Large-time limit of the quantum Zeno effect

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If very frequent periodic measurements ascertain whether a quantum system is still in its initial state, its evolution is hindered. This peculiar phenomenon is called quantum Zeno effect. We investigate the large-time limit of the survival probability as the total observation time scales as a power of the measurement frequency, $t \propto N^{\alpha}$. The limit survival probability exhibits a sudden jump from 1 to 0 at $\alpha = 1/2$, the threshold between the quantum Zeno effect and a diffusive behavior. Moreover, we show that for $\alpha \geq 1$, the limit probability becomes sensitive to the spectral properties of the initial state and to the arithmetic properties of the measurement periods. *Published by AIP Publishing.* [http://dx.doi.org/10.1063/1.4978851]

I. INTRODUCTION

The evolution of a quantum system is halted when many measurements are performed in a finite time, in order to check whether the system is still in its initial state. This phenomenon is called the Quantum Zeno Effect[1] (QZE): the survival probability at a given time goes to one as the measurement frequency increases.

The survival probability after N measurements in a time t is expressed by a product formula depending on t and N. In this paper we investigate the uniformity in time of the QZE, and study the behavior of the Zeno product formula for large N *and* t.

Let us first recall the basics of the QZE. Let a quantum system be prepared, at time $t = 0$, in the state ψ, a normalized vector in the separable Hilbert space $\mathcal{H}$. We denote by $\langle \cdot | \cdot \rangle$ the scalar product in $\mathcal{H}$. The system evolves under the action of the Hamiltonian H, a self-adjoint operator on $\mathcal{H}$, through the unitary group $t \mapsto \exp(-itH/\hbar)$. The quantities

$$\mathcal{A}(t) = \langle \psi | \exp\left(-\frac{it}{\hbar}H\right)\psi \rangle \tag{1}$$

and

$$p(t) = |\mathcal{A}(t)|^2 = \left|\langle \psi | \exp\left(-\frac{it}{\hbar}H\right)\psi \rangle\right|^2 \tag{2}$$

are called survival (or return) amplitude and probability, respectively, and represent the amplitude and probability that the quantum system is found back in the initial state ψ at time t.

If the state ψ is in the domain of the Hamiltonian H, we have for $t \to 0$

$$p(t) = 1 - \frac{t^2}{\hbar^2}\left(\langle H\psi | H\psi \rangle - \langle \psi | H\psi \rangle^2\right) + o\left(t^2\right), \tag{3}$$

where $\langle H\psi | H\psi \rangle - \langle \psi | H\psi \rangle^2$ is the variance of the Hamiltonian in the state ψ.

Let us now carry out N repeated measurements with period $\tau = t/N$, in order to check whether the system remains in its initial state. If at each and every time the measurement has a positive outcome

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and the system is found in its initial state, the state "collapses" and the evolution starts anew from ψ. Thus, the survival probability after N measurements reads

$$p^{(N)}(t) := p\left(\frac{t}{N}\right)^N = \left|\langle\psi|\exp\left(-\frac{it}{\hbar N}H\right)\psi\rangle\right|^{2N}. \tag{4}$$

This is called *Zeno product formula* and will be the subject of our investigation.

The limit of infinitely frequent measurements, $N \to +\infty$, of the Zeno product formula can be easily computed using the Taylor expansion in (3): if the initial state ψ is in the domain of the Hamiltonian H one gets

$$\lim_{N\to+\infty} p^{(N)}(t) = 1, \tag{5}$$

uniformly in t on compact subsets of $\mathbb{R}$, see Refs. 2–4. Therefore, if one performs frequent measurements on a quantum system in a given time interval $[0, t]$, a QZE takes place:[1] the transitions to states different from the initial one are hindered, despite the action of the Hamiltonian (in general the state ψ is not an eigenstate of the Hamiltonian H).

We have derived the QZE (5) from the Taylor expansion (3), which is valid for a state ψ in the domain of the Hamiltonian. However, (5) is of more general validity. For a full characterization of the limit (5) when ψ does not belong to the domain of the Hamiltonian, and/or the projective measurements are on a multidimensional subspace, see Ref. 4.

The QZE has been successfully demonstrated in a variety of physical systems, on experiments involving ionic hyperfine levels,[5] photons,[6] nuclear spins,[7] optical pumping,[8] ultracold atoms,[9] level dynamics of individual ions,[10] Bose-Einstein condensates,[11] optical systems,[12] and cavity quantum electrodynamics.[13] For a review on the mathematical and physical aspects of the subject see Ref. 2.

The QZE can be obtained both by pulsed and continuous measurements as well as by a strong interaction.[14–16] Recently it has been realized that by exploiting the quantum Zeno dynamics, one gets a powerful approach to control. The key idea is to engineer a given evolution by a rapid sequence of projections.[17–19] This can yield a Berry phase[20] or, more generally, non-Abelian geometric phases,[21] a resource for holonomic quantum computation. Moreover, the QZE can be seen as an effective way of imposing constraints and boundary conditions.[3,22,23] Finally, notice that the QZE is a purely quantum phenomenon: in classical mechanics it is not observed, since the measurement process can be conceived so that it does not interfere with the evolution of the system.

In this article, we want to investigate the behavior of the Zeno product formula (4) as the observation time becomes large, $t \to +\infty$, namely, the double limit

$$\lim_{\substack{t\to+\infty \\ N\to+\infty}} p^{(N)}(t). \tag{6}$$

Notice that, since the time dependence in (4) is given through the ratio $t/\hbar$, the long-time limit (6) is in fact a semiclassical limit, where the Planck constant $\hbar \to 0$, namely,

$$\lim_{\substack{t\to+\infty \\ N\to+\infty}} p^{(N)}(t) = \lim_{\substack{t\to+\infty \\ N\to+\infty}} \left|\langle\psi|\exp\left(-\frac{it}{\hbar N}H\right)\psi\rangle\right|^{2N} = \lim_{\substack{\hbar\to0 \\ N\to+\infty}} \left|\langle\psi|\exp\left(-\frac{it}{\hbar N}H\right)\psi\rangle\right|^{2N}. \tag{7}$$

In this respect, the limit (7) answers the following question: what happens to the evolution of the system when we compare the period between two successive measurements with the quantum scale given by $\hbar/E_0$, with E_0 being the relevant energy scale of the state?

For the analysis of the classical limit of the QZE see Ref. 24. By a semiclassical analysis on phase space,[25,26] it can be shown that the QZE vanishes at all orders in the Planck constant $\hbar$, in the limit $\hbar \to 0$, and thus it is a purely quantum phenomenon without classical analogue, at the same level of tunneling. (Notice, however, that at variance with Ref. 24, in the present situation the state ψ and the Hamiltonian H do not depend on $\hbar$.)

Heuristically, if we perform first the limit in N and then the limit in t we get QZE, namely,

$$\lim_{t\to+\infty}\lim_{N\to+\infty} p^{(N)}(t) = 1. \tag{8}$$

Conversely, for a decaying system, if we invert the order of the two limits we obtain a classical behavior, namely,

$$\lim_{N\to+\infty}\lim_{t\to+\infty} p^{(N)}(t)=0. \tag{9}$$

Therefore, the limit (6) does not exist because it depends on the way in which it is computed.

In order to better understand the transition from (8) to (9), we look at the double limit (6) when t diverges as a power of N, i.e.,

$$t=\tau N^{\alpha}, \tag{10}$$

where $\tau>0$ is a fixed time and $\alpha\geq 0$, see the right panel of Fig. 1. In this case, the survival probability $p^{(N)}(t)$ depends only on N, α, and τ. Therefore, in the following we will consider the product formula:

$$p_{N,\alpha}(\tau):=p^{(N)}(\tau N^{\alpha})=p(\tau N^{\alpha-1})^{N}=\left|\langle\psi|\exp\left(-\frac{i\tau}{\hbar}N^{\alpha-1}H\right)\psi\rangle\right|^{2N}, \tag{11}$$

and investigate the limit

$$p_{\alpha}^{(\infty)}(\tau)=\lim_{N\to+\infty} p_{N,\alpha}(\tau), \tag{12}$$

for different values of $\alpha\geq 0$.

The value $\alpha=0$ will correspond to the QZE limit (5),

$$p_{0}^{(\infty)}(\tau)=\lim_{N\to\infty} p\left(\frac{\tau}{N}\right)^{N}, \tag{13}$$

while the value $\alpha=1$ will correspond to the large-time limit of an evolution stroboscopically measured with period τ,

$$p_{1}^{(\infty)}(\tau)=\lim_{N\to\infty} p(\tau)^{N}. \tag{14}$$

The latter regime, describing a quantum system subject to periodic kicks, has become a paradigmatic example in the study of quantum chaos.[27,28] It represents a standard test bed for the investigation of different features of the quantum systems whose classical counterparts have a chaotic evolution. In Ref. 29, the dynamics of a kicked quantum system undergoing repeated measurements of momentum has been investigated. A diffusive behavior has been obtained, even when the dynamics of the classical counterpart is not chaotic and, in general, the system has been shown to have an anomalous diffusive behavior, characteristic of intermittent classical dynamical systems and random walks in random environments.[30]

Thus as α ranges from 0 to 1, one goes from QZE to a kicked dynamics, and for a decaying system the limit probability $p_{\alpha}^{(\infty)}(\tau)$ goes from 1 to 0. We will show that the transition is abrupt with a threshold at $\alpha=1/2$, as shown in Fig. 1.

Moreover, we will also consider larger values of the exponent, i.e., $\alpha>1$, which correspond to repeated measurements with a larger and larger period. In such a case, the limit probability $p_{\alpha}^{(\infty)}(\tau)$ of a decaying system is obviously 0, but interesting exceptions will occur at particular values of α and τ for systems with recurrences.

Notice also that, as a consequence of the previous discussion, the limit (12) can be viewed as the semiclassical limit (7) when $\hbar$ goes to zero as a power of N, i.e.,
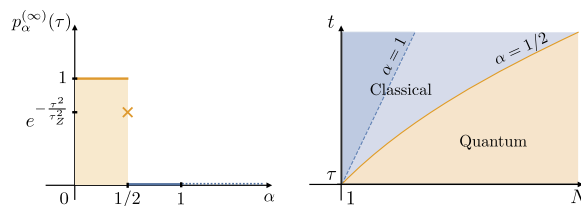
$$\hbar=\frac{\hbar_{0}}{N^{\alpha}}, \tag{15}$$

FIG. 1. Left panel: The value of the limit $p^{(\infty)}(\tau)$ in (12) as a function of the exponent α. Observe the abrupt transition at $\alpha=1/2$. Right panel: quantum and classical regimes in the $N-t$ plane. The QZE effect is along the horizontal axis, but it keeps holding along all curves under the critical parabola $t=\tau N^{1/2}$.

where $\hbar_0 > 0$ is a fixed constant. Therefore, for short in the following we will often refer to the regime with zero limit probability as the *classical* regime as opposed to a nonzero limit probability characteristic of a *quantum* regime, see Fig 1.

The article is organized as follows: in Section II, we discuss the case $\alpha < 1$ and we show that $\alpha = 1/2$ is the threshold exponent between quantum and classical behaviors; in Section III, we focus on the case $\alpha \geq 1$ and we prove that, essentially, the system exhibits always a classical behavior, but the limit becomes sensitive to the spectral properties of the state ψ and to some interesting arithmetical properties of τ and α.

II. THRESHOLD EXPONENT BETWEEN QUANTUM AND CLASSICAL BEHAVIORS

In this section, we discuss the case $0 \leq \alpha < 1$ and we show that $\alpha = 1/2$ is the threshold exponent between quantum and classical behaviors.

Theorem 1. *Let ψ be a normalized state in the domain of H, i.e., $\|H\psi\| < +\infty$. The limit (12) of the product formula (11) has the following behavior:*

(i)   *If $0 \leq \alpha < 1/2$ then*

$$p_\alpha^{(\infty)}(\tau) = 1, \tag{16}$$

uniformly in τ on compact subsets of $\mathbb{R}$.

(ii) *If $\alpha = 1/2$ then*

$$p_{1/2}^{(\infty)}(\tau) = \exp\left(-\frac{\tau^2}{\tau_Z^2}\right), \tag{17}$$

uniformly in τ on compact subsets of $\mathbb{R}$, where

$$\tau_Z^{-2} = \frac{1}{\hbar^2}\left(\langle H\psi | H\psi \rangle - \langle \psi | H\psi \rangle^2\right). \tag{18}$$

(iii) *If $1/2 < \alpha < 1$ and ψ is not an eigenstate of H, then*

$$p_\alpha^{(\infty)}(\tau) = 0, \tag{19}$$

uniformly in τ on compact subsets of $\mathbb{R}\backslash\{0\}$.

Remark 1. The time $\tau_Z > 0$ defined by (18) characterizes the initial quadratic behavior of the survival probability and is known in the literature as the Zeno time.[2] *Notice that the variance of H given by (18) is zero if and only if the state ψ is an eigenstate of H. In such a case, the Zeno time is $\tau_Z = +\infty$.*

Proof. If $0 \leq \alpha < 1/2$ then

$$p_{N,\alpha}(\tau) = p\left(\frac{\tau}{N^{1-\alpha}}\right)^N, \tag{20}$$

therefore using (3) we obtain that

$$p_{N,\alpha}(\tau) = \left[1 - \frac{\tau^2}{\tau_Z^2 N^{2(1-\alpha)}} + o\left(\frac{\tau^2}{N^{2(1-\alpha)}}\right)\right]^N = 1 - \frac{\tau^2}{\tau_Z^2 N^{1-2\alpha}} + o\left(\frac{\tau^2}{N^{1-2\alpha}}\right). \tag{21}$$

Since $1 - 2\alpha > 0$, we immediately obtain that

$$\lim_{N\to+\infty} p_{N,\alpha}(t) = 1, \tag{22}$$

uniformly in τ on compact subsets of $\mathbb{R}$.

Following the same procedure we obtain that for $\alpha = 1/2$,

$$p_{N,\alpha}(\tau) = \left[1 - \frac{\tau^2}{\tau_Z^2 N} + o\left(\frac{\tau^2}{N}\right)\right]^N, \tag{23}$$

therefore

$$\lim_{N\to+\infty} p_{N,\alpha}(\tau) = \exp\left(-\frac{\tau^2}{\tau_Z^2}\right), \tag{24}$$

uniformly in τ on compact subsets of $\mathbb{R}$.

Finally, we discuss the case $1/2 < \alpha < 1$. Notice that

$$p_{N,\alpha}(\tau) = \exp\left[N\log p(\tau/N^{1-\alpha})\right], \tag{25}$$

where p is the survival probability (2). By (3), we have

$$\log p(s) = \log\left(1 - \frac{s^2}{\tau_Z^2} + o(s^2)\right) = -\frac{s^2}{\tau_Z^2} + o(s^2), \tag{26}$$

as $s \to 0$, with a finite τ_Z, since ψ is not an eigenstate of H. Therefore for s sufficiently small, say $|s| \le \sigma$, one gets

$$\log p(s) \le -\frac{s^2}{2\tau_Z^2}, \tag{27}$$

whence

$$\log p(\tau/N^{1-\alpha}) \le -\frac{\tau^2}{2\tau_Z^2}\frac{1}{N^{2-2\alpha}}, \tag{28}$$

for $N \ge (|\tau|/\sigma)^{1/(1-\alpha)}$.

It follows that for $\tau \in [\tau_1, \tau_2] \subset \mathbb{R}\backslash\{0\}$ one gets

$$p_{N,\alpha}(\tau) \le \exp\left(-\frac{\tau_m^2}{2\tau_Z^2}N^{2\alpha-1}\right), \tag{29}$$

for $N \ge (\tau_M/\sigma)^{1/(1-\alpha)}$, where $\tau_m = \min\{|\tau_1|, |\tau_2|\} > 0$ and $\tau_M = \max\{|\tau_1|, |\tau_2|\}$. Therefore,

$$\lim_{N\to+\infty} p_{N,\alpha}(\tau) = 0, \tag{30}$$

uniformly in τ on compact subsets of $\mathbb{R}\backslash\{0\}$.   □

III. SENSITIVITY TO THE SPECTRAL PROPERTIES OF THE INITIAL STATE

In this section, we discuss the case $\alpha \ge 1$. We will show that in this regime the limit (12) exhibits always a classical behavior, but it becomes sensitive to the spectral properties of the state ψ and to the arithmetical nature of α.

We recall here some basic aspects of spectral theory; see, e.g., Ref. 31. Let φ be a vector in the Hilbert space $\mathcal{H}$, and let H be a self-adjoint operator. By the spectral theorem there exists a unique Borel measure μ_φ on $\mathbb{R}$ such that

$$\langle\varphi|f(H)\varphi\rangle = \int_{\sigma(H)} f(\lambda)\,\mathrm{d}\mu_\varphi(\lambda), \tag{31}$$

for all $f \in C_b(\mathbb{R})$, where $\sigma(H)$ denotes the spectrum of H and $C_b(\mathbb{R})$ denotes the space of bounded and continuous functions on $\mathbb{R}$ with complex values. The spectral properties of the Hamiltonian induce a canonical decomposition of the Hilbert space $\mathcal{H}$ into the direct sum

$$\mathcal{H} = \mathcal{H}_c \oplus \mathcal{H}_{pp}, \qquad \mathcal{H}_c = \mathcal{H}_{ac} \oplus \mathcal{H}_{sc}, \tag{32}$$

where

$$\mathcal{H}_c = \{\varphi \in \mathcal{H} : \mu_\varphi \text{ is a continuous measure}\}$$

is the *continuous subspace,*

$$\mathcal{H}_{pp} = \{\varphi \in \mathcal{H} : \mu_\varphi \text{ is a pure point measure}\}$$

is the *pure point subspace,*

$$\mathcal{H}_{ac} = \{\varphi \in \mathcal{H} : \mu_\varphi \text{ is absolutely continuous}\}$$

is the *absolutely continuous subspace*, and

$$\mathcal{H}_{\text{sc}} = \{\varphi \in \mathcal{H} : \mu_{\varphi} \text{ is singular continuous}\}$$

is the *singular continuous subspace*.

We recall that a Borel measure μ_{c} on $\mathbb{R}$ is continuous if it does not concentrate at any point, that is, if

$$\mu_{\text{c}}(\{x\}) = 0, \qquad \text{for all } x \in \mathbb{R}, \tag{33}$$

while a measure μ_{pp} is pure point (or discrete) if

$$\mu_{\text{pp}}(B) = \sum_{x \in B} \mu_{\text{pp}}(\{x\}), \tag{34}$$

for all measurable sets $B \subset \mathbb{R}$. Moreover, a measure μ_{ac} is absolutely continuous (with respect to the Lebesgue measure dx) if it has a density function ρ locally integrable so that

$$d\mu_{\text{ac}}(x) = \rho(x)dx. \tag{35}$$

Finally, a singular continuous measure μ_{sc} is continuous but not absolutely continuous; a paradigmatic example is the Cantor measure.[31]

In this section, we investigate how the limit (12) changes if the initial state ψ belongs to the spectral subspaces $\mathcal{H}_{\text{pp}}$, $\mathcal{H}_{\text{ac}}$, and $\mathcal{H}_{\text{sc}}$, which physically correspond to bound states (made up of eigenstates), scattering states, and unbound weakly recurrent states, respectively.

In the following theorem, we study the case $\alpha = 1$.

Theorem 2. *If $\alpha = 1$ the product formula (11) is given by*

$$p_{N,1}(\tau) = \left| \langle \psi | e^{-\frac{i\tau H}{\hbar}} \psi \rangle \right|^{2N} = p(\tau)^N \tag{36}$$

and its limit

$$p_1^{(\infty)}(\tau) = \lim_{N \to +\infty} p(\tau)^N \tag{37}$$

has the following behavior:

(i)   *If $p(\tau) < 1$ for all $\tau \in \mathbb{R}\backslash\{0\}$ then*

$$p_1^{(\infty)}(\tau) = 0, \tag{38}$$

   uniformly in τ on compact subsets of $\mathbb{R}\backslash\{0\}$.

(ii) *If there exists $\tau_0 \in \mathbb{R}\backslash\{0\}$ such that $p(\tau_0) = 1$, then $\psi \in \mathcal{H}_{\text{pp}}$ and there exists a positive integer m such that*

$$p_1^{(\infty)}(\tau) = \begin{cases} 1 & \text{if} \quad \tau/\tau_0 \in \frac{1}{m}\mathbb{Z} \\ 0 & \text{otherwise.} \end{cases} \tag{39}$$

Proof. The assertion (*i*) is obvious. Now assume that

$$p(\tau_0) = |\langle \psi | e^{-i\tau_0 H/\hbar} \psi \rangle|^2 = 1 \tag{40}$$

for some $\tau_0 \in \mathbb{R}\backslash\{0\}$. Then, we have that

$$\langle \psi | e^{-i\tau_0 H/\hbar} \psi \rangle = e^{-i\tau_0 a/\hbar} \tag{41}$$

for some $a \in \mathbb{R}$. By the spectral theorem, we have that

$$\int_{\sigma(H)} e^{-i\tau_0(\lambda-a)/\hbar} \, d\mu_{\psi}(\lambda) = 1, \tag{42}$$

and thus

$$\int_{\sigma(H)} (1 - \cos(\tau_0(\lambda - a)/\hbar)) \, d\mu_{\psi}(\lambda) = 0. \tag{43}$$

Since $(1 - \cos(\tau_0(\lambda - a)/\hbar)) \geq 0$, we have that

$$(1 - \cos(\tau_0(\lambda - a)/\hbar)) = 0 \tag{44}$$

almost everywhere with respect to μ_ψ. Therefore, there exist $r \in \mathbb{N} \cup \{+\infty\}$ and a subset of integers $\{k_j : j = 1, \ldots, r\} \subset \mathbb{Z}$ such that the support of the spectral measure μ_ψ is the set

$$\left\{ \lambda_j = a + \frac{2\pi k_j \hbar}{\tau_0} : j = 1, \ldots, r \right\}. \tag{45}$$

Therefore the state $\psi \in \mathcal{H}_{\mathrm{pp}}$ and has the form

$$\psi = \sum_{j=1}^{r} c_j \psi_j, \tag{46}$$

where ψ_j is a normalized eigenvector belonging to the eigenvalue λ_j and $c_j \in \mathbb{C}\backslash\{0\}$, for all $j = 1, \ldots, r$; see Ref. 31. Using (46) it is easy to check that $p(\tau)$ can be written as follows:

$$p(\tau) = \left| \int_{\sigma(H)} \mathrm{e}^{-\mathrm{i}\tau\lambda/\hbar} \, \mathrm{d}\mu_\psi(\lambda) \right|^2 = \left| \sum_{j=1}^{r} p_j \mathrm{e}^{-\mathrm{i}\tau\lambda_j/\hbar} \right|^2$$

$$= \sum_{j,l=1}^{r} p_j p_l \cos\left(\frac{2\pi(k_j - k_l)\tau}{\tau_0} \right), \tag{47}$$

where $p_j = |c_j|^2$, $j = 1, \ldots, r$, with

$$\sum_{j=1}^{r} p_j = 1, \tag{48}$$

since $\|\psi\| = 1$. Consider now the greatest common divisor of the k_js

$$m = \gcd\{k_j : j = 1, \ldots r\}, \tag{49}$$

so that $k_j = m\tilde{k}_j$ with $\tilde{k}_j \in \mathbb{Z}$ for $j = 1, \ldots r$. We get

$$p_{N,1}(\tau) = p(\tau)^N = \left[\sum_{j,l=1}^{r} p_j p_l \cos\left(\frac{2\pi(\tilde{k}_j - \tilde{k}_l)m\tau}{\tau_0} \right) \right]^N. \tag{50}$$

Therefore, if $m\tau/\tau_0$ is an integer then all cosines are equal to one and $p_{N,1}(\tau) = 1$ for all N. On the other hand, if $m\tau/\tau_0$ is not an integer then there exists at least a pair of integers $\tilde{k}_j$ and $\tilde{k}_l$ which are coprime, and thus $(\tilde{k}_j - \tilde{k}_l)m\tau/\tau_0$ is not an integer and the corresponding cosine is not 1. Therefore (39) holds.   □

Remark 2. Notice that $|\tau_0|/m$ is the first return time of the survival probability (2), that is, $p(|\tau_0|/m) = 1$ and $p(\tau) < 1$ for $0 < \tau < |\tau_0|/m$. Moreover, observe that in the proof of assertion (ii) of the previous theorem, we have retraced the proof of a well known result in probability, see Ref. 32, Theorem 5 p. 288.

Now we discuss the case $\alpha > 1$.

Theorem 3. *If $\alpha > 1$, the product formula (11) is given by*

$$p_{N,\alpha}(\tau) = \left| \langle \psi | \exp\left(-\frac{\mathrm{i}\tau}{\hbar} N^{\alpha-1} H \right) \psi \rangle \right|^{2N} = p(\tau N^{\alpha-1})^N \tag{51}$$

and its limit (12) has the following behavior:

(i)   *If H is bounded from below then*

$$p_\alpha^{(\infty)}(\tau) = \lim_{N \to +\infty} p_{N,\alpha}(\tau) = 0, \tag{52}$$

   almost everywhere in $\tau \in \mathbb{R}$.

(ii)   *If $\psi \in \mathcal{H}_{\mathrm{ac}}$ then the limit (52) holds uniformly in τ on compact subsets of $\mathbb{R}\backslash\{0\}$;*

(iii) *If there exists $\tau_0 \in \mathbb{R}\backslash\{0\}$ such that $p(\tau_0) = 1$, then $\psi \in \mathcal{H}_{\mathrm{pp}}$ and*

$$\limsup_{N \to +\infty} p_{N,\alpha}(M\,\tau_0) = 1, \tag{53}$$

   for all $\alpha \in \mathbb{Q}$ and all $M \in \mathbb{Z}$.

Proof. Notice first that if H is bounded from below, say $H \geq E_{\min}$, then $\tilde{H} = H - E_{\min} + 1 \geq 1$ is a strictly positive self-adjoint operator, and

$$|\langle\psi|e^{-it\tilde{H}/\hbar}\psi\rangle|^2 = |\langle\psi|e^{-itH/\hbar}\psi\rangle|^2 = p(t), \tag{54}$$

so we can assume that the Hamiltonian is strictly positive with spectrum

$$\sigma(H) \subset [1, +\infty). \tag{55}$$

Then observe that

$$\langle\psi|e^{-\frac{i\tau N^{\alpha-1}H}{\hbar}}\psi\rangle = \int_{\mathbb{R}} \exp\left(-\frac{i\tau N^{\alpha-1}\lambda}{\hbar}\right) d\mu_\psi(\lambda) = \hat{\mu}_\psi\left(\frac{N^{\alpha-1}\tau}{\hbar}\right), \tag{56}$$

where $\hat{\mu}$ denotes the Fourier transform of the measure μ. Let $\mu_{\psi,N}$ be the spectral measure of the self adjoint operator $N^{\alpha-1}H$ in the state ψ. Notice that the spectrum of the operator $N^{\alpha-1}H$ is

$$\sigma(N^{\alpha-1}H) = N^{\alpha-1}\sigma(H) = \{N^{\alpha-1}\lambda : \lambda \in \sigma(H)\}. \tag{57}$$

Using the property of the Fourier transform it is easy to see that

$$\left(\langle\psi|e^{-\frac{i\tau N^{\alpha-1}H}{\hbar}}\psi\rangle\right)^N = \left[\hat{\mu}_{\psi,N}\left(\frac{\tau}{\hbar}\right)\right]^N = \hat{\nu}_{\psi,N}\left(\frac{\tau}{\hbar}\right), \tag{58}$$

where

$$\nu_{\psi,N} := \underbrace{\mu_{\psi,N} * \ldots * \mu_{\psi,N}}_{N \text{ times}} \tag{59}$$

and $*$ denotes the convolution product, defined by

$$\int_{\mathbb{R}} f(\lambda) \, d\nu_{\psi,N}(\lambda) = \int_{\mathbb{R}^N} f(\lambda_1 + \ldots + \lambda_N) \, d\mu_{\psi,N}(\lambda_1) \ldots d\mu_{\psi,N}(\lambda_N), \tag{60}$$

for all $f \in C_b(\mathbb{R})$.

We prove assertion (*i*). First we prove that

$$\lim_{N\to+\infty} \int_{\mathbb{R}} f(\lambda) \, d\nu_{\psi,N}(\lambda) = 0, \tag{61}$$

for all $f \in C_0(\mathbb{R})$, where $C_0(\mathbb{R})$ denotes the space of continuous functions vanishing at infinity. Indeed, if we fix a function $f \in C_0(\mathbb{R})$, we have that

$$\int_{\mathbb{R}} f(\lambda) \, d\nu_{\psi,N}(\lambda) = \int_{\mathbb{R}^N} f(\lambda_1 + \ldots + \lambda_N) \, d\mu_{\psi,N}(\lambda_1) \ldots d\mu_{\psi,N}(\lambda_N)$$

$$= \int_{\mathbb{R}^N} f\left(N^{\alpha-1}\sum_{j=1}^N \lambda_j\right) d\mu_\psi(\lambda_1) \ldots d\mu_\psi(\lambda_N)$$

$$= \int_{\mathbb{R}^N} f\left(N^\alpha \sum_{j=1}^N \frac{\lambda_j}{N}\right) d\mu_\psi(\lambda_1) \ldots d\mu_\psi(\lambda_N)$$

$$= \int_{[1,+\infty)^N} f\left(N^\alpha \sum_{j=1}^N \frac{\lambda_j}{N}\right) d\mu_\psi(\lambda_1) \ldots d\mu_\psi(\lambda_N). \tag{62}$$

Therefore, by the mean value theorem, we have that

$$\int_{\mathbb{R}} f(\lambda) \, d\nu_{\psi,N}(\lambda) = f(N^\alpha \xi_N), \tag{63}$$

where

$$\xi_N = \frac{\tilde{\lambda}_1 + \ldots + \tilde{\lambda}_N}{N} \geq 1, \tag{64}$$

for some $(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_N) \in [1, +\infty)^N$ and thus $N^\alpha \xi_N \to +\infty$ as $N \to +\infty$. Therefore, the limit (61) holds.

Now we recall that, by the Riemann-Lebesgue lemma, for all $\phi \in L^1(\mathbb{R})$ its Fourier transform $\hat{\phi} \in C_0(\mathbb{R})$, and thus

$$\int_{\mathbb{R}} \phi(\tau)\, \hat{v}_{\psi,N}\left(\frac{\tau}{\hbar}\right) \mathrm{d}\tau = \int_{\mathbb{R}} \hat{\phi}\left(\frac{\lambda}{\hbar}\right) \mathrm{d}\nu_{\psi,N}(\lambda) \to 0 \tag{65}$$

as $N \to +\infty$. Therefore we have proved that

$$\lim_{N\to+\infty} \hat{v}_{\psi,N}\left(\frac{\tau}{\hbar}\right) = \lim_{N\to+\infty} \left(\langle\psi|\mathrm{e}^{-\frac{\mathrm{i}\tau N^{\alpha-1}H}{\hbar}}\psi\rangle\right)^N = 0 \tag{66}$$

almost everywhere in $\tau \in \mathbb{R}$, whence

$$\lim_{N\to+\infty} p_{N,\alpha}(\tau) = \lim_{N\to+\infty} \left|\hat{v}_{\psi,N}\left(\frac{\tau}{\hbar}\right)\right|^2 = 0 \tag{67}$$

almost everywhere in $\tau \in \mathbb{R}$.

Now we prove assertion (*ii*). If we assume that $\psi \in \mathcal{H}_{\mathrm{ac}}$, then $\mathrm{d}\mu_\psi(\lambda) = \rho(\lambda)\, \mathrm{d}\lambda$ (where the density $\rho(\lambda) = |\tilde{\psi}(\lambda)|^2 \in L^1(\mathbb{R})$ is the squared wave function of ψ in the energy representation), and thus

$$p(t) = |\langle\psi, \mathrm{e}^{-\frac{\mathrm{i}tH}{\hbar}}\psi\rangle|^2 = \left|\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\frac{t\lambda}{\hbar}}\rho(\lambda)\mathrm{d}\lambda\right|^2 \to 0 \tag{68}$$

as $t \to \pm\infty$, by the Riemann-Lebesgue lemma. Therefore we have that

$$p(\tau N^{\alpha-1}) < \beta < 1 \tag{69}$$

definitively in N, for $\tau \neq 0$, and thus

$$\lim_{N\to+\infty} p_{N,\alpha}(\tau) = \lim_{N\to+\infty} p(\tau N^{\alpha-1})^N = 0, \tag{70}$$

uniformly in τ on compact sets of $\mathbb{R}\backslash\{0\}$.

Finally we prove (*iii*). In the proof of Theorem 2 we have shown that if the survival probability

$$p(\tau_0) = |\langle\psi|\mathrm{e}^{-\mathrm{i}\tau_0 H/\hbar}\psi\rangle|^2 = 1 \tag{71}$$

for some $\tau_0 \neq 0$, then $\psi \in \mathcal{H}_{\mathrm{pp}}$ and the survival probability has the form (47). Therefore, we have that

$$p_{N,\alpha}(\tau) = p(\tau N^{\alpha-1})^N = \left|\sum_{j=1}^{r} p_j \mathrm{e}^{-\mathrm{i}\tau N^{\alpha-1}\lambda_j/\hbar}\right|^{2N}$$

$$= \left[\sum_{j,l=1}^{r} p_j p_l \cos\left(\frac{2\pi(k_j - k_l)N^{\alpha-1}\tau}{\tau_0}\right)\right]^N, \tag{72}$$

with $\sum p_j = 1$ and $k_j \in \mathbb{Z}$ for $j = 1, \ldots r$. Notice that if $\alpha \in \mathbb{Q}$, then $\alpha - 1 = n_1/n_2$ for some $n_1, n_2 \in \mathbb{N}\backslash\{0\}$. Therefore if we consider the subsequence $N_m := m^{n_2}$ we have that

$$p_{N_m,\alpha}(M\,\tau_0) = \left[\sum_{j=1}^{r} p_j p_l \cos\left(2\pi(k_j - k_l)m^{n_1}M\right)\right]^{N_m} = 1, \tag{73}$$

for all $m \in \mathbb{N}$ and $M \in \mathbb{Z}$, therefore we have that

$$\lim_{m\to+\infty} p_{N_m,\alpha}(M\,\tau_0) = 1. \tag{74}$$

Since $p_{N,\alpha}(\tau) \leq 1$ for all $\tau \in \mathbb{R}$ and $N \in \mathbb{N}$, we conclude that

$$\lim_{m\to+\infty} p_{N_m,\alpha}(M\,\tau_0) = \limsup_{N\to+\infty} p_{N,\alpha}(M\,\tau_0) = 1. \tag{75}$$

$\square$

Remark 3. In the case of a finite-dimensional Hilbert space, $\dim \mathcal{H} < +\infty$, all states ψ are bound, $\mathcal{H} = \mathcal{H}_{\mathrm{pp}}$, and are a (finite) linear combination of eigenstates of H. Therefore, the assertion (ii) of Theorem 3 is empty, while the assertions (ii) of Theorem 2 and (iii) of Theorem 3 correspond to the case of a closed orbit on a (finite-dimensional) torus, which happens if and only if the eigenvalue differences are in complete resonance (all their ratios are in $\mathbb{Q}$).

IV. CONCLUSIONS

Let us summarize the main results obtained in this article in more intuitive terms, by focusing on those quantities that are more directly related to physical intuition. We have analyzed the double limit

$$\lim_{\substack{t \to +\infty \\ N \to +\infty}} p^{(N)}(t) \tag{76}$$

in the case $t \propto N^\alpha$, for all possible values of $\alpha \geq 0$.

We have shown that if $0 \leq \alpha < 1/2$ the limit equals 1, namely, the system is frozen in its initial state and the QZE takes place. At $\alpha = 1/2$, the limit (76) is strictly smaller than 1 for all times and decays in time as a Gaussian (17). If $1/2 < \alpha < 1$ the limit (76) is equal to 0 for all times, thus we observe a classical behavior, see Fig. 1.

Moreover, if $\alpha \geq 1$ the limit probability is a strange beast and becomes sensitive to the spectral properties of the state ψ. In general, the limit (76) is 0 for almost all times, and if the state is decaying, $\psi \in \mathcal{H}_{ac}$, the limit (76) is always 0 for all times.

The existence of times t at which the limit (76) is nonzero has been clarified, at least for bound states, $\psi \in \mathcal{H}_{pp}$. In fact there are bound states with a periodic dynamics. Thus, if one performs repeated measurements at the natural period, namely, if one looks stroboscopically at the particle dynamics, the presence of the measurements becomes immaterial: the classical and the quantum behaviors simply coincide and the limit (76) at that time is equal to 1.

Concerning the existence of times at which the limit (76) is not 0 for states in the continuous singular spectrum $\psi \in \mathcal{H}_{sc}$, i.e., recurrent unbound states, we can only say that the set of those times is negligible. It is more difficult to grasp this situation by physical intuition and its full comprehension would require a further analysis which is beyond the scope of this paper.

Summarizing, we have unveiled the presence of two threshold exponents: the threshold between quantum and classical behaviors at $\alpha = 1/2$, and the threshold of sensitivity to the spectral properties of the initial state at $\alpha = 1$.

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