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Polarized ensembles of random pure states

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Abstract
A new family of polarized ensembles of random pure states is presented. These ensembles are obtained by linear superposition of two random pure states with suitable distributions, and are quite manageable. We will use the obtained results for two purposes: on the one hand we will be able to derive an efficient strategy for sampling states from isopurity manifolds. On the other, we will characterize the deviation of a pure quantum state from separability under the influence of noise.

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(Some figures may appear in colour only in the online journal)

1. Introduction

In the last few years many researchers have been investigating the typical properties of random pure states, i.e. unit vectors drawn at 'random' from the Hilbert space associated to a quantum system. This subject has attracted attention in several directions, and some important results have been achieved, mostly dealing with the characterization of entanglement [1–14].

The standard ensemble which has been intensively investigated is that of random pure states with measure induced by the Haar measure on the unitary group. This ensemble, being the maximally symmetric one, implements in a natural way the case of minimal knowledge on a quantum state [15]. It is structureless, in the sense that the induced measure only depends on the dimension of the total Hilbert space and it is not sensible to any tensor product structure [1, 16].

For this reason, the unitarily invariant ensemble is also known as the *unbiased ensemble* [14]. A natural question is whether this ensemble of pure random states can be used to construct different, more complicated, ones.

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Various approaches have been proposed independently by several groups that have introduced different physically motivated measures on the space of pure states. Recently, Zyczkowski *et al* [17] have analyzed some kind of *structured ensembles* of random pure states on composite systems that are invariant under local unitary transformations. From another perspective, De Pasquale *et al* [18] have proposed a classical statistical mechanics approach in order to explore the *isopurity manifolds* of random states. In the same spirit, Mueller *et al* [19] have recently investigated ensembles of *random pure states with fixed expectation value* of some observable, in the framework of the concentration of measure phenomenon. However, there are still many obstructions in carrying out these programs, and the links among them are not yet clear.

This paper is intended as another step toward new scenarios beyond the unbiased ensemble. This step is motivated as an operational way to capture the isopurity manifolds, and turns out to be in particular cases similar to the structured ensembles proposed in [17]. Our idea is to move beyond the unbiased ensemble by using a natural operation at hand in the Hilbert space, namely superposition of vector states. In this work we will show that unitarily invariant measures interact nicely with the operation of superposition of states.

As a remark, we want to stress that, in the large size limit, the robustness of Marčenko–Pastur's theorem [20] prevents many of the potentially workable ensembles to deviate from the Marčenko–Pastur law. The approach that we propose here is a way to circumvent such an obstruction and allows to obtain workable ensembles with entanglement spectra (i.e. the densities of Schmidt eigenvalues) that can sensibly differ from the Marčenko–Pastur law. In this way one can investigate, by varying the strength and/or the type of the polarization, the structure of the different spectral types and the possible emergence of phase transitions of entanglement [8].

The paper is organized as follows. In section 2 we introduce the concept of polarized ensembles of pure states that will play a central role in our work. In particular, we will consider the deviation from the unbiased ensemble induced by the Haar measure on the unitary group acting on the whole Hilbert space of the system. In section 3 we will characterize the polarized ensembles by using the local purity of a subsystem. This approach will be used both for the study of the ensembles and for the description of an efficient procedure for sampling typical states from an isopurity manifold (and, therefore, with a fixed value of bipartite entanglement). In section 4 we will apply these results to the characterization of separability of quantum states under the influence of noise. Finally, in section 5 we will draw some conclusions. Some technical points are discussed in two appendices for completeness and self-consistency. The appendices are dedicated respectively to the computation of the average purity and to the concentration around the average.

2. Polarized ensembles of pure states

Consider a bipartite quantum system described by a pure state in a finite-dimensional Hilbert space $\mathcal{H}$. The bipartition in two subsystems (A, B) will induce on $\mathcal{H}$ a tensor product structure such that

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B. \tag{1}$$

We will consider, without loss of generality, the case with

$$\dim \mathcal{H}_A = N \leqslant \dim \mathcal{H}_B = M, \tag{2}$$

whence $\dim \mathcal{H} = NM$.

Let us focus on the general situation in which the (pure) state of the quantum system has the form (up to normalization)

$$|\psi\rangle = \alpha|\psi_1\rangle + \beta|\psi_2\rangle, \tag{3}$$

where $|\psi_1\rangle, |\psi_2\rangle \in \mathcal{H}$ are random states, sampled according to arbitrary probability measures. Once the probability distributions of $|\psi_1\rangle$ and $|\psi_2\rangle$ are specified, the random vector $|\psi\rangle$ is characterized by a well defined distribution.

Due to its privileged role in many contexts of random matrix theory [21, 22], in the following we will make great use of the unitarily invariant (unbiased) probability measure on pure states, denoted by μ_{MN}, induced by the Haar measure on the unitary group $\mathcal{U}(\mathcal{H})$. Moreover, we will consider the product measure $\mu_N \times \mu_M$ on pure states, which is left invariant under the action of local unitaries $\mathcal{U}(\mathcal{H}_A) \times \mathcal{U}(\mathcal{H}_B)$. In the latter case, we recall that the space of pure states is foliated in orbits of local unitaries labeled by the degeneracy of the Schmidt coefficients. Each of these orbits is a natural domain for the locally invariant measure $\mu_N \times \mu_M$ [23, 24].

In general, depending on the ensembles chosen for sampling $|\psi_1\rangle$ and $|\psi_2\rangle$, the state $|\psi\rangle$ defined in (3) will exhibit very different properties. Let us briefly outline the relevant features in some cases of particular interest.

If both $|\psi_1\rangle$ and $|\psi_2\rangle$ are sampled according to the unbiased ensemble μ_{MN}, then $|\psi\rangle$ is also a random state whose distribution is invariant under the action of the unitary group $\mathcal{U}(\mathcal{H})$ (independently on the values of α and β).

The opposite situation occurs when $|\psi_1\rangle$ is a fixed pure state and $|\psi_2\rangle \sim \mu_{NM}$. In this case the weights α and β are relevant; if $|\alpha| \gg |\beta|$ the unbiased ensemble becomes 'polarized' along the direction defined by $|\psi_1\rangle$. This polarization phenomenon is of particular interest if one wants to study the deviation of the properties of an ensemble of quantum states from a fixed reference state. In particular, in this work we will specialize our analysis to the cases of $|\psi_1\rangle$ separable or maximally entangled (in the bipartite sense).

3. Typical local purity

3.1. Local purity of one-parameter polarized ensembles

In this section we will focus on the consequences of the polarization of the ensemble on the properties of bipartite entanglement between the subsystems A and B. As a measure of entanglement we will consider the local purity of one of the parties. Given a pure state $|\psi\rangle \in \mathcal{H}$, it is defined by

$$\pi_{AB}(|\psi\rangle) = \mathrm{Tr}_A \rho_A^2, \qquad \rho_A = \mathrm{Tr}_B |\psi\rangle\langle\psi|, \tag{4}$$

where ρ_A is the reduced density matrix of party A. The upper bound $\pi_{AB} = 1$ and the lower bound $\pi_{AB} = 1/\dim \mathcal{H}_A = 1/N$ correspond, respectively, to separable and maximally entangled states.

Let us consider a state $|\psi\rangle$ obtained as a superposition (3) where, in particular, a fixed pure state is superposed to an unbiased random one. We will get the following one-parameter ensemble

$$|\psi\rangle = \left[\epsilon \mathbb{1}_{AB} + \sqrt{1-\epsilon^2} U_{AB}\right]|\phi_0\rangle, \tag{5}$$

where the normalized state $|\phi_0\rangle \in \mathcal{H}$ is fixed, $\epsilon \in [0, 1]$ is a tunable parameter, $\mathbb{1}_{AB}$ is the identity operator, and $U_{AB} \in \mathcal{U}(\mathcal{H})$ is a random unitary acting on $\mathcal{H}$, sampled according to the Haar measure on the unitary group. The state

$$|\phi\rangle = U_{AB}|\phi_0\rangle \tag{6}$$

is therefore a (unit) random vector distributed according to the unitarily invariant measure on pure states μ_{NM}.

Notice that for $\epsilon = 0$ one recovers the unbiased ensemble. On the other hand, values of $\epsilon \neq 0$ play the role of an offset which parametrizes, as discussed in the previous section, the degree of polarization of the ensemble in the direction of $|\phi_0\rangle$. We also observe that, given two independent and symmetrically distributed random states $|\phi_1\rangle$, $|\phi_2\rangle$, the expectation value of their overlap vanishes, i.e.

$$\mathbb{E}[\langle\phi_1|\phi_2\rangle] = 0. \tag{7}$$

As a consequence, the normalization of the state $|\psi\rangle$ in (5) is assured on the average, in the sense that

$$\mathbb{E}\,[\langle\psi|\psi\rangle] = \mathbb{E}[\|\psi\|^2] = 1, \tag{8}$$

and deviations from the average are exponentially suppressed for large N, as shown in appendix B.

We emphasize that, since we are focusing on the typical features of an ensemble of random pure states, any statement in the paper has to be considered in the large size limit, $N \to \infty$. In this limit, the ensemble of vectors (5) is an ensemble of physical states, in the sense that it consists of unit vectors with overwhelming probability.

We are interested in the conditional expectation value of the purity $\pi_{AB}(|\psi\rangle)$ given a fixed state $|\phi_0\rangle$ and a bias ϵ. Due to concentration of measures, for large N this quantity will be the typical purity of the polarized ensemble (5).

The density operator associated to the random pure state $|\psi\rangle$ reads

$$|\psi\rangle\langle\psi| = \epsilon^2|\phi_0\rangle\langle\phi_0| + (1-\epsilon^2)|\phi\rangle\langle\phi| + \epsilon\sqrt{1-\epsilon^2}(|\phi_0\rangle\langle\phi| + |\phi\rangle\langle\phi_0|), \tag{9}$$

where $|\phi\rangle$ is given by (6). We will use the following notation:

$$\begin{aligned}\sigma &= \mathrm{Tr}_B|\phi\rangle\langle\phi|,\\ \sigma_0 &= \mathrm{Tr}_B|\phi_0\rangle\langle\phi_0|,\\ S_{0\phi} &= \mathrm{Tr}_B(|\phi_0\rangle\langle\phi| + |\phi\rangle\langle\phi_0|).\end{aligned} \tag{10}$$

By tracing over subsystem B and performing a straightforward calculation, we obtain the purity (which is a random variable)

$$\begin{aligned}\pi_{AB}(|\psi\rangle) &= \mathrm{Tr}_A\rho_A^2\\ &= \epsilon^4\mathrm{Tr}\,\sigma_0^2 + (1-\epsilon^2)^2\mathrm{Tr}\,\sigma^2 + \epsilon^2(1-\epsilon^2)\mathrm{Tr}\,S_{0\phi}^2 + 2\epsilon^2(1-\epsilon^2)\mathrm{Tr}(\sigma_0\sigma)\\ &\quad + 2\epsilon^3\sqrt{1-\epsilon^2}\mathrm{Tr}(\sigma_0 S_{0\phi}) + 2\epsilon(1-\epsilon^2)^{3/2}\mathrm{Tr}(\sigma S_{0\phi}).\end{aligned} \tag{11}$$

3.2. Typical local purity

We now evaluate the expectation value of the purity π_{AB}. The computation can be easily done by making use of a Gaussian approximation. More precisely, we will consider random vectors $|\phi\rangle \in \mathcal{H}$ whose components in an arbitrary basis are independent complex random variables normally distributed, $\mathcal{N}_{\mathbb{C}}(0, 1/NM)$ (the normalization of $|\phi\rangle$ is assured on the average).

The Gaussian approximation is fully justified for our purposes. Indeed, in the large size limit, concentration phenomena and the simultaneous convergence of the Gaussian measure to the unitarily invariant measure on the sphere [25] provide the typicality of our results (see appendix B for further details). Thus, averages on the unitary group can be substituted with

averages with respect to Gaussian measures. In this case, expectation values of any smooth quantity of interest $f(|\phi\rangle)$ can be easily estimated.

We claim that the typical local purity of (5) depends only on the local purity of the pure state $|\phi_0\rangle$,

$$\pi_0 = \pi_{AB}(|\phi_0\rangle), \tag{12}$$

and *not* on the particular vector $|\phi_0\rangle$ with the given purity. Indeed, a direct calculation with $|\phi\rangle$ a Gaussian vector shows (see appendix A for explicit calculations) that the only non-vanishing terms in the expectation value of (11) are

$$\mathbb{E}\left[\operatorname{Tr}\sigma_0^2\right] = \operatorname{Tr}\sigma_0^2 = \pi_0, \tag{13}$$

$$\mathbb{E}[\operatorname{Tr}\sigma^2] = \frac{M+N}{MN}, \tag{14}$$

$$\mathbb{E}[\operatorname{Tr}(\sigma_0\sigma)] = \frac{1}{N}, \tag{15}$$

$$\mathbb{E}\left[\operatorname{Tr}S_{0\phi}^2\right] = \frac{2}{M}. \tag{16}$$

By plugging (13)–(16) into (11) we finally get the conditional expectation value of the purity $\pi_{AB}(|\psi\rangle)$

$$\bar{\pi}_{AB} = \mathbb{E}[\pi_{AB} \,|\, |\phi_0\rangle, \epsilon] = \epsilon^4\pi_0 + (1-\epsilon^4)\frac{M+N}{MN}. \tag{17}$$

This is a central result of the paper.

Notice that if we had performed the average using the unitarily invariant measure μ_{NM} for $|\phi\rangle$, as in (6), the only difference with the above calculation would have been in the term

$$\mathbb{E}[\operatorname{Tr}\sigma^2] = \frac{M+N}{MN+1}, \tag{18}$$

as computed by Lubkin [26]. The relative difference with the typical purity of the unbiased Gaussian ensemble

$$\pi_{\text{unb}} = \bar{\pi}_{AB}|_{\epsilon=0} = \frac{M+N}{MN}, \tag{19}$$

is thus of order $O(1/(NM))$ and negligible for large systems.

We point out an important consequence of (17): even if $|\phi_0\rangle$ is substituted by a state $|\phi_0'\rangle = U_A \otimes U_B|\phi_0\rangle$, with $U_{A(B)} \in \mathcal{U}\left(\mathcal{H}_{A(B)}\right)$, belonging to its local orbit (therefore evolving on an isopurity manifold with arbitrary measure), the value of the typical purity given by (17) is not affected. In other words, the one-parameter ensemble of random states

$$|\psi\rangle = \left[\epsilon U_A \otimes U_B + \sqrt{1-\epsilon^2}U_{AB}\right]|\phi_0\rangle, \tag{20}$$

where $U_A \otimes U_B$ is a random local unitary, has average purity given by formula (17), with $\pi_0 = \pi_{AB}(|\phi_0\rangle) = \pi_{AB}(U_A \otimes U_B|\phi_0\rangle)$. The ensemble (20) is a linear superposition of two random pure states with suitable probability distributions.

Incidentally, this can also be seen as a consequence of a fundamental property of conditional expectations:

$$\mathbb{E}\,[f(X,Y)] = \mathbb{E}[\mathbb{E}[f(X,Y)\,|\,Y]], \tag{21}$$

where f is a function of two random variables X and Y. We will make frequent use of this property in the following.

3.3. Typical purity for a separable and a maximally entangled polarizing state $|\phi_0\rangle$

In this section we will discuss two interesting examples of the behavior of $\bar{\pi}_{AB}$, given the value of π_0. We will specialize our results about the generalized ensemble (20) to the extremal situations of a separable or maximally entangled polarizing state $|\phi_0\rangle$.

Let us start by considering the case of $|\phi_0\rangle$ separable with respect to the bipartition (A, B), so that $\pi_0 = 1$. According to the discussion at the end of the section 3.2, we can allow $|\phi_0\rangle$ to be a random pure separable state and consider the generalized ensemble

$$|\psi\rangle = \left[\epsilon U_A \otimes U_B + \sqrt{1-\epsilon^2} U_{AB}\right] |\phi_{\text{sep}}\rangle, \tag{22}$$

$$|\phi_{\text{sep}}\rangle = |\phi_0\rangle_A \otimes |\phi_0\rangle_B \tag{23}$$

where $|\phi_0\rangle_A$ and $|\phi_0\rangle_B$ are fixed states in $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, U_A and U_B are random local unitaries and U_{AB} is a random global unitary transformation. The typical purity (17) of the one-parameter ensemble with a separable polarization (23) reads

$$\mathbb{E}[\pi_{AB} \,|\, |\phi_{\text{sep}}\rangle, \epsilon] = \frac{M+N}{MN} + \epsilon^4 \frac{MN - M - N}{MN}. \tag{24}$$

The other extreme case is given by $|\phi_0\rangle = |\phi_{\text{ent}}\rangle$, a maximally entangled pure state, so that $\pi_0 = 1/N$. Then we are dealing with a polarized ensemble of the form

$$|\psi\rangle = \left[\epsilon U_A \otimes U_B + \sqrt{1-\epsilon^2} U_{AB}\right] |\phi_{\text{ent}}\rangle, \tag{25}$$

where the reference state is such that

$$\text{Tr}_B |\phi_{\text{ent}}\rangle\langle\phi_{\text{ent}}| = \frac{1}{N} \mathbb{1}_A. \tag{26}$$

Such a polarization decreases the typical purity of the unbiased ensemble to the value

$$\mathbb{E}\left[\pi_{AB} \,|\, |\phi_{\text{ent}}\rangle, \epsilon\right] = \frac{M+N}{MN} - \frac{\epsilon^4}{M}. \tag{27}$$

These results will be compared in the following section to a numerical Monte Carlo approach.

3.4. Generation of random pure states with fixed local purity

The results obtained in the previous sections suggest a very inexpensive strategy for generating random pure states with fixed value of the purity π_{AB}. Indeed, the numerical sampling of the uniform measure on the manifold of pure states $|\psi\rangle$ parametrized by a fixed value of π_{AB} will proceed through the following steps:

(1) Choose $\epsilon \in [0, 1]$ such that

$$\pi_{AB} = \epsilon^4 \pi_0 + (1 - \epsilon^4) \pi_{\text{unb}}, \tag{28}$$

where $\pi_0 = 1$ or $\pi_0 = 1/N$ if the desired value of π_{AB} is, respectively, larger or smaller than the unbiased typical value π_{unb} in (19).

(2) Generate a pure state $|\psi\rangle$ by superposition

$$|\psi\rangle = \epsilon |\phi_0\rangle + \sqrt{1-\epsilon^2} |\phi\rangle, \tag{29}$$

where $|\phi\rangle$ is sampled according to the unbiased measure μ_{NM} and $|\phi_0\rangle$ is a separable or maximally entangled pure state sampled randomly according to the invariant measure under local unitaries $\mu_N \times \mu_M$, depending on the value of π_0 chosen in (28).

In figure 1 (upper panel) the analytical formulas (24) and (27) are compared to the Monte Carlo results for the values of $\bar{\pi}_{AB}$ obtained by sampling pure states through the procedure outlined above. The comparison shows clearly the efficiency of the sampling procedure in providing the correct behavior of the typical purity versus the bias ϵ with quite small fluctuations already for dimensions $N = M = 30$. Such fluctuations around the average are more evident for smaller systems. See figure 1 (lower panel) for the case $N = M = 8$.

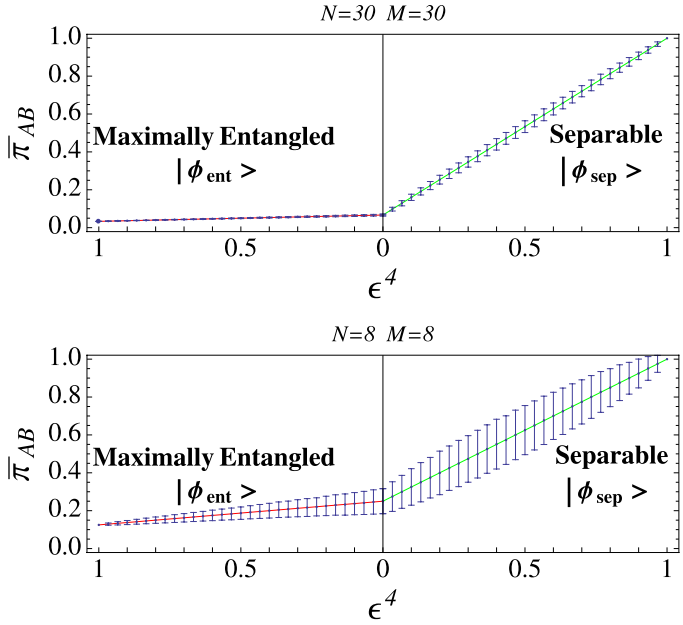

Figure 1. Typical purity of polarized ensembles versus ϵ^4, depending on the bias $|\phi_0\rangle$. We compare the analytical prediction (continuous lines) with the numerical values of π_{AB} (sample mean and error bars) obtained from the sampling procedure described in the text. We consider balanced bipartitions with size $N = M = 30$ (top) and $N = M = 8$ (bottom). In both cases the number of realizations used to perform the ensemble average is $n = 10^4$. Right side: the continuous line represents the analytical prediction for the purity of an ensemble polarized by a separable state (24). Depending on the value of the parameter ϵ the purity ranges from the unbiased value $\pi_{\text{unb}} = (M + N)/MN$ to the maximum $\pi_0 = 1$. Left side: the continuous line represents the analytical prediction for the purity of an ensemble polarized by a maximally entangled state (27). As ϵ increases the typical value of the purity decreases from π_{unb} to the minimum $\pi_0 = 1/N$. The error bars represent the standard deviations of the numerical simulations from the estimated average. Such fluctuations are exponentially suppressed as the dimensions N, M increase, according to equation (B.3) in appendix B.

4. Robustness of separability under random perturbations

As an application, in this section we will use the results obtained from the study of polarized ensembles to analyze the stability of 'separability' of quantum states with respect to random additive perturbations. More precisely, if the state of the system $|\phi_0\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ is separable, $|\phi_0\rangle = |\xi_0\rangle_A \otimes |\chi_0\rangle_B$, how much noise is necessary to make the reduced state $\rho_A = \text{Tr}_B[|\phi_0\rangle\langle\phi_0|]$ distinguishable from a pure state? We notice that the problem of the characterization of separability has attracted a lot of interest in the context of the analysis of ground states of quantum spin systems [27–29].

Let us consider the state of the bipartite system in the form

$$|\psi\rangle = \sqrt{1-\eta^2}|\xi_0\rangle_A \otimes |\chi_0\rangle_B + \eta|\phi\rangle, \quad 0 \leqslant \eta \leqslant 1, \tag{30}$$

where $|\phi\rangle \sim \mu_{NM}$ is an unbiased random perturbation, and η measures the strength of the noise. (Notice that in the language of polarized ensembles of the previous sections we have $\epsilon = \sqrt{1-\eta^2}$.)

In the context of spin systems, the parameter η should be related e.g. to the magnetic field that drives the ground state from a separable state to an entangled one. More generally, this setup models the following situation: some non-controllable noise prevents one to prepare with infinite precision a given state. The noise has the consequence that the reduced density matrix ρ_A is not a projection but is mixed. It is reasonable that, for weak noise ($\eta \ll 1$), we still obtain a reduced state ρ_A that is close to a projection. Then the question is: is there a threshold on the value of noise above which separability is appreciably destroyed?

In order to answer this question, it is convenient to introduce the notion of *effective dimension* (also known as 'participation ratio' [30]) of a state ρ defined as

$$d^{\text{eff}}(\rho) = \frac{1}{\operatorname{Tr}\rho^2} \in [1, N]. \tag{31}$$

The effective dimension of a mixed state quantifies how many pure states appreciably contribute to the mixture. Moreover, differently from the rank of ρ, d^{eff} captures the probabilistic weight of different states and is more manageable for explicit calculations.

For a separable pure state, $|\xi_0\rangle_A \otimes |\chi_0\rangle_B$, the reduced density matrix, $\rho_A = |\xi_0\rangle\langle\xi_0|$, has effective dimension $d^{\text{eff}}(\rho_A) = \operatorname{rank}(\rho_A) = 1$. A global perturbation acting on $|\xi_0\rangle_A \otimes |\chi_0\rangle_B$ can be appreciated locally if the reduced state ρ_A becomes a mixture. In order to obtain a mixture one needs at least two pure components. Therefore, we can say that a quantum state is distinguishable from a one-dimensional projection if its effective dimension d^{eff} is equal or larger than 2. As a consequence, we are led to the following criterion on the separability of the state averaged over the noise realization:

$$d^{\text{eff}}(\mathbb{E}[\rho_A]) < 2. \tag{32}$$

We get that

$$d^{\text{eff}}(\mathbb{E}[\rho_A]) = \frac{1}{\mathbb{E}\left[\operatorname{Tr}\rho_A^2\right]} = \frac{1}{\mathbb{E}\left[\pi_{AB} \,|\, |\phi_{\text{sep}}\rangle, \sqrt{1-\eta^2}\right]}, \tag{33}$$

and then, from (24), it is straightforward to prove that condition (32) is satisfied if

$$\eta^2 < \eta_\star^2(\pi_{\text{unb}}) = 1 - \sqrt{\frac{1-2\pi_{\text{unb}}}{2-2\pi_{\text{unb}}}}, \tag{34}$$

where π_{unb} is the typical purity of the unbiased ensemble (19). If $\eta \geqslant \eta_\star$ the local state ρ_A will be mixed with high probability.

In the limit of large system sizes, $N, M \to +\infty$, the threshold critical value becomes

$$\eta_\star^2 = \left(1 - \frac{1}{\sqrt{2}}\right) + O\left(\frac{1}{N}\right), \tag{35}$$

since $1/N \leqslant \pi_{\text{unb}} \leqslant 2/N$. Therefore, as long as the state $|\psi\rangle$ of the large quantum system has the form (30) with

$$\eta < \sqrt{1 - \frac{1}{\sqrt{2}}} \simeq 0.54, \tag{36}$$

one has $d^{\text{eff}}(\rho_A) < 2$, and separability will be (approximately) preserved. Notice that in this case, by applying perturbation theory, one gets that the spectrum of ρ_A is made of a large eigenvalue of order $O(1-\eta^2)$ and a sea of eigenvalues of order $O(1/N)$ that have a negligible influence on the reduced density matrix ρ_A.

5. Conclusions

In this paper we have shown that, using the superposition principle, we can take advantage of the knowledge of the unbiased ensemble of random pure states in order to explore new interesting ensembles. In particular, we have found that adding a bias in a suitable direction is enough to polarize the unitarily invariant ensemble. We stress that our approach has been oriented to the study of typical bipartite entanglement between subsystems, as measured by the local purity.

This strategy yields an efficient and simple sampling of random pure states with fixed value of purity, and paves the way to further explorations and a deeper characterization of the geometry of isopurity manifolds.

Finally, we have applied our results to the analysis of separability of quantum states under the influence of random perturbation modeled through a coherent superposition. For large systems, we have obtained a critical value of the noise strength, independent of the system size, beyond which the state is no longer separable, and the reduced state gets appreciably mixed.

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Appendix A. Evaluation of $\bar{\pi}_{AB}$

In this section we detail the derivation of the average purity $\bar{\pi}_{AB}$ given in (17). As mentioned in the text, the averages are more easily obtained by switching from the unitarily invariant measure to the Gaussian measure. The random state in (5) reads

$$|\psi\rangle = \epsilon|\phi_0\rangle + \sqrt{1-\epsilon^2}|\phi\rangle, \tag{A.1}$$

where $|\phi\rangle$ is a Gaussian random vector and $|\phi_0\rangle$ a reference vector. Let the NM complex components of $|\phi\rangle$ and $|\phi_0\rangle$ (in a given basis) be $X_{i\mu}$ and $A_{i\mu}$, respectively, where $1 \leqslant i \leqslant N$ and $1 \leqslant \mu \leqslant M$. In the following the $X_{i\mu}$s will be independent and identically distributed (iid) complex Gaussian variables with mean $\mathbb{E}\left[X_{i\mu}\right] = 0$ and variance $\mathbb{E}\left[|X_{i\mu}|^2\right] = 1/NM$. The one-dimensional projection $|\phi\rangle\langle\phi|$ has random entries $X_{i\mu}X^*_{jv}$, where the star denotes complex conjugation. The reduced state of subsystem A, obtained by partial trace, $\sigma = \mathrm{Tr}_B|\phi\rangle\langle\phi|$ is an N-dimensional square random matrix whose entries are $\sum_\mu X_{i\mu}X^*_{j\mu}$. We are interested in objects like

$$\mathrm{Tr}\,\sigma^2 = \sum_{i,j,\mu,v} X_{i\mu}X^*_{j\mu}X_{jv}X^*_{iv}. \tag{A.2}$$

Recall that the only ingredient necessary to deal with a collection of iid complex Gaussian random variables is the Wick formula for the expectation of the products

$$\mathbb{E}\big[X_{i_1}\cdots X_{i_n}X^*_{j_1}\cdots X^*_{j_n}\big] = \sum_p \mathbb{E}\big[X_{i_1}X^*_{j_{p(1)}}\big]\cdots\mathbb{E}\big[X_{i_n}X^*_{j_{p(n)}}\big] \tag{A.3}$$

where the sum is over all possible permutation p of n elements, and

$$\mathbb{E}\big[X_iX^*_j\big] = (NM)^{-1}\delta_{ij}. \tag{A.4}$$

The expectations of all other products vanish. Thus, the average value of the purity of state $|\phi\rangle$ reads

$$\begin{aligned}\mathbb{E}[\operatorname{Tr}\sigma^2] &= \mathbb{E}\left[\sum_{i,j,\mu,\nu} X_{i\mu}X^*_{j\mu}X_{j\nu}X^*_{i\nu}\right] \\ &= \sum_{i,j,\mu,\nu}\left\{\mathbb{E}\left[X_{i\mu}X^*_{j\mu}\right]\mathbb{E}\left[X_{j\nu}X^*_{i\nu}\right] + \mathbb{E}\left[X_{i\mu}X^*_{i\nu}\right]\mathbb{E}\left[X_{j\nu}X^*_{j\mu}\right]\right\} \\ &= \frac{1}{(MN)^2}\sum_{i,j,\mu,\nu}(\delta_{ij}+\delta_{\mu\nu}) = \frac{1}{(MN)^2}(M^2N+MN^2) \\ &= \frac{M+N}{MN}. \end{aligned} \tag{A.5}$$

Let us consider now the terms (15) and (16). The reduced state $\sigma_0 = \operatorname{Tr}_B|\phi_0\rangle\langle\phi_0|$ has components $\sum_\mu A_{i\mu}A^*_{j\mu}$. Then

$$\begin{aligned}\mathbb{E}[\operatorname{Tr}(\sigma_0\sigma)] &= \mathbb{E}\left[\sum_{i,j,\mu,\nu} A_{i\mu}A^*_{j\mu}X_{j\nu}X^*_{i\nu}\right] = \sum_{i,j,\mu,\nu} A_{i\mu}A^*_{j\mu}\mathbb{E}\left[X_{j\nu}X^*_{i\nu}\right] \\ &= \frac{1}{MN}\sum_{i,\mu,\nu}|A_{i\mu}|^2 = \frac{1}{MN}\sum_{\nu}1 = \frac{1}{N}. \end{aligned} \tag{A.6}$$

The last term $S_{0\phi} = \operatorname{Tr}_B(|\phi_0\rangle\langle\phi| + |\phi\rangle\langle\phi_0|)$ is a random matrix with entries $\sum_\mu A_{i\mu}X^*_{j\mu} + X_{i\mu}A^*_{j\mu}$. By squaring it and taking the trace we obtain

$$\begin{aligned}\mathbb{E}\left[\operatorname{Tr}S^2_{0\phi}\right] &= \mathbb{E}\left[\sum_{i,j,\mu,\nu}(A_{i\mu}X^*_{j\mu} + X_{i\mu}A^*_{j\mu})(A_{j\nu}X^*_{i\nu} + X_{j\nu}A^*_{i\nu})\right] \\ &= 2\sum_{i,j,\mu}|A_{i\mu}|^2\mathbb{E}[|X_{j\mu}|^2] \\ &= \frac{2}{MN}\sum_{i,j,\mu}|A_{i\mu}|^2 = \frac{1}{MN}\sum_{j}1 = \frac{2}{M}. \end{aligned} \tag{A.7}$$

Adding up all the pieces we obtain the result (17).

Finally, we notice that the computation with a Gaussian measure deviates from the computation with a uniform measure on the sphere only in the four-point correlation (A.5). Indeed for a unit vector $|\phi\rangle$ uniformly distributed on the unit sphere the Wick theorem (A.3) is no longer valid and the fourth moment is slightly modified into

$$\mathbb{E}\left[X_{i\mu}X^*_{j\mu}X_{j\nu}X^*_{i\nu}\right] = \frac{1}{MN(MN+1)}(\delta_{ij}+\delta_{\mu\nu}), \tag{A.8}$$

(see equation (53) of [31]) which gives (18) in place of (A.5).

Appendix B. Gaussian approximation and typicality

The typicality of the average purity in the polarized ensembles used in section 3.2 relies on the following concentration phenomenon for Gaussian variables [22]:

Lemma 1. *Let* $\boldsymbol{X} = (X_1, X_2, \ldots, X_k)$ *be a vector with iid Gaussian components, with distribution* $X_i \sim \mathcal{N}(0, \sigma^2)$. *Then, for any smooth function* $f : \mathbb{R}^k \to \mathbb{R}$, *with* $\eta = \sup |\nabla f| < \infty$, *the following concentration inequality holds*

$$\mathbf{Pr}\{|f(\boldsymbol{X}) - \mathbb{E}[f]| > \alpha\} \leqslant 2\exp\left(-\frac{\alpha^2}{4\eta^2\sigma^2}\right). \tag{B.1}$$

Let us now consider the polarized ensemble defined in (5). In the Gaussian approximation the $2MN$ real coordinates of the random vector state are Gaussian iid random variables with distribution $\mathcal{N}(0, 1/2MN)$, so that normalization is assured on average. However, since $\|\cdot\|$ has Lipschitz constant 1, $\|\psi\|$ has $\eta \leqslant \sqrt{1-\epsilon^2}$ and thus the ensemble (5) is composed of normalized vectors with overwhelming probability:

$$\mathbf{Pr}\{|\|\psi\|^2 - 1| > \alpha\} \leqslant 2\exp\left(-\frac{NM\alpha^2}{2(1-\epsilon^2)}\right). \tag{B.2}$$

Moreover, the local purity function π_{AB}, defined in (4), has Lipschitz constant bounded by $\eta \leqslant 4$ (see [32]). The Lipschitz constant of the purity of the polarized ensemble (5) is thus bounded by $\eta \leqslant 4\sqrt{1-\epsilon^2}$ and then from the lemma one gets

$$\mathbf{Pr}\{|\pi_{AB}(|\psi\rangle) - \mathbb{E}[\pi_{AB}]| > \alpha\} \leqslant 2\exp\left(-\frac{NM\alpha^2}{32(1-\epsilon^2)}\right). \tag{B.3}$$

Incidentally, we mention that a similar Gaussian tail can be derived for uniformly distributed unit vectors by Levy's lemma [32]. The proof relies on a judicious use of δ-nets. In a finite dimensional setting, any subset of the sphere of states is totally bounded, in the sense that it admits a finite δ-net, for all $\delta > 0$. What we are interested in is a bound on the cardinality of a δ-net $\mathcal{N}$ on manifolds of equal Schmidt rank k. For such manifolds a bound is given by

$$|\mathcal{N}| \leqslant \left(\frac{10}{\delta}\right)^{2k(N+M)}. \tag{B.4}$$

For a proof see [32]. In this work the authors bound the cardinality of δ-nets on set with fixed Schmidt rank. In fact, what they obtain are δ-nets on orbits of pure states under local unitaries, i.e. states with fixed Schmidt coefficients. Estimate (B.4) is good enough to control the probability of deviations of the purity from its average, so that a bound of the form (B.3) is obtained.

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