# Generic aspects of the resource theory of quantum coherence

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The class of incoherent operations induces a preorder on the set of quantum pure states, defined by the possibility of converting one state into the other by transformations within the class. We prove that if two $n$-dimensional pure states are chosen independently according to the natural uniform distribution, then the probability that they are comparable vanishes as $n \to \infty$. We also study the maximal success probability of incoherent conversions and find an explicit formula for its large-$n$ asymptotic distribution. Our analysis is based on the observation that the extreme values (largest and smallest components) of a random point uniformly sampled from the unit simplex are distributed asymptotically as certain explicit homogeneous Markov chains.

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## I. INTRODUCTION

The use of probability models and integral geometry to explain generic aspects of quantum states is now a well-established point of view and there are multiple lessons to learn from this approach [1–9]. The next logical step is to use the same set of probabilistic ideas to describe generic aspects of quantum resource theories. However, the difficulties in describing the generic or typical aspects of resource theories using probabilistic models remain considerable.

Perhaps the first question of this flavor arose in the resource theory of entanglement. After exposing a precise connection between the algebraic notion of majorization [10] and convertibility among pure states by local operations and classical communications (LOCC), Nielsen [11] made the remark that the set of pure states of a bipartite system has a natural order relation, induced by the majorization relation on their local spectra. This relation is *not* total: not all quantum states can be compared, i.e., connected by a LOCC transformation. Nielsen conjectured that for a bipartite system, the set of pairs of pure states that are LOCC convertible has relative volume asymptotically zero in the limit of large dimension: most quantum pure states are *incomparable*! He offered a probabilistic argument to justify the conjecture; shortly after, another heuristic explanation based on integral geometry was put forward in Ref. [12].

As far as we know, the only rigorous result around this question is the proof that for an infinite-dimensional system, the set of pairs of LOCC-convertible pure states is nowhere dense [13] (this statement though does not imply the conjecture). Recently, we have made some progress on Nielsen's conjecture by an extensive numerical analysis of the volume of LOCC-convertible pairs of pure states [14]. The results support the conjecture, provide some nontrivial quantitative measure of the volume of LOCC-convertible states, and suggest valuable connections with random matrix theory.

It is natural to ask whether the property of generic incomparability is a general feature shared by other quantum resource theories. Our attempt to answer this question starts from investigating this circle of ideas for the class of *quantum incoherent operations* (IOs) [15–24]. This choice is not arbitrary: the resource theory of coherence is sufficiently simple to be tractable and yet shares the connection with the algebraic notion of majorization that appears in the most interesting resource theories [25,26].

It is the purpose of this paper to present a complete analysis for the resource theory of coherence and indicate what might in the future be extended to other resource theories.

The structure of the paper is as follows. In Sec. II, we recall the definitions of incoherent, strictly incoherent and dephasing covariant incoherent operations, and the connection between incoherent convertibility and the majorization relation. In Sec. III, we present the distributions of the smallest and largest components of random pure quantum states; these are the main probabilistic properties relevant to our analysis. Section IV contains the main results: the set of comparable states in the resource theory of coherence has volume zero in the limit of large dimension $n \to \infty$ (this is the analog of Nielsen's conjecture in the theory of entanglement); this problem is related to the persistence probability of a non-Markovian random walk and we give numerical estimates on the rate of decay to zero; in the limit $n \to \infty$, the maximal success probability of incoherent conversion between two random independent pure states has a nontrivial asymptotic distribution that we characterize completely. We conclude the paper with some final remarks in Sec. V.

 

## II. RESOURCE THEORIES OF COHERENCE

Recall that a *resource theory* is defined by (i) a set of *free states* and (ii) a class of *free* or *allowed operations*. In this paper, we consider the resource theories of coherence introduced and studied by Åberg [27], Baumgratz *et al.* [15], Winter and Yang [18], and Chitambar and Gour [19,20].

### A. Free states and free operations

We denote by $\mathscr{H}_n$ a complex Hilbert space of dimension $n$, and by $\mathcal{S}_n$ the corresponding set of states $\rho$ (density matrices: $\rho \geqslant 0$, $\operatorname{tr}\rho = 1$). Fix a basis $\{|i\rangle\}_{i=1}^n$ in $\mathscr{H}_n$ to be called the *incoherent basis*. The choice may be dictated by physical considerations (for example, the eigenbasis of a particular observable).

The set of free states in the resource theory of coherence is the set of *incoherent states* $\mathcal{I}_n \subset \mathcal{S}_n$ defined as

$$\mathcal{I}_n := \left\{ \rho \in \mathcal{S}_n : \rho = \sum_{i=1}^n p_i |i\rangle\langle i| \right\}, \tag{1}$$

i.e., density matrices which are diagonal in the incoherent basis. Notice that $\mathcal{I}_n$ is the image of $\mathcal{S}_n$ under the *decohering map*, i.e., $\mathcal{I}_n = \mathcal{D}(\mathcal{S}_n)$, where

$$\mathcal{D}(\rho) := \sum_{i=1}^n \langle i|\rho|i\rangle |i\rangle\langle i|. \tag{2}$$

The specification of the free states alone does not completely determine a resource theory. It is indeed necessary to specify a class of free operations. For the resource theory of coherence, a number of different alternatives has been proposed, each yielding a different resource theory (see, e.g., Refs. [19,20]). Here we focus on three possible choices of free operations that allow for a criterion for convertibility between pure states in terms of the majorization relation.

Recall that any quantum channel that is a completely positive and trace preserving (CPTP) map $\mathcal{E} : \mathcal{S}_n \to \mathcal{S}_n$ can be characterized in terms of a Kraus representation,

$$\mathcal{E}(\rho) = \sum_\alpha \mathcal{K}_\alpha(\rho) = \sum_\alpha K_\alpha \rho K_\alpha^\dagger, \tag{3}$$

where $\mathcal{K}_\alpha(\cdot) = K_\alpha(\cdot)K_\alpha^\dagger$, and $\{K_\alpha\}$ is a set of (nonuniquely determined) operators on $\mathscr{H}_n$, with $\sum_\alpha K_\alpha^\dagger K_\alpha = \mathbb{I}$. We can then define three classes of CPTP maps on $\mathcal{S}_n$, representing three possible choices of free operations.

*Definition 1.* A quantum channel $\mathcal{E}$ is said to be an IO if it can be represented by Kraus operators $\{K_\alpha\}$ such that

$$\mathcal{D}(\mathcal{K}_\alpha(|i\rangle\langle i|)) = \mathcal{K}_\alpha(|i\rangle\langle i|) \tag{4}$$

for all $\alpha$, and for all the elements $|i\rangle$ of the incoherent basis.

Note that if $\mathcal{E}$ is an IO,

$$\rho \in \mathcal{I}_n \Rightarrow \mathcal{K}_\alpha(\rho) \in \mathbb{R}_+\mathcal{I}_n, \quad \text{for all } \alpha. \tag{5}$$

This restriction guarantees that, even if one has access to individual measurement outcomes $\alpha$ of the instrument $\{K_\alpha\}$, one cannot generate coherent states starting from an incoherent one. Notice that Eq. (4) can be interpreted as a requirement of commutation between the decohering operation $\mathcal{D}$ and the operation $\mathcal{K}_\alpha$ when acting on the set of incoherent states $\mathcal{I}_n$. One can also further restrict the allowed operations by requiring the validity of such commutativity on the whole set of states $\mathcal{S}_n$.

*Definition 2.* A quantum channel $\mathcal{E}$ is said to be a *strictly incoherent operation* (SIO) if it can be represented by Kraus operators $\{K_\alpha\}$ such that

$$\mathcal{D}(\mathcal{K}_\alpha(\rho)) = \mathcal{K}_\alpha(\mathcal{D}(\rho)) \tag{6}$$

for all $\alpha$, and for all $\rho \in \mathcal{S}_n$.

One can also consider a third class of IOs that satisfy the commutativity relation with $\mathcal{D}$ at a global level rather than at the level of Kraus operator representations.

*Definition 3.* A quantum channel $\mathcal{E}$ is said to be a *dephasing covariant incoherent operation* (DIO) if

$$\mathcal{D}(\mathcal{E}(\rho)) = \mathcal{E}(\mathcal{D}(\rho)) \tag{7}$$

for all $\rho \in \mathcal{S}_n$.

It is clear that SIO $\subsetneq$ IO and SIO $\subsetneq$ DIO, while the classes IO and DIO are incomparable [19,20]. It has been shown that transformations between *pure states* (i.e., rank-one projections $\psi = |\psi\rangle\langle\psi|$, with $\langle\psi|\psi\rangle = 1$) are fully governed by the same majorization criteria [17,21,24]. Therefore, although the three classes IO, SIO, and DIO are different from each other, they are operationally equivalent as far as pure-to-pure state transformations are concerned. We also mention that all these classes are subclasses of the *maximally incoherent operations* (MIOs), which is the largest possible class of operations not generating coherent states starting from incoherent ones [27]. However, our results do not apply to this class, since (as far as we know) pure state conversions under MIOs are not characterized by a majorization relation.

### B. Convertibility criterion and majorization relation

First, we need to introduce some notation. In this paper, $\Delta_{n-1}$ is the unit simplex, i.e., the set of $n$-dimensional probability vectors. For a vector $x$, we denote by $x^\downarrow$ the decreasing rearrangement of $x$, with $x_j^\downarrow \geqslant x_k^\downarrow$ for $j < k$. If $x, y$ are two vectors, we say that $x$ is *majorized* by $y$ — and write $x \prec y$ — if

$$\sum_{j=1}^k x_j^\downarrow \leqslant \sum_{j=1}^k y_j^\downarrow \tag{8}$$

for all $k = 1, \ldots, n$. For pure states $\psi = |\psi\rangle\langle\psi| \in \mathcal{S}_n$, we write

$$\delta(\psi) := (|\psi_1|^2, \ldots, |\psi_n|^2) \in \Delta_{n-1}, \tag{9}$$

where $\psi_j = \langle j|\psi\rangle$, i.e., the diagonal of the density matrix $\psi$, in the (fixed) incoherent basis.

The following results expose the connection between the resource theories of coherence and the majorization relation.

*Theorem 1* ([17,21,24]). A pure state $\psi$ can be transformed into a pure state $\psi'$ under IO, SIO, or DIO if and only if $\delta(\psi) \prec \delta(\psi')$.

This theorem allows us to endow the set of pure states on $\mathscr{H}_n$ with a natural preorder relation: we will write $\psi \prec \psi'$ whenever $\delta(\psi) \prec \delta(\psi')$.

*Theorem 2* ([21]).For two pure states $\psi$ and $\psi'$, the maximal conversion probability under IO is given by

$$\Pi(\delta(\psi), \delta(\psi')), \tag{10}$$

with $\Pi(\cdot,\cdot)$ being defined on pairs of probability vectors with $n$ nonzero components as

$$\Pi(x, y) = \min_{1\leqslant k\leqslant n} \frac{\sum_{j=k}^{n} x_j^{\downarrow}}{\sum_{j=k}^{n} y_j^{\downarrow}}. \tag{11}$$

Theorem 2 is the IO counterpart of an analogous result obtained by Vidal [28] for LOCC conversions. Note that $\Pi(\delta(\psi), \delta(\psi')) = 1$ if and only if $\psi \prec \psi'$. The theorem still holds if the class IO is replaced by SIO or DIO as a consequence of Theorem 1.

Summing up, the three classes of IOs considered are equivalent for manipulation of pure states, and they are all governed by majorization relations. In this paper, we only consider pure state transformations. For simplicity, we will always refer to the class IO, but all the results also hold for SIO and DIO.

## III. RANDOM PURE STATES

Let $\psi$ be a random pure state in $\mathcal{S}_n$ distributed according to the unitarily invariant measure. In the incoherent basis $\{|i\rangle\}$,

$$\psi = \sum_{ij} \psi_i \overline{\psi_j} |i\rangle\langle j|,$$

where $(\psi_1, \psi_2, \dots, \psi_n)$ is uniformly distributed in the $n$-dimensional complex unit sphere, $\sum_j |\psi_j|^2 = 1$.

Hence, the random vector $\delta(\psi) = (|\psi_1|^2, |\psi_2|^2, \dots, |\psi_n|^2)$ is *uniformly distributed* in the simplex $\Delta_{n-1}$:

$$p_{\delta(\psi)}(x) = (n-1)! 1_{x\in\Delta_{n-1}}. \tag{12}$$

If $\mu$ is a uniform point in $\Delta_{n-1}$, i.e., distributed according to (12), then the component $\mu_k$ is typically $O(1/n)$. The extreme components lie instead on very different scales. The largest components $\mu_1^{\downarrow}, \mu_2^{\downarrow}, \dots$ are of size $\ln n/n$ with fluctuations of $O(1/n)$; the smallest components $\mu_n^{\downarrow}, \mu_{n-1}^{\downarrow}, \dots$ are on the much smaller scale $1/n^2$, with fluctuations of $O(1/n^2)$. We now give a precise asymptotic descriptions of the extreme statistics of the uniform distribution on $\Delta_{n-1}$: they are distributed as time-homogenous Markov chains with explicit (and remarkably simple) transition densities. We must say that the uniform distribution on the simplex is one of the favorite topics in integral geometry [29] and its relevance in quantum applications has been already highlighted in the past [1,30]. The following result is probably folklore but we could not trace it in the literature. We report it here since it is crucial for the next analysis. The proof is given in Appendix B.

*Proposition 1.* Let $\mu = (\mu_1, \mu_2, \dots, \mu_n)$ be a uniform point in $\Delta_{n-1}$. Denote by $\mu^{\downarrow}$ the decreasing rearrangement of $\mu$. Then, for any fixed integer $k \geqslant 1$, the following hold as $n \to \infty$:

(i) The rescaled vector of the smallest components $(n^2\mu_{n-j+1}^{\downarrow})_{1\leqslant j\leqslant k}$ converges in distribution to $(V_1, V_2, \dots, V_k)$, where $(V_j)_{j\geqslant 1}$ is a Markov chain with initial and transition densities given by

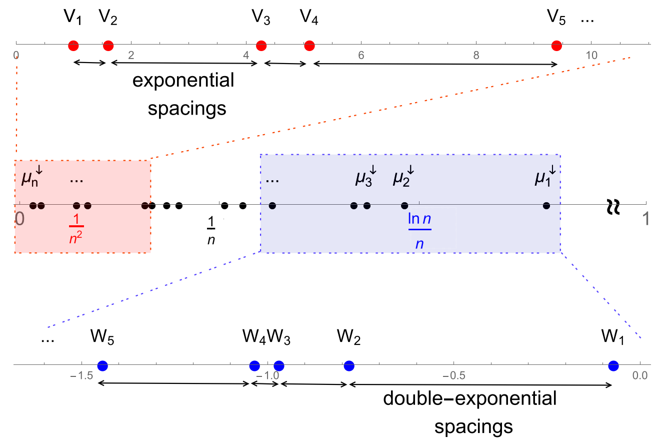


FIG. 1. Middle: The $n$ components of a uniform point $\mu$ in $\Delta_{n-1}$ lie in the unit interval [0,1]. Top: For large $n$, the smallest components $\mu_n^{\downarrow}, \mu_{n-1}^{\downarrow}, \dots, \mu_{n-k+1}^{\downarrow}$ after a proper rescaling behave statistically as the first $k$ points of a Poisson process $(V_j)_{j\geqslant1}$ with exponential spacings. Bottom: The largest components $\mu_1^{\downarrow}, \mu_2^{\downarrow}, \dots, \mu_k^{\downarrow}$ after a proper rescaling behave statistically as the first $k$ points of a Poisson process $(W_j)_{j\geqslant1}$ with double-exponential (or Gumbel) spacings.

$$f_{V_1}(v) = \exp(-v)1_{v\geqslant 0},$$
$$f_{V_{j+1}|V_j}(u|v) = \exp(v-u)1_{u\geqslant v}. \tag{13}$$

(ii) The rescaled vector of the largest components $(n\mu_j^{\downarrow} - \ln n)_{1\leqslant j\leqslant k}$ converges in distribution to $(W_1, W_2, \dots, W_k)$, where $(W_j)_{j\geqslant 1}$ is a Markov chain with initial and transition densities given by

$$f_{W_1}(w) = \exp(-\mathrm{e}^{-w} - w),$$
$$f_{W_{j+1}|W_j}(u|w) = \exp(\mathrm{e}^{-w} - \mathrm{e}^{-u} - u)1_{u\leqslant w}. \tag{14}$$

Note that, by the Markov property, we can write the joint density of $(V_1, \dots, V_k)$,

$$f_{V_1,\dots,V_k}(v_1, \dots, v_k) = \exp(-v_k)1_{0\leqslant v_1\leqslant v_2\leqslant\cdots\leqslant v_k}, \tag{15}$$

and the joint density of $(W_1, \dots, W_k)$,

$$f_{W_1,\dots,W_k}(w_1, \dots, w_k)$$
$$= \exp(-w_1 - w_2 - \cdots - w_k - \mathrm{e}^{-w_k})1_{w_1\geqslant w_2\geqslant\cdots\geqslant w_k}. \tag{16}$$

The next lemma gives a concrete realization of the Markov chains $(V_j)_{j\geqslant1}$ and $(W_j)_{j\geqslant1}$ in terms of discrete-time continuous random walks. In the following, $(X_j)_{j\geqslant1}$ is a sequence of independent exponential random variables with rate 1, i.e., $P(X_j \leqslant x) = 1 - \exp(-x)$.

*Lemma 1.* Let $(V_j)_{j\geqslant1}$ and $(W_j)_{j\geqslant1}$ be the Markov chains defined in (13) and (14), respectively. Then,

$$(V_j)_{j\geqslant1} \overset{D}{=} (X_1 + \cdots + X_j)_{j\geqslant1}, \tag{17}$$

$$(W_j)_{j\geqslant1} \overset{D}{=} (-\ln(X_1 + \cdots + X_j))_{j\geqslant1}, \tag{18}$$

where $\overset{D}{=}$ denotes equality in distribution.

See Fig. 1 for a pictorial illustration of Proposition 1 and Lemma 1.

## IV. VOLUME OF THE SET OF IO-CONVERTIBLE STATES

In 1999, Nielsen [11] conjectured that the relative volume of pairs of LOCC-convertible bipartite pure states vanishes in the limit of large dimensions. The precise statement of the conjecture is that for two independent random points in the simplex with a distribution of random matrix type (see Sec. V B below), the probability that they are in majorization relation is asymptotically zero. Here we pose a similar question in the theory of coherence: Is it true that most pairs of pure $n$-dimensional quantum states are *not* IO convertible if $n$ is large? The answer is yes.

### A. Asymptotics $n \to \infty$

*Theorem 3.* Let $\psi$ and $\psi'$ be independent random pure states in $\mathcal{S}_n$. Then,

$$\lim_{n\to\infty} P(\psi \prec \psi') = 0.$$

*Proof.* We use the shorter notation $\mu := \delta(\psi)$ and $\mu' := \delta(\psi')$. It turns out to be convenient to write the majorization relation $\mu \prec \mu'$ as

$$\sum_{i=n-j+1}^{n} \mu_i^{\downarrow} \geqslant \sum_{i=n-j+1}^{n} \mu_i'^{\downarrow}, \text{ for all } j = 1,\dots,n \tag{19}$$

by using the normalization condition $\sum_{i=1}^n \mu_i^{\downarrow} = \sum_{i=1}^n \mu_i'^{\downarrow}$. The idea of the proof, inspired by Ref. [31], is to show that

$$\lim_{k\to\infty}\lim_{n\to\infty} P[\mu, \mu' \text{ meet the first } k \text{ conditions in (19)}] = 0.$$

From Proposition 1,

$$(\mu_n^{\downarrow}, \mu_{n-1}^{\downarrow}, \dots, \mu_{n-k+1}^{\downarrow}) \text{ is asymptotic to } \left(n^{-2}V_j\right)_{1\leqslant j\leqslant k},$$

as $n \to \infty$. By Lemma 1, we have the representation $(V_j)_{j\geqslant 1} \overset{D}{=} (X_1 + \dots + X_j)_{j\geqslant 1}$. Analogous representation holds for the $k$ smallest components of $\mu'$ with their own sequence $(V_j')_{j\geqslant 1} \overset{D}{=} (X_1' + \dots + X_j')_{j\geqslant 1}$. Consider the probabilities ($1 \leqslant k \leqslant n$):

$$\pi_{n,k} := P\left(\sum_{i=n-j+1}^{n} \mu_i^{\downarrow} \geqslant \sum_{i=n-j+1}^{n} \mu_i'^{\downarrow}, \text{ for all } 1 \leqslant j \leqslant k\right).$$

Of course,

$$\pi_{n,n} = P(\mu \prec \mu') \quad \text{and} \quad \pi_{n,n} \leqslant \pi_{n,k}.$$

By taking the limit $n \to \infty$, we get

$$p_k := \lim_{n\to\infty} \pi_{n,k} = P\left(\sum_{i=1}^{j} V_i \geqslant \sum_{i=1}^{j} V_i', \text{ for all } 1 \leqslant j \leqslant k\right).$$

It is clear that

$$0 \leqslant \limsup_n \pi_{n,n} \leqslant \lim_{k\to\infty} p_k.$$

Hence, to prove that $\pi_{n,n} \to 0$ as $n \to \infty$, it is enough to show that $p_k \to 0$ as $k \to \infty$. The sequence $\widetilde{V}_k := (V_k - V_k') = \sum_{j=1}^k \widetilde{X}_j$, $k \geqslant 1$ is a time-discrete continuous random walk with independent steps $\widetilde{X}_j := (X_j - X_j')$ distributed according to the two-side exponential density $(1/2)\exp(-|x|)$. The process $I_k := \sum_{j=1}^k \widetilde{V}_j$, $k \geqslant 1$ is the corresponding *integrated random walk* (IRW). Hence, $p_k$ is the so-called *persistence probability* above zero of the IRW,

$$p_k = P\left(\min_{1\leqslant j\leqslant k} I_j \geqslant 0\right). \tag{20}$$

The proof that the persistence probability of the IRW asymptotically vanishes,

$$p_k \to 0, \quad \text{as } k \to \infty, \tag{21}$$

follows from the Lindeberg-Feller central limit theorem and the Kolmogorov 0-1 law, and is given in Appendix C. ■

It might seem that, having to deal with independent and identically distributed (i.i.d.) variables $\widetilde{X}_j$, the proof that $p_k \to 0$ is straightforward. Note however, that the IRW $(I_k)_{k\geqslant 1}$ is not Markov ($I_k$ depends on all variables $\widetilde{X}_j$, $j \leqslant k$) and this explains why some analysis is required.

We should also mention that (21) is a direct consequence of several persistence results for IRW [32–34], i.e., asymptotic estimates for the sequence $p_k$ in (20). For our purposes however, we do not really need the precision of those asymptotic statements and this is the reason for including in Appendix C a proof of (21) based on elementary probability.

The proof strategy in Theorem 3 is based on bounding $\pi_{n,n}$ by a sequence $p_k$ independent on $n$, and therefore gives no information on the rate of decay of $\pi_{n,n}$ to zero. Some insights can be obtained from the perspective of persistence probabilities as discussed in the next section.

### B. Majorization, persistence probabilities and the arcsine law

We next turn our attention to the convergence rate of $P(\psi \prec \psi')$ to 0. For two random pure states $\psi, \psi'$ in $\mathcal{S}_n$, the vector $\tilde{\delta} = (\tilde{\delta}_k)_{1\leqslant k\leqslant n}$ with

$$\tilde{\delta}_k = \delta(\psi')_k^{\downarrow} - \delta(\psi)_k^{\downarrow} \tag{22}$$

defines a continuous random walk $(S_k)_{0\leqslant k\leqslant n}$, started at $S_0 := 0$, with steps $\tilde{\delta}_k$'s,

$$S_k := \sum_{j=1}^{k} \tilde{\delta}_j \qquad 1 \leqslant k \leqslant n. \tag{23}$$

Note that $S_n = 0$ (the process is a random bridge). The majorization condition can be interpreted as the persistence (above zero) of the random walk. Hence,

$$P(\psi \prec \psi') = P\left(\min_{1\leqslant k\leqslant n} S_k \geqslant 0\right).$$

Persistence probabilities for random processes have been widely studied in statistical physics and probability. Certain exactly solvable models (that include symmetric random walks, classical random bridge, IRWs, etc.), and numerical study of many other models, showed that in the general case the persistence probability above zero decays *algebraically* as $bn^{-\theta}$, for large $n$. The so-called *persistence exponents* $\theta$ of a process is typically very difficult to compute explicitly if the process is not Markov, although $\theta$ is believed to be distribution free within a universality class.

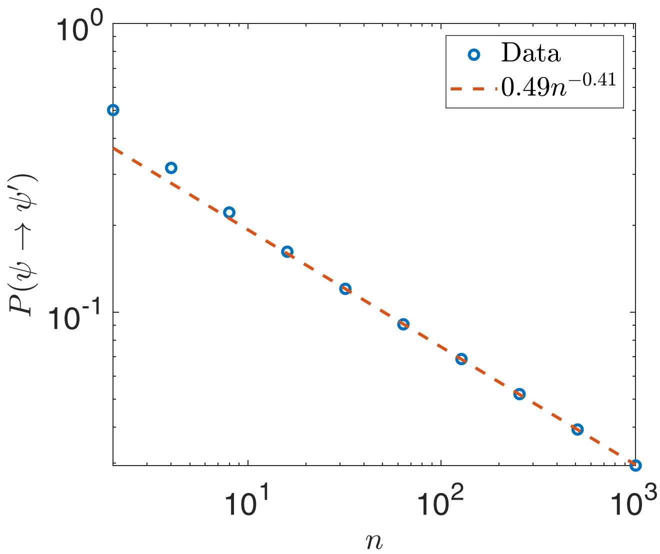


FIG. 2. The probability of incoherent convertibility $P(\psi \prec \psi')$ for $\psi$ and $\psi'$ independently chosen from the uniform distribution on pure states of $\mathcal{S}_n$. The fit (dotted line) confirms an algebraic decay (24) with exponent $\theta = 0.4052$. Here $n = 2, 4, 8, \dots, 1024$.

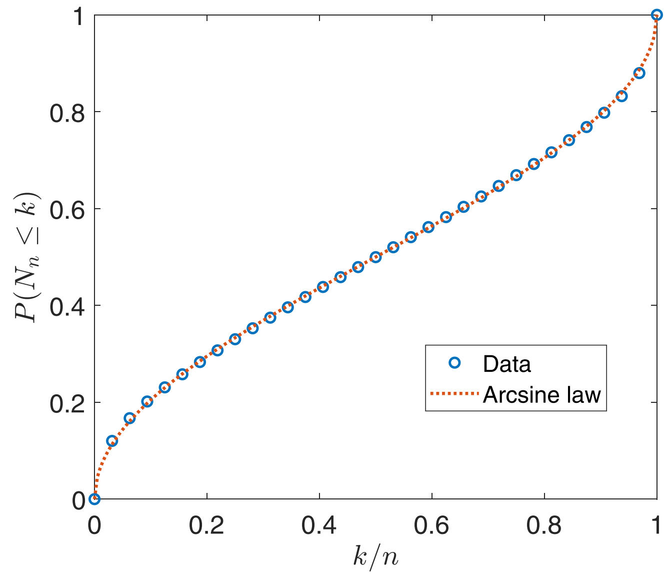


FIG. 3. Probability distribution of $N_n$, the time spent by the process $(S_k)$ above zero, compared with the arcsine law. Here $n = 32$.

It is therefore natural to expect that $P(\psi \prec \psi')$ also decays to zero as a power of $n$. In Fig. 2 we show the results of numerical estimates of $P(\psi \prec \psi')$ (obtained from $10^6$ realisations of $\psi, \psi'$) for increasing values of $n$. The plot in logarithmic scale shows quite convincing evidence that

$$P(\psi \prec \psi') \sim bn^{-\theta}. \tag{24}$$

The value of the persistence exponent obtained from a numerical fit is $\theta = 0.4052$, with $\sigma_\theta = 0.0028$.

Note that the process $(S_k)_{0 \leqslant k \leqslant n}$ is not Markov, and quite different from most familiar discrete-time random processes (the steps $\tilde{\delta}_k$'s in (22) are neither i.i.d, as in a classical random walk, nor distribution invariant under permutations, as in a classical random bridge). One can nevertheless try to compute certain statistics related to the persistence of $S_k$, for instance, the *time spent above zero*. Denote this time by $N_n := \#\{k \leqslant n : S_k \geqslant 0\}$. For classical symmetric random walks, the statistics of $N_n$ is universal, and its limit is the well-known *arcsine law* [35]. Surprisingly, numerical results (see Fig. 3) show that the fraction of time spent above 0 by $S_k$ is also asymptotically described by the arcsine law:

$$\lim_{n\to\infty} P\left(\frac{N_n}{n} \leqslant t\right) = \frac{2}{\pi} \arcsin\left(\sqrt{t}\right). \tag{25}$$

### C. Limit distribution of the maximal success probability of IO-conversion

The maximal success probability of IO conversion of state $\psi$ into $\psi'$ is $\Pi(\mu, \mu')$, where $\mu = \delta(\psi)$, $\mu' = \delta(\psi')$ are the diagonal entries of $\psi$, $\psi'$ (Theorem 2). Theorem 3 can be rephrased as the statement that if $\mu, \mu'$ are independent uniform points in $\Delta_{n-1}$, then

$$P(\Pi(\mu, \mu') = 1) \to 0, \quad \text{as } n \to \infty.$$

In our previous work [14] on the LOCC convertibility for random states, we conjectured a connection between the asymptotic fluctuations of the smallest component of random probability vectors $\lambda, \lambda'$ and the scaling limit of the variable $\Pi(\lambda, \lambda')$, when $\lambda$, $\lambda'$ are independent spectra of fixed-trace Wishart random matrices. Translated in this setting, the precise statement is that, if for some scaling constants $a_n, b_n$, the variable

$$a_n \frac{\mu_n^\downarrow}{\mathbb{E}[\mu_n^\downarrow]} + b_n \tag{26}$$

has a nontrivial limit in distribution, then, with the same constants,

$$P(a_n \Pi(\mu, \mu') + b_n \leqslant p) \tag{27}$$

has a nontrivial limit as $n \to \infty$.

The smallest component $\mu_n^\downarrow = \delta(\psi)_n^\downarrow$ has probability density

$$p_n(x) = n^2(1 - nx)^{n-1} 1_{0 \leqslant x \leqslant 1/n}.$$

The average and variance of $\mu_n^\downarrow$ are

$$\mathbb{E}[\mu_n^\downarrow] = \frac{1}{n(n+1)}, \quad \text{var}[\mu_n^\downarrow] = \frac{1}{n(n+1)^2(n+2)}.$$

Hence, the fluctuations of $\mu_n^\downarrow$ relative to the mean are asymptotically bounded, $\text{var}[\mu_n^\downarrow]^{\frac{1}{2}}/\mathbb{E}[\mu_n^\downarrow] = O(1)$, and therefore we can take $a_n = 1$ and $b_n = 0$ in (26). The conjectural statement (27) in this case says that the distribution function

$$F_n(p) = P(\Pi(\mu, \mu') \leqslant p) \tag{28}$$

should have a scaling limit. Indeed, we found numerically that, for large $n$, the function $F_n(p)$ tends to a limit distribution, as shown in Fig. 4.

Here we push further our previous conjecture and we propose that as $n \to \infty$, the distribution of the random variable $\Pi(\mu, \mu')$ is determined by the asymptotic behavior of the smallest components of $\mu$, $\mu'$ only. Any fixed block of the order statistics $n^2\mu_{n-j+1}^\downarrow$, $j = 1, \dots, k$ is asymptotic to the first $k$ components of a Poisson process $(V_j)_{j\geqslant 1}$; similarly, $n^2\mu_{n-j+1}'^\downarrow$ is asymptotic to its own independent copy

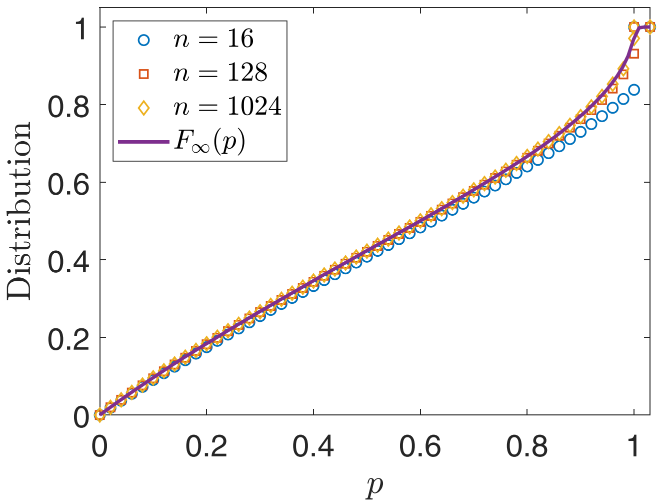


FIG. 4. Distribution $F_n(p)$ for various values of $n$ versus the limit distribution $F_\infty(p)$ (distribution functions computed from numerical simulations).

$(V'_j)_{j\geqslant 1}$. Hence, we conjecture that $\Pi(\mu,\mu')$ is asymptotically distributed as

$$\Pi^\infty(V,V') := \inf_{k\geqslant 1} \frac{\sum_{j=1}^k V_j}{\sum_{j=1}^k V'_j}, \tag{29}$$

where $(V_j)_{j\geqslant 1}$ and $(V'_j)_{j\geqslant 1}$ are two independent copies of a Poisson process with rate 1 (i.e., point processes with independent exponential spacings). In formulas, if we denote by $F_\infty(p) := P(\Pi^\infty(V,V') \leqslant p)$ the distribution function of $\Pi^\infty(V,V')$, we claim that

$$\lim_{n\to\infty} F_n(p) = F_\infty(p). \tag{30}$$

We report in Fig. 4 the result of numerical simulations of $\Pi(\mu,\mu')$ and $\Pi^\infty(V,V')$ on samples of $5\times 10^5$ pairs of random probability vectors $\mu$, $\mu'$ in $\Delta_{n-1}$, and pairs of random processes $V$ and $V'$. The agreement between the corresponding distributions $F_n(p)$ for large $n$, and $F_\infty(p)$ is quite convincing of the correctness of (29) and (30).

## V. CONCLUDING REMARKS

### A. Likelihood of comparability in algebraic combinatorics

In this paper, we proved that the probability that two independent random points uniformly distributed in the unit simplex are in the majorization relation is asymptotically zero as $n\to\infty$. A similar question in the discrete setting was posed back in 1979 by Macdonald [36, Ch.1.1, Ex.18]: For two integer partitions of $n$, chosen uniformly at random, and independently, is it true that the probability that they are in majorization relation (a.k.a. *dominance order*) is zero as $n\to\infty$? In 1999, Pittel [31] proved the positive answer to Macdonald's question. In the proof of Theorem 3 we emulated the main ideas exposed in Ref. [31]. We mention, however, a simplification that occurs in the continuous setting. Pittel considered the first $k$ conditions for majorization involving the $k$ largest components of random integer partitions. They are asymptotic to Markov chains $(W_j)_{j\geqslant 1}$ of Lemma 1 with double-exponential spacings. In the continuous setting of random points in the simplex, it is fairly easy to obtain the asymptotics of the smallest components. Hence, here we considered instead the first $k$ conditions (19) involving the smallest components of $\mu^\downarrow$, and this reduces the problem to the persistence probability of an IRW $I_k = \sum_{j=1}^k \sum_{i=1}^j \widetilde{X}_i$, where the increments $\widetilde{X}_i$ have two-sided exponential distribution. This choice makes the analysis simpler compared to the discrete setting for integer partitions.

### B. Entanglement theory and LOCC convertibility

There is a difference between the statement proved in Theorem 3 for the theory of coherence and Nielsen's conjecture for entanglement theory. The LOCC-convertibility criterion for bipartite systems is the majorization relation for spectra of reduced density matrices. In the random setting, those spectra are not uniformly distributed in the simplex; instead, they follow a random-matrix-type density in $\Delta_{n-1}$,

$$p_{\mathrm{RMT}}(x) = c_{n,m} \prod_{1\leqslant i<j\leqslant n} (x_i - x_j)^2 \prod_{k=1}^n x_k^{m-n} 1_{x\in\Delta_{n-1}},$$

where $n$ and $m$ are the dimensions of the subsystems (see Ref. [14, Eq. (30)]). In particular, for a point distributed according to $p_{\mathrm{RMT}}$, it is no longer true that the largest/smallest components are asymptotically described by point processes with independent spacings (their statistics is given instead by scaling limits at the edges of random matrices, known as Airy and Bessel point processes), and this complicates considerably the analysis.

We also note that the proof that $P(\mu \prec \mu') \to 0$ as $n\to\infty$ presented here can be adapted to the case where $\mu$, $\mu'$ are independent copies of points in the unit simplex picked according to a more general Dirichlet distribution:

$$p_{\mathrm{Dir}(\alpha)}(x) = \frac{\Gamma(n\alpha)}{\Gamma(\alpha)^n} \prod_{k=1}^n x_k^{\alpha-1} 1_{x\in\Delta_{n-1}}.$$

It would be interesting to see if this distribution appears naturally in the theory of random quantum states.

### C. Other resource theories

Majorization criteria also play an important role in other resource theories, such as the resource theory of purity, which is also closely connected to the resource theory of coherence (see e.g., Ref. [37]). It would be interesting to investigate the applicability of our methods to these other scenarios, even beyond the pure state case.

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## APPENDIX A: ORDER STATISTICS OF I.I.D. RANDOM VARIABLES AND MARKOV PROPERTY

We collect here a series of more or less known results about order statistics of independent random variables. In the following, $X_1, X_2, \dots$ are i.i.d. random variables with distribution function $F(x) := P(X_1 \leqslant x)$. We always assume that they have a density $f(x) = F'(x)$.

For a finite family $X_1, X_2, \dots, X_n$, the *order statistics* $X_k^{\downarrow}$, $k \leqslant n$, are the rearrangements of the variables in nonincreasing order, i.e., $X_1^{\downarrow} \geqslant X_2^{\downarrow} \geqslant \dots \geqslant X_n^{\downarrow}$. Of course, the order statistics are *not* i.i.d. random variables.

Under the previous assumptions on the distribution of the $X_i$'s, the order statistics have a density. The following lemma gives the explicit formulas that we need for our calculations.

*Lemma 2.* Let $X_1, X_2, \dots, X_n$ as above. Then,

(i) The density of $X_k^{\downarrow}$ is

$$f_{X_k^{\downarrow}}(x_k) = \frac{n!}{(n-k)!(k-1)!} F(x_k)^{n-k} f(x_k)(1-F(x_k))^{k-1}. \tag{A1}$$

(ii) The joint density of $(X_k^{\downarrow}, X_l^{\downarrow})$, for $k \leqslant l$ is

$$f_{X_k^{\downarrow}, X_l^{\downarrow}}(x_k, x_l) = \frac{n!}{(n-l)!(l-k-1)!(k-1)!} F(x_l)^{n-l} f(x_l)(F(x_k) - F(x_l))^{l-k-1} f(x_k)(1-F(x_k))^{k-1} \tag{A2}$$

for $x_k \geqslant x_l$, and zero otherwise.

(iii) The joint density of the $k$ largest variables $(X_1^{\downarrow}, X_2^{\downarrow}, \dots, X_k^{\downarrow})$ is

$$f_{X_1^{\downarrow}, X_2^{\downarrow}, \dots X_k^{\downarrow}}(x_1, x_2, \dots, x_k) = \frac{n!}{(n-k)!} f(x_1) f(x_2) \cdots f(x_k) F(x_k)^{n-k} \tag{A3}$$

for $x_1 \geqslant x_2 \geqslant \dots \geqslant x_k \geqslant 0$ and zero otherwise.

(iv) The joint density of the $k$ smallest variables $(X_{n-k+1}^{\downarrow}, \dots, X_{n-1}^{\downarrow}, X_n^{\downarrow})$ is

$$f_{X_{n-k+1}^{\downarrow}, \dots, X_{n-1}^{\downarrow}, X_n^{\downarrow}}(x_{n-k+1}, \dots, x_{n-1}, x_n) = \frac{n!}{(n-k)!}(1-F(x_{n-k+1}))^{n-k} f(x_{n-k+1}) \cdots f(x_{n-1}) f(x_n) \tag{A4}$$

for $x_{n-k+1} \geqslant x_{n-k+2} \geqslant \dots \geqslant x_n \geqslant 0$ and zero otherwise.

*Proof.* The proof is rather elementary (see, e.g., Ref. [38]). We sketch only the proof of part (i) to give a flavor of the type of arguments involved. The probability that $X_k^{\downarrow}$ is in $x_k$ is the probability that, among $X_1, \dots, X_n$, one is in $x_k$ [this gives a factor $f(x_k)$]; exactly $(k-1)$ are larger than $x_k$ (this gives the factor $(1-F(x_k))^{k-1}$); the remaining $(n-k)$ variables are smaller than $x_k$ (corresponding to the factor $F(x_k)^{n-k}$). There are $n\binom{n-1}{k-1}$ ways to partition the $n$ variables in that manner.

*Proposition 2.* Let $X_1, X_2, \dots, X_n$ be as above. Then,

(i) The vector $(X_1^{\downarrow}, X_2^{\downarrow}, \dots, X_n^{\downarrow})$ forms an inhomogeneous (finite) Markov chain with initial density

$$f_{X_1^{\downarrow}}(x) = nF(x)^{n-1} f(x) \tag{A5}$$

and transition densities given by

$$f_{X_{k+1}^{\downarrow}|X_k^{\downarrow}}(y|x) = \begin{cases} (n-k)\dfrac{F(y)^{n-k-1}}{F(x)^{n-k}} f(y) & \text{if } y \leqslant x \\ 0 & \text{otherwise.} \end{cases} \tag{A6}$$

(ii) The vector $(X_n^{\downarrow}, X_{n-1}^{\downarrow}, \dots, X_1^{\downarrow})$ forms an inhomogeneous (finite) Markov chain with initial density

$$f_{X_n^{\downarrow}}(x) = n(1-F(x))^{n-1} f(x) \tag{A7}$$

and transition densities given by

$$f_{X_{n-k}^{\downarrow}|X_{n-k+1}^{\downarrow}}(y|x) = \begin{cases} (n-k)\dfrac{(1-F(y))^{n-k-1}}{(1-F(x))^{n-k}} f(y) & \text{if } y \geqslant x \\ 0 & \text{otherwise.} \end{cases} \tag{A8}$$

*Proof.* We prove part (i). The density (A5) is a specialization of (A1). For $k \leqslant l$, the conditional density of $X_l^{\downarrow}$ given $X_k^{\downarrow}$ is

$$f_{X_l^{\downarrow}|X_k^{\downarrow}}(x_l|x_k) = \frac{f_{X_k^{\downarrow}, X_l^{\downarrow}}(x_k, x_l)}{f_{X_k^{\downarrow}}(x_k)} = \frac{(n-k)!}{(n-l)!(l-k-1)!} \frac{F(x_l)^{n-l}}{F(x_k)^{n-k}} (F(x_k) - F(x_l))^{l-k-1} f(x_l) \tag{A9}$$

for $x_k \geqslant x_l$ and zero otherwise. In particular, for $l = k+1$, we get (A6). Similarly, from (A3), we have, for all $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_k$,

$$f_{X_{k+1}^{\downarrow}|X_1^{\downarrow},X_2^{\downarrow}\dots,X_k^{\downarrow}}(x_{k+1}|x_1,x_2,\dots,x_k) = \frac{f_{X_1^{\downarrow},X_2^{\downarrow},\dots X_{k+1}^{\downarrow}}(x_1,x_2,\dots,x_{k+1})}{f_{X_1^{\downarrow},X_2^{\downarrow},\dots X_k^{\downarrow}}(x_1,x_2,\dots,x_k)} = (n-k)\frac{F(x_{k+1})^{n-k-1}}{F(x_k)^{n-k}}f(x_{k+1}) \tag{A10}$$

for $x_{k+1} \leqslant x_k$ and zero otherwise. Hence, we have proved that

$$f_{X_{k+1}^{\downarrow}|X_1^{\downarrow},X_2^{\downarrow}\dots,X_k^{\downarrow}}(x_{k+1}|x_1,x_2,\dots,x_k) = f_{X_{k+1}^{\downarrow}|X_k^{\downarrow}}(x_{k+1}|x_k). \tag{A11}$$

*Mutatis mutandi* we can prove part (ii).

*Remark 1.* The previous formulas for the densities can be rephrased in terms of the distribution functions:

$$F_{X_1^{\downarrow}}(x) = \int_{-\infty}^{x} f_{X_1^{\downarrow}}(z)dz = F(x)^n, \tag{A12}$$

$$F_{X_{k+1}^{\downarrow}|X_k^{\downarrow}}(y|x) = \int_{-\infty}^{y} f_{X_{k+1}^{\downarrow}|X_k^{\downarrow}}(z|x)dz = \left(\frac{F(\min(y,x))}{F(x)}\right)^{n-k}, \tag{A13}$$

$$F_{X_n^{\downarrow}}(x) = 1 - \int_{x}^{+\infty} f_{X_n^{\downarrow}}(z)dz = 1-(1-F(x))^n, \tag{A14}$$

$$F_{X_{n-k}^{\downarrow}|X_{n-k+1}^{\downarrow}}(y|x) = 1 - \int_{y}^{\infty} f_{X_{n-k}^{\downarrow}|X_{n-k+1}^{\downarrow}}(z|x)dz = 1 - \left(\frac{1-F(\max(y,x))}{1-F(x)}\right)^{n-k}. \tag{A15}$$

## APPENDIX B: PROOF OF PROPOSITION 1

*Proof.* We first recall a standard representation for the uniform distribution in $\Delta_{n-1}$ in terms of i.i.d. exponential random variables, and the classical asymptotic distributions of the extreme values for exponential random variables.

*Lemma 3.* Let $X_1, X_2, \dots$ be independent exponential random variables with rate 1, i.e., $P(X \leqslant x) = 1 - \mathrm{e}^{-x}$. Then, the vector

$$(\mu_1,\mu_2,\dots,\mu_n) := \left(\frac{X_1}{\sum_{i=1}^n X_i}, \frac{X_2}{\sum_{i=1}^n X_i}, \dots, \frac{X_n}{\sum_{i=1}^n X_i}\right) \tag{B1}$$

is uniformly distributed in $\Delta_{n-1}$.

*Lemma 4.* If $F(x) = 1 - \mathrm{e}^{-x}$, then

$$\lim_{n\to\infty} 1-(1-F(u/n))^n = 1-\exp(-u) \quad \text{(exponential distribution)}, \tag{B2}$$

$$\lim_{n\to\infty} F(\ln\ n+u)^n = \exp(-\mathrm{e}^{-u}) \quad \text{(Gumbel distribution)}. \tag{B3}$$

Let $\mu = (\mu_1,\mu_2,\dots,\mu_n)$ defined as in (B1) be a uniform point on $\Delta_{n-1}$. Combining Proposition 2, Remark 1, and the formula (B2), we see that for any fixed $k \geqslant 1$,

$$(nX_n^{\downarrow}, nX_{n-1}^{\downarrow}, \dots, nX_{n-k+1}^{\downarrow})$$

converges in distribution to the first $k$ components $(V_1, V_2, \dots, V_k)$ of the time-homogeneous Markov chain $(V_j)_{j\geqslant 1}$ with density of $V_1$ and transition density

$$f_{V_1}(v) = \exp(-v)1_{v\geqslant 0}, \qquad f_{V_{j+1}|V_j}(u|v) = \exp(v-u)1_{u\geqslant v},$$

respectively. To show the convergence for the order statistics of $\mu$, we simply observe that the vector

$$(n^2\mu_{n-j+1}^{\downarrow})_{1\leqslant j\leqslant k} = \left(\frac{n}{\sum_{i=1}^n X_i} nX_{n-j+1}^{\downarrow}\right)_{1\leqslant j\leqslant k}$$

has the same limit distribution of $(nX_{n-j+1}^{\downarrow})_{1\leqslant j\leqslant k}$. (Recall that $\mathbb{E}[\sum_{i=1}^n X_i] = n$; hence the factor $n^{-1}\sum_{i=1}^n X_i$ converges to 1 by the law of large numbers.)

Similarly, from Proposition 2, Remark 1, and the asymptotic formula (B3), we deduce that

$$(X_1^{\downarrow} - \ln n, X_2^{\downarrow} - \ln n, \dots, X_k^{\downarrow} - \ln n)$$

converges in distribution to the first $k$ components $(W_1, W_2, \dots, W_k)$ of the time-homogeneous Markov Chain $(W_j)_{j\geqslant 1}$ with density of $W_1$ and transition density

$$f_{W_1}(w) = \exp(-\mathrm{e}^{-w} - w), \quad f_{W_{j+1}|W_j}(u|w) = \exp(\mathrm{e}^{-w} - \mathrm{e}^{-u} - u)1_{u\leqslant w},$$

respectively. Denote by $\mu^{\downarrow}$ the decreasing rearrangement of $\mu$. For any $k$, we want to show that

$$(n\mu_1^{\downarrow} - \ln n, n\mu_2^{\downarrow} - \ln n, \dots, n\mu_k^{\downarrow} - \ln n)$$

converges in distribution to $(W_1, W_2, \dots, W_k)$. We write

$$(n\mu_j^{\downarrow} - \ln n)_{1\leqslant j\leqslant k} = \left(\frac{n}{\sum_{i=1}^n X_i}\big(X_j^{\downarrow} - \ln n\big) + \left(\frac{n}{\sum_{i=1}^n X_i} - 1\right)\ln n\right)_{1\leqslant j\leqslant k},$$

and we want to show that this vector has the same limit distribution of $(X_j^{\downarrow} - \ln n)_{1\leqslant j\leqslant k}$, as $n \to \infty$. The factor $n^{-1}\sum_{i=1}^n X_i$ converges to 1 by the law of large numbers. For all $\epsilon > 0$,

$$\begin{aligned} P\left(\left|\frac{n}{\sum_{i=1}^n X_i} - 1\right| > \frac{\epsilon}{\ln n}\right) &= P\left(\sum_{i=1}^n X_i < \frac{n\ln n}{\ln n + \epsilon} \quad \text{or} \quad \sum_{i=1}^n X_i > \frac{n\ln n}{\ln n - \epsilon}\right) \\ &= P\left(\sum_{i=1}^n X_i < n - \frac{n\epsilon}{\ln n + \epsilon} \quad \text{or} \quad \sum_{i=1}^n X_i > n + \frac{n\epsilon}{\ln n - \epsilon}\right) \\ &\leqslant P\left(\left|\sum_{i=1}^n X_i - n\right| > \min\left\{\frac{n\epsilon}{\ln n + \epsilon}, \frac{n\epsilon}{\ln n - \epsilon}\right\}\right). \end{aligned}$$

Recall that $\mathrm{var}[\sum_{i=1}^n X_i] = n$. Assuming $n > \exp(\epsilon)$, and using Chebyshev's inequality, we can estimate

$$P\left(\left|\frac{n}{\sum_{i=1}^n X_i} - 1\right| > \frac{\epsilon}{\ln n}\right) \leqslant \frac{\mathrm{var}[\sum_{i=1}^n X_i]}{\left(\frac{n\epsilon}{\ln n+\epsilon}\right)^2} = \frac{1}{n\epsilon^2}(\ln n + \epsilon)^2.$$

Hence, $(\frac{n}{\sum_{i=1}^n X_i} - 1)\ln n$ converges to 0 in probability as $n \to \infty$.

## APPENDIX C: VANISHING OF THE PERSISTENCE PROBABILITY ABOVE ZERO OF THE IRW. PROOF OF CLAIM (21) IN THEOREM 3

We want to prove that the persistence probability asymptotically vanishes:

$$\lim_{k\to\infty} P\left(\min_{1\leqslant j\leqslant k} I_j \geqslant 0\right) = 0.$$

Notice that

$$\lim_{k\to\infty} P\left(\min_{1\leqslant j\leqslant k} I_j \geqslant 0\right) = P\left(\inf_{k\geqslant 1} I_k \geqslant 0\right) \leqslant P\left(\liminf_{k\to\infty} I_k \geqslant 0\right) = P\left(\liminf_{k\to\infty} \frac{I_k}{k\ln k} \geqslant 0\right).$$

Therefore, it is sufficient to show that

$$P\left(\liminf_{k\to\infty} \frac{I_k}{k\ln k} \geqslant 0\right) = 0, \tag{C1}$$

and this follows from the Lindeberg-Feller central limit theorem as we outline now.

Denote by $A$ the event in (C1).

*Claim 1.* $P(A) \in \{0, 1\}$.

The proof of the Claim is almost verbatim the proof given by Pittel [31]. For the event

$$A = \left\{\liminf_{k\to\infty} \frac{I_k}{k\ln k} \geqslant 0\right\},$$

we want to show that $P(A) \in \{0, 1\}$. The key observation here is that the probability of the event $A$ does not depend on the variables of $\widetilde{X}_1, \widetilde{X}_2, \dots, \widetilde{X}_J$, no matter how large, albeit finite, $J$ is. Indeed, let $\widetilde{V}_k(J) = \sum_{j=J+1}^k \widetilde{X}_j$, for $k > J$ and $I_k(J) = \sum_{j=J+1}^k \widetilde{V}_j(J)$ for $k > J$ as well. Then,

$$I_k - I_k(J) = \sum_{j=1}^k \widetilde{V}_j - \sum_{j=J+1}^k \widetilde{V}_j(J) = \sum_{j=1}^k j\widetilde{X}_{k-j+1} - \sum_{j=1}^{k-J} j\widetilde{X}_{k-j+1} = \sum_{j=k-J+1}^k j\widetilde{X}_{k-j+1}.$$

Therefore, almost surely

$$\lim_{k\to\infty} \frac{1}{k\ln k}|I_k - I_k(J)| = 0, \quad \text{for all } J.$$

So, denoting

$$A_J = \left\{ \liminf_{k\to\infty} \frac{I_k(J)}{k \ln k} \geqslant 0 \right\},$$

we can write for the symmetric difference $A \triangle A_J$ of the events $A$ and $A_J$,

$$P(A \triangle A_J) = 0, \quad \text{for all} J.$$

Now observe that $A_J$ is measurable with respect to $(\widetilde{X}_j)_{j>J}$. (Informally, the event $A_J$ does not involve the first $J$ variables $\widetilde{X}_1, \dots, \widetilde{X}_J$). Then, writing a.a. for almost always and i.o. for infinitely often,

$$A_\infty = \liminf_J A_J = \bigcup_{J\geqslant 1} \bigcap_{m\geqslant J} A_m = \{A_J \text{ a.a.}\}$$

is a tail event, and

$$P(A \triangle A_\infty) = P(A \cap A_\infty^c) + P(A^c \cap A_\infty) = P(A \cap A_J^c \text{ i.o.}) + P(A^c \cap A_J \text{ a.a.})$$
$$\leqslant \sum_{J\geqslant 1} [P(A \cap A_J^c) + P(A^c \cap A_J)] = \sum_{J\geqslant 1} P(A \triangle A_J) = 0.$$

By the Kolmogorov 0-1 law, $P(A_\infty) \in \{0, 1\}$, so from the previous calculation we obtain $P(A) \in \{0, 1\}$, as well.

Given Claim 1, we can now complete the proof if we show that $P(A) < 1$. By the definition of $A$, to do so it suffices to show that

$$\lim_{k\to\infty} P\left(\frac{I_k}{k \ln k} \geqslant -b\right) < 1, \tag{C2}$$

for a constant $b > 0$. Writing $I_k = \sum_{j=1}^k j\widetilde{X}_{k-j+1}$, it is a routine matter to compute

$$\mathbb{E}[I_k] = 0, \ \ \mathrm{var}[I_k] = 2\sum_{j=1}^k j^2 = \frac{k(k+1)(2k+1)}{3} = O(k^3).$$

From this, one can check that the sequence $I_k$ satisfies the Lindeberg-Feller conditions, and thus $I_k/\sqrt{\mathrm{var}[I_k]}$ converges in distribution to the standard Gaussian variable as $k \to \infty$. Hence,

$$P\left(\frac{I_k}{k \ln k} \geqslant -b\right) = P\left(\frac{\sqrt{\mathrm{var}[I_k]}}{k \ln k} \frac{I_k}{\sqrt{\mathrm{var}[I_k]}} \geqslant -b\right) \xrightarrow{k\to\infty} \frac{1}{\sqrt{2\pi}} \int_0^\infty \mathrm{e}^{-x^2/2} dx = \frac{1}{2} < 1.$$

■


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