

Can Decay Be Ascribed to Classical Noise?

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Abstract. No.

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1. Introduction and Motivation

The dynamics of a dissipative quantum system, in the Markovian approximation, is governed by the Gorini-Kossakowski-Lindblad-Sudarshan (GKLS) equation [1–3]. This equation always admits a dilation to a stochastic differential equation [4–6] and can be read as a (quantum) Langevin equation [7]. In the mathematical physics literature, stochastic equations have been studied both for Gaussian processes [5] and for general combinations of Gaussian and Poisson processes [6]. In this paper, we shall limit our analysis to the Gaussian case, which is more relevant for physical applications, see for example the review [8] on derivations and applications of stochastic Schrödinger equations for quantum control and quantum information processing.

Quantum dissipation can take different forms, and is associated with different physical scenarios. Among these, there are genuine “dephasing” processes, as well as *bona-fide* “decay” processes (e.g., to the ground state). Accordingly, the GKLS equations have different mathematical features and physical meaning: for instance, some physical features of dephasing are often reflected in the self-duality of the quantum dynamical map.

We ask here the following questions: are these different physical and mathematical features mirrored in the Wiener process associated with the corresponding quantum Langevin equation? More specifically: can decay be ascribed to “classical” noise? Moreover: do the afore-mentioned features affect the Hamiltonians of the associated Ito and Stratonovich stochastic

equations, and if yes, how? The answers to the above questions will require definition and physical elucidation of these concepts. On this basis, we will endeavour to clarify the physical meaning of the corresponding stochastic Schrödinger equations.

This paper is organized as follows. In Sect. 2 we introduce notation and review paradigmatic examples of self-dual (SD) and non-self-dual (NSD) maps. We start with a few observations in Sect. 3, where we give a physical definition of classical noise. In Sect. 4 we look at a rather general example, that involves both SD and NSD components. We prove our first main result in Sect. 5, where the Stratonovich formulation is also discussed. The relation between self-duality and dephasing is elaborated in Sect. 6, where a definition is given of dephasing and decay channels and general conclusions are drawn. We put forward a few additional remarks in Sect. 7 and answer the question posed in the title in Sect. 8.

2. Generalities and Definitions

The GKLS evolution equation for the density matrix ϱ of a quantum system reads

$$\dot{\varrho}(t) = \mathcal{L}_{\text{tot}}\varrho(t), \quad \mathcal{L}_{\text{tot}} = \mathcal{L}_H + \mathcal{L}, \quad (1)$$

where \mathcal{L}_H and \mathcal{L} are the (time-independent) Hamiltonian and dissipative parts of the total map \mathcal{L}_{tot} , respectively, and the dot denotes derivative with respect to time d/dt . The solution is

$$\varrho(t) = e^{t\mathcal{L}_{\text{tot}}}\varrho(0) = \Lambda_t\varrho(0), \quad t \geq 0. \quad (2)$$

The adjoint dynamical equation for an observable A is given by

$$\dot{A}(t) = \mathcal{L}_{\text{tot}}^\# A(t), \quad (3)$$

whose solution is

$$A(t) = \Lambda_t^\# A(0), \quad t \geq 0. \quad (4)$$

The Dirac prescription [9]

$$\text{Tr}[\varrho(t)A(0)] = \text{Tr}[\varrho(0)A(t)], \quad \forall \varrho, A \quad (5)$$

connects the Schrödinger and Heisenberg pictures and consecrates their equivalence. The dissipative component \mathcal{L} of the map is said to be self-dual if

$$\mathcal{L} = \mathcal{L}^\# \iff \Lambda_t = \Lambda_t^\#, \quad (6)$$

while it is non-self-dual otherwise.

2.1. EXAMPLE OF SELF-DUAL MAP: PHASE DAMPING

Typical examples of self-dual maps are those describing phase damping. Consider for example the phase damping of a qubit performing Rabi oscillations. The evolution of the density matrix of the qubit is described by ($\hbar = 1$)

$$\dot{\varrho} = \mathcal{L}_{\text{tot}}\varrho = -i[\Omega\sigma_1, \varrho] - \gamma(\varrho - \sigma_3\varrho\sigma_3), \quad (7)$$

where $\gamma > 0$, and σ_α ($\alpha = 0, 1, 2, 3$) are the Pauli matrices (with $\sigma_0 \equiv \mathbf{1}$). The adjoint equation for an observable A reads

$$\dot{A} = \mathcal{L}_{\text{tot}}^\#A = i[\Omega\sigma_1, A] - \gamma(A - \sigma_3A\sigma_3) \quad (8)$$

and its dissipative part is SD, as $\mathcal{L} = \mathcal{L}^\#$. Physically, the above equation describes Rabi oscillations accompanied by a dephasing process: for example, if $\Omega = 0$, the asymptotic solution of (7) reads

$$\varrho = \frac{1}{2}(\sigma_0 + \mathbf{x} \cdot \boldsymbol{\sigma}) \xrightarrow{t \rightarrow \infty} \varrho(\infty) = \frac{1}{2}(\sigma_0 + x_3\sigma_3), \quad (9)$$

$\mathbf{x} = \text{Tr}(\rho\boldsymbol{\sigma})$ being the 3-dimensional Bloch vector, $|\mathbf{x}| \leq 1$. When $\Omega = 0$ populations do not change, but dephasing makes interference (between eigenstates of σ_3) impossible.

Equation (7) can be derived from the stochastic Hamiltonian [10]

$$H_\eta = \Omega\sigma_1 + \sqrt{\gamma}\eta(t)\sigma_3, \quad (10)$$

where η is a white noise (precise definitions are given later). The corresponding stochastic Schrödinger equation reads, by Ito calculus,

$$\begin{aligned} d\psi &= -i\Omega\sigma_1\psi dt - i\sqrt{\gamma}\sigma_3\psi \circ dW \\ &= [-i\Omega\sigma_1 - (\gamma/2)]\psi dt - i\sqrt{\gamma}\sigma_3\psi dW, \end{aligned} \quad (11)$$

where \circ denotes the Stratonovich product and $W = \int \eta dt$ is the Wiener process. Equation (11) yields (7) on average over the realizations of the Wiener process.

A similar example is the phase damping of a harmonic oscillator, whose dissipative part reads

$$\mathcal{L}\varrho = -\frac{\gamma}{2}(\{N^2, \varrho\} - 2N\varrho N), \quad (12)$$

where $N = a^\dagger a$ and $[a, a^\dagger] = 1$. Again, $\mathcal{L}^\# = \mathcal{L}$ and the map is SD. If the Hamiltonian is $H = \Omega a^\dagger a$, a generic density matrix becomes diagonal in the N -representation

$$\varrho = \sum c_{mn}|m\rangle\langle n| \xrightarrow{t \rightarrow \infty} \varrho(\infty) = \sum c_{nn}|n\rangle\langle n|, \quad (13)$$

so that populations do not change, but interference among eigenstates of the number operator becomes impossible.

2.2. EXAMPLE OF NON-SELF-DUAL MAP: ENERGY DAMPING

Typical examples of non-self-dual maps are those describing energy damping. Consider for example the energy damping of a qubit. Let

$$\dot{\varrho} = \mathcal{L}_{\text{tot}}\varrho = -i[H, \varrho] - \frac{\gamma'}{2}(\{\sigma_+\sigma_-, \varrho\} - 2\sigma_-\varrho\sigma_+), \quad (14)$$

where $\sigma_{\pm} = (\sigma_1 \pm i\sigma_2)/2$ and H is a Hamiltonian. One has

$$\mathcal{L}_{\text{tot}}^{\#}A = i[H, \varrho] - \frac{\gamma'}{2}(\{\sigma_+\sigma_-, A\} - 2\sigma_+A\sigma_-), \quad (15)$$

and the dissipative evolution is non-self-dual: $\mathcal{L}^{\#} \neq \mathcal{L}$. If $H = (\Omega/2)\sigma_3$, the solution of (14) reads

$$\varrho = \frac{1}{2}(\sigma_0 + \mathbf{x} \cdot \boldsymbol{\sigma}) \xrightarrow{t \rightarrow \infty} \varrho(\infty) = P_-, \quad (16)$$

so that the final state is the projection $P_- = (\sigma_0 - \sigma_3)/2$ over the ground state. Equation (14) must be derivable [4, 5] from a stochastic (non-Hermitian) Hamiltonian, through a term of the type

$$H_{\eta} = H + i\sqrt{\gamma'}\eta(t)\sigma_-. \quad (17)$$

However, such a derivation is not conceptually painless, as we shall see in the following.

A related example is the energy damping of a harmonic oscillator. For this dynamics, we have

$$\mathcal{L}\varrho = -\frac{\gamma'}{2}(\{a^{\dagger}a, \varrho\} - 2a\varrho a^{\dagger}), \quad (18)$$

whence

$$\mathcal{L}^{\#}A = -\frac{\gamma'}{2}(\{a^{\dagger}a, A\} - 2a^{\dagger}Aa), \quad (19)$$

which is NSD. The oscillator decays to the ground state (e.g., for $H = \Omega a^{\dagger}a$)

$$\varrho = \sum c_{mn}|m\rangle\langle n| \xrightarrow{t \rightarrow \infty} \varrho(\infty) = |0\rangle\langle 0|. \quad (20)$$

One of the main objectives of the present paper is to elucidate whether the difference between dephasing processes (and in general SD maps of the type shown in Sect. 2.1), and decay processes (and in general NSD maps of the type shown in this section) are reflected in some structural properties of the associated stochastic Schrödinger equation.

Incidentally, we observe that the infinite-time limit of the dissipative dynamics may lead to a contraction of the algebra of observables [11, 12]. We will not discuss in this paper whether such a contraction may bear signatures of the self-duality of the map (or lack thereof).

3. A Few Observations

Let η be a white noise

$$\langle \eta(t) \rangle = 0, \quad \langle \eta(t)\eta(t') \rangle = \delta(t - t'), \quad (21)$$

the brackets denoting the ensemble average over all possible realizations of the noise. The associated Wiener process reads

$$dW(t) \equiv W(t + dt) - W(t) = \int_t^{t+dt} \eta(s) ds, \quad (22)$$

$$\langle dW(t) \rangle = 0, \quad \langle dW(t)dW(t) \rangle = dt. \quad (23)$$

Consider the stochastic Schrödinger equation

$$d\psi = -i(H_1 - iH_2)\psi dt - iL\psi dW \quad (24)$$

to be understood in the Ito sense. The operators H_1 and H_2 are taken to be Hermitian while, in general, L is not.

The evolution of the density matrix $\rho = |\psi\rangle\langle\psi|$ is governed by

$$\begin{aligned} d\rho &= |\psi + d\psi\rangle\langle\psi + d\psi| - |\psi\rangle\langle\psi| \\ &= -i[H_1, \rho]dt - \{H_2, \rho\}dt - i(L\rho - \rho L^\dagger)dW + L\rho L^\dagger dt, \end{aligned} \quad (25)$$

where $[\cdot, \cdot]$ is the commutator and $\{\cdot, \cdot\}$ the anticommutator.

By taking the trace we get

$$\begin{aligned} \text{Tr}(d\rho) &= \|\psi + d\psi\|^2 - \|\psi\|^2 \\ &= \langle\psi|(-2H_2 + L^\dagger L)\psi\rangle dt - i\langle\psi|(L - L^\dagger)\psi\rangle dW, \end{aligned} \quad (26)$$

and, by taking the average over the noise, we see that a “weak” (i.e. on average) conservation of probability

$$\langle\|\psi + d\psi\|^2\rangle = \langle\|\psi\|^2\rangle = 1 \quad (27)$$

imposes a definite relation between the noise term and the non-Hermitian part of the Hamiltonian:

$$H_2 = \frac{1}{2}L^\dagger L. \quad (28)$$

This yields, by taking the average of (25), a GKLS equation

$$\frac{d}{dt}\rho = -i[H_1, \rho] - \frac{1}{2}\{L^\dagger L, \rho\} + L\rho L^\dagger. \quad (29)$$

Notice that, while relation (28) implies that probability is conserved on average, in general probability is *not conserved* along each *single realization* of the noise, which, therefore, does not represent a physical evolution. Every trajectory is physical and probability is (strictly) conserved in each individual realization if and only if

$$L = L^\dagger, \tag{30}$$

which, in turn, implies that the dissipative part of the generator is self-dual

$$\mathcal{L} = \mathcal{L}^\# . \tag{31}$$

When this happens, one can describe the dynamics in terms of a (Hermitian) time-dependent Hamiltonian

$$H_\eta = H_1 + \eta(t)L, \tag{32}$$

where H_1 and L are Hermitian, time-independent operators. Observe also that in this case the stochastic Schrödinger equation (24) admits a natural expression in terms of the Stratonovich product \circ

$$d\psi = -iH_1\psi dt - iL\psi \circ dW . \tag{33}$$

Equations (30)–(33) enable one to speak of a “classical” noise, in the sense that one can view the dynamics as arising from a classical noise source. An example will elucidate the concept. Consider a spin-1/2 particle in a magnetic field \mathbf{B} , described by the Hamiltonian

$$H = \boldsymbol{\mu} \cdot \mathbf{B}, \tag{34}$$

where $\boldsymbol{\mu} = \mu\boldsymbol{\sigma}$ is the magnetic moment. If the magnetic field has a random (white noise) component

$$\mathbf{B} = \mathbf{B}_0 + \delta\mathbf{B}(t), \tag{35}$$

then

$$H = \boldsymbol{\mu} \cdot \mathbf{B}_0 + \sqrt{\gamma}\eta(t)\boldsymbol{\sigma} \cdot \mathbf{n}, \tag{36}$$

\mathbf{n} being the unit vector (assumed to be time-independent) parallel to the random component of the field, and $\gamma \propto \delta B$. This Hamiltonian has the form (32) with $L = L^\dagger$, and describes the effect of a (classical) noisy magnetic field. From a physical perspective, the energy eigenvalues appear to be “shaken” by a random component.

Notice also that in this case there is no need of taking the average in (27), as

$$\|\psi + d\psi\|^2 = \|\psi\|^2 = 1 \tag{37}$$

in every individual realization of the stochastic process (provided (28) holds). Physically, one can view the quantum system as governed by a *bona fide* (Hermitian) time-dependent Hamiltonian (32) for all η . The dynamics is always unitary and probabilities are always conserved.

4. One Additional Example: Interaction with a Thermal Field

Let us look at one additional example: a two-level atom in interaction with a thermal field and subject to dephasing and decay. This example involves both SD and NSD components and puts together examples (7) and (14) of the preceding section, generalizing the latter to non-vanishing temperature.

The dynamics is given by $H_1 = (\Omega/2)\sigma_3$ and

$$\begin{aligned} \mathcal{L}\varrho &= -\frac{\gamma'}{2}(1+n)(\{\sigma_+\sigma_-, \varrho\} - 2\sigma_-\varrho\sigma_+) \\ &\quad -\frac{\gamma'}{2}n(\{\sigma_-\sigma_+, \varrho\} - 2\sigma_+\varrho\sigma_-) - \gamma(\varrho - \sigma_3\varrho\sigma_3), \end{aligned} \quad (38)$$

where $n = (e^{\beta\Omega} - 1)^{-1}$, with β the inverse temperature and Ω the energy difference of the two atomic states, and $\gamma \propto \beta^{-1}$. The constants γ and γ' are independent [13].

The asymptotic solution of (38) reads

$$\varrho = \frac{1}{2}(\sigma_0 + \mathbf{x} \cdot \boldsymbol{\sigma}) \xrightarrow{t \rightarrow \infty} \varrho(\infty) = \frac{P_- + e^{-\beta\Omega}P_+}{1 + e^{-\beta\Omega}}, \quad (39)$$

where the notation is the same as in (9), $P_{\pm} = \sigma_{\pm}\sigma_{\mp} = (\sigma_0 \pm \sigma_3)/2$ are the two projections, and Boltzmann's statistics is implied. The stochastic Ito-Schrödinger equation reads

$$d\psi = -i(H_1 - iH_2)\psi dt - i(L_-dW_- + L_+dW_+ + L_3dW_3)\psi, \quad (40)$$

where the noises are independent, $\langle dW_k \rangle = 0$, $\langle dW_k dW_l \rangle = \delta_{kl}dt$, with $k, l = \pm, 3$, and

$$\begin{aligned} L_- &= -i\sqrt{\gamma'(n+1)}\sigma_-, \\ L_+ &= -i\sqrt{\gamma'n}\sigma_+, \\ L_3 &= \sqrt{\gamma}\sigma_3, \end{aligned} \quad (41)$$

so that (weak) probability conservation implies

$$H_2 = \frac{1}{2} \sum_{k=\pm,3} L_k^\dagger L_k = \frac{\gamma'}{2}(1+n)P_+ + \frac{\gamma'}{2}nP_- + \frac{\gamma}{2}\sigma_0 \quad (42)$$

in agreement with the GKLS equation (38), as it should.

As in the examples considered in Sect. 2, similar comments apply to the thermal damping of a harmonic oscillator (with $H_1 = \Omega a^\dagger a$ and $N = a^\dagger a$)

$$\begin{aligned} \mathcal{L}\varrho &= -\frac{\gamma'}{2}(1+n)(\{a^\dagger a, \varrho\} - 2a\varrho a^\dagger) \\ &\quad -\frac{\gamma'}{2}n(\{aa^\dagger, \varrho\} - 2a^\dagger\varrho a) - \frac{\gamma}{2}(\{N^2, \varrho\} - 2N\varrho N). \end{aligned} \quad (43)$$

5. Generalization and First Main Theorem

We now generalize the observations of Sects. 3 and 4 to the case of a master equation with N GKLS operators L_k ($k = 1, \dots, N$). Notice that it is sufficient to consider $N \leq d^2 - 1$, where d is the dimension of the Hilbert space. A larger number of operators will be dependent and always reducible to this case.

In the Ito form, the stochastic Schrödinger equation reads

$$d\psi = -i(H_1 - iH_2)\psi dt - i \sum_{k=1}^N L_k \psi dW_k, \quad (44)$$

where $H_1 = H_1^\dagger$, $H_2 = H_2^\dagger$, and, in general, $L_k \neq L_k^\dagger$. Moreover, the noises are taken to be normalized and independent:

$$\langle dW_k \rangle = 0, \quad \langle dW_k dW_l \rangle = \delta_{kl} dt. \quad (45)$$

From the weak conservation of probability (27) we get

$$H_2 = \frac{1}{2} \sum_k L_k^\dagger L_k \quad (46)$$

and the ensuing master equation

$$\begin{aligned} \frac{d\rho}{dt} &= -i[H_1, \rho] - \{H_2, \rho\} + \sum_k L_k \rho L_k^\dagger \\ &= -i[H_1, \rho] - \frac{1}{2} \sum_k (\{L_k^\dagger L_k, \rho\} - 2L_k \rho L_k^\dagger). \end{aligned} \quad (47)$$

The Stratonovich form of the stochastic Schrödinger equation is instead

$$d\psi = -i(H_1^S - iH_2^S)\psi dt - i \sum_{k=1}^N L_k \psi \circ dW_k, \quad (48)$$

where $H_1^S - iH_2^S = H_1 - iH_2 + \frac{i}{2} \sum_k L_k^2$, that is,

$$H_1^S = H_1 - \frac{1}{2} \text{Im} \sum_k L_k^2 = H_1 - \frac{1}{2} \sum_k \frac{L_k^2 - L_k^{2\dagger}}{2i} \quad (49)$$

and

$$H_2^S = H_2 - \frac{1}{2} \text{Re} \sum_k L_k^2 = \frac{1}{2} \sum_k \left(L_k^\dagger L_k - \frac{L_k^2 + L_k^{2\dagger}}{2} \right). \quad (50)$$

Therefore, the total Hamiltonian reads

$$H_\eta = H_1^S - iH_2^S + \sum_k \eta_k(t)L_k. \quad (51)$$

This Hamiltonian is Hermitian if and only if

$$L_k = L_k^\dagger, \quad \forall k. \quad (52)$$

Indeed, this condition implies that the non-Hermitian time-independent Hamiltonian vanishes, namely,

$$H_2^S = \frac{1}{2} \sum_k \left(L_k^\dagger L_k - \frac{L_k^2 + L_k^{2\dagger}}{2} \right) = 0. \quad (53)$$

Remarkably, conditions (52) and (53) are in fact *equivalent*, as one can easily prove by setting $L_k = X_k + iY_k$ and taking the trace. This is an instance of the fluctuation-dissipation theorem: a (non-)Hermitian time-independent Hamiltonian (i.e. an imaginary optical potential *à la* Fermi [14, 15]) in (51) is accompanied by a (non-)Hermitian noise term.

Conversely, one can derive a *bona fide* GKLS equation from a non-Hermitian dissipative Hamiltonian $H - iV$ by adding an anti-Hermitian fluctuating term with $L = iV^{1/2}$. This is a way to cure the illness of an optical potential by restoring probability conservation through a fluctuation-dissipation mechanism.

The dissipative part of the master equation (47) reads

$$\mathcal{L}\rho = -\frac{1}{2} \sum_k (\{L_k^\dagger L_k, \rho\} - 2L_k \rho L_k^\dagger), \quad (54)$$

hence its dual is

$$\mathcal{L}^\sharp A = -\frac{1}{2} \sum_k (\{L_k^\dagger L_k, A\} - 2L_k^\dagger A L_k). \quad (55)$$

We recall that the trace conservation property, $\text{Tr}(\mathcal{L}\rho) = 0$, is equivalent to the unitality of the dual map, $\Lambda^\sharp \mathbb{I} = \mathbb{I}$ or $\mathcal{L}^\sharp \mathbb{I} = 0$.

By looking at the above expressions it is evident that $L_k = L_k^\dagger$ implies the self-duality of \mathcal{L} , say $\mathcal{L} = \mathcal{L}^\sharp$. The converse does not hold due to the non-uniqueness of the decomposition of \mathcal{L} in terms of the GKLS operators L_k : for example, $\mathcal{L}^\sharp = \mathcal{L}$ when $L_k^\dagger = e^{i\alpha_k} L_k$, with arbitrary phases α_k . Summarizing, we arrive at the following conclusion:

$$H_\eta = H_\eta^\dagger \iff H_2^S = 0 \iff L_k = L_k^\dagger \quad \forall k \implies \mathcal{L} = \mathcal{L}^\sharp. \quad (56)$$

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Conditions (51)–(53), together with (56), generalize (30)–(32). The validity of these conditions enable one to say that the noise sources in (51) are “classical”.

Let us check the main conclusions of this section by looking again at the example of a two-level atom in interaction with a thermal field, discussed in Sect. 4. Let us translate this example in Stratonovich form and check the chain of equivalence (56). In the case (40)–(42), (49)–(50) specialize to

$$H_1^S = 0 \quad (\text{since } H_1 = 0), \quad (57)$$

$$H_2^S = \frac{\gamma'}{2}(1+n)P_+ + \frac{\gamma'}{2}nP_-, \quad (58)$$

so that, according to (51),

$$H_\eta = iH_2^S + \sum_{k=\pm,3} \eta_k(t)L_k, \quad (59)$$

with L_k 's given in (41). Notice that H_2^S does not vanish, due to the presence of the NSD components L_\pm , that yield the terms in (58). This makes the interpretation of the Stratonovich “Hamiltonian” cumbersome for NSD equations. Incidentally, this example also clarifies that the “Hamiltonians” (10), (17) and (59) require different physical interpretations. In general, if condition (52) does not hold, then the Stratonovich Hamiltonian (53) does not vanish.

The following two sections are devoted to a thorough discussion of this result. In Sect. 6, we first define dephasing and decay channels, then discuss the self-duality of \mathcal{L} , for a single channel in Sect. 6.1 and for multiple channels in Sect. 6.2. We finally analyze how correlated noises give rise to equivalent forms of the master equation in Sect. 7.

6. Self-duality and Dephasing

In Sect. 5 we have proven our central result that only processes engendered by a self-dual generator $\mathcal{L} = \mathcal{L}^\sharp$ can be obtained as the average over a classical noise of a unitary evolution, engendered by a time-dependent self-adjoint Hamiltonian $H_\eta = H_\eta^\dagger$. In this section we shall investigate in detail the connection between the mathematical concept of self-duality and the physical notion of decay. By using the findings of Sect. 5, this will enable us to give an answer to the question posed in the title of this paper, and explain the laconic abstract.

We will first consider in Sect. 6.1 the situation of a single channel, that is a GKLS generator \mathcal{L} with a single term in the operator sum (54) and introduce the crucial definition of dephasing and decay channels, that we will need in

the following. Then, in Sect. 6.2, we will move to a generic (multichannel) generator \mathcal{L} and prove that a self-dual generator is the sum of dephasing, *i.e.* *nondecaying*, channels.

6.1. SINGLE CHANNEL

Let d be the dimension of the Hilbert space. We define a *single channel* as a GKLS generator \mathcal{L} with a single term in the operator sum (54), namely

$$\mathcal{L}\rho = L\rho L^\dagger - \frac{1}{2}\{L^\dagger L, \rho\}. \quad (60)$$

We shall say that a single channel \mathcal{L} is *dephasing* if

$$\text{there exists a basis } \{|e_n\rangle\}_{n=1,\dots,d} \text{ such that } \mathcal{L}(|e_n\rangle\langle e_n|) = 0 \text{ for all } n. \quad (61)$$

The basis $\{|e_n\rangle\}$ is named *stable basis* (under the action of \mathcal{L}). This definition is in accord with the general philosophy outlined in [16]. Observe also that some unitary dynamics are included as limiting cases of the above definition (when the dissipation is null).

A single channel \mathcal{L} that is not dephasing is called a *decay* channel. Therefore a decay channel admits *no stable basis*: physically, this implies that there are always population flows (except for special initial states).^a

We shall prove that

$$\mathcal{L} \text{ is a dephasing channel} \iff L \text{ is normal}, \quad (62)$$

that is $[L, L^\dagger] = 0$.

The proof goes as follows. Let $\{|e_n\rangle\}$ be a stable basis. \mathcal{L} dephasing implies that

$$L|e_n\rangle\langle e_n|L^\dagger - \frac{1}{2}(L^\dagger L|e_n\rangle\langle e_n| + |e_n\rangle\langle e_n|L^\dagger L) = 0 \quad (63)$$

for all n , and taking expectations over $|e_m\rangle$, $m \neq n$ gives

$$\langle e_m|L|e_n\rangle\langle e_n|L^\dagger|e_m\rangle = 0, \quad (64)$$

which implies $\langle e_m|L|e_n\rangle = 0$. We therefore conclude that L is diagonal in the stable basis, that is

$$L = \sum_{n=1}^d z_n |e_n\rangle\langle e_n|, \quad z_n \in \mathbb{C} \quad (65)$$

^aWe notice that these definitions of “channels” are in line with those adopted in the context of scattering theory.

and thus is normal, $[L, L^\dagger] = 0$.

Conversely, if L is normal then it can be diagonalized as in (65), where $\{|e_n\rangle\}$ is an orthonormal basis. Thus L commutes with $|e_n\rangle\langle e_n|$ for all n , and this implies (63). This proves the equivalence of the two statements in (62).

We now want to establish a link between dephasing (and thus normality of the operator) and self-duality, in the single-channel case. We shall prove that

$$\mathcal{L} = \mathcal{L}^\# \implies \mathcal{L} \text{ is a dephasing channel.} \quad (66)$$

In words, every single self-dual channel is a dephasing channel. The contrapositive of (66) reads

$$\mathcal{L} \text{ is a decay channel} \implies \mathcal{L} \neq \mathcal{L}^\# \quad (67)$$

meaning that every decay channel is non-self-dual.

The proof is straightforward. The dual of (60) reads

$$\mathcal{L}^\# \rho = L^\dagger \rho L - \frac{1}{2} \{L^\dagger L, \rho\}. \quad (68)$$

Therefore we get

$$\mathcal{L} \rho = \mathcal{L}^\# \rho \iff L \rho L^\dagger = L^\dagger \rho L, \quad (69)$$

and by taking $\rho = \mathbb{I}/d$, we have $[L, L^\dagger] = 0$, which in turn, by (62), implies (66).

Two comments are now in order. First, notice that normality of L , which is equivalent to a dephasing channel \mathcal{L} , does not imply self-duality, and the opposite implication of (66) is not true in general. However, it is possible to give a full characterization of single self-dual channels by strengthening the class of normal operators which act as GKLS operators. Indeed, one gets that

$$\mathcal{L} = \mathcal{L}^\# \iff L^\dagger = e^{i\alpha} L, \quad (70)$$

for some real α . That is, a single self-dual channel is characterized by a Hermitian operator modulo a phase: $L = e^{-i\alpha/2} X$, with $X = X^\dagger$.^b

The proof goes as follows. We have proved above that if $\mathcal{L} = \mathcal{L}^\#$ then L is normal, and thus is diagonal in some (stable) basis as in (65). By plugging (65) into the right-hand side of (69), we get

$$z_m z_n^* = z_m^* z_n, \quad (71)$$

for all m, n , which implies that $z_n = e^{-i\alpha/2} x_n$, with x_n real for all n and for some real (n -independent) phase α . The converse is immediate. The

^bRemember that a change of phase is a gauge freedom that does not change \mathcal{L} .

generalization of this characterization for dilations that include also Poisson processes has been proved by Kümmerer and Maassen in [6], where the detailed balance condition [17] plays a significant role.

Second, the extension of the above results to a multichannel process (54) has to confront the noncommutativity of the GKLS operators L_k . Suffice it to say that, in general, in the presence of several dephasing channels there exists *no* stable basis. Indeed, each single channel admits a stable basis, but two bases are incompatible if the corresponding GKLS operators do not commute. Therefore the net effect of several dephasing channels could be mistaken with decay by a naive application of definition (61). We will deal with such a general situation in the next subsection.

6.2. MULTIPLE CHANNELS

In the previous subsection we have proved that a single self-dual channel is a dephasing channel, by using the characterization of a dephasing channel in terms of the normality of its GKLS operator. Now we shall establish such a link in the general case of a multichannel process. We shall prove that

$$\mathcal{L} = \mathcal{L}^\sharp \implies \mathcal{L} = \sum_{k=1}^N \mathcal{L}_k \quad \text{with } \mathcal{L}_k \text{ a dephasing channel.} \quad (72)$$

In words, every self-dual process is composed by one or more dephasing channels. The contrapositive of (72) states that if a process cannot be decomposed into a sum of dephasing channels, then its generator is non-self-dual: the presence of a *bona fide* decay is symptomized by a non-self-dual \mathcal{L} .

We will get (72) by proving the stronger result that $\mathcal{L} = \mathcal{L}^\sharp$ implies each channel \mathcal{L}_k to have a Hermitian GKLS operator L_k and thus by (70) to be self-dual:

$$\mathcal{L}_k = \mathcal{L}_k^\sharp \quad \text{for all } k. \quad (73)$$

This in turn, by (66), implies (72).

Here is the proof. Consider a (multichannel) dissipative \mathcal{L} as in (54) and its dual (55). The self-duality condition, $\mathcal{L}\rho = \mathcal{L}^\sharp\rho$, requires that the following relation be satisfied

$$\sum_k L_k \rho L_k^\dagger = \sum_k L_k^\dagger \rho L_k \quad (74)$$

for any ρ . Notice that (74) implies the equality

$$\sum_k L_k L_k^\dagger = \sum_k L_k^\dagger L_k \quad (75)$$

and therefore the dual of \mathcal{L} reads

$$\begin{aligned}\mathcal{L}^\sharp \rho &= -\frac{1}{2} \sum_k \left(\{L_k^\dagger L_k, \rho\} - 2L_k^\dagger \rho L_k \right) \\ &= -\frac{1}{2} \sum_k \left(\{L_k L_k^\dagger, \rho\} - 2L_k^\dagger \rho L_k \right).\end{aligned}\quad (76)$$

We can write any self-dual \mathcal{L} as

$$\mathcal{L}\rho = \frac{1}{2}(\mathcal{L} + \mathcal{L}^\sharp)\rho, \quad (77)$$

which implies that every GKLS operator appears in pair with its Hermitian conjugate. Explicitly, we have

$$\begin{aligned}\mathcal{L}\rho &= -\frac{1}{4} \sum_k \left(\{L_k^\dagger L_k, \rho\} + \{L_k L_k^\dagger, \rho\} - 2L_k \rho L_k^\dagger - 2L_k^\dagger \rho L_k \right) \\ &= -\frac{1}{2} \sum_k \left(\{X_k^2, \rho\} - 2X_k \rho X_k + \{Y_k^2, \rho\} - 2Y_k \rho Y_k \right),\end{aligned}\quad (78)$$

where the Hermitian operators X_k and Y_k are defined by $L_k = X_k + iY_k$. The last expression means that a self-dual \mathcal{L} can always be described by the sum of Hermitian GKLS operators. This completes the proof.

It is worth noticing that the decomposition of a generator \mathcal{L} in terms of single channels is not unique, as for example is manifestly shown in the first and the second line of (78). Thus an evolution could be built up by a sum of decaying channels whose net effect is nevertheless self-dual, and thus, by (72), equivalent to a sum of purely dephasing channels. A paradigmatic example is the two-level atom in a thermal photon bath considered in (38): when the temperature goes to infinity it happens that $\gamma'(1+n) \sim \gamma'n$, and the net population transfer between the two atomic levels goes to zero. This again is related to a detailed balance condition [17].

Conclusions (56) and (72) are the central results of this paper. As stressed before, the contrapositive of (72) states that decay can only be obtained by a non-self-dual \mathcal{L} . Therefore an interpretation involving “classical noise”, in the sense of (51)–(53) and (56), is untenable.

7. Correlated Noises and Equivalent Forms of the Master Equation

We elaborate here on equivalent forms of the master equation and their corresponding stochastic Schrödinger equations. So far, our analysis has focused on noise terms of the type

$$\sum_{k=1}^N L_k dW_k, \quad (79)$$

with $N \leq d^2 - 1$, d being the dimension of the system, with generally non-Hermitian operators L_k and *real* independent noises dW_k such that

$$\langle dW_k dW_l \rangle = \delta_{kl} dt. \quad (80)$$

This ansatz yields a master equation with a diagonalized Kossakowski matrix, as in (47).

However, this is clearly not the only option. For example, one can decide to work with Hermitian GKLS operators and expand the L_k 's in terms of $d^2 - 1$ linearly independent *Hermitian* operators λ_j , e.g., $\text{su}(d)$ operators,

$$L_k = \sum_{j=1}^{d^2-1} c_{kj} \lambda_j, \quad (81)$$

where c_{kj} are the complex coefficients of the expansions. In such a case one ends up with $d^2 - 1$ *complex* noise terms

$$\sum_{j=1}^{d^2-1} \lambda_j dZ_j, \quad dZ_j = \sum_{k=1}^N c_{kj} dW_k, \quad (82)$$

that are in general not independent:

$$\langle dZ_i^* dZ_j \rangle = a_{ij} dt, \quad \langle dZ_i dZ_j \rangle = b_{ij} dt, \quad (83)$$

with

$$a_{ij} = \sum_{k=1}^N c_{ki}^* c_{kj}, \quad b_{ij} = \sum_{k=1}^N c_{ki} c_{kj}. \quad (84)$$

The covariance matrix a is positive semi-definite, $a = a^\dagger$ and $a \geq 0$, while the ‘‘relation matrix’’ b is symmetric, $b = b^T$, satisfying Picinbono’s condition $a^* - b^\dagger a^{-1} b \geq 0$ (with the inverse a^{-1} defined on the support of a) [18]. Such conditions guarantee the positivity of the complex noise matrix. The dissipative part of the corresponding master equation reads

$$\mathcal{L}\rho = -\frac{1}{2} \sum_{i,j} a_{ij} (\{\lambda_i \lambda_j, \rho\} - 2\lambda_j \rho \lambda_i), \quad (85)$$

instead of (47). Observe that the noise correlations yield the Kossakowski matrix a_{ij} .

In order to obtain the master equation (85) from a stochastic Schrödinger equation with the complex noise terms (82), the first condition in (83), $\langle dZ_i^* dZ_j \rangle = a_{ij} dt$, is crucial, while the second one, $\langle dZ_i dZ_j \rangle = b_{ij} dt$, is not needed and the relation matrix b can be arbitrary, as long as b satisfies Picinbono’s condition.

However, in order to go from the stochastic Schrödinger equation with the complex noises dZ_i in (82) to the one (44) with the real independent noises dW_i , by diagonalizing the covariance matrix a , the relation matrix b should be appropriately chosen in order to get the minimal number of real noises. Note that there are $2d^2 - 2$ real noises (real and imaginary parts) in the $d^2 - 1$ complex noises dZ_i , but only $d^2 - 1$ real noises dW_i suffice for the stochastic Schrödinger equation (44), with the rest of the degrees of freedom being redundant. The right choice of b is the following. We diagonalize a as $a_{ij} = \sum_k \gamma_k U_{ki}^* U_{kj}$, with a unitary matrix U and positive semi-definite eigenvalues γ_i . Then, we choose the relation matrix b as $b_{ij} = \sum_k \gamma_k U_{ki} U_{kj}$, which makes half of the real noises in dZ_i irrelevant (vanishing). On the other hand, while the choice of the relation matrix b does not affect the master equation (85), it does affect the Stratonovich Hamiltonian.

8. Answer(s)

Let us summarize the overall picture of our results. For a generic GKLS generator \mathcal{L} of a master equation (47) with multiple channels, by combining the implications (56) and (72), we get

$$\begin{aligned} H_\eta = H_\eta^\dagger &\iff H_2^S = 0 \iff L_k = L_k^\dagger \quad \forall k \\ &\implies \mathcal{L} = \mathcal{L}^\sharp \\ &\implies \mathcal{L} = \sum_{k=1}^N \mathcal{L}_k \text{ with } \mathcal{L}_k \text{ a dephasing channel.} \end{aligned} \tag{86}$$

For each single channel \mathcal{L}_k , by combining the implications (62), (66), and (70), we have

$$\begin{aligned} \mathcal{L}_k = \mathcal{L}_k^\sharp &\iff L_k^\dagger = e^{i\alpha_k} L_k \\ &\implies [L_k, L_k^\dagger] = 0 \iff \mathcal{L}_k \text{ is a dephasing channel.} \end{aligned} \tag{87}$$

The answer to the question posed in the title of this paper is negative: decay cannot be ascribed to a “classical” noise process, where the connotation of the term “classical” has to be understood according to (30)–(32), or in general to (51)–(53) and (56), without the *caveat* of taking the average over the noise, as e.g. in (27).

This is a consequence of the chain of equivalence (86), that is valid for master equations with an arbitrary finite number of multidimensional GKLS operators. Dephasing is inextricably related to self-dual generators and as a consequence the opposite process, decay, can only be ascribed to non-self-dual maps. Physical interpretations involving “classical noises” only apply to the former process. On the contrary, the latter process entails non-Hermitian

Hamiltonians (and imaginary optical potentials *à la* Fermi [14]): probability would no longer be conserved.

There is, however, a second possible answer to our question: yes, decay can be ascribed to a “classical” noise process, if we relax the condition (37) of probability conservation in individual realizations, and just require probability conservation on average: During the stochastic process, sometimes particles are absorbed by the environment, sometimes they are released, with a null average net flux. This is what we called *weak* conservation of probability before (27).

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