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Feynman graphs and the large dimensional limit of multipartite entanglement

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In this paper, we extend the analysis of multipartite entanglement, based on techniques from classical statistical mechanics, to a system composed of n d-level parties (qudits). We introduce a suitable partition function at a fictitious temperature with the average local purity of the system as Hamiltonian. In particular, we analyze the high-temperature expansion of this partition function, prove the convergence of the series, and study its asymptotic behavior as $d \to \infty$. We make use of a diagrammatic technique, classify the graphs, and study their degeneracy. We are thus able to evaluate their contributions and estimate the moments of the distribution of the local purity. *Published by AIP Publishing.* https://doi.org/10.1063/1.5019481

I. INTRODUCTION

Since its early origins,[1] entanglement has been considered as one of the most basic and intriguing features of quantum mechanics.[2] During the years, it has turned out to be a fundamental resource in quantum information[3–5] and has originated a large number of research topics in mathematical physics[6–13] and in applied science such as quantum teleportation technology[14] and quantum key distribution protocols.[15–17]

The characterization and quantification of quantum correlations are not simple tasks. Bipartite entanglement, i.e., the entanglement of two subsystems, denoted by A and $\bar{A}$, is well understood and can be completely characterized, for instance, using the von Neumann entropy[18] or the entanglement of formation.[19] Another possible measure is the so-called *purity* (of the relevant subsystem) π_A. Given an initial pure state, one can obtain the reduced density matrix of subsystem A performing a partial trace over the degrees of freedom of subsystem $\bar{A}$; the function π_A measures how "pure" is the reduced state, i.e., its "distance" from a completely mixed state. The more entangled the initial pure state, the smaller the value of this purity. In particular (see Lemma 1 and Definition 2), a maximally entangled (pure) state will be left in a completely mixed state after the partial trace.

On the other hand, multipartite entanglement is less understood and more elusive even if widely investigated.[20–23] These difficulties are deeply rooted both in the exponentially (with the system size) large number of measurements needed for its complete characterization and in new phenomena emerging from the complex interactions among the parties. Obviously, the choice of a particular measure, the dimension of the Hilbert space of local parties, and the symmetries imposed on quantum states will have an influence on the result. A natural question is whether it is possible to find maximally entangled states in the multipartite scenario. For instance, in Ref. 24, Gisin and Bechmann-Pasquinucci characterized pure and symmetric maximally entangled states of n qubits (i.e., an ensemble of n two-level systems) as the states such that all their partial traces are maximally mixed. The idea of characterizing *multipartite maximally entangled states* (MMESs) minimizing their average purity over different bipartitions of the system has been put forward in Ref. 25 where these states have been obtained as solutions of an optimization problem where the cost function is a proper average of purities, the *potential of multipartite entanglement*,

$$\pi_{\mathrm{ME}} = \frac{1}{\mathcal{N}_A} \sum_A \pi_A. \tag{1}$$

Here $\mathcal{N}_A$ denotes the number of terms in the summation, which can be restricted to a certain subset of partitions (in this paper, we will consider the number of balanced bipartitions, see Definition 3). It is interesting to notice that these states have been analyzed in different contexts. For instance, studies have been devoted to their connections with quantum secret sharing[26] and combinatorial designs.[27] Moreover, recent analyses have focussed attention on the so-called k-uniform states and their link to orthogonal arrays.[28,29]

As already mentioned, besides an interesting topic *per se*, the study of MMESs is important because of new intriguing phenomena arising in the multipartite scenario. A peculiar property of multipartite entanglement, the so-called *entanglement frustration*,[30] naturally appears when one tries to minimize the purity of all possible bipartitions at the same time. This subject has been explored because of its connection with self-dual codes,[31] and it has been possible to prove theorems which ensures the *impossibility* to reach the ideal minimum value of purity for all bipartitions for collections of $n \geq 7$ qubits[31,32] and even in the relatively simple case of $n = 4$.[22,33]

A possible approach to study the appearance of entanglement frustration has been introduced for qubits in Refs. 34–36. In particular, this approach is based on methods from classical statistical mechanics. One introduces a Hamiltonian representing the potential of multipartite entanglement,

$$H(z) = \pi_{ME}(z) = \binom{n}{\left[\frac{n}{2}\right]}^{-1} \sum_{|A|=n_A/2} \pi_A = \sum_{k,k',l,l' \in \mathbb{Z}_2^n} \Delta(k,k';l,l') z_k z_{k'} \bar{z}_l \bar{z}_{l'}, \tag{2}$$

for a normalized pure state written in the computational basis in terms of its Fourier coefficients $z = (z_k)$,

$$|\psi\rangle = \sum_{k \in \mathbb{Z}_2^n} z_k \, |k\rangle\,, \tag{3}$$

with *coupling function* Δ (see Theorem 2 for its complete general expression). By introducing the partition function

$$Z(\beta) = \int \mathrm{d}\mu(z) \mathrm{e}^{-\beta H(z)}, \tag{4}$$

with β as a Lagrange multiplier and μ as the unitarily invariant measure over pure states on the hypersphere $\{z \in \mathbb{C}^N \,|\, \|z\|^2 = \sum_k |z_k|^2 = 1\}$ induced by the Haar measure over the unitary group $\mathcal{U}(N)$,[37]

$$\mathrm{d}\mu(z) = \frac{(N-1)!}{\pi^N} \, \delta\left(1 - \|z\|^2\right) \prod_k \mathrm{d}z_k \mathrm{d}\bar{z}_k, \tag{5}$$

one can explore the configurations for $\beta \to +\infty$ where frustration appears and only MMESs are sampled. For qubits, it has been possible to use a high-temperature expansion technique and a diagrammatic evaluation of the cumulants of a probability density function. In principle, one should try to perform the re-summation of all diagrams. On the other hand, in order to better understand the meaning of entanglement frustration, it is interesting to analyze the contributions of different classes of diagrams. Unfortunately, the calculations are far from being simple and only a few number of cumulants are analytically known. In particular, the topology of diagrams is highly non-trivial and both analytical and numerical analyses suggest that the presence of frustration could be related to a precise class of graphs appearing in the cumulant expansion. One would like to find an objective procedure, if admissible, for discarding some graphs and resumming only a subset of them. Obviously, we would like to have a criterion for choosing diagrams which is not based only on simplicity. A possible way to tackle this problem has been introduced in Ref. 38 where the selection of graphs has been based on the introduction of a *color* index N_c and on the consideration of a field theory for the multipartite entanglement. An explicit calculation at leading order in N_c has given hints about the presence of a phase transition, and it has been possible to numerically observe that the limit of large values of the parameter N_c removes the frustration. On the other hand, it is difficult to give a direct physical interpretation to this approach though it is appealing from the mathematical point of view.

Following these motivations, in this paper, we want to explore another limit. In particular, after introducing a generalization of the previously sketched framework to a collection of d-level systems with $d \geq 2$, we want to study and characterize the behavior of the potential of multipartite entanglement in the limit of large values of d. In particular, we will find an explicit expression of the coupling function Δ (see Theorem 2), which generalizes the one obtained for qubits, and study its symmetries. Then we will examine the high-temperature expansion of the distribution function of the potential of multipartite entanglement, proving that the series characterizing the expansion converges, and observe that when d is large enough only a specific class of diagrams contributes to the partition function.

This paper is organized as follows. In Sec. II, we introduce the notation and give a detailed description of the problem. In Sec. III, we define and analyze the Hamiltonian function, introduce the statistical mechanics approach, and give the main results of the paper (Theorems 3–5). In Sec. IV, we introduce the diagrammatic technique used for the analysis of the cumulants. Finally, using this diagrammatic technique, in Sec. V, we give the proof of Theorem 5. Sections IV and V present the technical results of this paper and may be skipped by the eager reader who is interested on the main results without going into too many details. We add two appendices. In Appendix A, we exhibit numerical results about a state that, to the best of our knowledge, reaches the lowest value of the 7-qubit potential of multipartite entanglement. In Appendix B, we include for self-consistency some results about the relation between perfect MMESs and maximum distance separable codes.

II. BIPARTITE AND MULTIPARTITE ENTANGLEMENT

A. Bipartite entanglement and purity

Let us consider a collection of n d-dimensional quantum systems described by an N-dimensional Hilbert space $\mathcal{H} = \mathbb{C}^{d^n}$ (with $N = d^n$) and separate them into two disjoint sets of, respectively, n_A and $n_{\bar{A}}$ elementary systems ($n_A + n_{\bar{A}} = n$), thus defining a bipartition.

Definition 1. A bipartition of a system $S = \{1, 2, \ldots, n\}$ of n parties is a pair $(A, \bar{A})$ such that $A \cup \bar{A} = S$ and $A \cap \bar{A} = \phi$. Furthermore, if $|A| = n_A$ and $|\bar{A}| = n_{\bar{A}} = n - n_A$ are the dimensions of the two subsystems, then the bipartition is called balanced if

$$n_A = \left[\frac{n}{2}\right] \quad \text{and} \quad n_{\bar{A}} = \left[\frac{n+1}{2}\right], \tag{6}$$

where $[x]$ denotes the integer part of x (namely, the greatest integer that is less than or equal to x).

Notice that in the definition it is assumed, without loss of generality, that $n_A \leq n_{\bar{A}}$; indeed, the bipartitions $(A, \bar{A})$ and $(\bar{A}, A)$ will play the same role in our considerations.

With this definition, we can consider the Hilbert space $\mathcal{H}$ as a tensor product $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$, where $\mathcal{H}_A \simeq \mathbb{C}^{N_A}$ and $\mathcal{H}_{\bar{A}} \simeq \mathbb{C}^{N_{\bar{A}}}$ with dimensions $N_A = d^{n_A}$ and $N_{\bar{A}} = d^{n_{\bar{A}}}$, respectively. Every normalized vector $|\psi\rangle \in \mathcal{H}$, representing a pure state of the system, admits a Fourier expansion in terms of the orthonormal computational basis $\{|k\rangle\}_{k\in\mathbb{Z}_d^n}$,

$$|\psi\rangle = \sum_{k\in\mathbb{Z}_d^n} z_k\, |k\rangle\,, \tag{7}$$

where $z_k = \langle k|\psi\rangle \in \mathbb{C}$, $k \in \mathbb{Z}_d^n$, and $\mathbb{Z}_d = \mathbb{Z}/d\,\mathbb{Z}$ is the cyclic group with d elements. Indeed, there is a natural correspondence between the basis of the space and the strings of length n over d symbols.

A convenient measure of bipartite entanglement is the so-called *purity (of the reduced state)*

$$\pi_A(\psi) = \mathrm{tr}(\rho_A^2) = \mathrm{tr}(\rho_{\bar{A}}^2) = \sum_k \lambda_k^2, \tag{8}$$

where $\rho_A = \mathrm{tr}_{\bar{A}}(|\psi\rangle\langle\psi|)$ is the reduced density matrix of the subsystem A, with $\mathrm{tr}_{\bar{A}}$ denoting the partial trace over subsystem $\bar{A}$, and λ_k's are the eigenvalues of ρ_A. Since for the rest of this work we will consider only the purity of the relevant subsystem, we will refer to this quantity simply as purity.

Using this expansion, we can rewrite the purity as

$$\begin{aligned}\pi_A(\psi) &= \mathrm{tr}\left(\left(\mathrm{tr}_{\bar{A}}(|\psi\rangle\langle\psi|)\right)^2\right)\\ &= \mathrm{tr}\Bigg(\Big(\sum_{k,l\in\mathbb{Z}_d^n} z_k \bar{z}_l \delta_{k_{\bar{A}} l_{\bar{A}}} |k_A\rangle\langle l_A|\Big)^2\Bigg)\\ &= \mathrm{tr}\Bigg(\sum_{k,k',l,l'\in\mathbb{Z}_d^n} z_k z_{k'} \bar{z}_l \bar{z}_{l'} \delta_{k_{\bar{A}} l_{\bar{A}}} \delta_{k'_{\bar{A}} l'_{\bar{A}}} |k_A\rangle\langle l_A|k'_A\rangle |l'_A\rangle\Bigg)\\ &= \sum_{k,k',l,l'\in\mathbb{Z}_d^n} z_k z_{k'} \bar{z}_l \bar{z}_{l'} \delta_{k_{\bar{A}} l_{\bar{A}}} \delta_{k'_{\bar{A}} l'_{\bar{A}}} \delta_{k'_A l_A} \delta_{k_A l'_A},\end{aligned} \tag{9}$$

where we used the symbol k_A to denote the substring of k belonging to A, and where δ_{lm} is the Kronecker delta.

In the following, we will need this lemma of immediate proof.

Lemma 1. Given a state $|\psi\rangle \in \mathcal{H}$ of n qudits and a bipartition $(A, \bar{A})$, the following holds:

1. $\pi_A(\psi) = \pi_{\bar{A}}(\psi)$,
2. $1/N_A \le \pi_A(\psi) \le 1$,

where $N_A = d^{n_A}$, with $n_A = |A|$. Moreover, the upper and lower bounds in 2 are reached, respectively, only by separable states and maximally entangled states.

B. Multipartite entanglement and potential of multipartite entanglement

In this section, we will deeply extend the use of purity for the characterization of the *multipartite* entanglement of a generic system of n parties.

In general, a system composed of $n > 2$ parties has a number of different bipartitions that scale as $\mathcal{N}_A = O(2^n)$. The states that saturate the lower bound of the purity for some bipartitions are called *maximally entangled* with respect to these bipartitions. The Bell states are examples of maximally entangled states (here there is no need to specify the bipartition since it is unique in systems with only two components).

Definition 2. A state $|\psi\rangle$ such that $\pi_A(\psi) = 1/N_A$ with respect to every bipartition $(A, \bar{A})$, i.e., maximally entangled with respect to every bipartition of the system, is called a perfect multipartite maximally entangled state (perfect MMES).

To determine if a given state $|\psi\rangle$ is a perfect MMES, it is sufficient to check if it satisfies the minimization condition for all the balanced bipartitions, i.e., bipartitions with $n_A = |A| = [n/2]$. Indeed, if a state has a reduced density matrix of the form

$$\rho_A = \frac{\boldsymbol{I}}{N_A} \tag{10}$$

for subsystem A, then

$$\rho_B = \frac{\boldsymbol{I}}{N_B} \tag{11}$$

for every smaller subsystem $B \subset A$. Therefore, the problem of finding perfect MMESs can be tackled by studying the average purity over all the balanced bipartitions.

Definition 3. The average purity over all possible balanced bipartitions is called the potential of multipartite entanglement *and is given by*

$$\pi_{\mathrm{ME}}(\psi) = \binom{n}{\left[\frac{n}{2}\right]}^{-1} \sum_{|A|=[n/2]} \pi_A(\psi). \tag{12}$$

As for the purity, we can define a bound for the potential of multipartite entanglement.

Proposition 1. The potential of multipartite entanglement has the following bounds:

$$1/N_A \le \pi_{\mathrm{ME}}(\psi) \le 1, \tag{13}$$

with $N_A = d^{[n/2]}$*, for every state* $|\psi\rangle \in \mathcal{H}$.

The proof of this proposition is a straightforward consequence of Lemma 1.

In the case $d = 2$ (qubits), it is possible to obtain an explicit expression of π_{ME}.[35] We will extend this result for states of qudits. Let us recall that using the Fourier expansion of the state, we can write the purity of a pure state with respect to a given bipartition as (9),

$$\pi_A(\psi) = \sum_{k,k',l,l' \in \mathbb{Z}_d^n} z_k z_{k'} \bar{z}_l \bar{z}_{l'} \delta_{k'_A l_A} \delta_{k_A l'_A} \delta_{k_{\bar{A}} l_{\bar{A}}} \delta_{k'_{\bar{A}} l'_{\bar{A}}}. \tag{14}$$

If we average the purity over all the possible bipartitions with fixed dimension, we can define a *coupling function*[35]

$$\Delta(k, k'; l, l'; n_A) = \frac{1}{2}\tilde{\Delta}(k, k'; l, l'; n_A) + \frac{1}{2}\tilde{\Delta}(k', k; l, l'; n_{\bar{A}}), \tag{15}$$

where

$$\tilde{\Delta}(k, k'; l, l'; n_A) = \binom{n}{n_A}^{-1} \sum_{|A|=n_A} \delta_{k'_A l_A} \delta_{k_A l'_A} \delta_{k_{\bar{A}} l_{\bar{A}}} \delta_{k'_{\bar{A}} l'_{\bar{A}}}. \tag{16}$$

Definition 4. The Hamming weight of a string k over an alphabet Σ*, denoted by* $|k|$*, is the number of symbols that are different from the zero symbol of the alphabet used.*

If the alphabet is binary, i.e., it is $\mathbb{Z}_2 = \{0, 1\}$, then the Hamming weight is nothing but the number of occurrences of the symbol 1 in the string.

We can use the definition of Hamming weight to rewrite the coupling function in a more convenient form.

Theorem 2. *The coupling function* Δ *has the following expression:*

$$\Delta(k, k'; l, l'; n_A) = \delta_{k+k',l+l'} \ \delta_{0,(k-l)\wedge(k'-l)} f(k - l, k' - l, n_A), \tag{17}$$

where

$$f(k, l, n_A) = \frac{1}{2}\binom{n}{n_A}^{-1} \left[\binom{n - |k| - |l|}{n_A - |k|} + \binom{n - |k| - |l|}{n_A - |l|}\right], \tag{18}$$

and

$$k \pm l = (k_j \pm l_j)_j \quad \text{and} \quad k \wedge l = (\min\{k_j, l_j\})_j. \tag{19}$$

Remark 1. The binomial coefficients in Eq. (18) are intended to be zero if one of their arguments is negative. The sum and difference are in $\mathbb{Z}_d$*, and the minimum in the definition of* $\wedge$ *is taken on the (unique) representatives of* k_j *and* l_j *belonging to* $\{0, 1, 2, \dots, d-1\}$.

Proof. The coupling function

$$\Delta(k, k'; l, l') = \binom{n}{n_A}^{-1} \sum_{|A|=n_A} \frac{1}{2}(\delta_{k'_A l_A} \delta_{k_A l'_A} \delta_{k_{\bar{A}} l_{\bar{A}}} \delta_{k'_{\bar{A}} l'_{\bar{A}}} + \delta_{k'_A l'_A} \delta_{k_A l_A} \delta_{k'_{\bar{A}} l_{\bar{A}}} \delta_{k_{\bar{A}} l'_{\bar{A}}}) \tag{20}$$

is non-zero if and only if for some subset A of $S = \{1, 2, \dots, n\}$, with $|A| = n_A$, we have

$$k_A = l'_A, \quad k'_A = l_A, \quad k_{\bar{A}} = l_{\bar{A}}, \quad k'_{\bar{A}} = l'_{\bar{A}}. \tag{21}$$

This imposes that if $j \in A$ and $i \in \bar{A}$,

$$k_j = l'_j, \quad k'_j = l_j, \quad k_i = l_i, \quad k'_i = l'_i \tag{22}$$

or equivalently that for $j \in S$,

$$k_j - l'_j = 0 \quad \text{and} \quad k'_j - l_j = 0 \tag{23}$$

or

$$k_j - l_j = 0 \quad \text{and} \quad k'_j - l'_j = 0. \tag{24}$$

Putting these conditions together, we have that $\Delta \neq 0$ if and only if

$$k + k' = l + l' \text{ and } (k - l) \wedge (k' - l) = 0. \tag{25}$$

It remains to count the number of bipartitions $(A, \bar{A})$ that contribute to the sum in Eq. (20). For this aim, let us call

$$S_0 = \{i \in S \mid k_i = l_i = k'_i = l'_i\},$$
$$S_1 = \{i \in S \mid k_i \neq l_i \text{ or } k'_i \neq l'_i\},$$
$$S_2 = \{i \in S \mid k_i \neq l'_i \text{ or } k'_i \neq l_i\}.$$

From the previous discussion, it is easy to see that $S_1 \cap S_2 = \phi$ and that $S = S_0 + S_1 + S_2$. With this new notation, we can characterize a bipartition $(A, \bar{A})$ for which the contribution of the first term in the sum is non-zero, i.e.,

$$\delta_{k'_A l_A} \delta_{k_A l'_A} \delta_{k_{\bar{A}} l_{\bar{A}}} \delta_{k'_{\bar{A}} l'_{\bar{A}}} \neq 0, \tag{26}$$

as a bipartition such that $A \subset S_1 + S_0$ and $\bar{A} \subset S_2 + S_0$. Furthermore, since $A \cap \bar{A} = \phi$ and $A \cup \bar{A} = S$ then $A = S_1 + A \cap S_0$ and $\bar{A} = S_2 + \bar{A} \cap S_0$, we can conclude that their number is equal to the binomial coefficient

$$\binom{|S_0|}{|A - S_1|} = \binom{n - |S_1| - |S_2|}{n_A - |S_1|} = \binom{n - |k - l| - |k' - l|}{n_A - |k - l|}. \tag{27}$$

The same result can be obtained for the second term in the sum,

$$\delta_{k'_A l'_A} \delta_{k_A l_A} \delta_{k'_{\bar{A}} l_{\bar{A}}} \delta_{k_{\bar{A}} l'_{\bar{A}}}, \tag{28}$$

swapping the role of A and $\bar{A}$, and this ends the proof. □

Since we are going to focus on balanced bipartitions, from now on we will omit the dependence on n_A both in Δ and f, with the understanding that $n_A = \left[\frac{n}{2}\right]$. In this way, with the use of the coupling function, the potential of multipartite entanglement can be written as

$$\pi_{ME}(\psi) = \sum_{k,k',l,l' \in \mathbb{Z}_d^n} \Delta(k, k'; l, l') z_k z_{k'} \bar{z}_l \bar{z}_{l'}. \tag{29}$$

C. MMES, perfect MMES, and frustration

We will see that the lower bound $1/N_A = 1/d^{[n/2]}$ of the potential of multipartite entanglement is not always attained. This justifies the following.

Definition 5. A state $|\varphi\rangle$ *that minimizes* π_{ME}, *i.e.,* $\pi_{\mathrm{ME}}^{\min} = \pi_{\mathrm{ME}}(\varphi)$, *where*

$$\pi_{\mathrm{ME}}^{\min} = \min\{\pi_{\mathrm{ME}}(\psi) : |\psi\rangle \in \mathcal{H}, \langle\psi|\psi\rangle = 1\}, \tag{30}$$

is a multipartite maximally entangled state (MMES).

Let us stress, once again, that the difference between a MMES and a perfect MMES lies in the saturation of the lower bound of the potential of multipartite entanglement.

Example 1. The qubit GHZ state, i.e., the state $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, *is a perfect MMES; indeed it is easy to show that the purities with respect to all the possible bipartitions are* $\frac{1}{2}$.

One of the questions that arise naturally from the previous discussion is on the general structure of a perfect MMES for given values of d (the dimension of each subsystem) and n (the number of subsystems).

With an abuse of notation, we can say that the Bell states are perfect MMESs for systems of two qubits (there is no multipartite entanglement in this case), while for $n = 3$ qubits, the only perfect MMES, up to local and unitary transformations, is the GHZ state.

The problem of characterizing a perfect MMES does not always have such an easy solution. In Ref. 33, Gour *et al.* proved that for $n = 4$ qubits, a perfect MMES does not exist and that the minimum value the average purity can attain is

$$\pi_{ME}^{\min} = \frac{1}{3} > \frac{1}{4} = \frac{1}{N_A}. \tag{31}$$

When the lower bound of the potential of multipartite entanglement cannot be saturated, the system is said to be *frustrated*. If this is the case, the requirement that the purity be minimal for all the bipartitions generates conflicts among them.

For a system of $n = 5, 6$ qubits, there are examples of perfect MMESs, see Ref. 35, while for $n \geq 8$ qubits, a perfect MMES does not exist as proved by Scott in Ref. 31, using classical error correction theory. The case of $n = 7$ qubits has been recently shown to be frustrated.[32] On the other hand, the value of $\pi_{ME}^{\min}$ in this case is unknown and so is the structure of the associated MMES. Until now, only numerical estimates about the minimum of the potential of multipartite entanglement have been done. For a lower bound of $\pi_{ME}^{\min}$ for 7-qubits, see Appendix A.

Frustration appears when one or more bipartitions cannot reach their minima. Nevertheless, it can be proven that enlarging the dimension d of each subsystem, at fixed n, tends to eliminate this problem and in particular that there exist values of $d \geq n + 1$ for which it is possible to find a perfect MMES of n qudits. For a discussion on this statement, see Appendix B.

III. MAIN RESULTS

In this section, we want to go briefly through the main results of this paper, before going into the details of the proofs.

A. Symmetries of the coupling function Δ

We recall that, (20),

$$\Delta(k, k'; l, l') = \binom{n}{n_A}^{-1} \sum_{|A|=n_A} \frac{1}{2}(\delta_{k'_A l_A}\delta_{k_A l'_A}\delta_{k_{\bar{A}} l_{\bar{A}}}\delta_{k'_{\bar{A}} l'_{\bar{A}}} + \delta_{k'_A l'_A}\delta_{k_A l_A}\delta_{k'_{\bar{A}} l_{\bar{A}}}\delta_{k_{\bar{A}} l'_{\bar{A}}}). \tag{32}$$

Due to its form, it is invariant under the permutation of the qudits and under some swaps of the computational basis elements ($k \in \mathbb{Z}_d^n$).

It is well known that applying local unitary transformations to the system does not change its entanglement and, as a consequence, the local purity of any of its subsystem:

$$\pi_A(\psi) = \pi_A((U_1 \otimes U_2 \otimes \cdots \otimes U_n)\psi), \tag{33}$$

for all $\psi \in \mathcal{H}$, for all $A \subset S$, and for all $(U_1, \ldots, U_n) \in \mathrm{U}(d)^n$, with $\mathrm{U}(d)$ being the unitary group of degree d. Moreover, if we permute the order of the qudits, the global entanglement of the system is left invariant. The permutation group S_n of order n acts on $\mathcal{H}$ through (unitary) swap operators $p \in \mathrm{S}_n \to V_p \in \mathrm{U}(N)$,

$$V_p(|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle) = |\psi_{p(1)}\rangle \otimes |\psi_{p(2)}\rangle \otimes \cdots \otimes |\psi_{p(n)}\rangle. \tag{34}$$

For all $(U_1, \ldots, U_n) \in \mathrm{U}(d)^n$ and for all $p \in \mathrm{S}_n$, we get that

$$((U_1, \ldots, U_n; p)\pi_{\mathrm{ME}})(\psi) := \pi_{\mathrm{ME}}((U_1 \otimes U_2 \otimes \cdots \otimes U_n)V_p\psi) = \pi_{\mathrm{ME}}(\psi). \tag{35}$$

Therefore, the potential of multipartite entanglement (12) admits the semidirect product

$$\mathrm{SU}(d)^n \rtimes \mathrm{S}_n \tag{36}$$

as a symmetry group, whose product is easily seen to satisfy

$$(U_1, \ldots, U_n; p)(V_1, \ldots, V_n; q) = (U_1 V_{p(1)}, \ldots, U_n V_{p(n)}; p\, q). \tag{37}$$

As a consequence, for the symmetries of the coupling function Δ, we have the following:

Theorem 3. *The coupling function* Δ *in* (15) *is invariant under the action of the semidirect product group*

$$\mathrm{S}_d^n \rtimes \mathrm{S}_n, \tag{38}$$

whose action on $k \in \mathbb{Z}_d^n$ *is given by*

$$(p_1, \ldots, p_n; q)(k_1, \ldots, k_n) = (p_1(k_{q(1)}), \ldots, p_n(k_{q(n)})), \tag{39}$$

where $p_j \in \mathrm{S}_d$, $\forall j \in \{1, \ldots, n\}$, *and* $q \in \mathrm{S}_n$.

Proof. The proof is straightforward after observing that all the operations that characterize the coupling function act position-wise and the permutations are bijective maps. □

In Sec. IV, we will show that the presence of these symmetries of the coupling function has important consequences in the computation of the graphs in the diagrammatic technique we were mentioning in the Introduction.

B. Statistical mechanics approach and cumulant expansion

The minimization problem of the potential of multipartite entanglement can be handled following a statistical mechanics approach.[34] Roughly speaking, we will consider the free energy of a suitable classical system at a fictitious temperature, and we will recover the original problem in the zero temperature limit.

Considering the state

$$|\psi\rangle = \sum_{k \in \mathbb{Z}_d^n} z_k \, |k\rangle, \tag{40}$$

with $z = (z_k)_k$ as the vector of the Fourier coefficients in the expansion of the state, $\| z \|_2 = \sum_k |z_k|^2 = 1$, we define the Hamiltonian

$$H(z) = \pi_{ME}(\psi(z)), \tag{41}$$

where π_{ME} has been explicitly shown in Eq. (29). Let us consider M vectors and the ensemble $\{m_j\}$ of the number of vectors with fixed potential of multipartite entanglement, $H = \epsilon_j$. We want to find the distribution that maximizes the quantity

$$\Omega = \frac{M!}{\Pi_j m_j!} \tag{42}$$

under the constraints $\sum_j m_j = M$ and $\sum_j m_j \epsilon_j = ME$, where E is the average value of π_{ME}. In particular, if we let $M \to \infty$, we recover the canonical ensemble with the partition function

$$Z(\beta) = \int \mathrm{d}\mu(z) \mathrm{e}^{-\beta H(z)}, \tag{43}$$

where

$$\mathrm{d}\mu(z) = \frac{(N-1)!}{\pi^N} \, \delta\left(1 - \|z\|^2\right) \prod_k \mathrm{d}z_k \mathrm{d}\bar{z}_k \tag{44}$$

is the unitarily invariant measure over pure states induced by the Haar measure over $\mathcal{U}(\mathcal{H})$ through the mapping $|\psi\rangle = U\,|\psi_0\rangle$, for a given state $|\psi_0\rangle$.[37] Here β plays the role of an inverse temperature so that for $\beta \to +\infty$ only the configurations that minimize the Hamiltonian survive. In other words, we recover the MMES in the limit $\beta \to +\infty$. Moreover, if $\beta \to 0$, we recover the behavior of a typical state.

Using the partition function, the average energy can be written as

$$\langle H\rangle_\beta = \frac{1}{Z(\beta)}\int \mathrm{d}\mu(z)H(z)\mathrm{e}^{-\beta H(z)} = -\frac{\partial}{\partial\beta}\ln Z(\beta). \tag{45}$$

The high-temperature expansion of this energy distribution is

$$\langle H\rangle_\beta = \sum_{m=1}^{\infty}\frac{(-\beta)^{m-1}}{(m-1)!}\,\kappa_0^{(m)}[H], \tag{46}$$

where

$$\kappa_0^{(m)}[H] = (-1)^m\frac{\partial^m}{\partial\beta^m}\ln Z(\beta)\Big|_{\beta=0} = (-1)^{m-1}\frac{\partial^{m-1}}{\partial\beta^{m-1}}\langle H\rangle_\beta\Big|_{\beta=0} \tag{47}$$

is the mth cumulant, which is related to the moments $\langle H^m\rangle_0$ through the recursion formula

$$\kappa_0^{(m)} = \langle H^m\rangle_0 - \sum_{j=1}^{m-1}\binom{m-1}{j-1}\kappa_0^{(j)}\langle H^{m-j}\rangle_0, \tag{48}$$

with $\kappa_0^{(1)} = \langle H\rangle_0$.

This approach based on the methods from classical statistical mechanics has been applied to qubits both in the bipartite[39–41] and in the multipartite cases.[34,36] Here we want to analyze the general qudit case.

We observe that the series in Eq. (46) converges. Indeed, we can prove that it is majorized term by term by an absolutely convergent series.

Theorem 4. 1. *The partition function $Z(\beta)$ in (43) is an entire function of $\beta\in\mathbb{C}$.*
2. *The average energy (45) is holomorphic in a complex neighborhood of the real line.*
3. *Its high-temperature expansion (46) is a convergent series with a nonzero radius of convergence.*

Proof. 1. Notice that the measure in (43) has compact support $\{z\in\mathbb{C}^N, \|z\| = 1\}$, and $H(z)$ is a continuous function with $1/N_A \le H(z) \le 1$ for z in that support. Thus the integral converges for all $\beta\in\mathbb{C}$ and is differentiable with derivative given by

$$\frac{\mathrm{d}Z(\beta)}{\mathrm{d}\beta} = \int \mathrm{d}\mu(z)H(z)\mathrm{e}^{-\beta H(z)},$$

implying that $Z(\beta)$ is holomorphic in the whole complex plane.
2. This follows from the observation that the average energy is the ratio of two entire functions, $\langle H\rangle_\beta = Z'(\beta)/Z(\beta)$, and for $\beta\in\mathbb{R}$ and $\| z \| = 1$, one gets

$$\mathrm{e}^{-\beta H(z)} \ge \mathrm{e}^{-|\beta||H(z)|} \ge \mathrm{e}^{-|\beta|},$$

implying that

$$Z(\beta) \ge \mathrm{e}^{-|\beta|} > 0, \qquad \beta\in\mathbb{R}.$$

[In fact, for $\beta < 0$ one gets the stronger estimate $Z(\beta) \ge \mathrm{e}^{|\beta|/N_A} \ge 1$.] The statement follows by continuity.
3. This follows from statement 2. Notice in particular that $Z(0) = 1$. □

Finally, the following bounds hold:

Theorem 5. 1. *For all $m \ge 1$, the moment of the Hamiltonian has the form*

$$\langle H^m\rangle_0 = \langle H^m\rangle_{0,\mathrm{C}} + \langle H^m\rangle_{0,\mathrm{NC}}, \tag{49}$$

where

$$\langle H^m\rangle_{0,\mathrm{C}} = \frac{C_1(m)N!}{(N+2m-1)!}\left(\frac{N_A+N_{\bar{A}}}{2}\right)^m \tag{50}$$

and

$$0 \le \langle H^m\rangle_{0,\mathrm{NC}} \le \frac{C_2(m)N!}{(N+2m-1)!}\left(\frac{N_A+N_{\bar{A}}}{2}\right)^{m-1}, \tag{51}$$

with $C_1(m)$ and $C_2(m)$ being positive functions of the parameter m only that do not depend on d or n.

2. *The following bound holds*:

$$0 \leq \frac{\langle H^m \rangle_{0,\mathrm{NC}}}{\langle H^m \rangle_{0,\mathrm{C}}} \leq \frac{C(m)}{d^{[\frac{n}{2}]}}, \tag{52}$$

where $C(m) = C_2(m)/C_1(m)$.

This is the central result of our paper. In principle, this majorization allows us to evaluate the terms in series (46), using Eq. (48). In particular, we notice that since $C(m)$ does not depend on d, in the limit $d \to \infty$, the contribution of the second term, $\langle H^m \rangle_{0,\mathrm{NC}}$, in Eq. (49) is subdominant and

$$\langle H^m \rangle_0 \sim \langle H^m \rangle_{0,\mathrm{C}}, \qquad d \to \infty. \tag{53}$$

We will show in the following that this behavior can be interpreted in terms of the structure of graphs contributing to the moments.

We will give a proof of this theorem in Sec. V. Before doing this, we will introduce in Sec. IV the diagrammatic technique used for the majorization of the moments in Eq. (49).

IV. CACTUS AND OTHER DIAGRAMS

In this section, we will use the diagrammatic technique introduced in Ref. 36 for qubits, properly generalized for the case of qudits, in order to control each term of series (46).

First of all, let us consider the quantity

$$\langle H^m \rangle_0 = \Big\langle \Big(\sum_{k,k',l,l' \in \mathbb{Z}_d^n} \Delta(k,k';l,l') z_k z_{k'} \bar{z}_l \bar{z}_{l'} \Big)^m \Big\rangle_0. \tag{54}$$

An explicit form of this quantity requires the product of m coupling functions Δ. In order to simplify the notation, we introduce the vectors

$$\boldsymbol{k} = (k_1, \ldots, k_m, k_{1'}, \ldots, k_{m'}), \quad \boldsymbol{l} = (l_1, \ldots, l_m, l_{1'}, \ldots, l_{m'})$$

with $k_j, k_{j'}, l_j, l_{j'} \in \mathbb{Z}_d^n$. Therefore,

$$\langle H^m \rangle_0 = \sum_{\boldsymbol{k},\boldsymbol{l} \in \mathbb{Z}_d^{2mn}} \prod_{j=1}^m \Delta(k_j, k_{j'}; l_j, l_{j'}) \Big\langle \prod_{j=1}^m z_{k_j} z_{k_{j'}} \bar{z}_{l_j} \bar{z}_{l_{j'}} \Big\rangle_0. \tag{55}$$

Theorem 6. *The following equality holds:*

$$\langle H^m \rangle_0 = \frac{1}{N(N+1)\ldots(N+2m-1)} \sum_{\boldsymbol{k} \in \mathbb{Z}_d^{2mn}} \sum_{p \in \mathrm{S}_{2m}} \prod_{j=1}^m \Delta(k_j, k_{j'}; k_{p(j)}, k_{p(j')}), \tag{56}$$

with $p \in \mathrm{S}_{2m}$ being a permutation acting on the $2m$ elements

$$\{1, 2, \ldots, m, 1', 2', \ldots, m'\}.$$

This theorem was given in Ref. 36 for qubits $d = 2$. The proof of its extension to qudits $d > 2$ is a carbon copy of the proof for qubits.

By defining the square brackets

$$[p(1)\, p(1'), \ldots, p(m)\, p(m')] := \sum_{\boldsymbol{k} \in \mathbb{Z}_d^{2mn}} \prod_{j=1}^m \Delta(k_j, k_{j'}; k_{p(j)}, k_{p(j')}), \tag{57}$$

with $p \in \mathrm{S}_{2m}$, Eq. (56) becomes

$$\langle H^m \rangle_0 = \frac{1}{N(N+1)\ldots(N+2m-1)} \sum_{p \in \mathrm{S}_{2m}} [p(1)\, p(1'), \ldots, p(m)\, p(m')]. \tag{58}$$

As promised, we can give a diagrammatic representation of the terms in the sum. Each pair $(k_j, k_{j'})$ can be associated with a vertex of a graph from which two edges go out and two go in. The first two edges are labeled by $k_{p(j)}$ and $k_{p(j')}$, and the latter are k_j and $k_{j'}$; see Fig. 1(a).

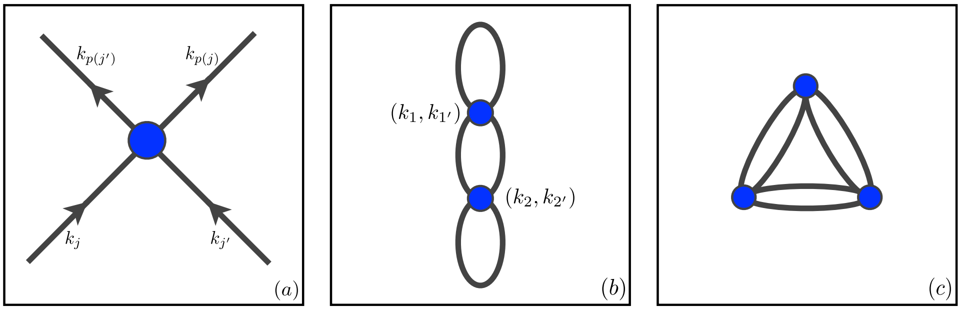


FIG. 1. (a) Graphical representation of the interaction of each pair $(k_i, k_{i'})$: each vertex with 4 edges, two going in and two going out. (b) Graph with two points representing [1 2, 1′ 2′]. (c) A graph with a cycle is a non-cactus.

Example 2. The square brackets [1 2, 1′ 2′] *lead to the graph in Fig. 1(b).*

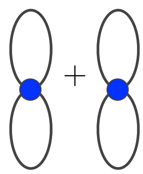
It is possible to rephrase some of the previous results in terms of these diagrams. Indeed, Eq. (25) can be interpreted as a current conservation law, i.e., the current going into a vertex has to be the same as the current that goes out; see Fig. 1(a). Moreover, the symmetries of the coupling function Δ, given in Theorem 3, are translated in the degeneracy of the graphs. For instance, the square brackets in example 2 lead to the same graph as

$$[1\ 2, 1'\ 2'],\ [2\ 1, 2'\ 1'],\ [1'\ 2', 1\ 2],\ [2'\ 1', 1\ 2],\ [1\ 2, 2'\ 1'], \tag{59}$$

and so on.

Example 3. In terms of Feynman graphs, we have

$$\langle H\rangle_0 = \frac{1}{N(N+1)}([1\ 1'] + [1'\ 1]) = \frac{1}{N(N+1)}(\quad + \quad). \tag{60}$$



Definition 6. A connected graph with $v \geq 2$ vertices is called a cactus if for every vertex there exists a pair of edges such that removing them the graph becomes disconnected, otherwise the graph is called a non-cactus. A graph with $v = 1$ is a cactus by definition.

Example 4. The graph in Fig. 1(b) is a cactus, while the graph in Fig. 1(c) is a non-cactus. Indeed, by removing a pair of edges from a vertex, the graph becomes one of the two graphs in Fig. 2.

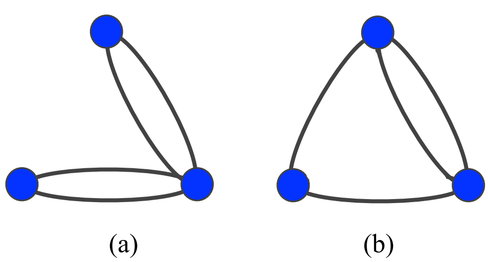


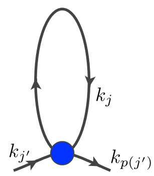
FIG. 2. Removing two edges from a vertex of the non-cactus in Fig. 1(c) leaves the graph connected.

A. Graph surgery

In this section, we will study in detail the graphs introduced at the beginning of Sec. IV. In particular, we will compute the contribution that each graph gives to moments (58) and their degeneracy. In order to do this, we will divide each graph into subgraphs and will compute the degeneracy and the contribution of each single subgraph.

Definition 7. We call a leaf the subgraph of a graph represented by

$$[\ldots, j\ p(j'), \ldots] = \quad . \tag{61}$$



Lemma 7. A leaf at the vertex $(k_j, k_{j'})$ *gives the contribution*

$$\delta_{k_{j'},k_{p(j')}} \frac{N_A + N_{\bar{A}}}{2}. \tag{62}$$

Proof. In a graph, the contribution of a leaf is

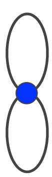
$$[\ldots, j\ p(j'), \ldots] = \sum_{\substack{k_i : i \neq j \\ k_{i'}}} \cdots \sum_{k_j} \Delta(k_j, k_{j'}; k_j, k_{p(j')}). \tag{63}$$

Since this is the only term in which the index j appears, it can be isolated from the rest,

$$\begin{aligned}\sum_{k_j} \Delta(k_j, k_{j'}; k_j, k_{p(j')}) &= \delta_{k_{j'},k_{p(j')}} \sum_{k_j} f(k_{j'} - k_j, 0) \\ &= \delta_{k_{j'},k_{p(j')}} \frac{N_A + N_{\bar{A}}}{2}.\end{aligned}$$

□

Example 5. As an example, consider the Feynman graph in Fig. 3(a). By computing the contribution of the upper leaf as in the previous lemma, we find



$$= \sum_{k_{1'}} \delta_{k_{1'},k_{1'}} \frac{N_A + N_{\bar{A}}}{2} = N\, \frac{N_A + N_{\bar{A}}}{2}. \tag{64}$$

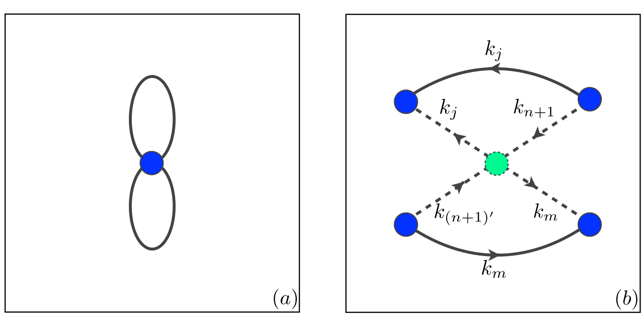
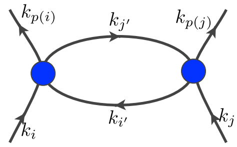


FIG. 3. (a) The only graph with a single vertex. (b) Pinching operation.

Definition 8. A loop is the subgraph of a graph represented by

$$[\dots, p(i)\ j', \dots, p(j)\ i', \dots] = \tag{65}$$



.

Theorem 8. *Each cactus graph gives a contribution*

$$N\left(\frac{N_A + N_{\bar{A}}}{2}\right)^{v}, \tag{66}$$

where v is the number of vertices in the graph.

Proof. We can compute the contribution of a graph decomposing it in its elementary subgraphs. From its definition, we can deduce that a cactus has at least one leaf. Moreover, notice that after we have computed the contribution of the leaf, the remaining terms in the square brackets correspond to a Feynman graph with $v - 1$ vertices. This new graph is essentially the same as the graph with v vertices but without a leaf and with a loop transformed into a leaf. Besides, the structure of the graph is left invariant after the removal of a leaf, i.e. this operation transforms a cactus in a cactus and a non-cactus in a non-cactus.

We can iterate the computation obtaining the contribution

$$\frac{N_A + N_{\bar{A}}}{2} \tag{67}$$

for each vertex. At the end of this computation, the remaining term will be $\sum_k 1 = N$, and this concludes the proof. □

The evaluation of the contribution of non-cactus graphs is not as simple as the one of the cactus. Nevertheless, we can give an upper bound for it by bounding the loop contributions.

Theorem 9. *A loop gives a contribution that is lower than or equal to* $(N_A + n_{\bar{A}})/2$.

Proof. We can isolate the contribution of each single loop obtaining

$$[\dots, p(i)\, j', \dots, p(j)\, i', \dots] = \sum_{k_{i'}, k_{j'}} \Delta(k_i, k_{i'}; k_{p(i)}, k_{j'})\Delta(k_j, k_{j'}; k_{p(j)}, k_{i'}). \tag{68}$$

If we substitute here the expression of the coupling function in Theorem 2, we find

$$\begin{aligned}
&\sum_{k_{i'}, k_{j'}} \delta_{k_i + k_{i'}, k_{p(i)} + k_{j'}}\ \delta_{k_j + k_{j'}, k_{p(j)} + k_{i'}}\ \delta_{0, (k_i - k_{p(i)}) \wedge (k_{i'} - k_{p(i)})} \\
&\quad \times \delta_{0, (k_j - k_{p(j)}) \wedge (k_{j'} - k_{p(j)})} f(k_i - k_{p(i)}, k_{i'} - k_{p(i)}) f(k_j - k_{p(j)}, k_{j'} - k_{p(j)}) \\
&= \delta_{k_i + k_j, k_{p(i)} + k_{p(j)}} \sum_l \delta_{0, (k_i - k_{p(i)}) \wedge\, l}\, f(k_i - k_{p(i)}, l)\, f(k_i - k_{p(i)}, l + k_{p(i)} - k_j) \\
&\quad \times \sum_{k_{j'}} \delta_{k_j + k_{j'}, k_{p(j)} + l + k_{p(i)}}\ \delta_{0, (k_j - k_{p(j)}) \wedge (k_{j'} - k_{p(j)})} \\
&= \delta_{k_i + k_j, k_{p(i)} + k_{p(j)}} \sum_l f(k_i - k_{p(i)}, l)\, f(k_i - k_{p(i)}, l + k_{p(i)} - k_j) \\
&\quad \times \delta_{0, (k_i - k_{p(i)}) \wedge l}\ \delta_{0, (k_i - k_{p(i)}) \wedge (l + k_{p(i)} - k_j)}.
\end{aligned} \tag{69}$$

It is straightforward to prove that $f(k, l) \le 1$. Moreover, the condition that the Kronecker deltas have to be different from zero and the assumption that the binomial coefficient is zero if one of its argument is negative, Remark 1, fix the positions in which l can be different from 0 and give the condition

$$|l| \leq n_A = \left[\frac{n}{2}\right], \tag{70}$$

and so the last expression in (69) can be bounded by $N_A = d^{\left[\frac{n}{2}\right]}$ and thus by $(N_A + N_{\bar{A}})/2$. □

In the next part of this section, we will give an explicit way to compute the degeneracy of graphs. In particular, we prove that we can compute the degeneracy of a generic $(v+1)$-vertex graph (we will call it daughter) knowing only the degeneracy $\mathcal{D}$ of a v-vertex graph from which the graph is generated (we will call it mother). In the following, given a $(v+1)$-vertex graph G, we will call $G^{(m)}$ its mother graph. In this notation, $G^{(m^2)}$ will be the mother of the mother and so on until the 1-vertex graph is obtained.

Definition 9. We define the pinching operation that connects two edges adding a vertex to the graph [see Fig. 3(b)].

The following proposition illustrates the degeneracy of this operation.

Proposition 2. Adding a vertex through pinching increases the degeneracy $\mathcal{D}(G)$ of a graph by factor

(a) 4 *if the four vertices are non-degenerate or if they degenerate into two but the edges have different directions;*
(b) 2 *if the four vertices degenerate into one;*
(c) 2 *if the four vertices degenerate into two and the directions of the two edges are the same.*

Remark 2. From now on, we suppose to start from a v-vertex graph and to add a vertex labeled by $(v+1, (v+1)')$.

Proof. In the first case [Fig. 3(b)], if the mother graph is of the form

$$[\ldots, j\, p(i'), \ldots, m\, p(l'), \ldots], \tag{71}$$

then the daughter graph can be represented by

$$[\ldots, v'\, p(i'), \ldots, (v+1)\, p(l'), \ldots, j\, m]. \tag{72}$$

After the exchange of v' and $v+1$ or j and m, the graph is left unchanged; therefore, the degeneracy of this new graph has an extra factor of 4 compared with the degeneracy of the mother, i.e., $\mathcal{D}(G) = 4\mathcal{D}(G^{(m)})$.

In the second case, see Fig. 4(a), the v-vertex graph can be represented by

$$[\ldots, j\, j', \ldots], \tag{73}$$

and the pinching leads to the representation

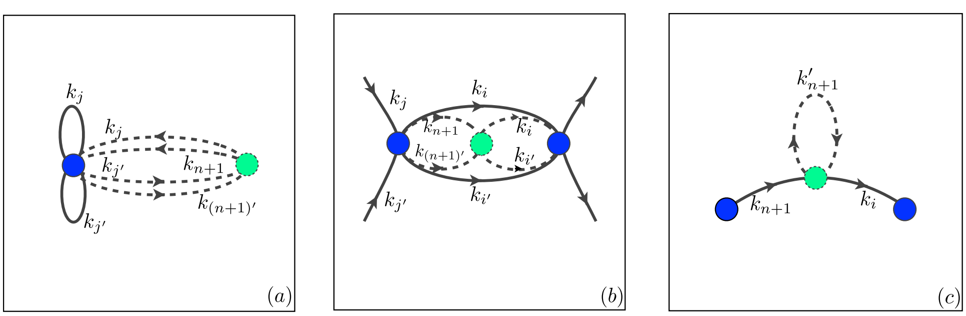


FIG. 4. (a) Four vertices degenerating into one. (b) Four vertices degenerating into two. (c) Germination of a leaf.

$$[\ \dots, (v+1)\ (v+1)', \dots, j\, j'\ ] \tag{74}$$

or equivalently to

$$[\ \dots, (v+1)'\ (v+1), \dots, j\, j'\ ]. \tag{75}$$

Since there are no other possibilities, $\mathcal{D}(G) = 2\mathcal{D}(G^{(m)})$.

In the last case, Fig. 4(b), we start from the graph

$$[\ \dots, i\, j', \dots\ ] \tag{76}$$

and arrive at

$$[\ \dots, (v+1)\ (v+1)', \dots, i\, i'\ ], \tag{77}$$

where again we have an extra factor 2 of degeneracy. □

Remark 3. In the previous proposition, there is no mention to the case in which the four vertices degenerate into two and the two edges degenerate into one. Nevertheless, this operation coincides with the germination of a leaf, Fig. 4(c). Indeed, before the creation of the leaf, the graph is associated with

$$[\ \dots, i\ p(j'), \dots\ ], \tag{78}$$

while after the pinching, the representation becomes

$$[\ \dots, (v+1)\ p(j'), \dots, i\ (v+1)'\ ], \tag{79}$$

and 3 *other combinations lead to the same graph, swapping* $v+1$ *and* $(v+1)'$ *or the elements in the two pairs.*

Remark 4. Remarkably, the pinching operation allows us to construct all $(v+1)$*-vertex graphs starting from the* v*-vertex graphs and in addition provides a practical way for computing the degeneracy. Moreover, we recall that cactus graphs are always generated by other cacti, and there is no way to transform a non-cactus into a simpler graph just by adding a vertex. However, it is not true in general that the degeneracy of the daughter graph is the degeneracy of the mother multiplied by the degeneracy of the pinching,* $\mathcal{D}_{\text{pinc}}$*. In fact, the addition of a vertex can break the symmetry of a graph, and when this happens, we have a factor lower than or equal to*

$$v+1 = \frac{(v+1)!}{v!} \tag{80}$$

so that the more symmetric the graph, the lower is its degeneracy. Furthermore, sometimes pinching different edges gives rise to the same graph so that in counting the degeneracy, we also have to consider all these possibilities.

Summing up the previous considerations, we can state the following.

Proposition 3. The degeneracy of a graph G is

$$\mathcal{D}(G) = \mathcal{D}(G^{(m)}) \times \mathcal{D}_{\text{pinc}} \times \tilde{\mathcal{D}}_{\text{pinc}} \times (v+1),$$

where the last factor is the symmetrization factor and $\tilde{\mathcal{D}}_{\text{pinc}}$ *are the different ways in which pinching the edges gives rise to the same graph.*

Remark 5. Pinching edges pointing in the same direction gives rise to different graphs from the ones obtained by pinching edges in the opposite direction. The difference between these two cases is shown in Figs. 5(b) and 5(c) where the two graphs differ in the direction of arrows.

We are now ready to give a bound for the degeneracy of the graphs.

Theorem 10. *The degeneracy of a* v*-vertex graph G satisfies*

$$\mathcal{D}(G) \le 2^{2v}\, v!. \tag{81}$$

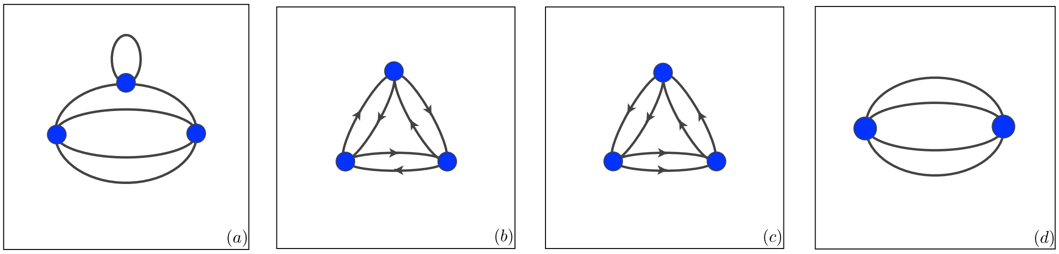


FIG. 5. (a)–(c) All connected non-cacti with three vertices. (d) A non-cactus with two vertices.

Proof. To compute the degeneracy of a graph in the worst case scenario, we have to compute all the configurations $[p(1)\,p(1'),\ldots,p(v)\,p(v')]$ that lead to the same graph. In the worst case, the pinching operation gives a factor of 4 for each point and every permutation of the vertices leaves the configuration unchanged so that the degeneracy in the worst case is $4^v v!$. □

Example 6. The only graph with one vertex is the one in Fig. 3(a) that has degeneracy 2*. From this graph, we can generate the two connected graphs with two vertices. The one in Fig. 1(a) is obtained by a non-degenerate pinching so that its degeneracy is* $2\times 4\times 2 = 16$*, while the one in Fig. 5(d) is generated by a degenerate pinching and so its degree of degeneracy is* $2\times 2 = 4$*.*

Example 7. The Feynman graph in Fig. 5(a) is generated from the one in Fig. 5(d) by the germination of a leaf. Its degeneracy is $4\times 4\times 4\times 3 = 192$*, where* 3 *is the symmetrization factor.*

Example 8. The graph in Fig. 5(c) is generated after a pinching of the graph in Fig. 5(d) so that its degeneracy is $16\times 4 = 64$*, while the graph in Fig. 5(b) is generated by pinching of the graph in Fig. 5(d); therefore, its degeneracy is* $4\times 2\times 2 = 16$*.*

V. HIGH-TEMPERATURE EXPANSION: PROOF OF THEOREM 5

The representation in terms of Feynman graphs introduced in Sec. IV gives important information about series (46) and in particular allows us to prove Theorem 5.

Proof of theorem 5. From Eq. (58), we can write $\langle H^m\rangle_0$ in terms of the square brackets,

$$\langle H^m\rangle_0 = \frac{1}{N(N+1)\ldots(N+2m-1)}\sum_{p\in S_{2m}}[p(1)p(1'),\ldots,p(m)p(m')]. \tag{82}$$

Let us now divide the permutation in S_{2m} into the ones that generate a cactus, $\mathcal{P}_1$, and the ones that generate a non-cactus, $\mathcal{P}_2$,

$$\langle H^m\rangle_0 = \frac{1}{N(N+1)\ldots(N+2m-1)} \times\left(\sum_{p\in\mathcal{P}_1}[p(1)p(1'),\ldots,p(m)p(m')] + \sum_{p\in\mathcal{P}_2}[p(1)p(1'),\ldots,p(m)p(m')]\right).$$

Applying Theorem 5, the first permutations give

$$[p(1)p(1'),\ldots,p(m)p(m')] = \left(\frac{N_A+N_{\bar{A}}}{2}\right)^m, \tag{83}$$

recalling that the number of vertices in a Feynman graph generated by permutations in S_{2m} is exactly m, while for permutation in $\mathcal{P}_2$ applying Theorem 9, we have

$$[p(1)p(1'),\ldots,p(m)p(m')] \le N\left(\frac{N_A+N_{\bar{A}}}{2}\right)^{m-1}. \tag{84}$$

Defining $\tilde{C}_1(m)$ and $\tilde{C}_2(m)$ as the number of nonequivalent cactus and non-cactus graphs (respectively) with m vertices, and recalling that according to theorem 10 every m-vertex graph has at most degeneracy $2^{2m}m!$, we eventually get

$$\langle H^m \rangle_0 \le \frac{2^{2m} m!}{N(N+1)\dots(N+2m-1)} \times \left[\tilde{C}_1(m)\, N \left(\frac{N_A + N_{\bar{A}}}{2}\right)^m + \tilde{C}_2(m)\, N \left(\frac{N_A + N_{\bar{A}}}{2}\right)^{m-1} \right],$$

and the first part of the theorem follows.

The second statement follows immediately from the first one by recalling that

$$\frac{N_A + N_{\bar{A}}}{2} = \frac{d^{[\frac{n}{2}]} + d^{[\frac{n+1}{2}]}}{2} \ge d^{[\frac{n}{2}]}.$$

This concludes the proof of theorem 5. □

Remark 6. A final remark is in order. As stated at the end of Sec. III, it is clear that since $C_1(m)$ and $C_2(m)$ do not depend on d, in the limit $d \to \infty$, the contribution to the sum due to the presence of the non-cactus graphs goes to zero and only the presence of a cactus becomes relevant. Heuristically, we can attribute the presence of frustration in the system to the relevance of the non-cactus graphs in the series.

Remark 7. We can apply theorem 5 from a different perspective, keeping d fixed and evaluating the limit $N = 2^n \to \infty$. In particular, let us consider the case of qubits ($d = 2$) where exact explicit expressions for the first, second, and third moments have been obtained using Feynman diagrams,[36] and one can evaluate the contributions from non-cactus diagrams in the limit $N = 2^n \to \infty$.

1. $m = 1$. *This case is trivial. Contributions to the moment only come from the cactus shown in Fig. 3(a).*
2. $m = 2$. *An explicit calculation shows that cactus diagrams are of the form in Fig. 3(a) (disconnected) whereas non-cactus diagrams are of the form in Fig. 5(d). We obtain*

$$\frac{\langle H^2 \rangle_{0,\mathrm{NC}}}{\langle H^2 \rangle_{0,\mathrm{C}}} = \frac{f_2(N)}{(N+4)(N_A + N_{\bar{A}})^2}, \tag{85}$$

with

$$f_2(N) = 2 \binom{n}{n_A}^{-1} \sum_{0 \le k \le n_A} \binom{n_A}{k} \binom{n_{\bar{A}}}{k} 2^{n/2} \left[4^{n/4-k} + 4^{-(n/4-k)} \right].$$

It is possible to prove[36] that in the limit $N \to \infty$,

$$f_2(N) \sim 3\sqrt{2} N^{\alpha}, \tag{86}$$

with

$$\alpha = \log_2 3 - 1 \simeq 0.5850. \tag{87}$$

Therefore, for $N \to \infty$ and $N_A \simeq N_{\bar{A}} = 2^{[\frac{n}{2}]}$, we have

$$\frac{\langle H^2 \rangle_{0,\mathrm{NC}}}{\langle H^2 \rangle_{0,\mathrm{C}}} \sim \frac{b(2)}{N^{2-\alpha}} \le \frac{C(2)}{2[\frac{n}{2}]}$$

with $b(2)$ as *a positive constant.*

3. $m = 3$. *We obtain*

$$\frac{\langle H^3 \rangle_{0,\mathrm{NC}}}{\langle H^3 \rangle_{0,\mathrm{C}}} = \frac{16 f_3^{(1)}(N) + 64 f_3^{(0)}(N) + 3(N+8)(N_A + N_{\bar{A}}) f_2(N)}{(40 + 12N + N^2)(N_A + N_{\bar{A}})^3},$$

where the role of $f_3^{(1)}$ and $f_3^{(0)}$ is analogous to that of f_2 for $m = 2$. Their complete expressions are not transparent and can be found in Ref. 36. We notice that the contributions of $f_3^{(0)}, f_3^{(1)}$ come *from non-cactus diagrams in Figs. 5(b) and 5(c), respectively. The contribution of f_2 comes from the non-cactus (connected) diagram in Fig. 5(a) and from (disconnected) contributions obtained from the non-cactus in Fig. 5(d) and the diagram in Fig. 3(a). In the limit $N \to \infty$, we obtain*

$$f_3^{(0)}(N) \sim cN^{5-\gamma}, \quad f_3^{(1)}(N) \sim N^{\alpha} \tag{88}$$

with

$$\gamma \simeq 4.1583, \quad c \simeq 1.053\,85. \tag{89}$$

Finally, for $N \to \infty$ and $N_A \simeq N_{\bar{A}} = 2^{[\frac{n}{2}]}$, we have

$$\frac{\langle H^3\rangle_{0,\mathrm{NC}}}{\langle H^3\rangle_{0,\mathrm{C}}} \sim \frac{b(3)}{N^{3-\alpha}} \leq \frac{C(3)}{2^{[\frac{n}{2}]}}$$

with $b(3)$ as a positive constant.

VI. CONCLUSION

The main aim of this paper is a deeper study of the potential of multipartite entanglement, i.e., the average purity over balanced partitions of the system, as a measure of multipartite entanglement. In particular, by introducing a properly defined Hamiltonian representing the multipartite entanglement and a fictitious inverse temperature, we have rephrased the problem in the framework of classical statistical mechanics for the case of d-level systems (qudits). Thus, we have examined the high-temperature expansion of the distribution function of the potential of multipartite entanglement. In Sec. III, we have included the main results of the paper. In particular, in Theorem 4 we have shown that the series characterizing the expansion converges and that the moments of the distribution of the Hamiltonian can be explicitly evaluated or estimated (Theorem 5). In order to prove these results, we have introduced a diagrammatic technique that leads to a natural classification of the graphs in two classes (Secs. IV and V). We have shown that when d is large enough only one of these classes (the so-called cactus diagrams) gives a contribution to the partition function. These results pave the way to further investigations about the properties of multipartite entanglement, such as the possible presence of phase transitions analogous to the case of bipartite entanglement.

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APPENDIX A: LOWER BOUND ON THE 7-QUBIT POTENTIAL OF MULTIPARTITE ENTANGLEMENT

The minimum of the potential of multipartite entanglement for 7 qubits is not known yet. Until now, only guesses have been proposed and some numerical bounds have been found. Here we construct a 7-qubit state with the lowest π_{ME} found until now, to the best of our knowledge. The state can be constructed starting from an orthonormal basis of 3- and 4-qubit MMESs.

Considering the computational basis, we will write the ith vector of the basis in terms of the Fourier coefficients $z^{(i)} = (z_k^{(i)})$,

$$|\psi\rangle_i = \sum_{k \in \mathbb{Z}_2^n} z_k^{(i)} \, |k\rangle \, . \tag{A1}$$

The 3-qubit MMES basis ($d = 2$, $n = 3$) is made of GHZ states $|\mathrm{GHZ}\rangle_i$ ($i = 0, \ldots, 7$) with Fourier coefficients

$$
\begin{aligned}
z^{(0)} &= \frac{1}{\sqrt{2}}\,(1,0,0,0,0,0,0,1)\,, & z^{(1)} &= \frac{1}{\sqrt{2}}\,(1,0,0,0,0,0,0,-1)\,,\\
z^{(2)} &= \frac{1}{\sqrt{2}}\,(0,1,0,0,0,0,1,0)\,, & z^{(3)} &= \frac{1}{\sqrt{2}}\,(0,1,0,0,0,0,-1,0)\,,\\
z^{(4)} &= \frac{1}{\sqrt{2}}\,(0,0,1,0,0,1,0,0)\,, & z^{(5)} &= \frac{1}{\sqrt{2}}\,(0,0,1,0,0,-1,0,0)\,,\\
z^{(6)} &= \frac{1}{\sqrt{2}}\,(0,0,0,1,1,0,0,0)\,, & z^{(7)} &= \frac{1}{\sqrt{2}}\,(0,0,0,1,-1,0,0,0)\,.
\end{aligned}
$$

The 4-qubit MMES basis ($d = 2$, $n = 4$) is made by the states $|\mathrm{MMES}_4\rangle_i$ ($i = 0, \ldots, 15$) with Fourier coefficients

$$
\begin{aligned}
z^{(0)} &= \frac{1}{4}\,(-1,-1,-1,-1,-1,-1,1,1,-1,1,-1,1,1,-1,-1,1)\,,\\
z^{(1)} &= \frac{1}{4}\,(-1,-1,-1,-1,-1,-1,1,1,1,-1,1,-1,-1,1,1,-1)\,,\\
z^{(2)} &= \frac{1}{4}\,(-1,-1,-1,-1,1,1,-1,-1,-1,1,-1,1,-1,1,1,-1)\,,\\
z^{(3)} &= \frac{1}{4}\,(-1,-1,-1,-1,1,1,-1,-1,1,-1,1,-1,1,-1,-1,1)\,,\\
z^{(4)} &= \frac{1}{4}\,(-1,-1,1,1,-1,-1,-1,-1,-1,1,1,-1,1,-1,1,-1)\,,\\
z^{(5)} &= \frac{1}{4}\,(-1,-1,1,1,-1,-1,-1,-1,1,-1,-1,1,-1,1,-1,1)\,,\\
z^{(6)} &= \frac{1}{4}\,(-1,-1,1,1,1,1,1,1,-1,1,1,-1,-1,1,-1,1)\,,\\
z^{(7)} &= \frac{1}{4}\,(-1,-1,1,1,1,1,1,1,1,-1,-1,1,1,-1,1,-1)\,,
\end{aligned}
$$

$$
\begin{aligned}
z^{(8)} &= \frac{1}{4}\,(-1,1,-1,1,-1,1,1,-1,-1,-1,-1,-1,1,1,-1,-1)\,,\\
z^{(9)} &= \frac{1}{4}\,(-1,1,-1,1,-1,1,1,-1,1,1,1,1,-1,-1,1,1)\,,\\
z^{(10)} &= \frac{1}{4}\,(-1,1,-1,1,1,-1,-1,1,-1,-1,-1,-1,-1,-1,1,1)\,,\\
z^{(11)} &= \frac{1}{4}\,(-1,1,-1,1,1,-1,-1,1,1,1,1,1,1,1,-1,-1)\,,\\
z^{(12)} &= \frac{1}{4}\,(-1,1,1,-1,-1,1,-1,1,-1,-1,1,1,1,1,1,1)\,,\\
z^{(13)} &= \frac{1}{4}\,(-1,1,1,-1,-1,1,-1,1,1,1,-1,-1,-1,-1,-1,-1)\,,\\
z^{(14)} &= \frac{1}{4}\,(-1,1,1,-1,1,-1,1,-1,-1,-1,1,1,-1,-1,-1,-1)\,,\\
z^{(15)} &= \frac{1}{4}\,(-1,1,1,-1,1,-1,1,-1,1,1,-1,-1,1,1,1,1)\,.
\end{aligned}
$$

We will look for a minimizing 7-qubit state expressed in terms of tensor products of the elements of the two bases

$$
|\sigma_7\rangle = \sum_{i,j} c_{i,j}\, |\mathrm{MMES}_4\rangle_i \otimes |\mathrm{GHZ}\rangle_j\,. \tag{A2}
$$

We have numerically evaluated the complex coefficients $c_{i,j}$ so that the state $|\sigma_7\rangle$ is a minimizer of the potential of multipartite entanglement π_{ME}. We have found a solution such that the only non-vanishing

TABLE I. Table of non-vanishing coefficients $c_{i,j}$ obtained from the minimization of π_{ME} in the case of 7 qubits.

(i,j)	$\phi_{i,j}^{(re)}$	$\phi_{i,j}^{(re)}$
(1,1)	0.313 685	−0.019 416
(1,4)	−0.124 963	0.007 514 04
(2,2)	6.168 768 05 ×10^{-6}	−0.000 116 371
(2,3)	−0.000 103 808	−0.000 069 107 2
(3,2)	0.000 046 695	−0.000 073 536 9
(3,3)	−0.000 243 151	−0.000 195 018
(4,1)	0.019 388 8	0.313 752
(4,4)	−0.007 777 71	−0.124 766
(5,5)	0.071 926 2	0.153 13
(5,8)	0.152 837	−0.072 125 4
(6,6)	−0.160 604	−0.052 677 1
(6,7)	0.052 874 4	−0.160 803
(7,6)	0.052 730 7	−0.160 861
(7,7)	0.161 076	0.052 717 9
(8,5)	0.153 033	−0.071 937 4
(8,8)	−0.072 319 4	−0.153 041
(9,5)	0.052 930 9	−0.160 616
(9,8)	0.160 688	0.052 632 1
(10,6)	−0.072 288	−0.153 269
(10,7)	−0.153 02	0.072 084 2
(11,6)	−0.152 985	0.071 988 8
(11,7)	0.071 915 7	0.153 028
(12,5)	−0.161 016	−0.052 708 3
(12,8)	0.052 609 4	−0.160 931
(13,1)	0.012 842 7	−0.081 243 7
(13,4)	0.032 297	−0.204 478
(14,2)	−0.087 611	−0.170 06
(14,3)	0.264 942	0.001 347 56
(15,2)	−0.245 851	−0.124 495
(15,3)	−0.132 874	0.115 682
(16,1)	−0.012 881 3	0.081 274 2
(16,4)	−0.032 362 8	0.204 452

$c_{i,j}$'s are tabulated in Table I and expressed in the form

$$c_{i,j} = \phi_{i,j}^{(re)} + i\phi_{i,j}^{(im)}, \tag{A3}$$

with $\phi_{i,j}^{(re)}, \phi_{i,j}^{(im)}$ denoting the real and imaginary parts of the coefficient, respectively.

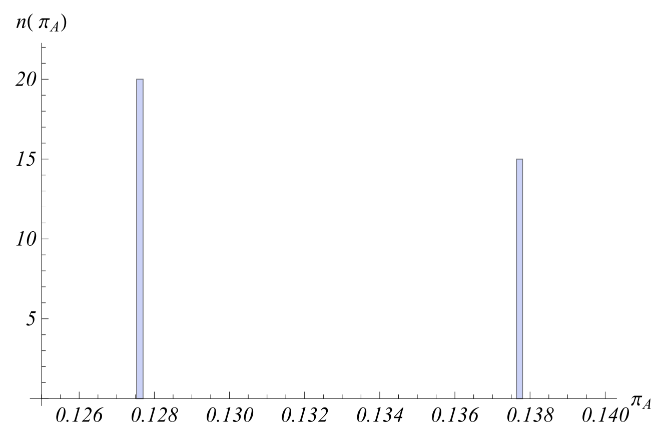

FIG. 6. Purity distribution among balanced bipartitions for the 7-qubit state $|\sigma_7\rangle$.

The value of π_{ME} in this case is

$$\pi_{\mathrm{ME}}(\sigma_7) = 0.131\,952. \tag{A4}$$

In Fig. 6, we plot the histogram characterizing the distribution of the purities among the balanced bipartitions. An interesting feature of this state is that all bipartitions are frustrated, i.e., none of them reaches the minimum for the corresponding purity, but the distribution is fairly well peaked in correspondence with two spikes only.

APPENDIX B: MAXIMUM DISTANCE SEPARABLE CODES AND PERFECT MMES

The definition of a (classical) code starts from the choice of a set of elements Σ that constitutes the alphabet of the code. Even if there is no restriction on this choice, since information theory is constructed around machines and computers, the most common set considered is $\Sigma = \{0, 1\}$, with the clear meaning that the information is encoded in bits.

Definition 10. Given an alphabet Σ, *a (classical) code* C *is a set of strings, called codewords, over* Σ *of fixed length.*

In the set of all possible codewords of fixed length n, Σ^n, it is possible to define a distance in the following.

Definition 11. The Hamming distance between two strings of the same length, $\mathrm{d}_H : \Sigma^n \times \Sigma^n \to \mathbb{R}$, *is the number of positions in which the corresponding symbols are different.*

Example 9. The Hamming distance between the two strings 0011 *and* 1001 *is* $\mathrm{d}_H(0011, 1001) = 3$.

Remark 8. The Hamming distance is a proper distance, i.e., it is positive, is symmetric, and satisfies the triangle inequality.

Definition 12. The minimal Hamming distance of the code, δ, *is defined as*

$$\delta = \min_{\{v,w \in C, v \neq w\}} \mathrm{d}_H(v, w). \tag{B1}$$

The following theorem gives a bound on the dimension of the code (the maximum number of codewords) and the minimal Hamming distance of the codewords.

Theorem 11 *(Singleton bound[42]). For any code* $C \subseteq \Sigma^n$, *the following inequality holds:*

$$M \geq q^{n-\delta+1}, \tag{B2}$$

with M *as the number of the codewords and* δ *as its minimum Hamming distance.*

Definition 13. A code for which the Singleton bound is saturated is called the Maximum Distance Separable (MDS) code.

Let us consider a code $C = \{c_j\}$, with N_A as codewords of length n and alphabet $\mathbb{Z}_d$. Using the codewords of C, we can construct the n-qudit state

$$|\psi\rangle = \frac{1}{\sqrt{N_A}} \sum_{j=1}^{N_A} |c_j\rangle. \tag{B3}$$

If the minimal Hamming distance of C is greater than $n_A + 1$, after the partial trace over a balanced bipartition, all the off-diagonal terms, $\mathrm{tr}_{\bar{A}}(|c_j\rangle\langle c_k|)$, vanish. Indeed

$$\mathrm{tr}_{\bar{A}}(|c_j\rangle\langle c_k|) = \sum_{l \in \mathbb{Z}_d^{n_{\bar{A}}}} \langle l|c_j\rangle\langle c_k|l\rangle \neq 0, \tag{B4}$$

if and only if c_j and c_k have at least $n_{\bar{A}}$ symbols in common. Moreover, the presence of $d^{\lfloor \frac{n}{2} \rfloor}$ terms in the sum is due to the necessity of having ρ_A proportional to identity, i.e., $\frac{I}{N_A}$ for every bipartition $(A, \bar{A})$. Therefore, for this state, π_{ME} reaches its minimum and the state in Eq. (B3) is a perfect MMES.

It remains to prove the existence of such a code. In particular, from the Singleton bound, $\delta \geq n\bar{A} + 1$, meaning that we are addressing the relation between n and d in order for a MDS code to exist.

Theorem 12. *If d is a prime or a prime power, a MDS code exists if* $n \leq d - 1$.

The MDS codes to which the theorem is referring are the Reed-Solomon codes.[43] This means that given n it is always possible to choose the first suitable $d \geq n + 1$ in order to construct a perfect MMES.

Remark 9. This bound gives only a bound on the point at which frustration disappears. Indeed, in the case of 5 qubits, for example, a perfect MMES exists, while according to the bound, we need $d \geq 4$.

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