Quantum parameter estimation affected by unitary disturbance

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We provide a general framework for handling the effects of a unitary disturbance on the estimation of the amplitude λ associated to a unitary dynamics. By computing an analytical and general expression for the quantum Fisher information, we prove that the optimal estimation precision for λ cannot be outperformed through the addition of such a unitary disturbance. However, if the dynamics of the system is already affected by an external field, increasing its strength does not necessarily imply a loss in the optimal estimation precision.

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I. INTRODUCTION

The quantum Cramér-Rao bound provides the proper theoretical framework for analyzing energy-time-like uncertainty relations [1] by setting limits on the precision attainable when estimating the parameters governing the dynamics of a physical system. Its application has profound consequences in quantum metrology [2], where it helps in identifying which resources (e.g., entanglement and squeezing) are useful to reach higher accuracy levels, and which are the proper procedures one needs to adopt to fully exploit them. For instance, in the absence of external noise, the quantum Cramér-Rao bound predicts [3] that in the process of estimating a relative phase, the use of entanglement between sequences of N independent probing systems allows one to gain a $\sqrt{N}$ improvement in precision (Heisenberg limit) over those procedures which, under the same experimental settings, adopt instead separable probes (standard quantum limit, or shot-noise limit in optical interferometry). More subtle is to establish the optimal performances in the presence of external disturbances. Many discouraging results attest to the fragility of entanglement which, in a noisy environment, limits any precision improvement at most to a constant factor independent of N [4], or to a superclassical precision scaling $N^{-5/6}$, achieved when the perturbation involves a preferential direction perpendicular to the unitary evolution governed by the parameter to be estimated [5]. Yet an exhaustive answer would require a systematic method for taking into account the presence of a disturbance in the system. The main obstacle is represented by the fact that the very fundamental tool needed to evaluate the quantum Cramér-Rao inequality, i.e., the quantum Fisher information (QFI) [6], apart from very simple cases, usually happens to be computationally cumbersome, especially for high dimensional systems. Recently, Escher *et al.* [7] proposed a strategy to circumvent this difficulty by introducing an upper bound to the QFI, which relies on the choice of a Kraus representation of the noisy evolution based on physical considerations. In this way, it was possible to propose a realistic example of optical lossy interferometry where the Heisenberg limit can be attained by properly tuning the number of input resources according to the noise level [7]. A generalization of the latter analysis for the single parameter estimation to the case of lossy optical wave-form reconstruction has been recently proposed [8]. Notwithstanding these bright techniques, a general prescription for computing the QFI of a generic dynamical process is still missing.

In this paper we consider the case of closed quantum systems and study the effects of a unitary disturbance on the estimation of a dynamical parameter λ. Specifically, we add a term to the generator of the dynamical evolution of the system to model the action of an external force that opposes the formation of the parametric trajectories governed by λ, and determine a compact analytical expression for the associated QFI. Starting from this result, we report several important facts. First of all, while for nonoptimal choices of model settings it is possible that the disturbance will improve the accuracy of the estimation procedure, we prove a *no-go theorem* which formalizes the rather intuitive fact that the *best* performances are always reached when no disturbance is present in the system. Most importantly we also notice that enhancing the level of a Hamiltonian disturbance which is already affecting the system does *not* necessarily yield a worse *optimal* estimation strategy and can reveal itself helpful in determining the value of λ. This is a rather counterintuitive finding which can be interpreted as the emergence of *dithering* [9] in the estimation process.

This paper is organized as follows. In Sec. II, after briefly reviewing the typical approach followed for the reconstruction of the global phase λ of a generic unitary evolution, we explicitly address the case in which the dynamics is affected by the presence of a unitary disturbance (Sec. II A). This technique is then reframed in the more general context of multiparametric estimation (Sec. II B). In Sec. III, we prove a *no-go theorem* comparing the optimal performances achievable with and without the unitary disturbance. In Sec. III we address the question of whether the additional term in the Hamiltonian is sufficient to induce a departure from the Heisenberg limit, and show that this is not the case. Finally, in Sec. IV, we specialize to the case of a single qubit, and gather evidence that if the system is already affected by such a unitary disturbance, the latter can be increased in order to achieve better estimation performances. Section V is devoted to final remarks.

II. THEORETICAL FRAMEWORK

A standard problem in quantum estimation theory is recovering a real parameter λ encoded in a set of states ϱ_λ of the system. The ultimate precision limit for such a task is given by the quantum Cramér-Rao bound [2] on the Root Mean Square Error (RMSE) $\Delta\lambda$ of a generic estimation strategy (the latter is defined as $\Delta\lambda = \sqrt{\mathbb{E}[(\lambda^{(\text{est})} - \lambda)^2]}$ where $\lambda^{(\text{est})}$ is the random variable which represents the estimation of λ extrapolated from the performed measurements, and $\mathbb{E}[x]$ indicates the expectation value of the random variable x). Accordingly, we have

$$\Delta\lambda \geqslant 1/\sqrt{\nu \mathcal{Q}}, \tag{1}$$

where $\mathcal{Q}$ is the QFI obtained by optimizing the Fisher information [10] over all the possible positive-operator valued measurements performed on the system encoding the parameter, and ν is the number of times the measurement is repeated (the threshold being reachable at least in the asymptotic limit of large ν—see however Ref. [11] for achievability at finite ν). The QFI is a function of the parameter λ which can be expressed in terms of the "instantaneous" velocity variation of the system, quantified by the Bures distance $\mathcal{D}_B$ [12],

$$\begin{aligned}\mathcal{Q} &= 4 \lim_{\delta\lambda\to 0} \frac{\mathcal{D}_B^2(\varrho_\lambda,\varrho_{\lambda+\delta\lambda})}{\delta\lambda^2} \\ &= 8 \lim_{\delta\lambda\to 0} \frac{1-\mathcal{F}(\varrho_\lambda,\varrho_{\lambda+\delta\lambda})}{\delta\lambda^2}\ ,\end{aligned} \tag{2}$$

where $\mathcal{F}(\varrho,\varrho') = \text{Tr}[\sqrt{\sqrt{\varrho}\,\varrho'\sqrt{\varrho}}]$ is the fidelity between the states ϱ and ϱ' [13].

A well studied case is the one in which the parameter λ is encoded into the state of a quantum system through a unitary transformation of the form

$$\varrho_\lambda = U_\lambda \varrho_0 U_\lambda^\dagger, \quad U_\lambda = \exp(-i\lambda H_I), \tag{3}$$

where ϱ_0 is the input state of the system, assumed to be controllable, and H_I is the generator of the parametric orbit, assumed to be assigned and independent from λ (as an example, consider the case of a massive particle that undergoes an abrupt translation induced by an external force whose intensity we wish to estimate by monitoring the particle). Under these conditions the QFI is also independent from λ, and is given by [14,15]

$$\mathcal{Q}[\varrho_0] = 4\sum_{j<j'} \frac{(\rho_j - \rho_{j'})^2}{\rho_j + \rho_{j'}} |\langle j|H_I|j'\rangle|^2\ , \tag{4}$$

where ρ_j and $|j\rangle$ are, respectively, the eigenvalues and the eigenvectors of ϱ_0. From the strong concavity of $\mathcal{F}$ it follows that the maximum of Eq. (4) is obtained for pure states $\varrho_0 = |\phi_0\rangle\langle\phi_0|$. In that case Eq. (4) becomes

$$\begin{aligned}\mathcal{Q} &= 4\langle\phi_0|\Delta^2 H_I|\phi_0\rangle \\ &= 4\big(\langle\phi_0|H_I^2|\phi_0\rangle - \langle\phi_0|H_I|\phi_0\rangle^2\big).\end{aligned} \tag{5}$$

Thereby the state ϱ_0 maximizing the value of $\mathcal{Q}$ [i.e., minimizing the RMSE threshold (1)] can be identified by observing that the maximum variance of the Hermitian operator H_I is proportional to the square of its spectral width:

$$\langle\Delta^2 H_I\rangle_{\max} = \frac{(h_{\max} - h_{\min})^2}{4}\ , \tag{6}$$

where $h_{\max/\min}$ is the maximum/minimum eigenvalue of H_I. Accordingly we have

$$\mathcal{Q}_{\max} := \max_{\varrho_0} \mathcal{Q}[\varrho_0] = (h_{\max} - h_{\min})^2\ , \tag{7}$$

the maximum being achieved by using as optimal input $\varrho_0^{(\text{opt})} = |\psi_0^{(\text{opt})}\rangle\langle\psi_0^{(\text{opt})}|$ the equally weighted superposition of the eigenvectors belonging to $h_{\max/\min}$, i.e.,

$$\big|\psi_0^{(\text{opt})}\big\rangle = \frac{1}{\sqrt{2}}(|h_{\max}\rangle + |h_{\min}\rangle)\ . \tag{8}$$

A. Phase estimation with unitary disturbance

Let us now consider the case where the above estimation process is disturbed by the presence of an additional contribution to the generator of the dynamics. Specifically we replace U_λ of Eq. (3) with the transformation

$$U_{\lambda,\eta} = \exp[-iH(\lambda,\eta)], \quad H(\lambda,\eta) = \lambda H_I + \eta H_0, \tag{9}$$

where H_0 is an Hermitian operator interfering with the parametric driving exerted by H_I, and where the real quantity η gauges the strength of the associated perturbation (in the example discussed previously, H_0 can be identified with a trapping potential that opposes the translation of the massive particle). In order to compute the QFI for λ, for any fixed η, we apply Uhlmann's theorem on the fidelity [16]:

$$\mathcal{F}(\varrho_\lambda,\varrho_{\lambda+\delta\lambda}) = \max_{|\varrho_\lambda\rangle,|\varrho_{\lambda+\delta\lambda}\rangle} |\langle\varrho_\lambda|\varrho_{\lambda+\delta\lambda}\rangle|\ , \tag{10}$$

the maximization being performed over all possible purifications $|\varrho_\lambda\rangle$ and $|\varrho_{\lambda+\delta\lambda}\rangle$ of ϱ_λ and $\varrho_{\lambda+\delta\lambda}$, respectively, through an ancillary system. By using the freedom in the purifications we write

$$\mathcal{F} = \max_V |\langle\varrho_0|\overleftarrow{\exp}[-i\,\delta\lambda\,\bar{H}_I(\lambda,\eta)]\otimes V|\varrho_0\rangle|, \tag{11}$$

where V belongs to the set of unitary transformations on the ancilla, while $|\varrho_0\rangle = \sum_j \sqrt{\rho_j}|j\rangle\otimes|j\rangle$ is a fixed purification of the initial state ϱ_0 (hereafter, by writing $A\otimes B$ we mean that A acts on the system and B on the ancilla). The average Hamiltonian

$$\bar{H}_I(\lambda,\eta) = \int_0^1 dt\ e^{iH(\lambda,\eta)t} H_I e^{-iH(\lambda,\eta)t} \tag{12}$$

emerges from the interaction picture representation of the evolution

$$U_{\lambda+\delta\lambda,\eta} = U_{\lambda,\eta}\overleftarrow{\exp}[-i\,\delta\lambda\,\bar{H}_I(\lambda,\eta)]\ , \tag{13}$$

with $\overleftarrow{\exp}[\ldots]$ denoting the time-ordered exponential (a similar approach was employed in Ref. [17]). Since we are interested in the limit of small $\delta\lambda$, without loss of generality we set $V = \exp(i\,\delta\lambda\,\Omega)$, with Ω a Hermitian operator on the ancillary system. It results that, up to corrections of order $O(\delta\lambda^4)$, the fidelity reads

$$\mathcal{F} \simeq 1 - \frac{\delta\lambda^2}{2}\min_\Omega[\langle\varrho_0|(\Delta\bar{H}_I\otimes\mathbb{I} - \mathbb{I}\otimes\Delta\Omega)^2|\varrho_0\rangle]\ , \tag{14}$$

where $\Delta\bar{H}_I = \bar{H}_I - \bar{h}$ and $\Delta\Omega = \Omega - \omega$, with $\bar{h} = \text{Tr}[\varrho_0\bar{H}_I]$ and $\omega = \text{Tr}[\varrho_0\Omega]$. Using the spectral decomposition of ϱ_0 introduced above, the QFI in Eq. (2) can be written as

$$\mathcal{Q}_{\lambda;\eta}[\varrho_0] = 4 \min_{\Omega} \mathrm{Tr}\Bigg[\bar{H}_I^2 \varrho_0 + \Omega^2 \varrho_0 - 2\sqrt{\varrho_0} \bar{H}_I^\top \sqrt{\varrho_0} \Omega - \bar{h}^2 \varrho_0 - \sum_{i,j} \rho_i \rho_j \Omega |i\rangle\langle j|\Omega|j\rangle\langle i| + 2\bar{h}\varrho_0 \Omega \Bigg], \tag{15}$$

where $\top$ denotes transposition. By differentiating the trace with respect to Ω we determine the minimization condition for it:

$$\varrho_0(\Omega - \omega) + (\Omega - \omega)\varrho_0 = 2\sqrt{\varrho_0}(\bar{H}_I^\top - \bar{h})\sqrt{\varrho_0}\,. \tag{16}$$

Its solution displays a translational invariance with respect to ω according to

$$\Omega - \omega = \Omega' - \omega', \quad \text{with} \quad \Omega' = \Omega + g\mathbb{I}, g \in \mathbb{C}\,. \tag{17}$$

Without loss of generality we can therefore fix $\omega = \bar{h}$ and write the solution of Eq. (16) in a basis for the ancilla isomorphic to the eigenbasis of ϱ_0 as

$$\Omega_{jj'} = 2[\bar{H}_I]_{j'j} \frac{\sqrt{\rho_j \rho_{j'}}}{\rho_j + \rho_{j'}}\,. \tag{18}$$

Finally, by substituting this solution into (15), we obtain the QFI for λ in the presence of an arbitrary disturbance ηH_0:

$$\mathcal{Q}_{\lambda;\eta}[\varrho_0] = 4 \sum_{j<j'} \frac{(\rho_j - \rho_{j'})^2}{\rho_j + \rho_{j'}} |\langle j|\bar{H}_I|j'\rangle|^2\,. \tag{19}$$

Notice that for $\eta = 0$, since $\bar{H}_I$ reduces to H_I, this expression gives back Eq. (4), i.e., $\mathcal{Q}_{\lambda;0}[\varrho_0] = \mathcal{Q}[\varrho_0]$. Furthermore in complete analogy to the latter case, if the initial state of the system is pure, Eq. (19) yields

$$\mathcal{Q}_{\lambda;\eta}[|\phi_0\rangle] = 4\langle\phi_0|\Delta^2 \bar{H}_I|\phi_0\rangle. \tag{20}$$

At variance with Eqs. (4) and (5), for $\eta \neq 0$, Eqs. (19) and (20) can exhibit an explicit dependence on λ via Eq. (12) (an example is provided below). In particular, this implies that the optimal states $\varrho_0^{(\mathrm{opt})}$ yielding the maximum of the QFI (and of course the QFI maximum itself) can now depend on the value of the parameter one wishes to estimate. Specifically, indicating with $\bar{h}_{\max/\min}(\lambda,\eta)$ the maximum/minimum eigenvalue of the average Hamiltonian $\bar{H}_I(\lambda,\eta)$ and with $|\bar{h}_{\max/\min}\rangle$ its corresponding eigenvector, we now get

$$\mathcal{Q}_{\lambda;\eta}^{(\max)} := \max_{\varrho_0} \mathcal{Q}_{\lambda;\eta}[\varrho_0] = [\bar{h}_{\max}(\lambda,\eta) - \bar{h}_{\min}(\lambda,\eta)]^2, \tag{21}$$

with the optimal state $\varrho_0^{(\mathrm{opt})} = |\psi_0^{(\mathrm{opt})}\rangle\langle\psi_0^{(\mathrm{opt})}|$ being the superposition [18]

$$|\psi_0^{(\mathrm{opt})}\rangle = \frac{1}{\sqrt{2}}[|\bar{h}_{\max}(\lambda,\eta)\rangle + |\bar{h}_{\min}(\lambda,\eta)\rangle]\,. \tag{22}$$

B. Multiparametric estimation

Equations (19) and (21) represent the central finding of our paper, and pave the way to a number of observations on the role played by a unitary disturbance in the estimation procedure. Before detailing them, we notice that the analysis presented so far can be naturally framed in the more general context of multiparametric estimation, where the family of states $\varrho_{\vec{\lambda}}$ now depends on a set of parameters $\vec{\lambda} = (\lambda_1, \ldots, \lambda_M)$, with $M \geqslant 2$. In this context the Cramér-Rao theorem is generalized to a bound

$$\mathbf{Cov}[\vec{\lambda}] \geqslant \boldsymbol{\mathcal{Q}}^{-1}/\nu \tag{23}$$

on the covariance matrix,

$$[\mathbf{Cov}[\vec{\lambda}]]_{j,k} = \mathbb{E}\big[\lambda_j^{(\mathrm{est})}\lambda_k^{(\mathrm{est})}\big] - \lambda_j \lambda_k\,, \tag{24}$$

with $\boldsymbol{\mathcal{Q}}$ being the QFI matrix of the problem. While referring the reader to the formal expression of $\boldsymbol{\mathcal{Q}}$, we remind that its diagonal elements coincide with the QFI of the corresponding parameter λ_j, at fixed values of the others. The off-diagonal terms can be evaluated in a similar way by observing that for any other set of parameters $\vec{\mu} = \vec{\mu}(\vec{\lambda})$, which is an invertible function of $\vec{\lambda}$, the associated QFI matrix can be computed as $\tilde{\boldsymbol{\mathcal{Q}}} = \boldsymbol{J}\boldsymbol{\mathcal{Q}}\boldsymbol{J}^\top$, where $\boldsymbol{J}$ is the Jacobian matrix with elements $[\boldsymbol{J}]_{jk} = \partial\lambda_k/\partial\mu_j$.

Let us consider, for example, the case of two parameters $\vec{\lambda} = (\lambda,\eta)$. The diagonal elements become, respectively,

$$[\boldsymbol{\mathcal{Q}}]_{\lambda\lambda} = \mathcal{Q}_{\lambda;\eta}[\varrho_0], \quad [\boldsymbol{\mathcal{Q}}]_{\eta\eta} = \mathcal{Q}_{\eta;\lambda}[\varrho_0], \tag{25}$$

given by Eq. (19) and its analog obtained by substituting $\bar{H}_I$ with $\bar{H}_0$ [whose definition is exactly specular to that in Eq. (12)]. By defining

$$\mu_1 = \frac{\lambda + \eta}{\sqrt{2}}\,, \quad \mu_2 = \frac{\lambda - \eta}{\sqrt{2}}\,, \tag{26}$$

one can also compute $[\boldsymbol{\mathcal{Q}}]_{\lambda\eta}$ as

$$[\boldsymbol{\mathcal{Q}}]_{\lambda\eta} = [\boldsymbol{\mathcal{Q}}]_{\eta\lambda} = [\tilde{\boldsymbol{\mathcal{Q}}}]_{\mu_1\mu_1} - ([\boldsymbol{\mathcal{Q}}]_{\lambda\lambda} + [\boldsymbol{\mathcal{Q}}]_{\eta\eta})/2. \tag{27}$$

From the previous analysis of the QFI at $M = 1$ for a system affected by a unitary disturbance, it follows that $[\tilde{\boldsymbol{\mathcal{Q}}}]_{\mu_1\mu_1}$ can be smoothly determined by rewriting the global Hamiltonian of the system as

$$H = \mu_1 \frac{(H_I + H_0)}{\sqrt{2}} + \mu_2 \frac{(H_I - H_0)}{\sqrt{2}}\,, \tag{28}$$

and by using Eq. (19) upon substituting $\bar{H}_I$ with $(\bar{H}_I + \bar{H}_0)/\sqrt{2}$. It follows that the off-diagonal terms of the QFI matrix are

$$[\boldsymbol{\mathcal{Q}}]_{\lambda\eta} = 4 \sum_{j<j'} \frac{(\rho_j - \rho_{j'})^2}{\rho_j + \rho_{j'}} \mathrm{Re}[\langle j|\bar{H}_I|j'\rangle\langle j'|\bar{H}_0|j\rangle]\,. \tag{29}$$

This technique can be naturally extended to the case of an arbitrary number of parameters.

III. NO-GO THEOREM

A question which spontaneously arises from the similarity between the expressions for the QFI with and without a unitary disturbance, i.e., Eqs. (4) and (19), concerns the possibility to compare the performances of an estimation procedure in the two cases. First of all, it is evident that for nonoptimal choices of the input state ϱ_0, it is indeed possible that a nonzero value of η could help the estimation process (for an explicit example, take ϱ_0 to be an eigenvector of H_I [19]). However, in terms of *optimal* estimation thresholds, the following no-go theorem can be derived:

No-go theorem. It is not possible to outperform the optimal estimation strategy for the amplitude λ of the unitary dynamics (3) through the addition of any linear contribution to its generator, namely,

$$\mathcal{Q}^{(\max)}_{\lambda;\eta} \leqslant \mathcal{Q}^{(\max)}_{\lambda;0} = \mathcal{Q}_{\max}\,. \tag{30}$$

In order to prove this inequality we observe that Eqs. (7) and (21) allow us to equivalently rewrite it in terms of a contraction of the spectral width of the Hamiltonian $\bar{H}_I$ with respect to H_I, i.e.,

$$\bar{h}_{\max}(\lambda,\eta) - \bar{h}_{\min}(\lambda,\eta) \leqslant h_{\max} - h_{\min}\,. \tag{31}$$

The latter can then be proved by observing that the operator $\bar{H}_I$ is obtained from H_I via a weighted convex sum of random unitaries. Therefore, according to Uhlmann's majorization theorem [20], $\bar{H}_I$ is majorized by H_I. This, in particular, implies

$$\bar{h}_{\max}(\lambda,\eta) \leqslant \bar{h}_{\max}(\lambda,0) \quad \text{and} \quad \bar{h}_{\min}(\lambda,\eta) \geqslant \bar{h}_{\min}(\lambda,0) \tag{32}$$

from which the contraction of the spectral width, and hence Eq. (30), is derived.

Heisenberg limit

Once established that $\mathcal{Q}^{(\max)}_{\lambda;\eta}$ is always smaller than $\mathcal{Q}^{(\max)}_{\lambda;0}$, one might ask whether or not this implies a departure from the Heisenberg limit of the optimal accuracy. We remind the reader that the latter is associated with the case where (say) N independent probes are prepared in entangled states before being acted upon by the generator of the dynamics. Formally this can be accounted for by replacing H_I of Eqs. (3) to (5) with the operator

$$H_I^{(N)} = \sum_{j=1}^{N} H_I^{(j)} \tag{33}$$

with $H_I^{(j)}$ being the local generator acting on the jth probe (see Ref. [3] for details). As a result, $\mathcal{Q}_{\max}$ of Eq. (7) becomes

$$\mathcal{Q}^{(N)}_{\max} = N^2 (h_{\max} - h_{\min})^2 \tag{34}$$

with $h_{\max/\min}$ being still the extremal eigenvalues of the local (single probe) Hamiltonian H_I (the N^2 dependence certifying the arising of the Heisenberg limit in the RMSE accuracy). If the Hamiltonian disturbance H_0 is acting locally on the individual probes, it is immediate to see that the same dependence upon N remains also for $\eta \neq 0$. Indeed in this case $H(\lambda,\eta)$ is replaced by

$$H^{(N)}(\lambda,\eta) = \sum_{j=1}^{N} \left(\lambda H_I^{(j)} + \eta H_0^{(j)}\right), \tag{35}$$

which is still given by a sum of N independent, local contributions yielding

$$\mathcal{Q}^{(N,\max)}_{\lambda;\eta} = N^2[\bar{h}_{\max}(\lambda,\eta) - \bar{h}_{\min}(\lambda,\eta)]^2\,, \tag{36}$$

where bars refer to eigenvalues of the single probe Hamiltonian $\bar{H}_I$. The situation becomes more complex when H_0 is nonlocal and $H^{(N)}(\lambda,\eta)$ acquires coupling terms between the N probes. As the overall evolution is still unitary, one is tempted to conjecture that the same N^2 scaling for $\mathcal{Q}^{(N,\max)}_{\lambda;\eta}$ should survive in typical situations. A rigorous proof of this fact is left to a future investigation.

IV. OPTIMAL ACCURACY IMPROVEMENT VIA DISTURBANCE

Inequality (30) compares the best achievable performance with and without the addition of a linear disturbance ηH_0 to the generator of a given unitary dynamics (3). From this relation one could be tempted to conclude that the maximum QFI is a monotonic decreasing function of η; that is, the larger the disturbance is, the worse the estimation of λ. In general, however, this is not true: once the threshold $\eta = 0$ has been crossed, Eq. (30) does not provide a recipe for comparing the response of the system to increasing or decreasing values of η [21]. This opens the possibility of dithering effects.

We now provide an explicit example of such phenomenon in a qubit system. Let us adopt the Bloch sphere formalism and set

$$H_I = \boldsymbol{a}\cdot\boldsymbol{\sigma}\,, \quad H_0 = \boldsymbol{b}\cdot\boldsymbol{\sigma}\,, \tag{37}$$

where, without any loss of generality, $\boldsymbol{a}$ and $\boldsymbol{b}$ are unit (three-dimensional) real vectors, and $\boldsymbol{\sigma}$ is the vector of Pauli matrices. In this case the average Hamiltonian is given by $\bar{H}_I = \boldsymbol{m}\cdot\boldsymbol{\sigma}$, with

$$\boldsymbol{m} = [1+\operatorname{sinc}(2\theta)]\boldsymbol{a}/2 - \eta(\boldsymbol{b}\wedge\boldsymbol{a})\operatorname{sinc}^2\theta + \frac{1-\operatorname{sinc}(2\theta)}{2\theta^2}[(\boldsymbol{n}\cdot\boldsymbol{a})\boldsymbol{n} - \eta(\boldsymbol{b}\wedge\boldsymbol{a})\wedge\boldsymbol{n}], \tag{38}$$

where $\wedge$ is the vector product, $\operatorname{sinc} x = x^{-1}\sin x$, and

$$\boldsymbol{n} = \lambda\boldsymbol{a} + \eta\boldsymbol{b}, \quad \theta = |\boldsymbol{n}|. \tag{39}$$

From Eq. (21) it immediately follows that

$$\mathcal{Q}^{(\max)}_{\lambda;\eta} = 4|\boldsymbol{m}|^2\,. \tag{40}$$

In Fig. 1 we plot Eq. (40) as a function of λ for different values of $\eta \geqslant 0$, the case $\eta < 0$ being symmetric with respect to $\lambda = 0$. For $\eta = 0$ (no unitary disturbance) we have $|\boldsymbol{m}|^2 = 1$ for all λ's: as already observed, the optimal choice for the initial state of the system does not depend on the amplitude of the unitary dynamics. On the other hand, according to the no-go

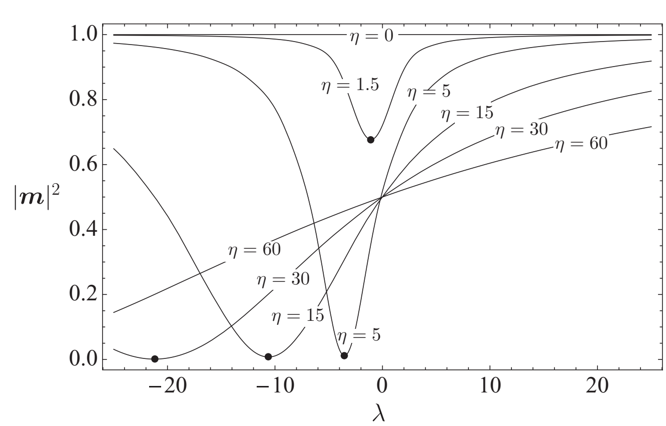

FIG. 1. Plot of $\mathcal{Q}^{(\max)}_{\lambda;\eta}/4$ of Eq. (40) as a function of λ for different values of η, and for $\boldsymbol{a}\cdot\boldsymbol{b} = 1/\sqrt{2}$. The dots signal the minima at $\lambda_{\min} = -\eta\,\boldsymbol{a}\cdot\boldsymbol{b}$.

theorem, for all $\eta \neq 0$ we always have that $|\boldsymbol{m}|^2 < 1$. This function shows a minimum at $\lambda_{\min} = -\eta\, \boldsymbol{a} \cdot \boldsymbol{b}$, and asymptotically reaches 1 for $\lambda \to \pm\infty$ (in this regime the effects of the unitary disturbance ηH_0 can be considered negligible). The antilinear relation between $\lambda_{\min}$ and η determines the following behavior of $|\boldsymbol{m}|^2$: for η large enough, there exists an interval I such that for $\lambda \in I$,

$$\mathcal{Q}^{(\max)}_{\lambda;\eta} < \mathcal{Q}^{(\max)}_{\lambda;\tilde{\eta}}, \quad \text{for} \quad \tilde{\eta} > \eta \tag{41}$$

(see Fig. 1). Inequality (41) establishes that, for sufficiently large η, there exists a finite interval of λ's whose values, by properly choosing the state of the input probe, can be estimated better than the values achievable with *any* possible choice of ϱ_0, when the unitary disturbance is smaller.

V. CONCLUSIONS

The optimal estimation precision for the amplitude of a unitary dynamics cannot be enhanced by switching on an external field, or more generally by adding a linear term to the generator of the dynamical process. However, we have shown that if the system is already affected by such a unitary disturbance, enhancing its strength does not necessarily imply a loss in the estimation precision of the other dynamical parameter(s). These results have been achieved by explicitly computing the quantum Fisher information for an arbitrary system in a generic mixed state, thus generalizing the already known expression for the case of a unitary dynamics (3). Furthermore, reframed into the more general context of multiparametric estimation, this analysis enabled us to easily determine a compact analytical expression for all the elements of the quantum Fisher information matrix.

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[18] The explicit dependence of $\varrho_0^{(\mathrm{opt})}$ upon λ requires a comment on the achievability of the accuracy bound (1) associated with the maximum QFI value of Eq. (21). In this case, in fact, the standard argument [2] of the asymptotic optimality of (1) for large enough ν seems to include a circular reasoning [in brief, to reach the specific threshold dictated by Eq. (21), one would need to *first* identify the state $\varrho_0^{(\mathrm{opt})}$ which in turn depends upon the value we wish to determine]. This however can be avoided by using adaptive, recursive protocols where the information about λ recovered from an initial set of measurements performed with a nonoptimal input state ϱ_0 is used to refine the state preparation of the probe in the subsequent trials.
[19] For such a choice, $\mathcal{Q}_{\lambda;0}[\varrho_0] = 0$, implying that *no* information can be recovered on λ when $\eta = 0$ [the state of the probe being invariant under the mapping (3)]. On the contrary, as in general for $\eta \neq 0$ the operator $\bar{H}_I$ differs from H_I, the same result does not necessarily hold when the disturbance is on (indeed in this case we can have $\mathcal{Q}_{\lambda;0}[\varrho_0] > 0$).
[20] M. Nielsen (unpublished), http://michaelnielsen.org/blog/talks/2002/maj/book.ps.
[21] Notice that, from the definition (12) of $\bar{H}_I$, we cannot assert that $\bar{H}_I(\lambda,\eta + \Delta\eta)$ majorizes $\bar{H}_I(\lambda,\eta)$ for arbitrary η and $\Delta\eta \neq 0$.