will be considered in the next section. In this section we will fix our attention on (10).

The eq.(3) is no longer satisfied by the $\widetilde{Z}_n$'s. Therefore the determination of A and B depends on the pair of distributions which we consider. We start from the simplest ones, that is $\widetilde{P}_1$ and $\widetilde{P}_2$. Then, on the basis of (9), we would obtain the two equations for A and B

$$\frac{\partial}{\partial A} \frac{\widetilde{Z}_2}{\widetilde{Z}_1} = 0 \, , \quad \frac{\partial}{\partial B} \frac{\widetilde{Z}_2}{\widetilde{Z}_1} = 0, \tag{12}$$

which are equivalent to the requirement of the equality for $\widetilde{P}_1$ and $\widetilde{P}_2$ of the short distances correlation functions $\langle s_i s_{i+1} \rangle$, $\langle u_i u_{i+1} \rangle = \langle s_i s_{i+1} \rangle$ and $\langle s_i u_i \rangle$

on Σ_v. However, we have many different marginality conditions which can be imposed and these are faced with only two parameters A and B.

Of course, having only two couplings, $\widetilde{\Phi}_{\rm I}(\sigma,\tau)$ cannot be a globally correct description of the averaging process. However an actual proper choice of A and B, can give a reliable *local* description, in connection with a specific property of the model. Even properties of different nature could be properly obtained from the formal structure (10) of $\widetilde{\Phi}_{\rm I}$, by changing the numerical values of A and B. So, before fixing A and B, we have to decide what quantity we wish to calculate. If the choice is a quantity α, related to some expectation values, then a minimal requirement of consistency demands these expectation values to be the same for $\widetilde{P}_1$ and $\widetilde{P}_2$, with the result of a common answer for α. In this way A and B are adapted to the property α. If the free energy F is chosen, then (12) is the proper condition. This is the content of the standard variational method, where, in a strict sense, A and B are given by the point where $\sup(\widetilde{Z}_2/\widetilde{Z}_1)$ is attained. However we are more interested in the second derivatives of F, which, unlike eq.(12), are related to long distances behaviour. In the marginality conditions, it would be better to have at least quantity which has a direct physical relevance and does not require any further operation.

The main advantage of considering strips infinite in one direction, which is the starting point of our approach, is that one can study correlations even at large distances in this direction. So we can consider a quantity which has an important direct relevance, that is the correlation length ξ along the vertical sections Σ_v, Σ'_v, ..., of our film. Then, due to the particular role of this quantity, we impose, as a basic consistency condition, the equality of the correlation length (CLE) for $\widetilde{P}_1$ and $\widetilde{P}_2$ [7,9,10]. As a matter of fact, this equality involves a large number of expectation values, that is the two-point correlation functions at various large distances.

Suppose that the CLE has been solved with respect to the parameter A, giving a function $A(B)$. We need a futher independent condition in order to fix B. Even if we are not interested in the free energy, we can impose, as a stability requirement, the equation

$$\frac{\mathrm{d}}{\mathrm{d}B}\frac{\widetilde{Z}_2(B)}{\widetilde{Z}_1(B)} = 0, \tag{13}$$

(or consider $\sup_{B\geq 0}\widetilde{Z}_2/\widetilde{Z}_1$) which is the content of the constrained variational method (CVM).

We call $\xi_{\rm I}^{(1,2)}$ the common correlation length of $\widetilde{P}_1$ and $\widetilde{P}_2$ obtained through the CVM, by making use of the ansatz (10); $\xi_{\rm I}^{(1,2)}$ is an estimate of ξ. The other distributions $\widetilde{P}_3, \widetilde{P}_4, \ldots$ have the role to test the reliability of this estimate and to get further improvements [9]. We can limit ourselves to the pairs $\widetilde{P}_n, \widetilde{P}_{n+1}$. The next step would be to consider $\widetilde{P}_2$ and $\widetilde{P}_3$, to impose the CLE to these distributions and to fix B through

$$\frac{\mathrm{d}}{\mathrm{d}B}\frac{\widetilde{Z}_3(B)}{\widetilde{Z}_2(B)} = 0 \quad (\text{or} \sup_{B\geq 0} \frac{\widetilde{Z}_3}{\widetilde{Z}_2}) \tag{14}$$

We obtain a new determination of A and B, and a new estimate $\xi_{\mathrm{I}}^{(2,3)}$ of ξ, given by the common correlation length of $\widetilde{P}_2$ and $\widetilde{P}_3$. We call $\xi_{\mathrm{I}}^{(n,n+1)}$ the evaluation of ξ obtained by the CVM applied to $\widetilde{P}_n, \widetilde{P}_{n+1}$.

The distributions $\widetilde{P}_n$, can be obtained explicitly from (1) and(10), through

$$\begin{aligned} P_1^0(\sigma,\tau) &= \exp\left[K_1\sum_{i=1}^{M}(s_is_{i+1}+u_iu_{i+1})+K_2\sum_{i=1}^{M}s_iu_i\right] \\ P_2^0(\sigma,\tau;\sigma',\tau') &= P_1^0(\sigma,\tau)\ Q(\sigma,\tau;\sigma',\tau')\ P_1^0(\sigma',\tau') \\ P_3^0(\sigma,\tau;\sigma',\tau';\sigma'',\tau'') &= P_1^0(\sigma,\tau)\ Q(\sigma,\tau;\sigma',\tau')\ P_2^0(\sigma',\tau';\sigma'',\tau'') \\ \ldots &= \ldots \end{aligned} \tag{15}$$

where $K_1 = J_1/kT$, $K_2 = J_2/kT$ and

$$Q(\sigma,\tau;\sigma'\tau') \equiv \exp\left[K_1\sum_{i=1}^{M}(s_is_i'+u_iu_i')\right] \tag{16}$$

The calculation of the correlation length as a function of A and B, for each $\widetilde{P}_n$, can be performed by making use of the transfer matrix formalism. To $\widetilde{P}_n$ is associated a $4^n\times 4^n$ transfer matrix, which can also be reduced considerably. Of this matrix we have to calculate the largest and next-to-largest eigenvalues.

By considering $\widetilde{P}_1, \widetilde{P}_2, \widetilde{P}_3$ and $\widetilde{P}_4$, we have determined $\xi_{\mathrm{I}}^{(1,2)}$, $\xi_{\mathrm{I}}^{(2,3)}$ and $\xi_{\mathrm{I}}^{(3,4)}$. These correlation lengths are obtained as functions of the two parameters K and ρ, defined by

$$K = K_1 \quad , \quad \rho = \frac{K_2}{K_1} \tag{17}$$

In Figure 2, we give $1/\xi_{\mathrm{I}}^{(1,2)}(K,\rho)$ and $1/\xi_{\mathrm{I}}^{(2,3)}(K,\rho)$ as functions of K, for several values of ρ.

The points where

$$1/\xi_{\mathrm{I}}^{(n,n+1)}(K,\rho) = 0 \tag{18}$$

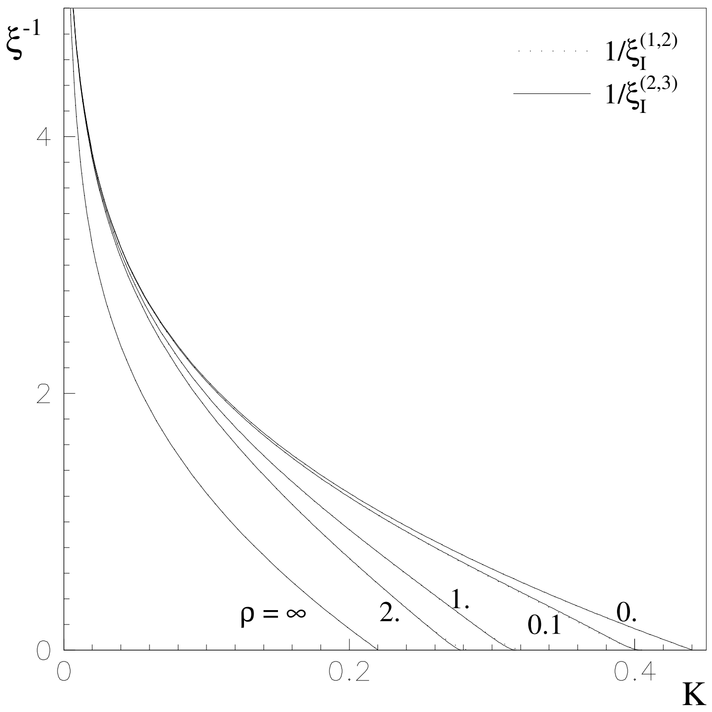

Fig. 2. The inverse correlation length ξ^{-1}, as a function of K, for several values of ρ, obtained by the CVM through $\widetilde{\Phi}_{\mathrm{I}}$.

allow to obtain an estimate $K_{\mathrm{Ic}}^{(n,n+1)}(\rho)$ of the critical curve $K_{\mathrm{c}}(\rho)$.

The function $\xi(K,\rho)$ is known exactly in the two cases $\rho = 0$ and $\rho = +\infty$. If we denote by $\xi_{\mathrm{ex}}(K)$ the correlation length along a row, obtained by Onsager for the isotropic square lattice Ising model with coupling $J = kTK$, we have $\xi(K,0) = \xi_{\mathrm{ex}}(K)$ and $\xi(K,+\infty) = \xi_{\mathrm{ex}}(2K)$. In these two limiting cases, our calculations reproduce the exact curves (and so the exact critical point). We have

$$\begin{aligned} \xi_{\mathrm{I}}^{(1,2)}(K,0) = \xi_{\mathrm{I}}^{(2,3)}(K,0) &= \xi_{\mathrm{ex}}(K) \\ \xi_{\mathrm{I}}^{(1,2)}(K,+\infty) = \xi_{\mathrm{I}}^{(2,3)}(K,+\infty) &= \xi_{\mathrm{ex}}(2K) \end{aligned} \tag{19}$$

in the whole intervals $0 \le K \le K_{\mathrm{c}}^{d=2} = K_{\mathrm{c}}(0)$, $0 \le K \le \frac{1}{2}K_{\mathrm{c}}^{d=2} = K_{\mathrm{c}}(+\infty)$ respectively, where $K_{\mathrm{c}}^{d=2} = \frac{1}{2}\ln(1+\sqrt{2}) \simeq 0.440687$, as is well known. The same property is shared by $\xi_{\mathrm{I}}^{(3,4)}(K,\rho)$.

For $0 < \rho < +\infty$ we obtain approximate, but quite accurate results for every K, except in the neighbourhood of the critical points, where we can detect some small differences between $1/\xi_{\mathrm{I}}^{(1,2)}(K,\rho)$ and $1/\xi_{\mathrm{I}}^{(2,3)}(K,\rho)$. At the scale of Figure 2, there is practically no difference between $\xi_{\mathrm{I}}^{(2,3)}(K,\rho)$ and $\xi_{\mathrm{I}}^{(3,4)}(K,\rho)$.

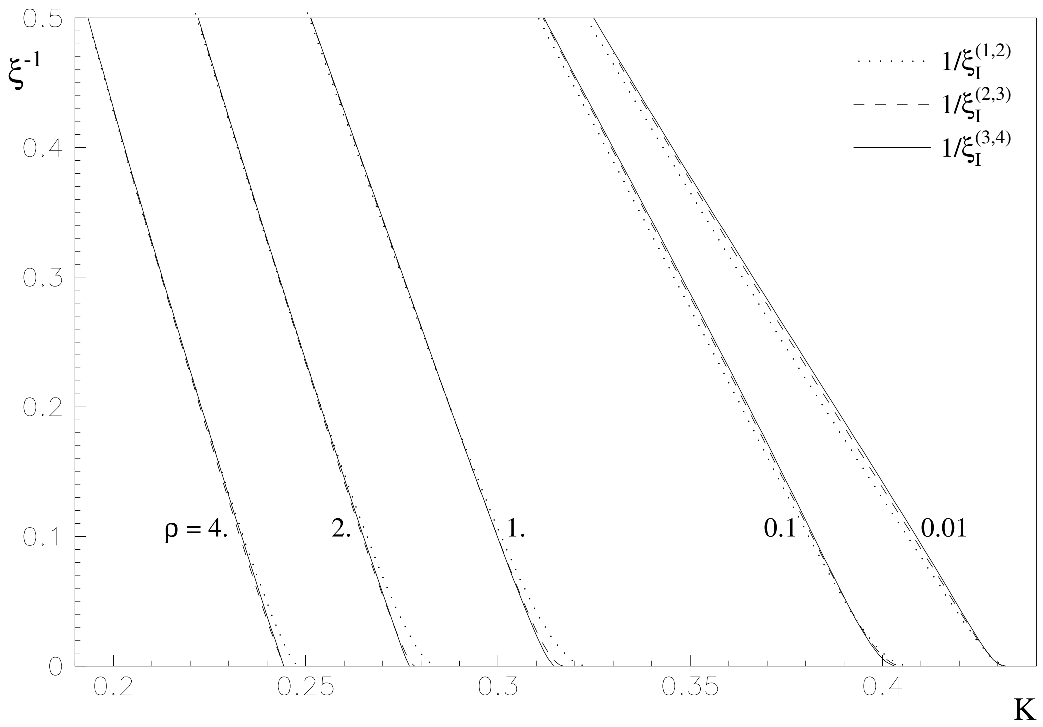

Fig. 3. The inverse correlation length ξ^{-1}, for several values of ρ, in the critical regions, obtained by the CVM through $\widetilde{\Phi}_{\mathrm{I}}$.

The sequence $1/\xi_{\mathrm{I}}^{(n,n+1)}$ $(n = 1, 2, 3)$ is reported in Figure 3, where the critical region has been amplified considerably in order to show the type of convergence of our estimates and the relevant critical features of the CVM within the ansatz (10).

Let us fix our attention on $\xi_{\mathrm{I}}^{(1,2)}(K, \rho)$ first. For every fixed ρ, we see from Figure 3 that there is always a significant interval $K_1(\rho) < K < K_2(\rho)$ where $1/\xi_{\mathrm{I}}^{(1,2)}$ is practically linear. This behaviour leads to predict that the value of the critical exponent $\nu = +1$, in agreement with universality and the value at $\rho = 0$ and $\rho = +\infty$. There is however a crossover phenomenon between $K_2(\rho)$ and $K_{\mathrm{Ic}}^{(1,2)}(\rho)$, which requires some remarks. Near the critical point there is a competition between two complementary behaviours, associated to the CLE and the variational method respectively. The final result depends on the type of ansatz. The variational method tends to give smaller correlation lengths, and within the ansatz (10), dominates very near the critical point, leading to the crossover shown in Figure 3. By making use of a diffuse terminology, we can say that, in the case of (10), the role of the variational method is relevant. In the next section, by introducing an alternative and more appropriate ansatz, we will show that the above crossover phenomenon disappears.

As a matter of fact, when we go to the higher approximations $1/\xi_{\mathrm{I}}^{(2,3)}$ and $1/\xi^{(3,4)}$ the crossover is reduced (in a drastic way for $\rho > 1$), confirming more and more that $\nu = 1$. At the same time we get more stability in our calculations.

It is interesting to see the values of $K_{\rm Ic}^{(1,2)}(\rho)$ obtainedby our first approximation. We have

$$K_{\rm Ic}^{(1,2)}(+1) = 0.32350\ , \quad K_{\rm Ic}^{(1,2)}(+2) = 0.28454\ ,$$

which are 3-4% higher than the correct estimates (see the next section). We also point out that at $\rho = +1$ and $\rho = +2$, $K_{\rm Ic}^{(3,4)}$ gives an error of 0.8% and 0.3%, respectively. For small and high values of ρ, we obtain

$$\begin{aligned} K_{\rm c}(0) - K_{\rm Ic}^{(1,2)}(\rho) &= 0.18352\ \rho^{2/3} + O(\rho) \\ K_{\rm Ic}^{(1,2)}(\rho) - K_{\rm c}(+\infty) &= 0.20711\ e^{-2\rho K_{\rm c}(+\infty)} + O(e^{-4\rho K_{\rm c}(+\infty)}) \end{aligned} \tag{20}$$

We find an estimate $\bar{\psi} = \frac{2}{3}$ for the shift exponent, while the expected value is $\frac{4}{7}$ [4].

The above results show that $K_{\rm Ic}^{(1,2)}(\rho)$ coincides essentially with the estimate of the critical points obtained by Lipowski and Suzuki ([8], and references therein), by their extension of the Muller-Hartmann–Zittartz method to the two-layer model. This method is based on an approximation of the interfacial tension and is designed only to the calculation of the critical point. Our approach allows to see the meaning of $K_{\rm Ic}^{(1,2)}(\rho)$ from another point of view; furthermore the origin of the slightly higher values can be explained on the basis of the crossover phenomenon for $1/\xi_{\rm I}^{(1,2)}(K, \rho)$.

The ansatz (10) is easy enough to perform calculations without much effort, and to illustrate in a simple way some basic aspects of our method. In the next section we will introduce an alternative and more appropriate description $\widetilde{\Phi}_{\rm II}(\sigma, \tau)$ of $\Phi(\sigma, \tau)$, and show how very accurate results for the correlation length over the whole range $K < K_{\rm c}(\rho)$ and the critical curve can be obtained.

3 The correlation length and critical curve by an alternative ansatz. The CVM method and the double CLE condition

Through (10), $h(\sigma, \tau)$ is described approximately by the effective "boundary" hamiltonian

$$\widetilde{h}_{\rm I}(\sigma, \tau) = A \sum_{i=1}^{M} (s_i s_{i+1} + u_i u_{i+1}) + B \sum_{i=1}^{M} s_i u_i \tag{21}$$

One possible further improvement in the description of the averaging process, is to consider again a truncation of the expansion (8), but adding to (21) a further coupling term, for example

$$C \sum_{i=1}^{M} (s_i u_{i+1} + u_i s_{i+1}).$$

We have then three free parameters to be handled. Proceeding in this way, we have the problem that some new parameters could not be significant, and, at the same time, the calculations become more involved. In [7] the above additional term was considered. However the analysis was carried out only in the framework of $\widetilde{P}_1$ and $\widetilde{P}_2$, by making use, besides of the CVM, of an independent analytic assumption which cannot be mantained when the higher order distributions are compared.

Another procedure is to avoid the truncation of (8), and to describe approximately all the couplings A_i through functions $\widetilde{A}_i$ depending on a fixed small number of free parameters. As a matter of fact, a way to realize this last procedure, is provided by the higher order distributions $\widetilde{P}_n$. Take for example $\widetilde{P}_3$ and $\widetilde{P}_4$, and let us start from $\widetilde{\Phi}_{\mathrm{I}}(\sigma,\tau)$. The problem of computing the correlation length and the free energy by $\widetilde{P}_3$ and $\widetilde{P}_4$, is equivalent to considering the two distributions $\widetilde{P}'_1(\sigma,\tau)$ and $\widetilde{P}'_2(\sigma,\tau;\sigma',\tau')$,

$$\begin{aligned} \widetilde{P}'_1(\sigma,\tau) &= \frac{\widetilde{P}^0_1(\sigma,\tau)\ [\widetilde{\Phi}'_{\mathrm{I}}(\sigma,\tau)]^2}{\widetilde{Z}'_1} \\ \widetilde{P}'_2(\sigma,\tau;\sigma',\tau') &= \frac{\widetilde{\Phi}'_{\mathrm{I}}(\sigma,\tau)\ \widetilde{P}^0_2(\sigma,\tau;\sigma',\tau')\ \widetilde{\Phi}'_{\mathrm{I}}(\sigma',\tau')}{\widetilde{Z}'_2} \end{aligned} \tag{22}$$

with the modified ansatz $\widetilde{\Phi}'_{\mathrm{I}}(\sigma,\tau)$ given by

$$\widetilde{\Phi}'_{\mathrm{I}}(\sigma,\tau) = \sum_{(\overline{\sigma},\overline{\tau})} Q(\sigma,\tau;\overline{\sigma},\overline{\tau})\ P^0_1(\overline{\sigma},\overline{\tau})\ \widetilde{\Phi}_{\mathrm{I}}(\overline{\sigma},\overline{\tau}) \tag{23}$$

The "boundary" hamiltonian $\widetilde{h}'_{\mathrm{I}}(\sigma,\tau)$ associated to $\widetilde{\Phi}'_{\mathrm{I}}(\sigma,\tau)$, contains the various couplings A_i in the form of functions $\widetilde{A}_i(A,B,K_1,K_2)$. The consideration of the higher order distributions is equivalent to further modifications of $\widetilde{\Phi}_{\mathrm{I}}(\sigma,\tau)$, obtained by the application to this function of powers of the transfer matrix

$$T = T(\sigma,\tau;\overline{\sigma},\overline{\tau}) = Q(\sigma,\tau;\overline{\sigma},\overline{\tau})\ P^0_1(\overline{\sigma},\overline{\tau}), \tag{24}$$

or, in symmetrized form,

$$L = L(\sigma,\tau;\overline{\sigma},\overline{\tau}) = [P_1^0(\sigma,\tau)]^{1/2}\ Q(\sigma,\tau;\overline{\sigma},\overline{\tau})\ [P_1^0(\overline{\sigma},\overline{\tau})]^{1/2}. \tag{25}$$

From a mathematical point of view, considering wider strips amounts to projecting $(P_1^0)^{1/2}\widetilde{\Phi}_{\mathrm{I}}$ more and more, by means of higher powers of L, on its principal eigenvector.

However the procedure of avoiding the truncation of (8), can be realized in a direct way, without the intermediate action of the transfer matrix. The problem is to give an approximate description not of the result of the averaging process, but of the process itself. The simplest ansatz which can be introduced according to this spirit, is given by

$$\widetilde{\Phi}_{\mathrm{II}}(\sigma,\tau) = \sum_{\{\mu_i\}} \exp\left[a\sum_{i=1}^{M}\mu_i\mu_{i+1} + b\sum_{i=1}^{M}\mu_i(s_i+u_i)\right] \tag{26}$$

in terms of two parameters a and b. The averaging process is described through the averaging on a random external field of Ising spins $\{\mu_i\}$, coupled to the spins of our vertical section. This coupling and the correlations in the field $\{\mu_i\}$ due to the parameter a, which we take positive, give rise to the various couplings $\widetilde{A}_i$ on Σ_v, functions of a and b. The sign of b is irrelevant; we fix the attention on $b > 0$.

Let us consider the CVM method in the framework of $\widetilde{\Phi}_{\mathrm{II}}(\sigma,\tau)$. We denote by $\widetilde{P}_n^{\mathrm{II}}$ the distributions (6) obtained through (26). We have considered explicitly $\widetilde{P}_1^{\mathrm{II}}, \widetilde{P}_2^{\mathrm{II}}$ and $\widetilde{P}_3^{\mathrm{II}}$. The calculation of the correlation lengths as functions of a and b, for each $\widetilde{P}_n^{\mathrm{II}}$, can be carried again by the transfer matrix technique. To $\widetilde{P}_n^{\mathrm{II}}$ we can associate a $4^{n+1}\times 4^{n+1}$ transfer matrix, so that we have to handle matrices of the same order as those considered for $\widetilde{P}_2, \widetilde{P}_3, \widetilde{P}_4$ within the old ansatz $\widetilde{\Phi}_{\mathrm{I}}(\sigma,\tau)$. We call $\xi_{\mathrm{II}}^{(n,n+1)}$ the estimate of ξ obtained by the CVM applied to $\widetilde{P}_n^{\mathrm{II}}, \widetilde{P}_{n+1}^{\mathrm{II}}$. In Figure 4 we give $\xi_{\mathrm{II}}^{(1,2)}(K,\rho)$ and $\xi_{\mathrm{II}}^{(2,3)}(K,\rho)$ for several values of ρ, in regions near the critical points. For smaller values of K, we have essential agreement with the previous calculations. The relevant feature of both $\xi_{\mathrm{II}}^{(1,2)}(K,\rho)$ and $\xi_{\mathrm{II}}^{(2,3)}(K,\rho)$ is that, for any ρ, the type of crossover phenomenon described in the previous section, is absent; both $\xi_{\mathrm{II}}^{(1,2)}(K,\rho)$ and $\xi_{\mathrm{II}}^{(2,3)}(K,\rho)$ give, the critical exponent $\nu = +1$, irrespective of ρ. One can say that, within (26), the role of the variational method is not relevant. As a matter of fact, the CVM, as we see from Figure 4, gives very satisfactory results for any $\rho \gtrsim 0.2$. The small difference between $\xi_{\mathrm{II}}^{(1,2)}(K,\rho)$ and $\xi_{\mathrm{II}}^{(2,3)}(K,\rho)$ indicates that already the first approximation $\xi_{\mathrm{II}}^{(1,2)}(K,\rho)$ is quite accurate when ρ is not small ($\rho \gtrsim 0.2$).

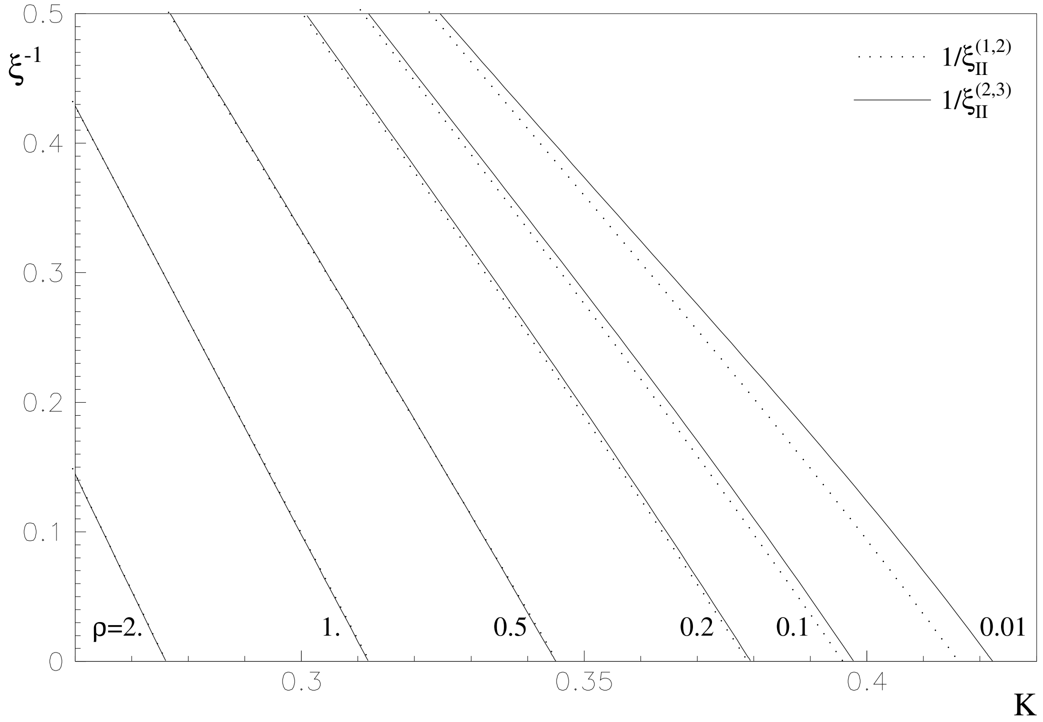

Fig. 4. The inverse correlation length ξ^{-1}, for several values of ρ, in the critical regions, obtained by the CVM through $\widetilde{\Phi}_{\mathrm{II}}$.

For small ρ, the difference between $\xi_{\mathrm{II}}^{(1,2)}(K,\rho)$ and $\xi_{\mathrm{II}}^{(2,3)}(K,\rho)$ in the critical region, becomes more and more significant. This signals that, as $\rho \to 0$, also the estimate $\xi_{\mathrm{II}}^{(2,3)}(K,\rho)$ must be improved. As a matter of fact, while for $\rho \to +\infty$ $\xi_{\mathrm{II}}^{(1,2)}(K,\rho)$ and $\xi_{\mathrm{II}}^{(2,3)}(K,\rho)$ give the exact answer, this is not true in the limit $\rho \to 0$, in contrast with what happens for the old $\xi_{\mathrm{I}}^{(n,n+1)}$. The problem is that, after the solution of the CLE with respect to the parameter a, the variational method leads to a determination $b(K,\rho)$ of b which remains finite when $\rho \to 0$. As a consequence, within the ansatz (26) the CVM does not describe correctly the decoupling of the two Ising planes in the limit $\rho \to 0$. In the framework of the CVM, we would need higher order distributions in order to get the proper behaviour. Alternatively, we have to renounce, for small ρ, to the contribution of the variational method.

It is useful to remark that the exact ξ is not only common to P_1 and P_2, or P_2 and P_3, but is the same for all the distributions P_n. So, if $\widetilde{\Phi}_{\mathrm{II}}$ is a more appropriate description of the averaging process and the quantity which we are interested in is the correlation length, we can require, as a more complete consistency condition, $\widetilde{P}_1^{\mathrm{II}}, \widetilde{P}_2^{\mathrm{II}}$, and $\widetilde{P}_3^{\mathrm{II}}$ to have the same correlation length. This requirement, which we call double CLE (DCLE), leads to two coupled equations involving a and b. This can be a stringent condition, so that it may have no solution. In fact we have difficulties with the DCLE, if we make use

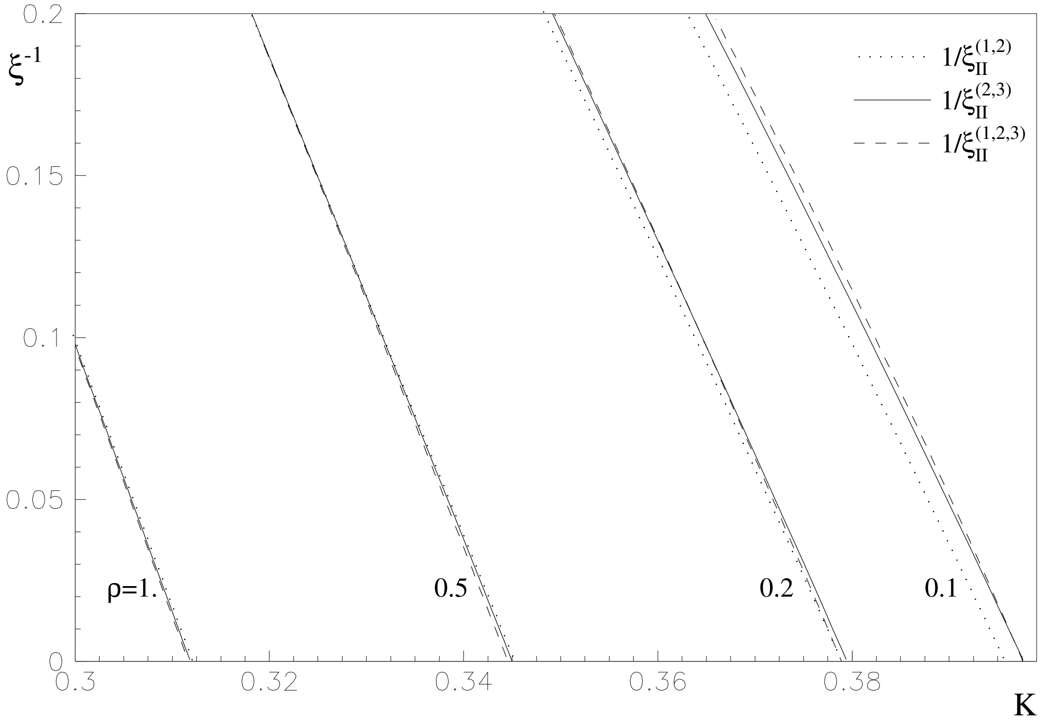

Fig. 5. The inverse correlation length ξ^{-1}, in the critical regions, as given by the DCLE through $\widetilde{\Phi}_{\rm II}$, compared with the CVM.

of the old ansatz (10). The same happens for $\widetilde{\Phi}_{\rm II}$, only for $\rho \gtrsim 2$; however, in this interval, the CVM works very well. For $\rho \lesssim 2$, on the contrary, the DCLE allows to obtain a unique determination of a and b, and hence a new proper estimate of ξ, which we denote $\xi_{\rm II}^{(1,2,3)}$.

In Figure 5, where we have enlarged the scale of the critical region, we give $\xi_{\rm II}^{(1,2,3)}(K,\rho)$ for some values of $\rho \leq 1$; it is compared with $\xi_{\rm II}^{(1,2)}$ and $\xi_{\rm II}^{(2,3)}$. The quantity $\xi_{\rm II}^{(1,2,3)}$ confirms essentially the CVM, when it is sufficiently stable. More significant corrections are obtained for ρ very small, as is shown in Figure 6a. In Figure 6a and 6b we summarize the behaviour of our calculations obtained through matrices of the same order, by comparing near the critical points $\xi_{\rm I}^{(3,4)}$, $\xi_{\rm II}^{(2,3)}$, $\xi_{\rm II}^{(1,2,3)}$ for some typical values of ρ. The DCLE allows to obtain a quite accurate determination of ξ, uniformily in the whole range $0 \leq K \leq K_{\rm c}(\rho)$, for $\rho \lesssim 2$, giving the critical exponent $\nu = 1$ and the exact correlation length as $\rho \to 0$. The same type of results can be obtained for $\rho \gtrsim 1$, through the CVM and the calculation of $\xi_{\rm II}^{(2,3)}$. However, if we are not near the critical region, then also (10), through the CVM, leads to quite reliable estimates.

In the limit $\rho \to 0$, the DCLE gives the right ξ through an interesting mechanism. Let us fix the attention on a distribution $\widetilde{P}_n^{\rm II}$ $(n = 1, 2, 3)$ and call $K_{\rm IIc}^{(1,2,3)}(\rho)$ the point where, at fixed ρ, $\xi_{\rm II}^{(1,2,3)}(K,\rho)$ diverges. For $\rho \neq 0$, the

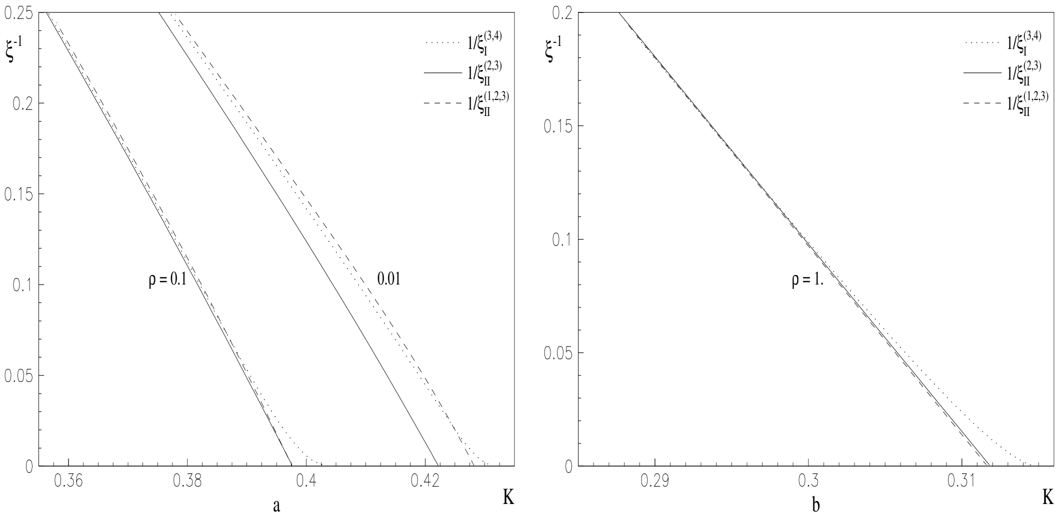

Fig. 6. Summary of results for the inverse correlation length ξ^{-1}, for typical values of ρ, near the critical points, as given by the CVM and DCLE within $\widetilde{\Phi}_{\rm I}$ and $\widetilde{\Phi}_{\rm II}$.

DCLE allows to fix the parameters a and b, obtaining two unique functions $a(K,\rho)$ and $b(K,\rho)$. It turns out that $a(K,\rho)$, defined for $K < K_{\rm IIc}^{(1,2,3)}(\rho)$, approaches $+\infty$ as $K \to K_{\rm IIc}^{(1,2,3)}(\rho)$, obtaining then strong correlations in the random external field $\{\mu_i\}$. Furthermore we have that $b(K,\rho) \neq 0$, so that these correlations propagate on Σ_v, or Σ_v', Σ_v'', giving strong correlations in our spins $\{s_i, u_i\}$. We notice that P_n^0 in no way can lead to a diverging correlation length, so that if b were exactly zero, the correlation length along Σ_v, Σ_v', Σ_v'', would be always small.

In Figure 7 we give $\exp[-b(K,\rho)]$, obtained by the DCLE, as a function of K, for some small values of ρ. We see that, as ρ becomes smaller and smaller, $b(K,\rho)$ is finite, but very small, allowing in any case the propagation of the strong correlations from $\{\mu_i\}$ to $\{s_i, u_i\}$. In the limit $\rho \to 0$ we have that $b(K,\rho) \to 0$, obtaining in this way the decoupling of the spins $\{s_i\}$ from the spins $\{u_i\}$, but the strong correlations shared with the field $\{\mu_i\}$ are not removed, with the result of common diverging correlation length.

As a by-product of the calculation of the correlation length, we obtain the critical curve $K_{\rm c}(\rho)$. Our numerical values of $K_{\rm c}$, for several values of ρ, are given in Table 1. These estimates are the points $K_{\rm IIc}^{(1,2,3)}(\rho)$, for $\rho \lesssim 2$, while, for $\rho \gtrsim 2$, they coincide with $K_{\rm IIc}^{(2,3)}(\rho)$ as obtained through the CVM; $K_{\rm IIc}^{(1,2,3)}(\rho)$ and $K_{\rm IIc}^{(2,3)}(\rho)$ agree up to the fourth decimal place in the range $0.2 \lesssim \rho \lesssim 2$.

It is interesting to compare the present results on $K_{\rm c}(\rho)$, with previous calculations. The series analysis of high temperature expansions gives $K_{\rm c}$ for $\rho = 1$ and $\rho = 2$ [1,2],

$$[\tanh K_{\rm c}(+1)]_{\rm SA} = 0.3020(6) \ , \quad [\tanh K_{\rm c}(+2)]_{\rm SA} = 0.2692(11) \ , \qquad (27)$$

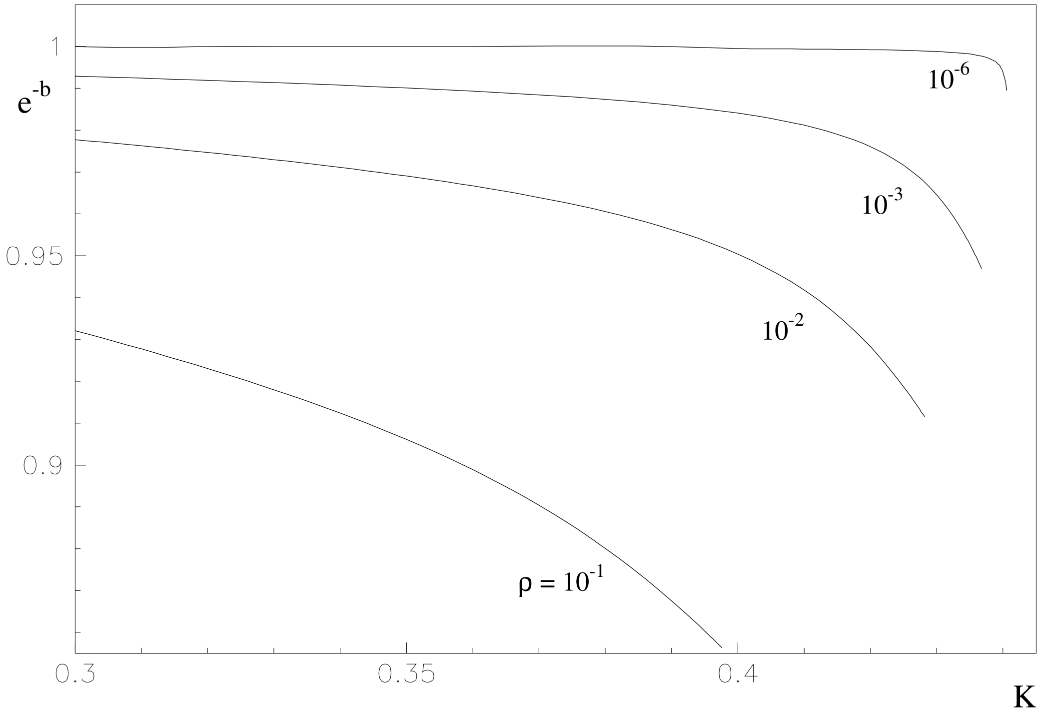

Fig. 7. Exp$[-b(K,\rho)]$, obtained through DCLE, as a function of K, for some small values of ρ.

while, from Table 1, we obtain

$$\tanh K_c(+1) = 0.30196 \ , \quad \tanh K_c(+2) = 0.26919 \ . \tag{28}$$

In the previous work [7], we obtained 0.30141 and 0.27147 respectively. There are also results from the Monte Carlo method. An old estimate of $K_c(+1)$ by Binder [5], gives $[\tanh K_c(+1)]_{MC} = 0.2980$, which is slightly smaller than the values in (27) and (28). Recent calculations by Li, Schulke and Zheng [6] get some improvement. They obtain

$$[\tanh K_c(+1)]_{MC} = 0.30145 \ , \quad [\tanh K_c(+2)]_{MC} = 0.26918. \tag{29}$$

In reference 6, there are also estimates of $K_c(\rho)$ for several ρ. For example, the Monte Carlo method gives

$$\begin{gathered}[K_c(0.12953)]_{MC} = 0.38837(3) \ , \quad [K_c(0.27397)]_{MC} = 0.36634(12) \\ [K_c(0.53097)]_{MC} = 0.34107(10) \ ,\end{gathered} \tag{30}$$

while our results, for these ρ-values, are 0.39131, 0.36841 and 0.34193 respectively. The estimates (29) and (30) are in a good agreement with our

Table 1
Numerical values of the critical points $K_c(\rho)$, for several values of ρ, deduced from $\widetilde{\Phi}_{II}$, by the DCLE and CVM.

ρ	K_c
0.00	0.44069
0.01	0.42826
0.10	0.39761
0.20	0.37892
0.40	0.35399
0.60	0.33642
0.80	0.32278
1.00	0.31167
1.40	0.29437
1.60	0.28745
2.00	0.27599
3.00	0.25590
4.00	0.24409
8.00	0.22493
∞	0.22034

calculations and (27), but they are always slightly smaller (except for $\rho \simeq 2$).

Recently, strips of finite width of the two-layer Ising film, have been also considered by Lipowski and Suzuki [8], but within the scheme of the mean field theory [12]. The consistency conditions, in a slightly different form, involve the magnetization. This approach is concerned only with the calculation of the critical temperature; strong correlations are neglected and the proper bulk magnetization cannot be evaluated. However, having cluster of infinite spins, the determination of K_c is greatly improved with respect to the standard mean field results, obtaining in the case of the two dimensional Ising model the exact value [12]. By considering strips made of $n = 1, 2, \ldots, 6$ vertical sections, Lipowski and Suzuki [8] obtain the estimates

$$[\tanh K_c(+1)]_{LS} = 0.30204 \,, \quad [\tanh K_c(+2)]_{LS} = 0.26923. \tag{31}$$

Our results for $K_c(\rho)$ agree with (31) up to the fourth decimal place.

The study of the two-layer Ising film shows a very interesting property which will be discussed in the following. First of all, let us fix the attention on the critical curve in the weak ($\rho \to 0$) and strong ($\rho \to \infty$) interlayer coupling limits.

On the basis of perturbation theory and scaling laws of correlation functions, it has been found [4,13] that $K_c(\rho)$ is singular at $\rho = 0$, the singularity being described through the shift exponent ψ

$$K_c(0) - K_c(\rho) \sim \rho^{\psi}, \qquad (\rho \to 0) \tag{32}$$

with ψ related to the properties of the unperturbed system ($\rho = 0$), that is of the square Ising model. It turns out [4,13] that ψ is equal to the inverse of the susceptibility exponent, $\psi = \frac{1}{\gamma} = \frac{4}{7}$. Renormalization group arguments [14,15] lead to the same conclusion.

On the other hand, as $\rho \to +\infty$, we can consider as the unperturbed system the square Ising model with coupling $2K$. From the structure of the partition function it can be deduced that we can make an expansion in the parameter $\exp(-2K_2) = \exp[-2\rho K]$. Then, on the critical curve, we have

$$K_c(\rho) - K_c(+\infty) \sim e^{-2\rho K_c(\rho)} \tag{33}$$

for large ρ.

By making use of our determination of $K_c(\rho)$, we have calculated the quantities

$$f(\rho) \equiv \log \frac{K_c(0) - K_c(\rho)}{K_c(\rho)}, \tag{34}$$

and

$$g(\rho) \equiv \frac{K_c(\rho) - K_c(+\infty)}{K_c(+\infty) e^{-2\rho K_c(\rho)}} \tag{35}$$

In Figure 8 we give $f(\rho)$ (points) as a function of $\log \rho$, for several ρ in the range $10^{-4} \leq \rho \leq 1$. The relevant behaviour of $f(\rho)$ is that it is essentially linear in two intervals, having there two different slopes. The points lie practically on the straight line (full line in Figure 8)

$$f(+1) + \frac{4}{7} \log \rho, \tag{36}$$

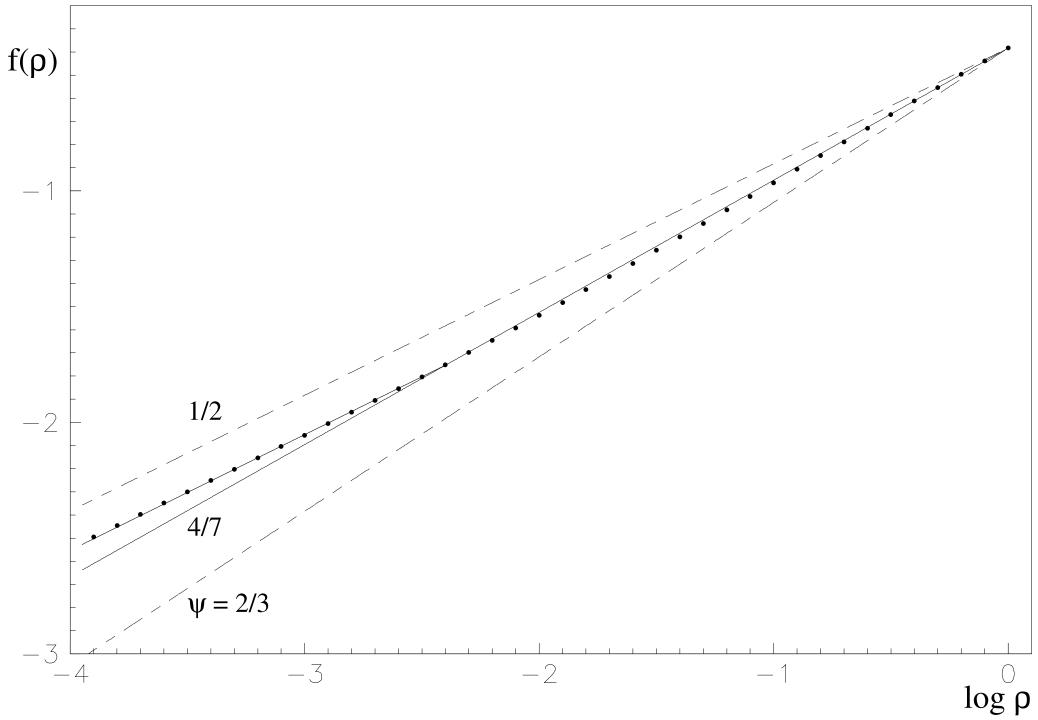

Fig. 8. The quantity $f(\rho) \equiv \frac{\log(K_c(0)-K_c(\rho))}{K_c(\rho)}$ vs $\log\rho$ (discrete points). Straight lines with slopes $\frac{1}{2}$, $\frac{4}{7}$ and $\frac{2}{3}$ are also plotted.

for ρ in the interval $10^{-2.5} \lesssim \rho \lesssim 1$, while for smaller values of ρ they lie on the straight line

$$f(10^{-2.5}) + \frac{1}{2}\log\rho \tag{37}$$

The two dashed straight lines in Figure 8 are

$$f(+1) + \frac{1}{2}\log\rho\,, \quad f(+1) + \frac{2}{3}\log\rho. \tag{38}$$

Our results for ρ not very small, on one hand are in agreement with the prediction based on the scaling laws or the renormalization group, on the other show the remarkable property that this prediction, in fact, can be extended also up to $\rho \simeq 1$. For very small ρ, there is a crossover mechanism which leads to $\psi = \frac{1}{2}$, as found in the previous work [7]. The origin of this crossover within our approach will discussed in the last section. In any case the value $\psi = \frac{2}{3}$ of section 2 is excluded.

The behaviour of our $f(\rho)$ is less reliable for $\rho \lesssim 10^{-2.5}$. However the equation (36) can be matched with the proper asymptotic prediction, by extending its validity to all $\rho \lesssim 1$. As a consequence we deduce from (34) that a piece of the critical curve is given, with a good accuracy, by the analytic expression

$$K_{\rm c}^{\rm w}(\rho) = \frac{K_{\rm c}(0)}{1 + C\rho^{4/7}} \ , \tag{39}$$

for $\rho \lesssim 1$, where

$$C = \frac{K_{\rm c}(0) - K_{\rm c}(+1)}{K_{\rm c}(+1)} = 0.413695 \ .$$

The function $K_{\rm c}^{\rm w}(\rho)$ is given in Figure 9 (full line). As we see, $K_{\rm c}^{\rm w}(\rho)$ reproduces very well the critical points up to $\rho \leq 1.4$.

On the other hand, by considering $g(\rho)$, given by (35), we obtain that this function is essentially constant

$$g(\rho) = D, \tag{40}$$

for $\rho \gtrsim 2$, with $D = 0.7647$. In particular, for large ρ, we deduce that

$$\begin{aligned} K_{\rm c}(\rho) &= K_{\rm c}(+\infty) + DK_{\rm c}(+\infty)\ e^{-2\rho K_{\rm c}(+\infty)} + \dots \\ &= \ K_{\rm c}(+\infty) + 0.1685\ e^{-2\rho K_{\rm c}(+\infty)} + \dots \ , \end{aligned} \tag{41}$$

to be compared with (20). The coefficient 0.1685 agrees completely with the calculations of ref. [8]. However (40) allows to obtain $K_{\rm c}(\rho)$ from an analytic equation, also for ρ not very large. We call $K_{\rm c}^{\rm s}(\rho)$ the solution (in fact it is unique) of the equation

$$\frac{K_{\rm c}^{\rm s}(\rho) - K_{\rm c}(+\infty)}{K_{\rm c}(+\infty)e^{-2\rho K_{\rm c}^{\rm s}(\rho)}} = D \tag{42}$$

The function $K_{\rm c}^{\rm s}(\rho)$ is shown in Figure 9 (dashed line). As we see, the critical points lie very well on the dashed line, for $\rho \gtrsim 1.4$. Around $\rho \simeq 1.4, K_{\rm c}^{\rm w}(\rho)$ and $K_{\rm c}^{\rm s}(\rho)$ match very well with each other. This result suggests that the full equation of the critical points breaks up into two factors, which originate from the scaling behaviour for $\rho \to 0$ and $\rho \to +\infty$. As a matter of fact, (32) is a consequence of the fact that, for small ρ and $t = (K_{\rm c}(0) - K)/K$, the free energy $F(t, \rho)$ is given by the scaling law [4,13]

$$F(t, \rho) = t^{\nu d} \phi(\frac{\rho^{1/\gamma}}{t}), \tag{43}$$

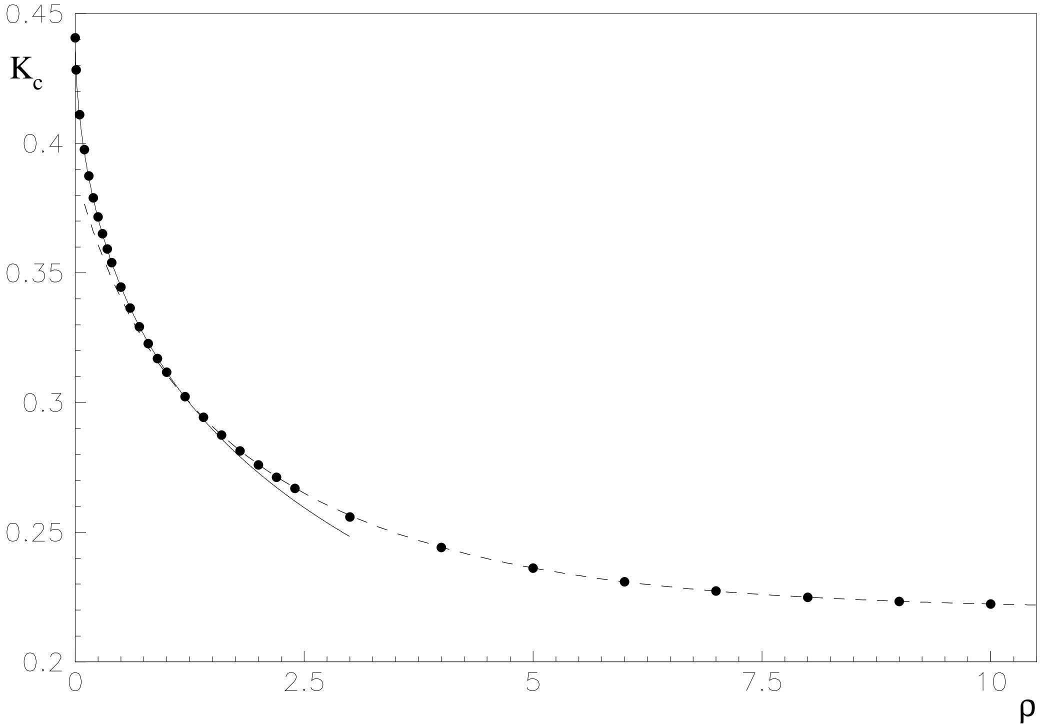

Fig. 9. Critical points $K_c(\rho)$ with the functions $K_c^{\rm w}(\rho)$ (full line) and $K_c^{\rm s}(\rho)$ (dashed line) super-imposed.

where $\nu = 1$ and $d = 2$. We thus expect that

$$\frac{1}{\xi}(t, \rho) = t\phi'(\frac{\rho^{1/\gamma}}{t}) = \rho^{1/\gamma}\widetilde{\phi}(\frac{\rho^{1/\gamma}}{t}), \tag{44}$$

The zero of the scaling function $\widetilde{\phi}$ gives (32). From (44) we have that, for small ρ and t, the scaled inverse correlation length,

$$\widetilde{\xi}^{-1} = (\xi\rho^{1/\gamma})^{-1} \ ,$$

is function only of the scaled, reduced inverse temperature

$$\widetilde{k} = \frac{\rho^{1/\gamma}}{t} \ .$$

In Figure 10 we give $\widetilde{\xi}^{-1}$ as a function of $\widetilde{k}$, for $\rho = 10^{-2}, 10^{-\frac{3}{2}}, 10^{-1}$. The results, obtained through the DCLE, show that, within a good approximation, near the critical points, $\widetilde{\xi}^{-1}$ does not depend on ρ.

We expect that also (42) is related to a zero of some scaling function, describing the behaviour for $\rho \to +\infty$. Our results on the critical curve suggest that in

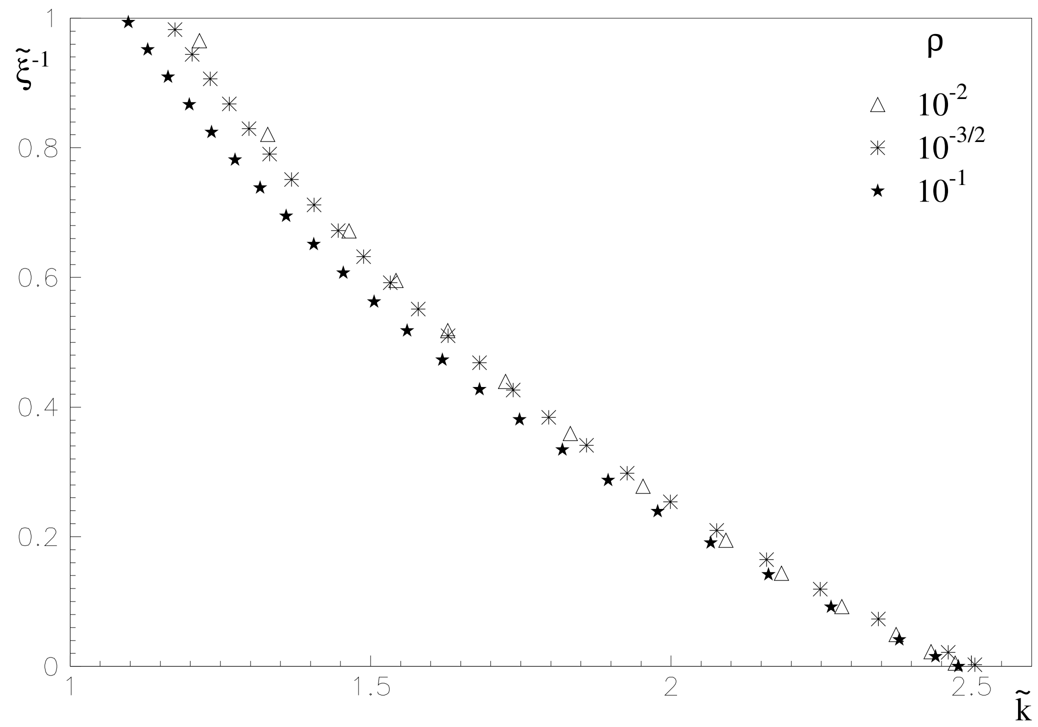

Fig. 10. Scaled inverse correlation length *vs* scaled inverse temperature in the critical region, for some small ρ-values.

the critical region $1/\xi(K,\rho)$, for any ρ, is given by the product of two scaling functions, which depend on different scaling variables.

4 Conclusions

As we have seen, for $\rho \lesssim 10^{-2.5}$, we have a crossover in the behaviour of $f(\rho)$, which leads to $\psi = \frac{1}{2}$. This value is a consequence of the structure of our ansatz and confirms the scaling law

$$\gamma = \nu(2-\eta) \tag{45}$$

in our case. We have already pointed out that to $\widetilde{P}_n^{\mathrm{II}}$ is associated a transfer matrix of finite order. Then, at $K_{\mathrm{c}}(\rho)$, we obtain

$$\langle s_i s_{i+r} \rangle_{\widetilde{P}_n^{\mathrm{II}}} \longrightarrow \text{constant}\,, \quad \text{as } r \to \infty \tag{46}$$

for any ρ. So, while the ratio of the largest eigenvalues of the exact transfer matrix can be well reproduced by a transfer matrix of finite order, the asymptotic behaviour of the two point function at $K_{\mathrm{c}}(\rho)$ is not adequately described. From (46) we deduce $\eta = 0$. On the other hand, having obtained correctly $\nu = 1$, the validity of (45) leads to $\gamma = 2$ for every ρ, $\rho = 0$ included. Therefore ψ, which is related to the unperturbed system, attains the value $\frac{1}{2}$. What is relevant is that our determination of $K_c(\rho)$ through $\xi_{\rm II}^{(1,2,3)}$ is reliable up to such small ρ, that the crossover appears very near to $\rho = 0$, so that the proper asymptotic behaviour of $K_c(\rho)$ can be matched or else estimated.

The correlation length which we have determined, is a fundamental property of the model considered in this paper. In order to obtain proper estimates of other properties, these should be involved into the marginality conditions, after the introduction of an ansatz of the averaging process. So, in connection with the last discussion, the study of the two point correlation function $g(r) \equiv \langle s_i s_{i+r} \rangle$, at $K_c(\rho)$ for example, requires $g(r)$, at various r, to enter in the consistency conditions, adapting the parameters of an ansatz to such a situation.

For $\rho \neq 0$, we can think of (1) as a model of two interacting fermionic fields. In this connection, having determined ξ, it is relevant to calculate also the higher excitations and to study the continuum limit.It is also very interesting to consider antiferromagnetic vertical coupling or else different horizontal intralayer couplings. These aspects will be investigated elsewhere.

Acknowledgements

This research was supported by Ministero dell'Università e della Ricerca Scientifica (40% and 60% funds) and Istituto Nazionale di Fisica Nucleare.

References

[1] G. A. T. Allan, Phys. Rev. B **1**, 352 (1970).

[2] M. E. Fisher, in *Critical Phenomena*, edited by M. S. Green (Academic, New York, 1971).

[3] T. W. Capehart and M. E. Fisher, Phys. Rev. B **13**, 5021 (1976).

[4] R. Abe, Progr. Theor. Phys., **44**, 339 (1970).

[5] K. Binder, Thin Solid Films, **20**, 367 (1974).

[6] Z. B. Li, L. Schulke and B. Zheng, University of Siegen (FRG) preprint, 1994.

[7] L. Angelini, D. Caroppo, M. Pellicoro and M. Villani, J. Phys. A: Math. Gen., **25**, 5423 (1992).

[8] A. Lipowski and M. Suzuki, Physica (Utrecht) A, **198**, 227 (1993).

[9] L. Angelini, M. Pellicoro, I. Sardella and M. Villani, J. Phys. A: Math. Gen., **26**, 5207 (1993).

[10] M. Villani, J. Phys. A: Math. Gen., **23**, 4977 (1990).

[11] T. Morita, J. Math. Phys.,**13**, 115 (1972).

[12] A. Lipowski and M. Suzuki, J. Phys. Soc. Japan, **61**, 4356 (1992).

[13] M. Suzuki, Progr. Theor. Phys., **46**, 1054 (1971).

[14] M. K. Grover, Phys. Lett., **44A**, 253 (1973).

[15] T. S. Chang and H. E. Stanley, Phys. Rev. B, **8**, 4435 (1973).