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Dynamical decoupling of unbounded Hamiltonians

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We investigate the possibility to suppress interactions between a finite dimensional system and an infinite dimensional environment through a fast sequence of unitary kicks on the finite dimensional system. This method, called dynamical decoupling, is known to work for bounded interactions, but physical environments such as bosonic heat baths are usually modeled with unbounded interactions; hence, here, we initiate a systematic study of dynamical decoupling for unbounded operators. We develop a sufficient decoupling criterion for arbitrary Hamiltonians and a necessary decoupling criterion for semibounded Hamiltonians. We give examples for unbounded Hamiltonians where decoupling works and the limiting evolution as well as the convergence speed can be explicitly computed. We show that decoupling does not always work for unbounded interactions and we provide both physically and mathematically motivated examples. *Published by AIP Publishing.* https://doi.org/10.1063/1.5016495

I. INTRODUCTION AND OVERVIEW

A powerful strategy to protect a quantum system from decoherence is dynamical decoupling.[19] The application of frequent and instantaneous unitary operations (kicks), which correspond to strong classical pulses applied to the system, makes it possible to average the system-environment interactions to zero. Originally, dynamical decoupling dates back to the pioneering work of Haeberlen and Waugh,[11,28] who developed pulse sequences, such as spin-echo techniques, in order to increase the resolution in nuclear magnetic resonance. Later, these schemes were generalized by Viola and Lloyd,[24–26] establishing a theoretical framework that allows to suppress generic system-environment interactions. Its particular strength is that it is applicable even if the details of the system-environment coupling are unknown.

Since perfect decoupling only happens in the limit of infinitely frequent kicks, in practice, it is important to understand the convergence speed. In finite dimensions, error estimates are given in terms of the higher orders of the Magnus expansion or the Dyson series.[19,25] Here, the existence and the speed of convergence to the decoupled dynamics rely on norm bounds of the Hamiltonian,[17] allowing one to prove that dynamical decoupling works arbitrarily well on a finite time scale.

However, real physical environments, such as the free electromagnetic field, are (to a good approximation) infinite dimensional. In particular, the description of system-environment interactions through potentially unbounded operators makes it challenging to decide whether dynamical decoupling works and, moreover, to estimate the time scales necessary to efficiently dynamically decouple the system from the environment. Commutativity and, in particular, series expansions are a very delicate matter[22] and norm bounds diverge (Ref. 22, Sec. VIII.5). The main purpose of this paper is to establish criteria and examples for dynamical decoupling of unbounded Hamiltonians.

From a physical perspective, dynamical decoupling has to be faster than the fastest time scale of the overall dynamics[24] and it is typically argued that dynamical decoupling only works for

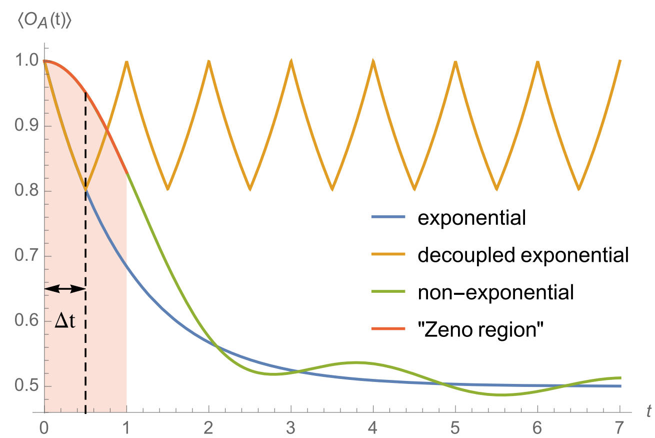

FIG. 1. Exponential and non-exponential decay of some expectation $\langle O_A(t)\rangle$ and the quadratic "Zeno region." In the case of the shallow pocket model, the observable shown is $p_+(t)/p_+(0)$, with $p_+(t) = \mathrm{tr}\,\{|+\rangle_A\,\langle+|\Lambda_t(\rho_A)\}$, and the dynamics in presence of decoupling pulses with pulse time steps $\Delta t = 0.5$ is shown. The non-exponential curve is an approximation of the shallow pocket model, with cutoff parameter $|x| \leq 2$ in the Cauchy distribution. See Example 3.3 for details.

environments yielding non-exponential decay.[19] It is argued that a "Zeno" region of non-exponential decay (Fig. 1) determines the time scale for dynamical decoupling. However, this is a heuristic argument rather than a rigorous mathematical conclusion, and we will provide several counterexamples to it below. In fact, it is interesting to note that to decide whether dynamical decoupling works for infinite dimensional environments, the full Hamiltonian must be provided. That is, the reduced dynamics does not provide enough information, and for the same reduced dynamics, there can be dilations (given by system-environment Hamiltonians and environment initial states), which can be decoupled, whereas others cannot. An example is given by qubit dephasing, for which the shallow pocket model[2] provides a dilation that can be decoupled, whereas its Cheborev-Gregoratti dilation[27] was recently shown to be not amenable to decoupling.[10] These two dilations can be considered as two extreme cases: the former being highly non-Markovian and the latter being very singular with built-in Markovian properties. The true physical models are likely to be found in between such extremes, and it is important to find general criteria for decoupling.

In Sec. III we give a sufficient criterion for dynamical decoupling (Theorem 3.1), which is based on Trotter's product formula and generalizes the findings of Ref. 2. As an example, we discuss the shallow pocket model[2] that yields exponential decay but can be decoupled on arbitrary time scales. Then we provide several generalizations that can be decoupled, but for which, the time scale of decoupling is non-trivial. Here, we explicitly provide the corresponding time scales in order to dynamically decouple the system from the environment and show that the efficiency depends on the initial bath state. Finally, we provide an example showing that Theorem 3.1 is sufficient but not necessary for successful decoupling.

In Sec. IV, we discuss lower bounded Hamiltonians, for which, more can be said about the convergence of the Trotter limit. Theorem 4.1 provides a necessary condition for dynamical decoupling of such Hamiltonians. This is physically relevant, because most reasonable interaction Hamiltonians are unbounded above but bounded below. We provide an abstract example of a Hamiltonian where dynamical decoupling does not work. Finally, in Sec. V, we provide a generalization of the Friedrichs-Lee model, which gives rise to an amplitude damping channel. We find that this model cannot be dynamically decoupled and provide a physical interpretation.

II. PREREQUISITES

Consider a quantum mechanical system that is coupled to an environment. We suppose the system Hilbert space $\mathcal{H}_s \simeq \mathbb{C}^d$ to be d-dimensional (with d finite) and the environment Hilbert space $\mathcal{H}_e$ infinite dimensional and separable. We write $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_e$ for the total Hilbert space and H for the total Hamiltonian, a self-adjoint operator with domain $\mathcal{D}(H) \subset \mathcal{H}$.

We assume that the initial state is uncorrelated $\rho = \rho_s \otimes \rho_e$, where ρ_s and ρ_e are non-negative trace-class operators with tr ρ_s = tr $\rho_e = 1$. Notice that in order to relate the unitary with the reduced dynamics, we have to fix some state ρ_e for the environment, and we suppose that a physically realistic choice can be made here. With $U(t) = e^{-itH}$, we refer to

$$\rho_s \mapsto \Lambda_t(\rho_s) := \mathrm{tr}_{\mathcal{H}_e}\,\{U(t)(\rho_s \otimes \rho_e)U^*(t)\} \tag{1}$$

as the dynamical map on state ρ_s at time t. Here, $\mathrm{tr}_{\mathcal{H}_e}$ denotes the partial trace with respect to the factor $\mathcal{H}_e$. The typical feature of reduced dynamics is that certain expectation values

$$\langle O_s(t)\rangle := \mathrm{tr}\,\{O_s \Lambda_t(\rho_s)\}$$

of observables $O_s \in B(\mathcal{H}_s)$, the space of (bounded) linear operators on $\mathcal{H}_s$, can *decay irreversibly*. This is particularly so if the dynamical map Λ_t has the semigroup property $\Lambda_{t_1} \circ \Lambda_{t_2} = \Lambda_{t_1+t_2}$, for all $t_1, t_2 \geq 0$. In such a case, the reduced dynamics $\rho_s(t) := \Lambda_t(\rho_s)$ is described by a Gorini-Kossakowski-Lindblad-Sudarshan (GKLS) master equation $\dot{\rho}_s(t) = \mathcal{L}\rho_s(t)$, with generator

$$\mathcal{L}(\rho_s) = -\,\mathrm{i}[H_s, \rho_s] + \sum_{i=1}^{d^2} \gamma_i \Big(L_i \rho_s L_i^* - \frac{1}{2} L_i^* L_i \rho_s - \frac{1}{2} \rho_s L_i^* L_i \Big), \tag{2}$$

for all system density matrices ρ_s, where $\gamma_i \geq 0$ and $L_i, H_s \in B(\mathcal{H}_s)$ and H_s, moreover, self-adjoint.[9,20]

Definition 2.1. A *decoupling set for* $\mathcal{H}_s$ is a finite group of unitary operators $V \subset \mathrm{U}(\mathcal{H}_s)$, such that

$$\frac{1}{|V|} \sum_{v \in V} v x v^* = \frac{1}{d} \mathrm{tr}(x) \mathbf{1}_{\mathcal{H}_s}, \quad \text{for all } x \in B(\mathcal{H}_s).$$

Let N be a multiple of the cardinality $|V|$. A *decoupling cycle* of length N is a cycle $(v_1, v_2, \ldots, v_N)$ through V, which reaches each element of V the same number of times.

In Ref. 1, it is shown that such a decoupling set always exists, but it is usually not unique. Obviously, given a decoupling cycle, one gets

$$\frac{1}{N} \sum_{k=1}^{N} v_k x v_k^* = \frac{1}{|V|} \sum_{v \in V} v x v^*.$$

Dynamical decoupling on $\mathcal{H}_s \otimes \mathcal{H}_e$ is now implemented by applying the decoupling operations $v \otimes \mathbf{1}_{\mathcal{H}_e}$ instantaneously in time steps $\tau > 0$. To shorten notation, we shall simply write v instead of $v \otimes \mathbf{1}_{\mathcal{H}_e}$, when confusion is unlikely. In Refs. 12 and 1, we discuss a random implementation of these decoupling operations, while here, we restrict ourselves to a deterministic implementation since our focus is rather on the unboundedness of H. To be precise, consider a decoupling cycle of unitaries $(v_1, v_2, \ldots, v_N)$ and apply them to the system periodically; so the total time evolution unitary after one decoupling cycle will be given by

$$v_N\, \mathrm{e}^{\mathrm{i}\tau H}\, v_N^* v_{N-1}\, \mathrm{e}^{\mathrm{i}\tau H}\, v_{N-1}^* \cdots v_2\, \mathrm{e}^{\mathrm{i}\tau H}\, v_2^* v_1\, \mathrm{e}^{\mathrm{i}\tau H}\, v_1^*. \tag{3}$$

We can now split a given time interval $[0, t]$ into nN steps and apply the decoupling cycle of length N there n times. Thus, the following definition makes sense:

Definition 2.2. For given Hamiltonian H and decoupling set V, we say that *dynamical decoupling works specifically* if there is a decoupling cycle $(v_1, v_2, \ldots, v_N)$ and there is a self-adjoint operator B on $\mathcal{H}_e$, such that

$$\operatorname*{s-lim}_{n\to\infty} \Big(\mathrm{e}^{\mathrm{i}\,\frac{t}{nN} v_1 H v_1^*} \cdots \mathrm{e}^{\mathrm{i}\,\frac{t}{nN} v_N H v_N^*} \Big)^n = \mathrm{e}^{\mathrm{i}\,t(\mathbf{1}_{\mathcal{H}_s} \otimes B)},$$

uniformly for t in compact intervals of $\mathbb{R}$.

We say that *dynamical decoupling works uniformly* if there is a self-adjoint operator B on $\mathcal{H}_e$, such that, for every decoupling cycle $(v_1, v_2, \ldots, v_N)$,

$$\operatorname*{s-lim}_{n\to\infty} \left(\mathrm{e}^{\mathrm{i}\,\frac{t}{nN} v_1 H v_1^*} \cdots \mathrm{e}^{\mathrm{i}\,\frac{t}{nN} v_N H v_N^*} \right)^n = \mathrm{e}^{\mathrm{i}\,t(\mathbf{1}_{\mathcal{H}_s}\otimes B)}, \tag{4}$$

uniformly for t in compact intervals of $\mathbb{R}$.

The physical interpretation is that, in the limit where time steps go to 0, only the environment evolves. From the physical point of view, the strong topology is satisfactory. Indeed, one gets norm convergence (that is, uniform rate) on $\mathcal{H}_s$ for a fixed environment state ρ_e, for example, a thermal state.

It is unclear whether "specifically" in Definition 2.2 is really weaker than "uniformly" or whether the existence of one decoupling cycle that works would in fact imply that all decoupling cycles work. Intuitively, one might expect that the order is irrelevant as a consequence of the homogenisation effect of the limit.

To conclude the prerequisites, we will frequently use the following convention: if a_k, with $k = 1, \ldots, N$, are in $B(\mathcal{H})$, then we write

$$\prod_{k=1}^{N} a_k := a_N \cdots a_1$$

for the product in $B(\mathcal{H})$ with this specific order.

III. A SUFFICIENT CONDITION FOR DYNAMICAL DECOUPLING

Theorem 3.1. *Let V be a decoupling set for $\mathcal{H}_s$ and $H : \mathcal{D}(H) \to \mathcal{H}$ be self-adjoint. If the sum $\sum_{v\in V} vHv^*$ is essentially self-adjoint on the intersection of the domains, $\mathcal{D} = \bigcap_{v\in V} v\mathcal{D}(H)$, then dynamical decoupling works uniformly for H.*

Proof. The theorem follows from a straight-forward generalisation of the Trotter product formula (Ref. 14, Corollary 11.1.6) to N factors. More precisely, given a decoupling cycle $(v_1, \ldots, v_N)$ in V, define the function

$$F(t) := \prod_{k=1}^{N} \mathrm{e}^{\mathrm{i}\,\frac{t}{N} v_k H v_k^*}, \qquad t \in \mathbb{R}_+.$$

Then $F : \mathbb{R} \to B(\mathcal{H})$ is a strongly continuous function with $F(0) = \mathbf{1}_{\mathcal{H}}$. Moreover, we get

$$\frac{F(t)\xi - F(0)\xi}{t} = \frac{\prod_k \mathrm{e}^{\mathrm{i}\,\frac{t}{N} v_k H v_k^*}\,\xi - \xi}{t} \to \frac{\mathrm{i}}{|V|} \sum_{v\in V} vHv^*\xi, \qquad t \to 0$$

for all $\xi \in \mathcal{D}$.

Now, we claim that the closure

$$\frac{1}{|V|} \overline{\sum_{v\in V} vHv^*} = \mathbf{1}_{\mathcal{H}_s} \otimes B,$$

with some self-adjoint B on $\mathcal{H}_e$. Indeed, since the left-hand side is commuting with all $v \in V$, the group $(U_t)_{t\in\mathbb{R}}$ generated by it satisfies the relation

$$U_t = \frac{1}{|V|} \sum_{v\in V} vU_t v^* = \mathbf{1}_{\mathcal{H}_s} \otimes \frac{1}{d}\operatorname{tr}_{\mathcal{H}_s}(U_t), \qquad \text{for all } t \in \mathbb{R},$$

by Definition 2.1 of decoupling set, and thus must be of the form

$$U_t = \mathbf{1}_{\mathcal{H}_s} \otimes \mathrm{e}^{\mathrm{i}\,Bt},$$

by Stone's theorem.

We apply Chernoff's product formula (Ref. 14, Theorem 11.1.2) to this and obtain that

$$F(t/n)^n = \Big(\prod_{k=1}^{N} \mathrm{e}^{\mathrm{i}\,\frac{t}{Nn} v_k H v_k^*} \Big)^n \to \mathrm{e}^{\mathrm{i}(\mathbf{1}_{\mathcal{H}_s} \otimes B)t}, \quad n \to \infty$$

in the strong operator topology and uniformly for t in compact intervals in $\mathbb{R}$. This verifies condition (4). □

Example 3.2 (Qubit). The following construction is a building block that will allow us to create several examples at increasing complexity and transfer results about the Trotter formula to the context of dynamical decoupling. The idea is to study the space

$$\mathcal{H} = \mathbb{C}^2 \otimes \mathcal{H}_e \simeq \mathcal{H}_e \oplus \mathcal{H}_e,$$

describing a qubit system coupled to an environment $\mathcal{H}_e$. Suppose our Hamiltonian, expressed in the decomposition $\mathcal{H}_e \oplus \mathcal{H}_e$ of $\mathcal{H}$, is of the form $A \oplus B$, i.e.,

$$H = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \tag{5}$$

on $\mathcal{D}(A) \oplus \mathcal{D}(B)$, with both A and B self-adjoint. The standard decoupling set for $\mathbb{C}^2$ consists of the Pauli group: (multiples by 1, i, −1, −i) of the four Pauli matrices $\mathbf{1}$, X, Y, and Z. Now if we take the Pauli matrix

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then

$$H = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \quad XHX^* = \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix}.$$

The adjoint action of other Pauli matrices to H produces one of these two matrices, so we can reduce our situation down to a group with two elements $V = \{\mathbf{1}, X\}$. As decoupling cycles in the examples here, we consider simply $(\mathbf{1}, X)$, although an analogous reasoning holds for any other cycle in V. Thus, though we prove everything only for this specific cycle, one can actually show that decoupling works uniformly in all of the following examples. □

Example 3.3 (Shallow pocket model). See Ref. 2, Sec. 3. In the setting of the preceding Example 3.2 with one qubit, we consider $\mathcal{H}_e = L^2(\mathbb{R})$ and $A = -B = q$, the position operator, $q\xi(x) = x\xi(x)$, with $\mathcal{D}(q) = \{\xi \in L^2(\mathbb{R}) : q\xi \in L^2(\mathbb{R})\}$, in (5):

$$H = \begin{pmatrix} q & 0 \\ 0 & -q \end{pmatrix}. \tag{6}$$

Theorem 3.1 applies, and the model can be dynamically decoupled. In fact, we get that $XHX = -H$, so the Trotter limit is trivial and decoupling works uniformly and perfectly at all the time scales.

We can study the reduced dynamics as well. Let us assume that the environment initial state is

$$\xi_C(x) = \left(\frac{2}{\pi} \frac{1}{x^2 + 4} \right)^{1/2}, \quad x \in \mathbb{R}. \tag{7}$$

The spectrum of H is the full line $\mathbb{R}$. The state ξ_C does not belong to the domain $D(q)$, and then, any initial factorized state $\psi \otimes \xi_C$ does not belong to the domain of H.

The dynamical map Λ_t in (1) has the semigroup property and its generator is the dephasing GKLS operator

$$\mathcal{L}\rho = -[Z, [Z, \rho]]$$

for all system density matrices ρ, where

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

giving rise to exponential decay of the coherences. See Fig. 1. □

Example 3.4 ($q \oplus p$ example). Again in the setting of Example 3.2, we choose $\mathcal{H}_e = L^2(\mathbb{R})$ and $A = q$ and $B = p$, the position and the momentum operator, respectively, in (5):

$$H = \begin{pmatrix} q & 0 \\ 0 & p \end{pmatrix}, \tag{8}$$

where $q = M_x$ and $p = -\mathrm{i}\,\mathrm{d}/\mathrm{d}x$ are self-adjoint on their natural domains $\mathcal{D}(q) = \{\xi \in L^2(\mathbb{R}) : q\xi \in L^2(\mathbb{R})\}$, $\mathcal{D}(p) = H^1(\mathbb{R})$, the first Sobolev space. The sum of the two is essentially self-adjoint on Schwartz space $\mathcal{S}(\mathbb{R})$. This can be shown along the lines of Ref. 23, Sec. X.6. According to Example 3.2, it is sufficient to consider the group $V = \{\mathbf{1}, X\}$ and

$$H + XHX^* = \mathbf{1}_{\mathbb{C}^2} \otimes (q + p),$$

which is essentially self-adjoint on $\mathbb{C}^2 \otimes \mathcal{S}(\mathbb{R})$. According to Theorem 3.1, dynamical decoupling works uniformly.

We can study the reduced dynamics as follows. As environment initial state, let us consider

$$\xi = \mathrm{e}^{-\mathrm{i}\,\frac{\pi}{8}(q^2+p^2)}\,\xi_C,$$

with ξ_C as in (7). Then the dynamical map Λ_t in (1) is generated by the dephasing GKLS operator plus a time-dependent Hamiltonian

$$\mathcal{L}_t\rho = -[Z,[Z,\rho]] - \mathrm{i}\,\frac{t}{2}[Z,\rho],$$

giving rise to exponential decay of the coherences.

In order to determine the unitary evolution after n decoupling cycles and the decoupling error explicitly, we need some prerequisites. Consider the 3-dimensional real Lie algebra $\mathfrak{g} = \mathrm{span}_{\mathbb{R}}\{E, P, Q\}$ with commutation relations

$$[E,P] = 0, \quad [E,Q] = 0, \quad [P,Q] = E$$

and its representation π by unbounded skew-symmetric operators defined by linear continuation of

$$E \mapsto \mathrm{i}\,\mathbf{1}, \quad P \mapsto \mathrm{i}\,p, \quad Q \mapsto \mathrm{i}\,q,$$

where all operators here act on the Schwartz space $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R})$ as common invariant domain. Notice that the Nelson-Laplacian $\mathbf{1} + p^2 + q^2$ of this representation π of $\mathfrak{g}$ is essentially self-adjoint on $\mathcal{S}(\mathbb{R})$ (cf. Ref. 23, Sec. X.6). Thus, the conditions in Nelson's criterion (Ref. 21, Theorem 5) are fulfilled, and the representation π exponentiates to the Lie group G of $\mathfrak{g}$, such that

$$\mathrm{e}^{\pi(X)} = \pi(\exp(X)),$$

where exp denotes the exponential map of G. This means that the Baker-Campbell-Hausdorff formula holds in the representation π as well, namely,

$$\begin{aligned}\mathrm{e}^{\mathrm{i}\,q}\,\mathrm{e}^{\mathrm{i}\,p} &= \pi(\exp(Q))\pi(\exp(P)) = \pi(\exp(Q)\exp(P)) = \pi(\exp(Q + P + \tfrac{1}{2}[Q,P])) \\ &= \pi(\exp(Q + P - \tfrac{1}{2}E)) = \mathrm{e}^{\mathrm{i}(q+p) - \frac{\mathrm{i}}{2}};\end{aligned}$$

all higher order commutators in $\mathfrak{g}$ vanish.

We apply this now to dynamical decoupling. The time evolution after time t with n decoupling cycles of length $N = |V| = 2$ then reads

$$U_n(t) = \left(\mathrm{e}^{\mathrm{i}\,\frac{t}{2n}H}\,\mathrm{e}^{\mathrm{i}\,\frac{t}{2n}XHX}\right)^n = \exp\left(-\mathrm{i}\,\frac{t^2}{8n}Z\right) \otimes \exp\left(\mathrm{i}\,\frac{t}{2}(q+p)\right). \tag{9}$$

Interestingly, the decoupling error—the deviation from Eq. (4)—is a unitary on the system only, and the convergence is uniform for t in compact intervals in $\mathbb{R}$:

$$\|U_n(t) - \mathbf{1}_{\mathcal{H}_s} \otimes \mathrm{e}^{\mathrm{i}\,Bt}\| = \|\,\mathrm{e}^{-\mathrm{i}\,\frac{t^2}{2n}Z} - \mathbf{1}_{\mathcal{H}_s}\| \to 0,$$

as $n \to +\infty$, with $B = \frac{1}{2}(q+p)$. □

Example 3.5 ($q \oplus p^2$ example). Again in the setting of Example 3.2, we now choose $\mathcal{H}_e = L^2(\mathbb{R})$ and $A = q$ and $B = p^2$ in (5):

$$H = \begin{pmatrix} q & 0 \\ 0 & p^2 \end{pmatrix}. \tag{10}$$

The operators $q = M_x$ and $p^2 = -\mathrm{d}^2/\mathrm{d}x^2$ are self-adjoint on their natural domains $\mathcal{D}(q) = \{\xi \in L^2(\mathbb{R}) : q\xi \in L^2(\mathbb{R})\}$, $\mathcal{D}(p^2) = H^2(\mathbb{R})$, the second Sobolev space. The sum of the two is essentially self-adjoint on the Schwartz space $\mathcal{S}(\mathbb{R})$ (cf. Ref. 23, Sec. X.6). According to Example 3.2, it is sufficient to consider $V = \{\mathbf{1}, X\}$ and

$$H + XHX^* = \mathbf{1}_{\mathbb{C}^2} \otimes (q + p^2),$$

which is essentially self-adjoint on $\mathbb{C}^2 \otimes \mathcal{S}(\mathbb{R})$. According to Theorem 3.1, dynamical decoupling works uniformly.

In order to study the decoupling error, let us consider a Cauchy distribution in momentum space

$$\xi(p) = \left(\frac{\gamma}{2\pi}\frac{1}{p^2 + \frac{\gamma^2}{4}}\right)^{1/2}, \quad p \in \mathbb{R}$$

as environment initial state. Then for the qubit state $\rho_s(t)$ at time t, the decoupling error becomes

$$\epsilon(t) = \|\rho_s(t) - \rho_s(0)\|_2^2$$

as a function of the decoupling steps n, with $\|\cdot\|_2$ being the Hilbert Schmidt norm, and we assume that the qubit is initially prepared in $\rho_s(0) = |+\rangle\langle+|$, where $|+\rangle = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, yielding $\epsilon(t) = 2(1 - \langle + |\rho_s(t)| + \rangle)$.

We proceed in analogy with the previous example, verifying Nelson's criterion for the Lie algebra $\mathfrak{g} = \mathrm{span}_{\mathbb{R}}\{E, P, Q, R\}$ with commutation relations

$$[E,P] = [E,R] = [E,Q] = 0, \quad [P,R] = 0, \quad [P,Q] = E, \quad [R,Q] = 2P.$$

We represent $\mathfrak{g}$ by unbounded skew-symmetric operators

$$E \mapsto \mathrm{i}\,\mathbf{1}, \quad P \mapsto \mathrm{i}\,p, \quad Q \mapsto \mathrm{i}\,q, \quad R \mapsto \mathrm{i}\,p^2$$

on $\mathcal{S}(\mathbb{R})$. The Nelson-Laplacian $\mathbf{1} + p^2 + q^2 + p^4$ is essentially self-adjoint on $\mathcal{S}(\mathbb{R})$ (cf. Ref. 23, Sec. X.6); thus, the Lie algebra representation exponentiates to a Lie group representation again (cf. Ref. 21, Theorem 5), and the Baker-Campbell-Hausdorff and the Zassenhaus formula hold. Since nested commutator expressions vanish after depth 3, the Baker-Campbell-Hausdorff and the Zassenhaus formula show that the unitary evolution after n decoupling cycles $U_n(t)$ takes the form of

$$U_n(t) = \mathrm{e}^{-\mathrm{i}\,\frac{t}{2}(\mathbf{1}\otimes(q+p^2)+t^2/(24n^2))}\,\mathrm{e}^{-\mathrm{i}\,\frac{t^3}{16n}(Z\otimes\mathbf{1})}\,\mathrm{e}^{\mathrm{i}\,\frac{t^2}{4n}(Z\otimes p)}.$$

This evolution, for finite n, leads to dephasing in Z direction of the qubit. Therefore the choice for $\rho_s(0)$ as above describes the worst-case scenario, i.e., the supremum of ϵ over all initial states of the qubit. After tracing out the environmental degrees of freedom, we obtain

$$\epsilon(t) = 1 - \cos(t^3/(16n))\,\mathrm{e}^{\frac{-t^2}{4n}\gamma}, \tag{11}$$

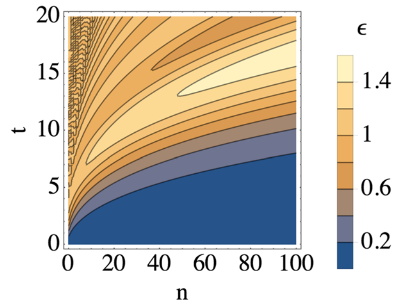

FIG. 2. Decoupling error (11) for the $q \oplus p^2$ model given by (10) as a function of the total evolution time t and the decoupling steps n for a fixed $\gamma = 1$.

which vanishes for $n \to \infty$. We thus have found the explicit form of the decoupling error for the $q \oplus p^2$ model, which is plotted as a function of t and n in Fig. 2 for a fixed $\gamma = 1$. The form of the decoupling error (11) also shows the dependency of the initial state of the environment on the efficiency of dynamical decoupling. For fixed $t > 0$ and $n < \infty$, we can always find an environment initial state, i.e., some $\gamma > 0$, such that decoupling becomes arbitrarily bad.

Example 3.6 ($q^2 \oplus p^2$ example). We choose $\mathcal{H}_e = L^2(\mathbb{R})$ and $A = q^2$ and $B = p^2$, in (5):

$$H = \begin{pmatrix} q^2 & 0 \\ 0 & p^2 \end{pmatrix}, \tag{12}$$

where q^2 and p^2 are self-adjoint on their natural domains $\mathcal{D}(q^2) = \{\xi \in L^2(\mathbb{R}), q^2\xi \in L^2(\mathbb{R})\}$, $\mathcal{D}(p^2) = H^2(\mathbb{R})$, the second Sobolev space. The sum of the two is essentially self-adjoint on the Schwartz space $\mathcal{S}(\mathbb{R})$, (cf. Ref. 23, Sec. X.6). Theorem 3.1 applies, and the model can be dynamically decoupled.

The unitary evolution after n decoupling steps can be obtained using a symplectic representation,[6] and reads for $n > t$

$$U_n(t) = \exp\left(\mathrm{i}\, t f\left(\frac{t}{n}\right)\overline{\left(\mathbf{1} \otimes (q^2 + p^2) + \frac{t}{2n} X \otimes (qp + pq)\right)}\right), \tag{13}$$

where

$$f(t) = \frac{2}{t\sqrt{4 - t^2}} \arctan\left(\frac{t\sqrt{4 - t^2}}{2 - t^2}\right).$$

Notice that $f(t) = 1 + t^2/6 + O(t^4)$ for $t \to 0$.

In this case, the decoupling error (on the system) is not unitary, and the convergence is non-uniform. Interestingly, the original Hamiltonian has an absolutely continuous spectrum on the positive real line, while the limit is the Harmonic oscillator, which has a purely point spectrum; this may be regarded as a consequence of "averaging" between p^2 and q^2.

□

Example 3.7 (Spin-boson model). Consider again $\mathcal{H}_s = \mathbb{C}^2$ and $\mathcal{H}_e = L^2(\mathbb{R})$ as in the preceding examples; but now

$$H = \overline{\omega_c \mathbf{1} \otimes a^\dagger a + \frac{\omega_a}{2} Z \otimes \mathbf{1} + \frac{\Omega}{4}\left((X - \mathrm{i}\, Y) \otimes a + (X + \mathrm{i}\, Y) \otimes a^\dagger\right)}.$$

Here, $a = \frac{1}{\sqrt{2}}(q + \mathrm{i}\, p)$ and $a^\dagger = \frac{1}{\sqrt{2}}(q - \mathrm{i}\, p)$ (the formal adjoint of a) are the harmonic oscillator ladder operators on the common invariant domain $\mathcal{S}(\mathbb{R})$; ω_c, ω_a, and Ω are real constants; and $\overline{}$ denotes the closure of the above sum on $\mathbb{C}^2 \otimes \mathcal{S}(\mathbb{R})$, where it is essentially self-adjoint (again along the lines of Ref. 23, Sec. X.6), so that H is self-adjoint. This way the interaction part of the Hamiltonian is not block-diagonal, in contrast to Example 3.2. This model can be decoupled using the full Pauli group V because the sum $\sum_{v \in V} vHv^*$ is essentially self-adjoint on the Schwartz space $\mathcal{S}(\mathbb{R})$ which is contained in the domain intersection. This example can be easily generalized to a finite number of bosonic modes. □

One could argue whether the conditions of Theorem 3.1 are also necessary. This is not the case, as shown in the following slightly artificial example by Chernoff.[3] Later we will give a necessary condition for semibounded Hamiltonians.

Example 3.8 (Non-overlapping domains). We use again the setting Example 3.2. In (5), let $\mathcal{H}_e = L^2(\mathbb{R})$ and consider $A = p$, the momentum operator, which is self-adjoint on $\mathcal{D}(p) = H^1(\mathbb{R})$, the first Sobolev space. Let B be the multiplication operator

$$B\psi(x) = f(x)\psi(x), \qquad \mathcal{D}(B) = \{\psi \in L^2(\mathbb{R}), f\psi \in L^2(\mathbb{R})\},$$

where $f \in L^1_{\mathrm{loc}}(\mathbb{R}) \setminus L^2_{\mathrm{loc}}(\mathbb{R})$ is locally integrable but is not locally square-integrable. For example, we can take

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n!} |x - r_n|^{-1/2},$$

where $(r_n)_{n\in\mathbb{N}}$ is some enumeration of the rationals. One can prove that (Ref. 3, Proposition 5.1)

$$\operatorname*{s-lim}_{n\to+\infty} \left(\mathrm{e}^{\mathrm{i}\,\frac{t}{n}p}\,\mathrm{e}^{\mathrm{i}\,\frac{t}{n}B}\right)^n = \mathrm{e}^{-\mathrm{i}\,C}\,\mathrm{e}^{\mathrm{i}\,tp}\,\mathrm{e}^{\mathrm{i}\,C},$$

where C is the multiplication operator

$$C\psi(x) = F(x)\psi(x), \qquad \mathcal{D}(C) = \{\psi \in L^2(\mathbb{R}), F\psi \in L^2(\mathbb{R})\},$$

by an antiderivative of f,

$$F(x) = \int_0^x f(y)\,\mathrm{d}y.$$

The unitary evolution after n decoupling cycles $(\mathbf{1}, X)$ converges

$$\operatorname*{s-lim}_{n\to+\infty} U_n(t) = \operatorname*{s-lim}_{n\to+\infty} \left(\mathrm{e}^{\mathrm{i}\,\frac{t}{2n}H}\,\mathrm{e}^{\mathrm{i}\,\frac{t}{2n}XHX}\right)^n = \mathbf{1}_{\mathcal{H}_s} \otimes \exp\left(\mathrm{i}\,t\,\mathrm{e}^{-\mathrm{i}\,C}\,p\,\mathrm{e}^{\mathrm{i}\,C}\right), \tag{14}$$

and therefore, decoupling works specifically, and a similar reasoning applies to any decoupling cycle, so dynamical decoupling works uniformly.

Notice that if $\psi \in C(\mathbb{R})$ is continuous, and thus locally bounded, then $f\psi \notin L^2_{\mathrm{loc}}(\mathbb{R})$. Therefore, $\mathcal{D}(B)$ does not contain any nonzero continuous function, and thus,

$$\mathcal{D}(A) \cap \mathcal{D}(B) = \{0\},$$

since $H^1(\mathbb{R}) \subset C(\mathbb{R})$. Thus, the Trotter formula for unitary groups can converge when the operator sum is not essentially self-adjoint, and even in the extreme case of a trivial domain intersection. Dynamical decoupling works even though $\mathcal{D} = \bigcap_{v\in V} v\mathcal{D}(H) = \{0\}$.

□

IV. A NECESSARY CONDITION FOR DYNAMICAL DECOUPLING OF NON-NEGATIVE HAMILTONIANS

In Theorem 3.1, we have established a sufficient condition for dynamical decoupling to work uniformly. However, Example 3.8 showed that it is not at all necessary; so let us now turn to our promised necessary condition, under the additional assumption of a non-negative Hamiltonian:

Theorem 4.1. *Let V be a decoupling set for $\mathcal{H}_s$, and suppose that H is non-negative. If dynamical decoupling works uniformly, then for all $v, w \in V$, the form domain intersections*

$$v\mathcal{D}(H^{1/2}) \cap w\mathcal{D}(H^{1/2}) \subset \mathcal{H}$$

must be dense.

Before starting the proof, let us quickly recall something about form domains. Every densely defined operator A on $\mathcal{H}$ gives rise to a bilinear form $\mathcal{D} \times \mathcal{D} \to \mathbb{C}$ with some form domain $\mathcal{D} \subset \mathcal{H}$, in general not unique. In the case A is non-negative, this form domain is defined as $\mathcal{D}(A^{1/2})$. Notice that $\mathcal{D}(A^{1/2}) \supset \mathcal{D}(A)$, so the form domain of (the bilinear form of) a non-negative operator is always dense. Given two non-negative operators, which might have trivial domain intersection and therefore no sum but whose form domains intersect densely, it is possible to define a sum of the two forms; this new bilinear form corresponds to a new self-adjoint operator which is generally called the *form sum* of the two initial operators. For a proper introduction to form domains, we refer the reader to Ref. 22, Sec. 8.6 or Ref. 14, Sec. 10.3.

Proof. Suppose that dynamical decoupling works, but not all of the form domain intersections are dense. Choose $v_1, v_2 \in V$, such that for these two elements,

$$\mathcal{H}_0 := \overline{v_1\mathcal{D}(H^{1/2}) \cap v_2\mathcal{D}(H^{1/2})} \neq \mathcal{H}.$$

Choose $\xi \in \mathcal{H}_0^\perp$ with $\|\xi\| = 1$. Extend the two elements v_1 and v_2 to a palindromic cycle of length $N = 2|V|$ in V, say $(v_1, v_2, \ldots, v_{|V|}, v_{|V|}, \ldots v_2, v_1)$. We can then define the following continuous functions

$$f_n : \overline{\mathbb{C}_+} \to \mathcal{H}, \quad f_n(z) = \left(\mathrm{e}^{-\frac{z}{nN} v_1 H v_1^*} \cdots \mathrm{e}^{-\frac{z}{nN} v_{|V|} H v_{|V|}^*} \mathrm{e}^{-\frac{z}{nN} v_{|V|} H v_{|V|}^*} \cdots \mathrm{e}^{-\frac{z}{nN} v_1 H v_1^*} \right)^n \xi$$

which are analytic on $\mathbb{C}_+$, for every $n \in \mathbb{N}$; here, $\mathbb{C}_+ = \{z \in \mathbb{C} : \Re(z) > 0\}$ denotes the open complex right half-plane. Since we assumed dynamical decoupling to work uniformly, we know from (4) that f_n converges on the boundary and there is a self-adjoint B on $\mathcal{H}_e$, such that

$$f_n(-\mathrm{i}\, t) \to \mathrm{e}^{\mathrm{i}\, t(\mathbf{1}_{\mathcal{H}_s} \otimes B)}\, \xi,$$

as $n \to \infty$, uniformly for t in compact intervals in $\mathbb{R}$. Moreover, Ref. 3, Theorem 7.2 shows that, since vHv^* is non-negative for every $v \in V$, B is non-negative as well and

$$f_n(z) \to f(z) := \mathrm{e}^{-z(\mathbf{1}_{\mathcal{H}_s} \otimes B)}\, \xi, \quad n \to \infty, \tag{15}$$

uniformly for z in compact subsets of $\mathbb{C}_+$, and f is continuous on $\overline{\mathbb{C}_+}$ and analytic on $\mathbb{C}_+$. It is obvious that

$$f(0) = \xi \neq 0. \tag{16}$$

We now claim that

$$f_n(t) \to 0, \tag{17}$$

as $n \to \infty$, uniformly for t in compact intervals in $(0, \infty)$. To this end, we make use of the proof in Ref. 15. Following the notation there, let us write

$$F'_t = \mathrm{e}^{-tv_1Hv_1^*}\, \mathrm{e}^{-tv_2Hv_2^*}\, \mathrm{e}^{-tv_2Hv_2^*}\, \mathrm{e}^{-tv_1Hv_1^*}, \quad t \in [0, \infty).$$

This is precisely the quantity defined in Ref. 15, (3.6). Moreover, let us define

$$G_t = \mathrm{e}^{-\frac{t}{N} v_1 H v_1^*} \cdots \mathrm{e}^{-\frac{t}{N} v_{|V|} H v_{|V|}^*} \mathrm{e}^{-\frac{t}{N} v_{|V|} H v_{|V|}^*} \cdots \mathrm{e}^{-\frac{t}{N} v_1 H v_1^*}, \quad t \in [0, \infty).$$

Then it follows that $0 \le G_t \le F'_{t/N} \le \mathbf{1}$, for all $t \in [0, \infty)$, and

$$0 \le G_t = \mathbf{1} - (\mathbf{1} - G_t) \le (\mathbf{1} + (\mathbf{1} - G_t))^{-1}.$$

We are interested in the limit of $G_{t/n}^n \xi$. We have

$$0 \le G_{t/n}^{2n} \le (\mathbf{1} + (\mathbf{1} - G_t))^{-2n} \le (\mathbf{1} + 2n(\mathbf{1} - G_t))^{-1}.$$

Using the fact that $x \mapsto -\frac{1}{x}$ is operator-monotone on $(0, \infty)$[13] and the fact that $0 \le G_t \le F'_{t/N} \le \mathbf{1}$, so $\mathbf{1} \le \mathbf{1} + 2n(\mathbf{1} - F'_{t/N}) \le \mathbf{1} + 2n(\mathbf{1} - G_t)$, we get

$$(\mathbf{1} + 2n(\mathbf{1} - G_t))^{-1} \le (\mathbf{1} + 2n(\mathbf{1} - F'_{t/N}))^{-1}.$$

Then it follows from Ref. 15, (3.11) that

$$\|G^n_{t/n}\xi\|^2 = \langle \xi, G^{2n}_{t/n}\xi\rangle \le \langle \xi, (\mathbf{1} + 2n(\mathbf{1} - F'_{t/N}))^{-1}\xi\rangle \to 0, \quad n \to \infty,$$

uniformly for t in compact intervals in $(0, \infty)$. This proves our claim in (17), namely, $f_n(t) \to 0$, uniformly for t in compact intervals of $(0, \infty)$.

On the other hand, (15) shows that $f_n(t) \to f(t)$ as $n \to \infty$, which means that $f(t) = 0$, for $t \in (0, \infty)$. By the identity theorem for analytic functions, we get that $f(z) = 0$, for all $z \in \mathbb{C}_+$, and since f is continuous on $\overline{\mathbb{C}_+}$, we must have $f(0) = 0$ as well. This is in contradiction with (16). Thus, dynamical decoupling cannot work uniformly if the form domain intersections are not dense. □

The preceding theorem provides a necessary condition for dynamical decoupling to work uniformly, namely, that the form domain intersections $v\mathcal{D}(H^{1/2}) \cap w\mathcal{D}(H^{1/2})$ are dense in $\mathcal{H}$, for every two v, $w \in V$. We believe that it should be possible to strengthen this as follows, though in order to prove this, we would require a generalisation of Ref. 15 to Trotter products of arbitrarily many semigroups rather than only two, which is currently an open problem.

Conjecture 4.2. Let V be a decoupling set for $\mathcal{H}_s$, and suppose that H is non-negative. If dynamical decoupling works uniformly, then the total form domain intersection

$$\bigcap_{v\in V} v\mathcal{D}(H^{1/2}) \subset \mathcal{H}$$

must be dense.

In the case where V consists of two elements, the conjecture reduces to Theorem 4.1, and we can realize the relevance of the condition in the following example.

Example 4.3 (Non-overlapping form domains). Assume that in (5), both $A, B \ge 0$ but vanishing form domain intersection, $\mathcal{D}(A^{1/2}) \cap \mathcal{D}(B^{1/2}) = \{0\}$. Then, applying the decoupling operations on $\mathbb{C}^2$ as in Example 3.2 leads to

$$\mathcal{D}(H^{1/2}) \cap \mathcal{D}(vH^{1/2}v^*) = (\mathcal{D}(A^{1/2}) \oplus \mathcal{D}(B^{1/2})) \cap (\mathcal{D}(A^{1/2}) \oplus \mathcal{D}(B^{1/2})) = \{0\}.$$

According to the criterion in Theorem 4.1, this system cannot be decoupled from the environment.

Now in order to find such operators, let us modify Example 3.8 (see also Ref. 3, Example 5.6 or Ref. 14, Example 10.3.21). Namely, consider $\mathcal{H}_e = L^2(\mathbb{R})$, and $A = p^2$ the negative second derivative operator $-\frac{d^2}{dx^2}$ on $\mathbb{R}$ and B the multiplication, with a certain positive measurable function f yet to be determined. The domain of A is the second Sobolev space, $\mathcal{D}(A) = H^2(\mathbb{R})$, and the form domain is the first Sobolev space, $\mathcal{D}(A^{1/2}) = H^1(\mathbb{R})$. Instead, for B, we find

$$\mathcal{D}(B) = \{\xi \in L^2(\mathbb{R}) : f\xi \in L^2(\mathbb{R})\}$$

and

$$\mathcal{D}(B^{1/2}) = \{\xi \in L^2(\mathbb{R}) : \sqrt{f}\xi \in L^2(\mathbb{R})\}.$$

Now we take f in such a way that it is nowhere locally integrable. For example, we can take

$$f(x) = \Big(\sum_{n=1}^{\infty} \frac{1}{n!}|x - r_n|^{-1/2}\Big)^2,$$

where $(r_n)_{n\in\mathbb{N}}$ is a complete enumeration of the set of rational numbers.

With this choice, one can prove that A and B are densely defined self-adjoint operators on $L^2(\mathbb{R})$ but with trivial form domain intersection:

$$\mathcal{D}(A^{1/2}) \cap D(B^{1/2}) = \{0\},$$

which concludes our example. □

Remark 4.4 (Some variations). In order to allow for a wider selection of models that can be dynamically decoupled, we might try to relax the condition of dynamical decoupling in (4) a bit. One way forward would be to say that as we fix t and let $n \to \infty$, we no longer require (4) for the whole sequence but instead for a subsequence only. In other words, we could say that dynamical decoupling works if, for any $t \in [0, \infty)$, there is a subsequence $(n_k)_{k\in\mathbb{N}}$, such that

$$\operatorname*{s-lim}_{k\to\infty} \Big(\prod_{j=1}^{N} \mathrm{e}^{\mathrm{i}\,\frac{t}{n_k N} v_j H v_j^*} \Big)^{n_k} = \mathrm{e}^{\mathrm{i}\,t(\mathbf{1}_{\mathcal{H}_s}\otimes B)}. \tag{18}$$

Interestingly enough, it follows from Ref. 16 using the argument in Ref. 14, Proposition 11.7.4 that for any system with decoupling set V and $H \geq 0$, the condition of dense total form domain intersection in Conjecture 4.2 is sufficient in order for (18) to hold for almost all t (though *not* for all t and *not* uniformly in compact intervals). Other variations may be derived in a similar manner from the ideas and results collected in Ref. 5, Sec. 1.

We end this section with a difficult open problem:

Problem 4.5. Let V be a decoupling set for $\mathcal{H}_s$, and suppose that H is non-negative. Is it true that dynamical decoupling works (uniformly) if and only if the total form domain intersection

$$\bigcap_{v\in V} v\mathcal{D}(H^{1/2}) \subset \mathcal{H}$$

is dense?

Notice that an affirmative answer would prove Conjecture 4.2.

V. GENERALIZED FRIEDRICHS-LEE MODEL

The aim here is to provide a physically realistic model where dynamical decoupling does not work. The environment is described by the standard Friedrichs-Lee model,[7,8,18] which we briefly recall here; to some extent, we also follow the lines in Ref. 27, Sec. 4.2.2-3. In this section, the Hamiltonian will no longer be non-negative, so Theorem 4.1 becomes irrelevant here.

The Hilbert space is $\mathcal{H}_e = \mathbb{C} \oplus L^2(\mathbb{R})$. It will be convenient to use a matrix notation and write the vectors $\psi = x \oplus \xi \in \mathcal{H}_e$ in the form

$$\psi(\omega) = \begin{pmatrix} x \\ \xi(\omega) \end{pmatrix}, \tag{19}$$

where $\omega \in \mathbb{R}$, $x \in \mathbb{C}$, and $\xi \in L^2(\mathbb{R})$. Then, given a function $g \in L^2(\mathbb{R})$, the Friedrichs-Lee Hamiltonian has a block matrix form defined by[7]

$$(H_g\psi)(\omega) = \begin{pmatrix} 0 & \langle g| \\ g(\omega) & \omega \end{pmatrix} \begin{pmatrix} x \\ \xi(\omega) \end{pmatrix} = \begin{pmatrix} \langle g|\xi\rangle \\ xg(\omega) + \omega\xi(\omega) \end{pmatrix}, \tag{20}$$

on the domain $\mathcal{D}(H_g) = \mathbb{C} \oplus \mathcal{D}(q)$, where $\mathcal{D}(q) = \{\xi \in L^2(\mathbb{R}), q\xi \in L^2(\mathbb{R})\}$ is the domain of the position operator, $(q\xi)(\omega) = \omega\xi(\omega)$.

This Hamiltonian is the restriction to the vacuum/one-particle sector of the quantum field Hamiltonian

$$H_{\mathrm{L}} = \int_{\mathbb{R}} \mathrm{d}\omega\, \omega\, a_\omega^* a_\omega + \int_{\mathbb{R}} \mathrm{d}\omega\, (g(\omega) a_\omega^* + \overline{g(\omega)} a_\omega) \tag{21}$$

on the symmetric Fock space $\mathcal{F}_s(L^2(\mathbb{R}))$, where a_ω and a_ω^* are the bosonic annihilation and creation operators.[22] Indeed, $\mathcal{F}_{n\leq 1} = \mathbb{C} \oplus L^2(\mathbb{R}) = \mathcal{H}_e$, and by noting that $|\,\mathrm{vac}\,\rangle = (1, 0)$ and $a^*(\xi)\,|\mathrm{vac}\rangle = (0, \xi)$, the restriction of (21) to $\mathcal{H}_e$ gives (20). This model, introduced by Lee[18] as a solvable quantum field model for studying the renormalisation problem, describes the decay of an unstable vacuum into the one-particle sector, due to an interaction term with coupling function g. When the coupling becomes flat, on physical ground one expects that the decay will be purely exponential.

While it is tempting to simply put g = constant in Eq. (21), this does not result in a self-adjoint Hamiltonian.[27] Instead, one can show that, given a uniformly bounded sequence of positive coupling functions $(g_n)_{n\in\mathbb{N}} \subset L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, with

$$g_n(\omega) \to 1/\sqrt{2\pi}$$

pointwise as $n \to \infty$, there exists a self-adjoint operator $H_\star$, which is the limit

$$H_{g_n} \to H_\star$$

in the strong-resolvent sense, and thus,[4]

$$\mathrm{e}^{\mathrm{i}\, tH_{g_n}} \to \mathrm{e}^{\mathrm{i}\, tH_\star}, \quad n \to \infty,$$

strongly, for each $t \in \mathbb{R}$.

One can explicitly write the unitary time evolution under the limit Hamiltonian $H_\star$. It is convenient to consider the Fourier transform on the second component of (19), which leads us to

$$\begin{pmatrix} x \\ \xi \end{pmatrix} \mapsto \mathrm{e}^{-\mathrm{i}\, tH_\star} \begin{pmatrix} x \\ \xi \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{-t/2}\, x - \mathrm{i}\, \mathrm{e}^{-t/2} \displaystyle\int_0^t \mathrm{e}^{s/2}\, \xi(s)\, \mathrm{d}s \\ \xi(t+\cdot) - \mathrm{i}\, \chi_{[-t,0]}(\cdot)\, \mathrm{e}^{-(t+\cdot)/2}\, x - \chi_{[-t,0]}(\cdot) \displaystyle\int_0^{t+\cdot} \mathrm{e}^{-(t+\cdot)/2}\, \mathrm{e}^{s/2}\, \xi(s)\, \mathrm{d}s \end{pmatrix},$$

for $t \geq 0$. Here, χ_Ω is the characteristic function of set $\Omega \subset \mathbb{R}$, i.e., $\chi_\Omega(t) = 1$ if $t \in \Omega$ and $= 0$ otherwise. In particular, the vacuum state $|\mathrm{vac}\rangle = (1, 0)$ will exponentially decay into a one-photon state as

$$\mathrm{e}^{-\mathrm{i}\, tH_\star} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{-t/2} \\ -\mathrm{i}\, \chi_{[-t,0]}(\cdot)\, \mathrm{e}^{-(t+\cdot)/2}\, x \end{pmatrix}.$$

This implies that the spectrum of $H_\star$ is the whole real line. The coupling between a single qubit and the singular Friedrichs-Lee model $H_\star$ provides a reasonably realistic model of a quantum system in interaction with a Markovian environment. It is the dilation of the amplitude damping Lindbladian discussed in Ref. 2, Sec. 3, and it exhibits exponential decay. The single qubit is our system $\mathcal{H}_s = \mathbb{C}^2$ introduced in Example 3.2. This way the total Hilbert space is then $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_e$. We can write this out as $\mathcal{H} = \mathbb{C} \oplus L^2(\mathbb{R}) \oplus \mathbb{C} \oplus L^2(\mathbb{R})$. This way the free time evolution U_t, for $t \geq 0$, of the coupled total system can be defined as

$$\begin{pmatrix} x_1 \\ \xi_1 \\ x_2 \\ \xi_2 \end{pmatrix} \mapsto \begin{pmatrix} \mathrm{e}^{-t/2}\, x_1 - \mathrm{i}\, \mathrm{e}^{-t/2} \displaystyle\int_0^t \mathrm{e}^{s/2}\, \xi_2(s)\, \mathrm{d}s \\ \xi_1 \\ x_2 \\ \xi_2(t+\cdot) - \mathrm{i}\, \chi_{[-t,0]}(\cdot)\, \mathrm{e}^{-(t+\cdot)/2}\, x_1 - \chi_{[-t,0]}(\cdot) \displaystyle\int_0^{t+\cdot} \mathrm{e}^{-(t+\cdot)/2}\, \mathrm{e}^{s/2}\, \xi_2(s)\, \mathrm{d}s \end{pmatrix}.$$

From this time evolution, one could now compute the Hamiltonian, following the lines of Ref. 27, Sec. 4.2.2, and show that the conditions in Theorem 3.1 are not fulfilled. But since we want to show that decoupling does not work for this model, we proceed differently and compute the evolution explicitly.

Consider as initial state the vector $(1, 0) \otimes (1, 0)$, i.e.,

$$\begin{pmatrix} x_1 \\ \xi_1 \\ x_2 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

we get the free time evolution from time 0 to t as

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} \mathrm{e}^{-t/2} \\ 0 \\ 0 \\ -\mathrm{i}\, \chi_{[-t,0]}(\cdot)\, \mathrm{e}^{-(t+\cdot)/2} \end{pmatrix}.$$

This shows that the state of the qubit system decays exponentially due to interaction with the environment.

We would now like to show that this still happens when dynamical decoupling is applied. We choose the group generated by the four Pauli matrices as decoupling set V. Following (3), at time 4τ, we find the total perturbed time evolution

$$\begin{pmatrix} x_1 \\ 0 \\ 0 \\ \xi_2 \end{pmatrix} \mapsto \begin{pmatrix} \mathrm{e}^{-\tau} x_1 - \mathrm{i}\,\mathrm{e}^{-\tau} \int_0^{\tau} \mathrm{e}^{s/2}\, \xi_2(s)\, \mathrm{d}s + \mathrm{i}\,\mathrm{e}^{-\tau} \int_{\tau}^{2\tau} \mathrm{e}^{s/2}\, \xi_2(s)\, \mathrm{d}s \\ 0 \\ 0 \\ \xi_2(2\tau + \cdot) - \mathrm{i}\, \chi_{[-2\tau,-\tau]}(\cdot)\, \mathrm{e}^{-(2\tau+\cdot)/2} x_1 + \mathrm{i}\, \chi_{[-\tau,0]}(\cdot)\, \mathrm{e}^{-(2\tau+\cdot)/2} x_1 \\ -\chi_{[-2\tau,-\tau]}(\cdot) \int_0^{2\tau+\cdot} \mathrm{e}^{-(2\tau+\cdot)/2}\, \mathrm{e}^{s/2}\, \xi_2(s)\, \mathrm{d}s \\ +\chi_{[-\tau,0]}(\cdot) \int_0^{\tau} x_1\, \mathrm{e}^{-(2\tau+\cdot)/2}\, \mathrm{e}^{s/2}\, \xi_2(s)\, \mathrm{d}s - \chi_{[-\tau,0]}(\cdot) \int_{\tau}^{2\tau+\cdot} \mathrm{e}^{-(2\tau+\cdot)/2}\, \mathrm{e}^{s/2}\, \xi_2(s)\, \mathrm{d}s \end{pmatrix}.$$

Iterating this cycle now n times, we can compute the perturbed time evolution at time $t = 4\tau n$ from this formula. However, we do not require such a level of generality since we are mainly interested in the evolution of the state $(1, 0, 0, 0)$. A closer inspection under the assumption that the initial state is $(1, 0, 0, 0)$ shows the following: the second and the third component remain 0 at time t; the fourth component evolves to a function with support on $[-2\tau n, 0]$, so on the negative half-axis, so that the term

$$-\mathrm{i}\,\mathrm{e}^{-\tau} \int_0^{\tau} \mathrm{e}^{s/2}\, \xi_2(s)\, \mathrm{d}s + \mathrm{i}\,\mathrm{e}^{-\tau} \int_{\tau}^{2\tau} \mathrm{e}^{s/2}\, \xi_2(s)\, \mathrm{d}s$$

vanishes. Therefore, we get

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} \mathrm{e}^{-t/4} \\ 0 \\ 0 \\ \mathrm{i}\, \phi_{n,t} \end{pmatrix},$$

where

$$\phi_{n,t}(s) = \mathrm{e}^{-(t/2+s)} \sum_{k=1}^{n} \Big(\chi_{[-tk/2n, -tk/2n+t/4n]}(s) - \chi_{[-tk/2n+t/4n, -t(k-1)/4n]}(s) \Big), \tag{22}$$

so we still get exponential decay on the system if the initial state was $(1, 0, 0, 0)$. We are interested in the limit $n \to \infty$ with $\tau \to 0$, such that $t = 4\tau n$ remains fixed. Then we should get

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} \mathrm{e}^{-t/4} \\ 0 \\ 0 \\ \phi_t \end{pmatrix}, \tag{23}$$

with some function $s \mapsto \phi_t(s)$, but it turns out impossible to obtain the limit ϕ_t because (22) does not converge as $n \to \infty$. See Fig. 3.

Notice, however, that $\phi_{n,t}$ converges weakly to zero, that is, $\langle f \mid \phi_{n,t} \rangle \to 0$ as $n \to \infty$ for all $f \in L^2(\mathbb{R})$. Physically, one can interpret the behaviour of the wave function $\phi_{n,t}$ as the result of pumping larger and larger energy in the system through the decoupling pulses. In the limit $n \to \infty$, the pumped energy becomes infinite and $\phi_{n,t}$ gets orthogonal to any given wave function.

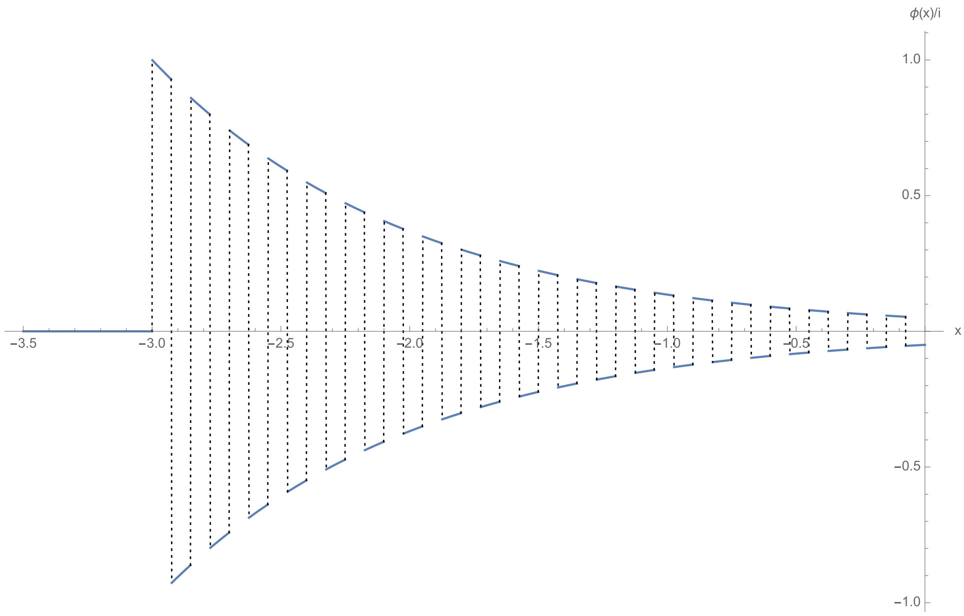

FIG. 3. Plot of the function $\phi_{n,t}(s)$ versus s. We set $t = 6$ and $n = 20$.

In any case, if dynamical decoupling worked, then there would be a self-adjoint B, such that the time evolution of $(1, 0, 0, 0) = (1, 0) \otimes (1, 0)$ is given by

$$\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} \mapsto \begin{pmatrix}1\\0\end{pmatrix} \otimes \mathrm{e}^{\mathrm{i}\,tB} \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \otimes v(t),$$

with some $v(t) \in \mathcal{H}_e$ of norm 1; this in turn would be a vector of the form

$$\begin{pmatrix}*\\ *\\ 0\\ 0\end{pmatrix}.$$

Given that the first component was $e^{-t/4}$, we see that the second should be nonzero, which contradicts (23), where the second component is 0. Therefore, there cannot be such B, and thus, dynamical decoupling does not work for this model.

VI. CONCLUSIONS

We have provided criteria and examples for dynamical decoupling of unbounded Hamiltonians. From a mathematical perspective, the ability to decouple is essentially a question of approximate commutativity and operator domains. From a physical perspective, it is a question of interaction time scales, but we saw that such time scales cannot be revealed by looking at the reduced dynamics only. Moreover, even if the complete model is known and can be decoupled, such time scales are very hard to compute in practice, because they depend explicitly on the environment initial state.

In practice, our results imply that many more systems can be protected from environmental noise than previously thought. In particular, seeing exponential decay in the laboratory should not stop one from trying to apply decoupling. Whether or not it works on a feasible time scale can be decided experimentally.

This paper is probably the first to discuss dynamical decoupling of unbounded Hamiltonians in a systematic way. We saw that the question of whether decoupling works is a very hard and delicate one, and we believe a precise characterization, e.g., an affirmative answer to Problem 4.5, is still a long way off; yet we provided some new and very useful methods to start with.

Apart from the physical motivation, our work also offers a refreshed view on the mathematics of Trotter product limits. On the one hand, established results on convergence can be embedded into a dynamical decoupling model through the construction (3.3) and get a physical meaning. On the other hand, a proof of long-standing conjectures such as the generalization of Ref. 15 to more than two generators would be highly relevant in dynamical decoupling.

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