Heisenberg scaling precision in the estimation of functions of parameters in linear optical networks

Danilo Triggiani,[1,*] Paolo Facchi,[2,3] and Vincenzo Tamma[1,4,†]
[1]*School of Mathematics and Physics, University of Portsmouth, Portsmouth PO1 3QL, United Kingdom*
[2]*Dipartimento di Fisica and MECENAS, Università di Bari, I-70126 Bari, Italy*
[3]*INFN, Sezione di Bari, I-70126 Bari, Italy*
[4]*Institute of Cosmology and Gravitation, University of Portsmouth, Portsmouth PO1 3FX, United Kingdom*

(Received 17 March 2021; revised 29 September 2021; accepted 17 November 2021; published 3 December 2021)

We propose a metrological strategy reaching Heisenberg-scaling precision in the estimation of functions of any fixed number p of arbitrary parameters encoded in a generic M-channel linear network. This scheme is experimentally feasible since it only employs a single-mode squeezed vacuum and homodyne detection on a single output channel. Two auxiliary linear networks are required and their role is twofold: to refocus the signal into a single channel after the interaction with the interferometer, and to fix the function of the parameters to be estimated according to the linear network analyzed. Although the refocusing requires some knowledge on the parameters, we show that the required precision on the prior measurement is achievable with a classic measurement. We conclude by discussing two paradigmatic schemes in which the choice of the auxiliary stages allows us to change the function of the unknown parameter to estimate.

DOI: 10.1103/PhysRevA.104.062603

I. INTRODUCTION

The estimation of physical properties has always played a central role in the development of science, engineering, technologies, and ultimately human knowledge. At the same time, advances in technologies and a better understanding of nature allow for improvements in the sensing protocols and, occasionally, for breakthroughs on the ultimate precisions fundamentally achievable. One of the most recent breakthroughs is the discovery of the advantage that quantum strategies can bring to metrology: it has been shown that the ultimate precision achievable employing N entangled probes for the estimation of a single phase (or more in general the amplitude of a unitary evolution) exceeds the precision of any classical strategy [1–4]. In particular, it is possible to conceive quantum estimation strategies which lead to errors that scale as fast as $1/N$, a bound usually called the *Heisenberg limit* (HL), whereas the error of any classical strategy is bounded by the so-called *shot-noise limit* (SNL) and cannot scale faster than $1/\sqrt{N}$. Since these initial works, considerable effort has been put into the development of quantum estimation protocols reaching the HL [5–9], with applications in imaging [10,11], thermometry [12,13], and magnetic field [14,15] and gravitational waves detection [16], among others.

A difficulty often encountered in early protocols is the fragility of the required probes state, which are usually entangled. To overcome this drawback, the employment of Gaussian states as probes and *squeezing* as resource has recently started to be considered, since these states are feasible to be experimentally produced and robust against noise [17–27].

In some recent works [26,27], a protocol has been proposed for the estimation of a single parameter encoded in a generic passive linear network, reaching Heisenberg-scaling sensitivity by employing a single squeezed-vacuum state and a single homodyne detection. In this scheme it has been found that employing two auxiliary passive and linear stages opportunely engineered, one at each side of the main network, suffices for the estimation protocol to reach Heisenberg-scaling sensitivity with only classical prior knowledge on the single unknown parameter [27]. Moreover, it is shown that the two auxiliary stages introduce some redundant degrees of freedom, that typically do not affect the ability of the setup to reach Heisenberg-scaling precision [26]. The advantage of this approach, where the overall network is affected by a single global parameter, is to give a general result irrespective of the way the unknown parameter is encoded in the linear transformation, that is how it is physically implemented in the network.

The scope of work of this article is the study of the very structure of the linear network, and in particular how the different passive components, phase shifters and beam splitters, might concur to encode the parameter to be estimated and how they might affect the precision in the estimation. In other words, our goal here is to answer the following questions: What happens if the network is affected locally by an arbitrary number of unknown parameters? Is it possible to measure a given function of such parameters?

Due to the presence of many independent unknown parameters, each one representing, for example, local properties of the components of the network, and as such being independent sources of uncertainty in the estimation process, the transition from the estimation of a single parameter to a single function

*danilo.triggiani@port.ac.uk
†vincenzo.tamma@port.ac.uk

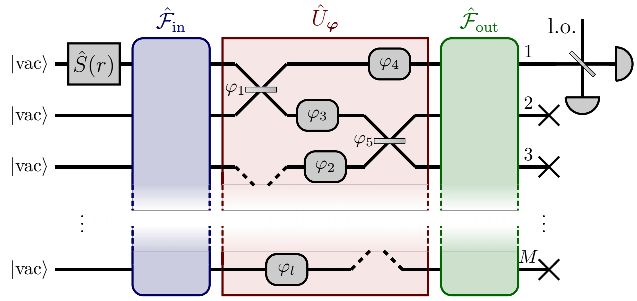

FIG. 1. Optical scheme for Heisenberg limited estimation of a function $f(\boldsymbol{\varphi})$ of an arbitrary number of parameters φ_i, with $i = 1, \dots, p$, encoded in an arbitrary M-channel linear optical network $\hat{U}_{\boldsymbol{\varphi}}$ by using either two auxiliary optical stages $\hat{\mathcal{F}}_{\text{in}}$ and $\hat{\mathcal{F}}_{\text{out}}$ or only one of them. Such parameters can be optical phases or phase-like parameters of beam-splitters. By injecting a single-mode squeezed-vacuum state with N mean photons in (say) the first channel, through single-mode homodyne detection in (say) the first channel and ignoring the others, it is possible to tune the phase of the local oscillator (l.o) to estimate the phase acquired by the probe through the whole network, which is a function of the p parameters and of the specifics of the auxiliary stages $\hat{\mathcal{F}}_{\text{in}}$ and $\hat{\mathcal{F}}_{\text{out}}$. When an auxiliary stage ($\hat{\mathcal{F}}_{\text{in}}$ or $\hat{\mathcal{F}}_{\text{out}}$) is suitably chosen, this setup allows Heisenberg scaling precision in the photon number N. Furthermore, the extra degrees of freedom in either of the two auxiliary gates could be used to manipulate the functional dependence of $f(\boldsymbol{\varphi})$ on the parameters.

of multiple parameters is not straightforward, nor it is granted to be applicable [28]. A feasible and efficient solution to this problem would present the compelling advantage of allowing the estimation of global properties (e.g., spatial average of a field, field gradients, or nonlinear functions) without wasting resources to measure each local unknown parameter, at the expense of the use of two auxiliary networks and a more thorough theoretical analysis to guarantee that results similar to those of the scenario of a single-parameter network hold true for the estimation of a function of multiple parameters affecting the network, employing the more complete multiparameter formalism, such as the use of the Fisher information matrix [29–32]. However, this would require overcoming a number of constraints on the parameters and on the network that are usually required in other protocols for the estimation of functions found in the literature, such as commuting generators, reduced working range, scarce freedom on the structure of the networks and on the nature of the parameters [8,9,24,25,28,33–36], which pose a problem for certain practical applications, including scenarios where small or no control is possible on the structure and parameter dependence of a given arbitrary multiparameter network.

In the present work we demonstrate that this transition is indeed possible: we show that Heisenberg scaling precision in the estimation of an arbitrary number of either linear or nonlinear functions of $p > 1$ independent parameters, which can be encoded with great freedom in a passive linear optical network, is possible by employing a single squeezed state, a single homodyne detection, and two auxiliary stages. Indeed, for any given parameter-dependent network, the function of the parameters to be estimated can be manipulated by using simple linear optical stages (see Fig. 1). Furthermore, no constraint in the values of the parameters and no entangled probes are needed. In particular, the p unknown parameters can be both optical phases or reflectivities of beam splitters which can depend in principle also on external parameters such as pressure, electromagnetic fields, or temperature. In this scheme, the two auxiliary linear networks serve a twofold purpose: they act on the probe to scatter first and then refocus the photons in the only channel measured, and they can also be opportunely engineered to manipulate how the information about the parameters is encoded, and ultimately to choose the function of the unknown parameters to be estimated with Heisenberg scaling precision. Although these auxiliary stages might depend on the value of the unknown parameters, requiring thus a prior knowledge in order to correctly implement our scheme, we show that classical precision on the prior knowledge of the parameters is sufficient, meaning that a prior classical measure of each parameter is enough to correctly select the auxiliary stages. Finally, we apply our general scheme to the case of two different multiparameter interferometers: the first is the simplest nontrivial example of multiparameter estimation, which allows us to estimate a whole family of (in general) nonlinear functions: a linear interferometer with three parameters encoded in two phase shifts and a beam splitter; the second allows us to estimate any linear combination with positive coefficients of phase shifts and beam-splitter parameters.

II. LINEAR NETWORK WITH AN ARBITRARY NUMBER OF UNKNOWN PARAMETERS

Let us consider an $M \times M$ passive and linear optical network which depends on p unknown parameters that, in a compact form, can be written as components of an p-dimensional vector $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_p)$. These parameters can be, for example, local independent properties of the optical components of the network (e.g., phase shifts, reflectivities of beam splitters), or the values of an external nonuniform field (magnetic field, temperature) affecting the components. The network is described by a unitary operator $\hat{U}_{\boldsymbol{\varphi}}$ which depends smoothly on $\boldsymbol{\varphi}$, whose action on the M bosonic annihilation operators $\hat{a}_i$, for $i = 1, \dots, M$ is given by the unitary matrix $U_{\boldsymbol{\varphi}}$ such that

$$\hat{U}_{\boldsymbol{\varphi}}^{\dagger} \hat{a}_i \hat{U}_{\boldsymbol{\varphi}} = \sum_{j=1}^{M} (U_{\boldsymbol{\varphi}})_{ij} \hat{a}_j. \tag{1}$$

The input probe in our scheme consists in a single-mode squeezed-vacuum state with $N = \sinh^2 r$ mean number of photons, where r is the real squeezing parameter of the probe. This is injected into one of the input ports, namely, the first, of an auxiliary linear and passive optical network $\hat{\mathcal{F}}_{\text{in}}$ (described by the unitary matrix $\mathcal{F}_{\text{in}}$), whose role is to distribute the probe photons among all the channels of the network $\hat{U}_{\boldsymbol{\varphi}}$, and ultimately to control the function of the parameters $f(\boldsymbol{\varphi})$ to estimate. Then, the photons go through a second auxiliary linear and passive network $\hat{\mathcal{F}}_{\text{out}}$, described by $\mathcal{F}_{\text{out}}$, which refocuses the photons into a single output port, say the first, so that all the information on the parameters acquired by the probe can be read by performing homodyne detection on a single channel. The probability amplitude $\chi_{\boldsymbol{\varphi}}$ associated with the transition of a single photon from the first input port to the first output port of the whole setup reads

$$\chi_{\boldsymbol{\varphi}} \equiv \sqrt{P_{\boldsymbol{\varphi}}} e^{if(\boldsymbol{\varphi})} = (\mathcal{F}_{\text{out}} U_{\boldsymbol{\varphi}} \mathcal{F}_{\text{in}})_{11} \quad (2)$$

and represents the only relevant quantity in the presented scheme. If the refocusing process is perfect, the probability $P_{\boldsymbol{\varphi}}$ will be exactly equal to one, and all the information on $\boldsymbol{\varphi}$ is encoded in the phase acquired by the probe $f(\boldsymbol{\varphi})$, whose functional dependence on the parameters $\boldsymbol{\varphi}$ can be partially controlled through the auxiliary gates $\hat{\mathcal{F}}_{\text{in}}$ and $\hat{\mathcal{F}}_{\text{out}}$. In the next section we show that Heisenberg-limited sensitivity can be achieved even for an imperfect refocusing as long as the probability for the photons to exit from a different channel scales as $1/N$, which is attainable without any additional quantum resource.

Since the initial squeezing parameter is real, and the probe acquires a total phase-shift of $f(\boldsymbol{\varphi})$, the quadrature field with minimum variance at the first output port of the interferometer corresponds to $\hat{x}_{f(\boldsymbol{\varphi})\pm\pi/2}$. Eventually, the quadrature field $\hat{x}_\theta$ is measured through homodyne detection on the first channel, where θ is the reference phase of the homodyne local oscillator, in order to infer the value of $f(\boldsymbol{\varphi})$. From Eq. (2) it is possible to make explicit the dependence of the acquired phase $f(\boldsymbol{\varphi})$ from the elements of the scattering matrix $U_{\boldsymbol{\varphi}}$,

$$f(\boldsymbol{\varphi}) = \arctan\left(\frac{\text{Im}(\mathcal{F}_{\text{out}} U_{\boldsymbol{\varphi}} \mathcal{F}_{\text{in}})_{11}}{\text{Re}(\mathcal{F}_{\text{out}} U_{\boldsymbol{\varphi}} \mathcal{F}_{\text{in}})_{11}}\right), \quad (3)$$

and the influence of both the networks $\mathcal{F}_{\text{in}}$ and $\mathcal{F}_{\text{out}}$. We show how, once the expression of the scattering matrix $U_{\boldsymbol{\varphi}}$ is specified, the phase (3) can be specialized into various functions of the parameters (both linear and nonlinear), useful for applications, such as function interpolation [9] or gradient-fields estimation [34]. In the next section, we show that the precision reached with this setup in the estimation of the function $f(\boldsymbol{\varphi})$ of the parameters asymptotically reaches the Heisenberg limit.

III. HEISENBERG LIMITED ESTIMATION OF A FUNCTION OF THE NETWORK PARAMETERS

The probability density function $p(x|\boldsymbol{\varphi})$ that the outcome x_θ of the homodyne detection falls between x and $x + dx$ is a centered Gaussian distribution

$$p(x|\boldsymbol{\varphi}) = \frac{1}{\sqrt{2\pi\sigma_{\boldsymbol{\varphi}}^2}} \exp\left[-\frac{x^2}{2\sigma_{\boldsymbol{\varphi}}^2}\right], \quad (4)$$

whose variance

$$\sigma_{\boldsymbol{\varphi}}^2 = \frac{1}{2} + P_{\boldsymbol{\varphi}}\{\sinh^2 r + \cos[2f(\boldsymbol{\varphi}) - 2\theta] \sinh r \cosh r\} \quad (5)$$

depends on the parameters $\boldsymbol{\varphi}$ through $P_{\boldsymbol{\varphi}}$ and $f(\boldsymbol{\varphi})$ (see Appendix A). Due to the presence of multiple independent parameters in the optical network $U_{\boldsymbol{\varphi}}$, the theoretical analysis of the precision of this setup has to make use of the multiparameter formalism. The ultimate precision achievable on the estimation of any function $\alpha(\boldsymbol{\varphi})$ of the parameters is regulated by the Fisher information matrix [37]

$$\mathcal{I} = \int dx\, p(x|\boldsymbol{\varphi})[\nabla_{\boldsymbol{\varphi}} \ln p(x|\boldsymbol{\varphi})][\nabla_{\boldsymbol{\varphi}} \ln p(x|\boldsymbol{\varphi})]^{\text{T}}, \quad (6)$$

where $\nabla_{\boldsymbol{\varphi}} = (\partial_{\varphi_1}, \dots, \partial_{\varphi_p})^{\text{T}}$ is the gradient in parameter space. In particular, substituting the distribution (4) in this expression, we obtain

$$\mathcal{I} = \frac{1}{2\sigma_{\boldsymbol{\varphi}}^4} (\nabla_{\boldsymbol{\varphi}} \sigma_{\boldsymbol{\varphi}}^2)(\nabla_{\boldsymbol{\varphi}} \sigma_{\boldsymbol{\varphi}}^2)^{\text{T}}. \quad (7)$$

We assume now that the refocusing performed by the stage $\hat{\mathcal{F}}_{\text{out}}$ is such that the probability $P_{\boldsymbol{\varphi}}$ differs from unity by a small quantity of $O(N^{-1})$, namely,

$$P_{\boldsymbol{\varphi}} \sim 1 - \frac{\ell}{N}, \quad \ell \geqslant 0, \quad (8)$$

with ℓ independent of N. This condition is a requirement on the precision of the refocusing performed through the auxiliary networks: only a small portion of the signal [i.e., an average number of photons $N(1 - P_{\boldsymbol{\varphi}})$ which does not grow with N] can be scattered into unmeasured output channels. For an arbitrary $\boldsymbol{\varphi}$ dependent network $\hat{U}_{\boldsymbol{\varphi}}$ this condition can only be satisfied with a prior knowledge of the parameters in order to suitably chose the auxiliary gates. Remarkably, in Appendix B we show that a classical prior knowledge $\boldsymbol{\varphi}_{\text{cl}}$ of the parameters, corresponding to a shot-nose precision $\delta\boldsymbol{\varphi} = \boldsymbol{\varphi} - \boldsymbol{\varphi}_{\text{cl}} = O(N^{-1/2})$, suffices to satisfy (8).

Moreover, we impose that the local oscillator phase θ is experimentally tuned on a value $\theta_{\boldsymbol{\varphi}}$ of the asymptotic form

$$\theta_{\boldsymbol{\varphi}} \sim f(\boldsymbol{\varphi}) \pm \frac{\pi}{2} + \frac{k}{N}, \quad k \neq 0, \quad (9)$$

which differs from the phases $f(\boldsymbol{\varphi}) \pm \pi/2$ of the quadrature field with minimum variance only by a quantity k/N, with k independent of N. This condition assures that the right quadrature field $\hat{x}_\theta$ is measured, namely, a quadrature that simultaneously has a squeezed variance, so that $\sigma_{\boldsymbol{\varphi}}^2 = O(N^{-1})$, but that still is sensible to variations of the parameter, namely, $\nabla_{\boldsymbol{\varphi}}\sigma_{\boldsymbol{\varphi}}^2 = O(1)$. When conditions (8) and (9) hold, in the large-N limit, the Fisher information matrix reads (see Appendix C)

$$\mathcal{I} \sim 8\varrho(k, \ell)N^2[\nabla_{\boldsymbol{\varphi}} f(\boldsymbol{\varphi})][\nabla_{\boldsymbol{\varphi}} f(\boldsymbol{\varphi})]^{\text{T}}, \quad (10)$$

with

$$\varrho(k, \ell) = \left(\frac{8k}{1 + 16k^2 + 4\ell}\right)^2. \quad (11)$$

A thorough analysis of the matrix $\mathcal{I}$ in (10) (see Appendix D) shows that only $f(\boldsymbol{\varphi})$, or functions of $f(\boldsymbol{\varphi})$, admit estimators with finite variances. In particular, any unbiased estimator $\tilde{f}$ of $f(\boldsymbol{\varphi})$ is characterized by a variance which satisfies the Cramér-Rao bound (see Appendix D)

$$\text{Var}(\tilde{f}) \geqslant \frac{1}{8\varrho(k, \ell)N^2}, \quad (12)$$

and thus our scheme allows Heisenberg scaling precision in N through Bayesian analysis [18,38,39] or employing the maximum-likelihood estimator (obtained in Appendix E), which asymptotically saturate the bound.

From Eq. (2) we notice that the functional dependence of $f(\boldsymbol{\varphi})$ on the parameters $\boldsymbol{\varphi}$ changes for different choices of $\hat{\mathcal{F}}_{\text{in}}$ and $\hat{\mathcal{F}}_{\text{out}}$. In particular, after the optimization required to satisfy condition (8), the remaining degrees of freedom on the stages $\hat{\mathcal{F}}_{\text{in}}$ and $\hat{\mathcal{F}}_{\text{out}}$ can be employed to manipulate

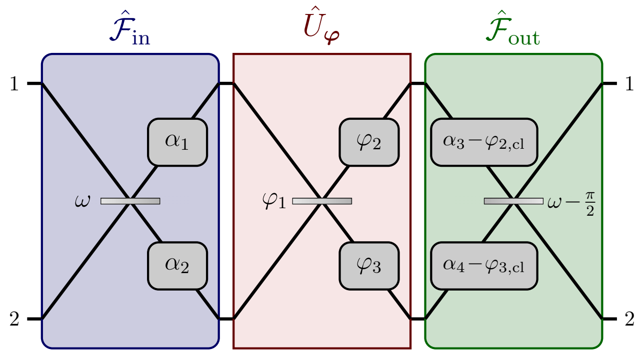

FIG. 2. Example of an M-channel linear network $\hat{U}_{\boldsymbol{\varphi}}$ in (1) for $M = 2$ with suitable auxiliary stages $\hat{\mathcal{F}}_{\rm in}$ and $\hat{\mathcal{F}}_{\rm out}$ for the estimation of a family of functions of two optical phases and a beam-splitter parameter within $\hat{U}_{\boldsymbol{\varphi}}$. The function of the parameters $\boldsymbol{\varphi} \equiv (\varphi_1, \varphi_2, \varphi_3)$ estimated depends on a control parameter $\Delta\alpha = \alpha_1 - \alpha_2$ which can be arbitrarily chosen. To correctly tune the auxiliary stage $\hat{\mathcal{F}}_{\rm out}$ to achieve Heisenberg-limited sensitivity, only a classical prior knowledge $\boldsymbol{\varphi}_{\rm cl}$ on the parameter is required, namely, the error $\delta\boldsymbol{\varphi} = \boldsymbol{\varphi} - \boldsymbol{\varphi}_{\rm cl}$ in a preparative coarse estimation must be of order of $1/\sqrt{N}$

the function of the parameters $f(\boldsymbol{\varphi})$ that we can estimate with Heisenberg-scaling precision. Hereafter we are going to show some examples for which the acquired phase $f(\boldsymbol{\varphi})$ to be estimated assumes simple functional dependencies of the parameters $\boldsymbol{\varphi}$, which can be manipulated through simple changes in the two auxiliary stages, useful for practical purposes.

IV. EXAMPLES OF SETUPS FOR THE ESTIMATION OF FUNCTIONS OF PARAMETERS

A. Functions of parameters in a two-channel network

We consider a two-mode network (see Fig. 2), which allows a Heisenberg scaling precision in the estimation of a family of functions $f(\boldsymbol{\varphi}; \Delta\alpha)$ of the reflectivity φ_1 of a beam splitter, described by a scattering matrix

$$U_{\rm BS}(\varphi_1) = e^{i\varphi_1\sigma_y}, \tag{13}$$

with σ_y being the second Pauli matrix, and of two phase shifts described by

$$U_{\rm PS}(\varphi_2, \varphi_3) = \begin{pmatrix} e^{i\varphi_2} & 0 \\ 0 & e^{i\varphi_3} \end{pmatrix}. \tag{14}$$

The family of functions is parametrized by the relative phase shift $\Delta\alpha = \alpha_1 - \alpha_2$ in the arms of an input auxiliary gate $\hat{\mathcal{F}}_{\rm in} \equiv \hat{\mathcal{F}}_{\rm in}(\alpha_1, \alpha_2)$, where α_1 and α_2 can be arbitrarily chosen. The probe photons, in a single-mode squeezed state $\hat{S}_1(r)|{\rm vac}\rangle = e^{\frac{r}{2}(\hat{a}_1^2 - \hat{a}_1^{\dagger 2})}|{\rm vac}\rangle$ with an average number of photons $N = \sinh^2 r$, are first injected into the first channel of the auxiliary input linear network $\hat{\mathcal{F}}_{\rm in}(\alpha_1, \alpha_2)$ which, in general, may require some prior classical knowledge $\varphi_{1,\rm cl}$ of the beam-splitter parameter φ_1, namely, such that the error $\delta\varphi_1 = \varphi_1 - \varphi_{1,\rm cl} = k_1/\sqrt{N}$ in the prior coarse estimation is of order $O(1/\sqrt{N})$, where k_1 is a constant value not depending on N. In particular, in order to satisfy condition (8), a possible choice for this stage consists in a beam splitter $U_{\rm BS}(\omega)$, whose reflectivity ω is tuned according to

$$\omega = \frac{1}{2} \arctan\left(\frac{\cos(\varphi_{1,\rm cl})}{\sin(\varphi_{1,\rm cl})\cos\Delta\alpha}\right), \tag{15}$$

and in the two arbitrary and $\boldsymbol{\varphi}$-independent phase shifts α_1 and α_2, so that the scattering matrix of this stage is

$$\mathcal{F}_{\rm in} = U_{\rm PS}(\alpha_1, \alpha_2) U_{\rm BS}(\omega). \tag{16}$$

Then, the probe goes through the passive linear network described by the matrix

$$U_{\boldsymbol{\varphi}} = U_{\rm PS}(\varphi_2, \varphi_3) U_{\rm BS}(\varphi_1), \tag{17}$$

and finally through a second auxiliary linear network $\hat{\mathcal{F}}_{\rm out}(\alpha_3, \alpha_4)$, depending on two $\boldsymbol{\varphi}$-independent parameters α_3 and α_4 such that $\alpha_4 - \alpha_3 = \alpha_1 - \alpha_2 \equiv \Delta\alpha$, and a prior classical knowledge $\boldsymbol{\varphi}_{\rm cl}$ on the three unknown parameters, namely, such that the errors $\delta\varphi_i = \varphi_i - \varphi_{i,\rm cl} = k_i/\sqrt{N}$ in the prior estimations are of order of $O(1/\sqrt{N})$, with k_i constants not depending on N for $i = 1, 2, 3$. In particular, this stage is composed of a phase-shift in each channel of values $\alpha_3 - \varphi_{2,\rm cl}$ and $\alpha_4 - \varphi_{3,\rm cl}$, and of a beam splitter with reflectivity $\omega - \pi/2$. The scattering matrix of the whole output stage thus reads

$$\mathcal{F}_{\rm out}(\alpha_3, \alpha_4) = U_{\rm BS}(\omega - \pi/2) U_{\rm PS}(\alpha_3 - \varphi_{2,\rm cl}, \alpha_4 - \varphi_{3,\rm cl}). \tag{18}$$

Finally, a homodyne detection is performed on the first output port of the interferometer according to condition (9).

A straightforward calculation shows that, for this setup, the one-photon transition amplitude (2) reads

$$\begin{aligned} \chi_{\boldsymbol{\varphi}} = {} & e^{i\left(\alpha_1 + \alpha_3 + \frac{\delta\varphi_2 + \delta\varphi_3}{2}\right)} \\ & \times \left[\cos\left(\frac{\delta\varphi_2 - \delta\varphi_3}{2}\right) \sin 2\omega \cos\varphi_1 \right. \\ & + \cos\left(\Delta\alpha - \frac{\delta\varphi_2 - \delta\varphi_3}{2}\right) \cos 2\omega \sin\varphi_1 \\ & \left. + \, i \sin\left(\Delta\alpha - \frac{\delta\varphi_2 - \delta\varphi_3}{2}\right) \sin\varphi_1 \right], \end{aligned} \tag{19}$$

and is such that condition (8) on the transition probability $P_{\boldsymbol{\varphi}} = |\chi_{\boldsymbol{\varphi}}|^2$ is satisfied for large N and with a reflectivity ω given by (15). The acquired phase $f(\boldsymbol{\varphi}; \Delta\alpha)$ through the interferometer reads

$$\begin{aligned} f(\boldsymbol{\varphi}; \Delta\alpha) = {} & \alpha_1 + \alpha_3 + \frac{\delta\varphi_2 + \delta\varphi_3}{2} \\ & + \arctan\left(\frac{\sin\varphi_1 \sin\left(\Delta\alpha - \frac{\delta\varphi_2 - \delta\varphi_3}{2}\right)\sqrt{1 - \sin^2(\varphi_{1,\rm cl})\sin^2\Delta\alpha}}{\cos\varphi_1 \cos\varphi_{1,\rm cl} \cos\left(\frac{\delta\varphi_2 - \delta\varphi_3}{2}\right) + \sin\varphi_1 \sin\varphi_{1,\rm cl} \cos\Delta\alpha \cos\left(\Delta\alpha - \frac{\delta\varphi_2 - \delta\varphi_3}{2}\right)}\right), \end{aligned} \tag{20}$$

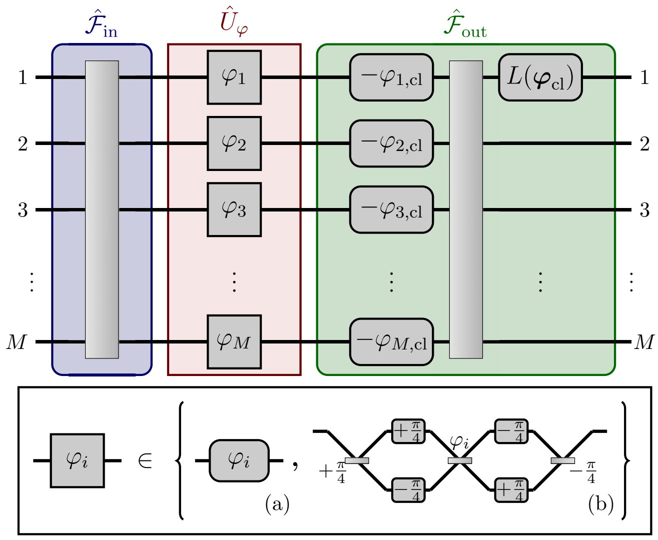

FIG. 3. An M-channel linear and passive network which allows Heisenberg scaling precision in the estimation of any linear combination of M unknown parameters $\boldsymbol{\varphi}$, all encoded within U_φ (red box). The auxiliary stages defined in (24) and (27) are composed of multichannel beam splitters encoding the weights of the linear combination and of phase shifts depending on the classical knowledge of the parameter. As shown in the lower panel, each parameter φ_i is (a) an optical phase acquired through a single-mode phase-shift, or (b) the reflectivity of a lossless beam splitter which can be readily imprinted into an optical phase by means of a two-channel parameter-independent local network. Since the probability that a photon injected into the first port of each local network comes out from the second channel is exactly zero, each local network acts as a single-channel phase delay with magnitude φ_i.

and in general is not linear in the parameters $\boldsymbol{\varphi}$. It can be arbitrarily tuned by changing the value of $\Delta\alpha$ and estimated with Heisenberg scaling precision through homodyne measurement and the maximum-likelihood estimator (shown in Appendix E) which saturates asymptotically the Cramér-Rao bound (12).

For certain choices of $\Delta\alpha$, the function $f(\boldsymbol{\varphi};\Delta\alpha)$ becomes linear in $\boldsymbol{\varphi}$. For example, for $\Delta\alpha=\pi/2$, the beam splitters needed in the input stage $\hat{\mathcal{F}}_{\rm in}$ and output stage $\hat{\mathcal{F}}_{\rm out}$ are balanced, since from condition (15) one gets $\omega=\pm\pi/4$. For $\omega=+\pi/4$, the acquired phase becomes

$$f(\boldsymbol{\varphi};\pi/2)=\alpha_1+\alpha_3+\varphi_1+\frac{\delta\varphi_2+\delta\varphi_3}{2}, \tag{21}$$

while the value of ℓ in the Cramér-Rao bound (12) can be found by expanding for large N the transition probability $P_{\boldsymbol{\varphi}}=|\chi_{\boldsymbol{\varphi}}|^2$, with $\chi_{\boldsymbol{\varphi}}$ given in (19), $\ell=(k_2-k_3)^2/4$. In particular, if both φ_2 and φ_3 are perfectly known (i.e., $\varphi_2=\varphi_{2,\rm cl}$ and $\varphi_3=\varphi_{3,\rm cl}$) so that $\delta\varphi_2=\delta\varphi_3=0$, then the network in Fig. 2 reduces to the one in Fig. 3(b) and can be employed for the estimation of φ_1, namely, the parameter associated with the beam splitter, without requiring any prior information.

For $\Delta\alpha=0$ instead, condition (15) reads $\omega=-\pi/4-\varphi_{1,\rm cl}/2$, and the phase acquired becomes

$$\begin{aligned} f(\boldsymbol{\varphi};0) &= \alpha_1+\alpha_3+\frac{\delta\varphi_2+\delta\varphi_3}{2} \\ &\quad -\arctan\left(\tan\left(\frac{\delta\varphi_2-\delta\varphi_3}{2}\right)\frac{\sin(\varphi_1)}{\cos(\delta\varphi_1)}\right) \\ &= \alpha_1+\alpha_3+\frac{\delta\varphi_2+\delta\varphi_3}{2} \\ &\quad -\frac{\delta\varphi_2-\delta\varphi_3}{2}\sin(\varphi_1)+O(N^{-3/2}), \end{aligned} \tag{22}$$

where we exploited the fact that the errors $\delta\varphi_2$ and $\delta\varphi_3$ are of order $O(1/\sqrt{N})$, with $\ell=k_1^2+(k_2-k_3)^2\cos^2(\varphi_1)/4$. Noticeably, this is an example of estimation of a nonlinear function of the unknown parameters, namely, the transmittivity amplitude of the unknown beam splitter multiplied by an unknown parameter-dependent factor $(\delta\varphi_2+\delta\varphi_3)/2$ up to a constant, provided that $\delta\varphi_2\neq\delta\varphi_3\neq 0$.

If instead $\varphi_1=0$, this setup reduces to a Mach-Zehnder interferometer with balanced beam splitters, and the function (20) becomes the average of the two remaining unknown parameters $\delta\varphi_2$ and $\delta\varphi_3$. In the following section, we show a generalization of this case, in which an M-channel Mach-Zehnder-like interferometer can be exploited for the estimation of an arbitrary linear combination with positive weights of phase shifts, or beam-splitter reflectivities as displayed in (21) when φ_2 and φ_3 are known.

B. Convex sum of any number of phase shifts and of beam-splitter reflectivities

We finally show an interferometer (see Fig. 3) which allows Heisenberg-scaling precision in the estimation of any linear combination with non-negative weights

$$L(\boldsymbol{\varphi})=\sum_{i=1}^{M}\omega_i\varphi_i \tag{23}$$

of M independent and unknown quantities $\boldsymbol{\varphi}=(\varphi_1,\dots,\varphi_M)$, which can be both optical phases acquired through single-channel phase shifts $U_{\rm PS}(\varphi_i)=e^{i\varphi_i}$, or reflectivities of beam splitters $U_{\rm BS}(\varphi_i)=e^{i\varphi_i\sigma_y}$, with M non-negative coefficients $\boldsymbol{\omega}=(\omega_1,\dots,\omega_M)$. Applications for this type of estimations can be found in field-gradient inference or in the study of spatial properties or fluctuations of fields [8,34]. In our discussion, we will additionally suppose that the positive coefficients $\boldsymbol{\omega}$ sum to one, without any loss of generality: rescaling the estimated quantity by a constant factor would change the error by the same factor, hence without ruining the Heisenberg scaling.

As previously discussed, the probe employed is a single-mode squeezed state $\hat{S}_1(r)|{\rm vac}\rangle$, with $N=\sinh^2 r$ average number of photons. The probe is injected in the first port of a first auxiliary linear network $\hat{\mathcal{F}}_{\rm in}$ which scatters the photons into each channel of $\hat{U}_{\boldsymbol{\varphi}}$ with probabilities

$$|(\mathcal{F}_{\rm in})_{i1}|^2=\omega_i, \tag{24}$$

where the normalization of the positive coefficients allows for the unitarity of $\mathcal{F}_{\rm in}$. The network $\hat{U}_{\boldsymbol{\varphi}}$ encodes all the unknown parameters $\boldsymbol{\varphi}$, each one associated with a different channel. For each beam splitter with an unknown reflectivity, an auxiliary two-channel $\boldsymbol{\varphi}$-independent network V [see Fig. 3(b)] is employed in order to turn its parameter into an optical phase. In particular, the network

$$V^\dagger U_{\mathrm{BS}}(\varphi)V = U_{\mathrm{PS}}(\varphi, -\varphi), \tag{25}$$

with

$$V = U_{\mathrm{PS}}(\pi/4, -\pi/4)U_{\mathrm{BS}}(\pi/4), \tag{26}$$

acts as a phase shift on both channels. Thus, when a signal is fed into the first port of this network, all the light comes out from the first output port shifted by a phase φ, so it behaves as a single-channel phase shift of magnitude φ. Noticeably, the network V needed for this purpose is the same as employed in Sec. IV A, in the case of $\varphi_2 = \varphi_3 = 0$ for $\Delta\alpha = \pi/2$. It is worth mentioning that, if the signal is injected into the second channel of this local network, it will come out only from the second output channel shifted by $-\varphi$, hence allowing in this way a negative weight in the linear combination (23). At the output of $\hat{U}_{\boldsymbol{\varphi}}$, a second auxiliary passive and linear network $\hat{\mathcal{F}}_{\mathrm{out}}$ is employed, whose preparation requires a prior classical knowledge $\boldsymbol{\varphi}_{\mathrm{cl}} = (\varphi_{1,\mathrm{cl}}, \dots, \varphi_{M,\mathrm{cl}})$ of the unknown parameters, meaning that the errors $\delta\varphi_i = \varphi_i - \varphi_{i,\mathrm{cl}} = k_i/\sqrt{N}$ are of order of $O(1/\sqrt{N})$, with k_1 constants independent of N, a transformation that can be regarded as a single-step phase feedback operation [40]. In particular, the signal in the ith channel must undergo a phase shift of $-\varphi_{i,\mathrm{cl}}$. Then, the probe is refocused on a single channel by inverting the action of $\hat{\mathcal{F}}_{\mathrm{in}}$, and it undergoes a phase shift of constant and known magnitude $L(\boldsymbol{\varphi}_{\mathrm{cl}})$, so that the scattering matrix of the output stage reads

$$\mathcal{F}_{\mathrm{out}} = U_{\mathrm{PS}}(L(\boldsymbol{\varphi}_{\mathrm{cl}}), 0, \dots, 0)\mathcal{F}_{\mathrm{in}}^\dagger U_{\mathrm{PS}}(-\boldsymbol{\varphi}_{\mathrm{cl}}), \tag{27}$$

where we denoted with $U_{\mathrm{PS}}(\lambda_1, \dots, \lambda_l) = \mathrm{diag}(e^{i\lambda_1}, \dots, e^{i\lambda_l})$. Finally, single-mode homodyne detection is performed according to condition (9).

With this setup, the probability amplitude, shown in (2), reads

$$\chi_{\boldsymbol{\varphi}} = e^{iL(\boldsymbol{\varphi}_{\mathrm{cl}})}\sum_{i=1}^{M}\omega_i e^{i\delta\varphi_i} = \\ = e^{iL(\boldsymbol{\varphi}_{\mathrm{cl}})}\left(1 + \sum_{i=1}^{M} i\omega_i\delta\varphi_i - \frac{1}{2}\sum_{i=1}^{M}\omega_i\delta\varphi_i^2\right) + O(N^{-3/2}). \tag{28}$$

The shot-noise scaling $\delta\varphi_i = O(N^{-1/2})$ implies that the one-photon probability

$$P_{\boldsymbol{\varphi}} = \left|e^{iL(\boldsymbol{\varphi}_{\mathrm{cl}})}\sum_{i=1}^{M}\omega_i e^{i\delta\varphi_i}\right|^2 \\ = 1 + \left(\sum_{i=1}^{M}\omega_i\delta\varphi_i\right)^2 - \sum_{i=1}^{M}\omega_i\delta\varphi_i^2 + O(N^{-3/2}) \\ \equiv 1 - \frac{\ell}{N} + O(N^{-3/2}) \tag{29}$$

satisfies condition (8), with $\ell = (\sum_i \omega_i k_i)^2 - \sum_i \omega_i k_i^2$, so that Heisenberg scaling sensitivity in the estimation of the total acquired phase $f(\boldsymbol{\varphi}) = \arg(\chi_{\boldsymbol{\varphi}})$ can be achieved. In particular,

$$f(\boldsymbol{\varphi}) = L(\boldsymbol{\varphi}_{\mathrm{cl}}) + \sum_{i=1}^{M}\omega_i\delta\varphi_i + O(N^{-3/2}) \\ = L(\boldsymbol{\varphi}) + O(N^{-3/2}), \tag{30}$$

so that it is possible to recover the linear combination (23) with Heisenberg-scaling precision from the estimation of $f(\boldsymbol{\varphi})$. Notice that, although $L(\boldsymbol{\varphi})$ in (23) and $f(\boldsymbol{\varphi})$ in (30) are not exactly equal, they differ by a quantity of order $O(N^{-3/2})$, which is beyond the Heisenberg resolution and thus negligible for our estimation purposes.

We conclude this section with an insightful observation. As already discussed, the local networks (25) shown in Fig. 3(b), appearing inside $U_{\boldsymbol{\varphi}}$, whose purpose is to translate the unknown beam-splitter reflectivities into optical phase shifts, can be obtained from the setup in Fig. 2 by imposing $\Delta\alpha = \pi/2$, when $\varphi_2 = \varphi_3 = 0$. It is indeed possible to further generalize this scheme by replacing the local networks (25) with the more general network in Fig. 2, and only requiring conditions (15) to hold for each local network (see Appendix F). This allows us to estimate with Heisenberg-scaling precision linear combinations of functions of unknown local parameters of the type shown in (20), with each function parametrized by an arbitrary local quantity $\Delta\alpha_i$.

V. CONCLUSIONS

We provided a feasible metrologic strategy to estimate functions of multiple parameters encoded arbitrarily in a M-channel linear network with Heisenberg-scaling precision in the average number of photons injected. Our scheme is experimentally feasible since it only requires a single-mode squeezed-vacuum state which, in general, is scattered by a first auxiliary network, interacts with the interferometer, and eventually is refocused by a second auxiliary network in a single output channel, where homodyne detection takes place. We show that, in order for the refocusing to be successful, a prior knowledge of the unknown parameters is required for the realization of only one of the two stages (i.e., $\hat{\mathcal{F}}_{\mathrm{out}}$) and only with a precision at shot-noise level, so that a classical estimation strategy is sufficient. Remarkably, the remaining degrees of freedoms in the stage $\hat{\mathcal{F}}_{\mathrm{out}}$ and all those in the parameter-independent stage, in this case $\hat{\mathcal{F}}_{\mathrm{in}}$, can be used to manipulate the functional expression of the unknown parameters depending on which function we are interested to estimate with Heisenberg-limited precision. We provide as examples two interferometric schemes, and we demonstrate how the choice of the auxiliary stages influences the functions of the parameters that is possible to estimate with Heisenberg-scaling precision: in the first, we examine a nontrivial two-channel linear network with unknown phase shifts and beam splitters, and we show how a whole family of functions, in general nonlinear in the parameters and parametrized by an arbitrary relative phase in the auxiliary stages, can be estimated with Heisenberg-scaling precision. In the second, we show that it is possible to estimate any linear combination of phase shifts and beam-splitter reflectivities, with the only requirement being for the linear combination coefficients to be positive. Our results are strongly relevant in experimental scenarios where we are interested to measure the global properties of a given network associated with particular functional dependencies from its parameters with no constraints on their values.

ACKNOWLEDGMENTS

We thank Frank A. Narducci for useful discussions. This work was supported by the Office of Naval Research Global (N62909-18-1-2153). P.F. is partially supported by Istituto Nazionale di Fisica Nucleare (INFN) through the project QUANTUM, and by the Italian National Group of Mathematical Physics (GNFM-INdAM).

APPENDIX A: VARIANCE OF THE QUADRATURE $\hat{x}_\theta$

To evaluate the variance (5) along the quadrature $\hat{x}_\theta$ of the squeezed vacuum state after the action of the interferometer, we first recall the covariance matrix Γ_0 of the input state $\hat{S}_1(r)|\text{vac}\rangle$, which reads

$$\Gamma_0 = \frac{1}{2}\begin{pmatrix} e^{2\mathcal{R}} & 0 \\ 0 & e^{-2\mathcal{R}} \end{pmatrix}, \tag{A1}$$

where $\mathcal{R}$ is the $M \times M$ diagonal matrix $\mathcal{R} = \text{diag}(r, 0, \dots, 0)$. After the action of the interferometer, the covariance matrix transforms into

$$\Gamma_{\boldsymbol{\varphi}} = R_{\boldsymbol{\varphi}} \Gamma_0 R_{\boldsymbol{\varphi}}^T, \tag{A2}$$

where $R_{\boldsymbol{\varphi}}$ is the orthogonal and symplectic matrix associated with the interferometer unitary matrix $u_{\boldsymbol{\varphi}} = \mathcal{F}_{\text{out}} U_{\boldsymbol{\varphi}} \mathcal{F}_{\text{in}}$:

$$R_{\boldsymbol{\varphi}} = \begin{pmatrix} \text{Re}u_{\boldsymbol{\varphi}} & -\text{Im}u_{\boldsymbol{\varphi}} \\ \text{Im}u_{\boldsymbol{\varphi}} & \text{Re}u_{\boldsymbol{\varphi}} \end{pmatrix}. \tag{A3}$$

Thus $\Gamma_{\boldsymbol{\varphi}}$ in (A2) reads

$$\Gamma_{\boldsymbol{\varphi}} = \begin{pmatrix} \Delta X_{\boldsymbol{\varphi}}^2 & \Delta XP_{\boldsymbol{\varphi}} \\ \Delta XP_{\boldsymbol{\varphi}}^T & \Delta P_{\boldsymbol{\varphi}}^2 \end{pmatrix}, \tag{A4}$$

where

$$\begin{aligned}\Delta X_{\boldsymbol{\varphi}}^2 &\equiv \frac{1}{2}[\text{Re}u_{\boldsymbol{\varphi}} e^{2\mathcal{R}} \text{Re}u_{\boldsymbol{\varphi}}^\dagger - \text{Im}u_{\boldsymbol{\varphi}} e^{-2\mathcal{R}} \text{Im}u_{\boldsymbol{\varphi}}^\dagger] \\ &= \frac{1}{2}\big\{\text{Re}[u_{\boldsymbol{\varphi}} \cosh{(2\mathcal{R})} u_{\boldsymbol{\varphi}}^\dagger] + \text{Re}\big[u_{\boldsymbol{\varphi}} \sinh{(2\mathcal{R})} u_{\boldsymbol{\varphi}}^\text{T}\big]\big\},\end{aligned} \tag{A5}$$

$$\begin{aligned}\Delta P_{\boldsymbol{\varphi}}^2 &\equiv \frac{1}{2}[-\text{Im}u_{\boldsymbol{\varphi}} e^{2\mathcal{R}} \text{Im}u_{\boldsymbol{\varphi}}^\dagger + \text{Re}u_{\boldsymbol{\varphi}} e^{-2\mathcal{R}} \text{Re}u_{\boldsymbol{\varphi}}^\dagger] \\ &= \frac{1}{2}\big\{\text{Re}[u_{\boldsymbol{\varphi}} \cosh{(2\mathcal{R})} u_{\boldsymbol{\varphi}}^\dagger] - \text{Re}\big[u_{\boldsymbol{\varphi}} \sinh{(2\mathcal{R})} u_{\boldsymbol{\varphi}}^\text{T}\big]\big\},\end{aligned} \tag{A6}$$

$$\begin{aligned}\Delta XP_{\boldsymbol{\varphi}} &\equiv \frac{1}{2}[-\text{Re}u_{\boldsymbol{\varphi}} e^{2\mathcal{R}} \text{Im}u_{\boldsymbol{\varphi}}^\dagger - \text{Im}u_{\boldsymbol{\varphi}} e^{-2\mathcal{R}} \text{Re}u_{\boldsymbol{\varphi}}^\dagger] \\ &= \frac{1}{2}\big\{-\text{Im}[u_{\boldsymbol{\varphi}} \cosh{(2\mathcal{R})} u_{\boldsymbol{\varphi}}^\dagger] + \text{Im}\big[u_{\boldsymbol{\varphi}} \sinh{(2\mathcal{R})} u_{\boldsymbol{\varphi}}^\text{T}\big]\big\}.\end{aligned} \tag{A7}$$

In the second lines of each of the previous expression, we have used the fact that $\mathcal{R}$ is real.

The 2×2 reduced covariance matrix $\Gamma'_{\boldsymbol{\varphi}}$ of the first mode reads

$$\Gamma'_{\boldsymbol{\varphi}} = \begin{pmatrix} (\Delta X_{\boldsymbol{\varphi}}^2)_{11} & (\Delta XP_{\boldsymbol{\varphi}})_{11} \\ (\Delta XP_{\boldsymbol{\varphi}})_{11} & (\Delta P_{\boldsymbol{\varphi}}^2)_{11} \end{pmatrix}. \tag{A8}$$

Our final step is to recover the variance of the quadrature $\hat{x}_\theta$. To do that, we introduce the 2×2 orthogonal and symplectic matrix

$$O_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \tag{A9}$$

representing the action of a phase shift $e^{-i\theta}$, namely, a clockwise rotation of an angle θ in the first mode phase space. The variance $\sigma_{\boldsymbol{\varphi}}^2$ in (5) is finally obtained by a direct computation

$$\begin{aligned}\sigma_{\boldsymbol{\varphi}}^2 &= \big(O_\theta \Gamma'_{\boldsymbol{\varphi}} O_\theta^\text{T}\big)_{11} \\ &= \frac{1}{2} + P_{\boldsymbol{\varphi}}\{\sinh^2{(r)} + \cos{[2f(\boldsymbol{\varphi}) - 2\theta]} \cosh{(r)} \sinh{(r)}\}.\end{aligned} \tag{A10}$$

APPENDIX B: PRIOR KNOWLEDGE OF THE PARAMETERS

In this Appendix we show that, with the setup presented in this work, a classical prior knowledge $\boldsymbol{\varphi}_\text{cl}$, with an error $\delta\boldsymbol{\varphi} = \boldsymbol{\varphi} - \boldsymbol{\varphi}_\text{cl}$ scaling as $1/\sqrt{N}$, is enough to correctly optimize our setup, satisfy condition (8), and thus ultimately reach Heisenberg scaling.

In general, in order to perform the optimization, the auxiliary stages $\hat{\mathcal{F}}_\text{in}$ and $\hat{\mathcal{F}}_\text{out}$ are chosen accordingly to a prior knowledge on the parameter, so that once the setup is correctly optimized, we can write $\hat{\mathcal{F}}_\text{in} \equiv \hat{\mathcal{F}}_\text{in}(\boldsymbol{\varphi}_\text{cl})$ and $\hat{\mathcal{F}}_\text{out} \equiv \hat{\mathcal{F}}_\text{out}(\boldsymbol{\varphi}_\text{cl})$. The one-photon transition probability $P_{\boldsymbol{\varphi}}$ is by definition the square modulus of the (complex) scalar product of the two M-dimensional normalized vectors $U_{\boldsymbol{\varphi}} \mathcal{F}_\text{in}(\boldsymbol{\varphi}_\text{cl}) \boldsymbol{e}_1$ and $\mathcal{F}_\text{out}^\dagger(\boldsymbol{\varphi}_\text{cl}) \boldsymbol{e}_1$, namely,

$$P_{\boldsymbol{\varphi}} = \big|\boldsymbol{e}_1^\dagger \mathcal{F}_\text{out}(\boldsymbol{\varphi}_\text{cl}) U_{\boldsymbol{\varphi}} \mathcal{F}_\text{in}(\boldsymbol{\varphi}_\text{cl}) \boldsymbol{e}_1\big|^2 \equiv \eta(\boldsymbol{\varphi}, \boldsymbol{\varphi}_\text{cl}), \tag{B1}$$

with $\boldsymbol{e}_1 = (1, 0, \dots, 0)^T$, and η is a smooth function of $\boldsymbol{\varphi}$ and $\boldsymbol{\varphi}_\text{cl}$ with global maxima along the condition $\boldsymbol{\varphi}_\text{cl} = \boldsymbol{\varphi}$ since, with a perfect prior knowledge of the parameters $\boldsymbol{\varphi}$, the auxiliary stages are chosen so that $|\boldsymbol{e}_1^\dagger \mathcal{F}_\text{out}(\boldsymbol{\varphi}) U_{\boldsymbol{\varphi}} \mathcal{F}_\text{in}(\boldsymbol{\varphi}) \boldsymbol{e}_1| = 1$. If a small uncertainty $\delta\boldsymbol{\varphi} = \boldsymbol{\varphi} - \boldsymbol{\varphi}_\text{cl}$ is present due to an imperfect prior knowledge on the parameter, then

$$\begin{aligned}P_{\boldsymbol{\varphi}} &= \eta(\boldsymbol{\varphi}, \boldsymbol{\varphi} - \delta\boldsymbol{\varphi}) \\ &= 1 + \frac{1}{2}\sum_{i,j=1}^{p} \big(\partial_{i,j}^2 \eta\big)_{\boldsymbol{\varphi}} \delta\varphi_i \delta\varphi_j + O(|\delta\boldsymbol{\varphi}|^3),\end{aligned} \tag{B2}$$

where $(\partial_{i,j}^2 \eta)_{\boldsymbol{\varphi}}$ is the second derivative of $\eta(\boldsymbol{\varphi}, \cdot)$ with respect of the ith and jth components evaluated at $\boldsymbol{\varphi}$. Comparing the expression of $P_{\boldsymbol{\varphi}}$ in (B2) with the condition (8), it is evident that the uncertainty allowed on the prior estimation, in order to correctly optimize $\hat{\mathcal{F}}_\text{out}$ and ultimately reach Heisenberg scaling independently of the value of $\boldsymbol{\varphi}$, must be of order $1/\sqrt{N}$, namely, it must happen that $\delta\varphi_i = O(N^{-1/2})$, for $i = 1, \dots, p$. It is straightforward to notice that these results also hold if one of the two auxiliary gates is chosen independently from $\boldsymbol{\varphi}_\text{cl}$, including the case of an identity operation corresponding to the absence of one gate.

APPENDIX C: DERIVATION OF THE FISHER INFORMATION MATRIX IN (10)

In this Appendix we obtain the expression of the Fisher information matrix (10) from the general Fisher information matrix for a Gaussian distribution (7) when conditions (8) and (9) hold.

Since the dependence of σ_{φ}^2 in (5) on the parameters $\boldsymbol{\varphi}$ only appears through P_{φ} and $f(\boldsymbol{\varphi})$, the gradient of the variance,

$$\begin{aligned}\nabla_{\varphi}\sigma_{\varphi}^2 &= \{[\nabla_{\varphi}P_{\varphi}]\partial_P + [\nabla_{\varphi}f(\boldsymbol{\varphi})]\partial_f\}\sigma_{\varphi}^2 \\ &= \nabla_{\varphi}P_{\varphi}\{\sinh^2 r + \cos[2f(\boldsymbol{\varphi}) - 2\theta]\sinh r\cosh r\} \\ &\quad - 2P_{\varphi}\nabla_{\varphi}f(\boldsymbol{\varphi})\sin[2f(\boldsymbol{\varphi}) - 2\theta]\sinh r\cosh r, \end{aligned} \tag{C1}$$

can be written in terms of $\nabla_{\varphi}P_{\varphi}$ and $\nabla_{\varphi}f(\boldsymbol{\varphi})$. We now impose condition (9) and fix $\theta = \theta_{\varphi}$, and evaluate the variance in (5) and its gradient (C1) in the large-N limit

$$\begin{aligned}\sigma_{\varphi}^2 &= \frac{1}{2} + NP_{\varphi}\left[1 - \cos\left(\frac{2k}{N}\right)\sqrt{1 + \frac{1}{N}}\right] \\ &= \frac{1}{2} + NP_{\varphi}\left[1 - \left(1 - \frac{2k^2}{N^2}\right)\left(1 + \frac{1}{2N} - \frac{1}{8N^2}\right)\right] \\ &\quad + O\left(\frac{1}{N^2}\right) \\ &= \frac{1 - P_{\varphi}}{2} + P_{\varphi}\left(\frac{2k^2}{N} + \frac{1}{8N}\right) + O\left(\frac{1}{N^2}\right),\end{aligned} \tag{C2}$$

$$\begin{aligned}\nabla_{\varphi}\sigma_{\varphi}^2 &= N\nabla_{\varphi}P_{\varphi}\left[1 - \cos\left(\frac{2k}{N}\right)\sqrt{1 + \frac{1}{N}}\right] \\ &\quad + 2NP_{\varphi}\nabla_{\varphi}f(\boldsymbol{\varphi})\sin\left(\frac{2k}{N}\right)\sqrt{1 + \frac{1}{N}} \\ &= N\nabla_{\varphi}P_{\varphi}\left[1 - \left(1 + \frac{1}{2N}\right)\right] \\ &\quad + 2NP_{\varphi}\nabla_{\varphi}f(\boldsymbol{\varphi})\frac{2k}{N} + O\left(\frac{1}{N}\right) \\ &= -\frac{1}{2}\nabla_{\varphi}P_{\varphi} + 4kP_{\varphi}\nabla_{\varphi}f(\boldsymbol{\varphi}) + O\left(\frac{1}{N}\right).\end{aligned} \tag{C3}$$

Then, by imposing condition (8) on P_{φ}, we get

$$\sigma_{\varphi}^2 = \left(2k^2 + \frac{1}{8} + \frac{\ell}{2}\right)\frac{1}{N} + O\left(\frac{1}{N^2}\right), \tag{C4}$$

$$\nabla_{\varphi}\sigma_{\varphi}^2 = 4k\nabla_{\varphi}f(\boldsymbol{\varphi}) + O\left(\frac{1}{N}\right). \tag{C5}$$

Therefore, the Fisher information matrix (7) can be asymptotically written as

$$\mathcal{I} = 8\varrho(k,\ell)N^2[\nabla f(\boldsymbol{\varphi})][\nabla f(\boldsymbol{\varphi})]^T, \tag{C6}$$

with

$$\varrho(k,\ell) = \left(\frac{8k}{16k^2 + 1 + 4\ell}\right)^2 \tag{C7}$$

a positive and N-independent prefactor.

APPENDIX D: PROOF OF HEISENBERG SCALING

In this Appendix we show that the Fisher information matrix $\mathcal{I}$ shown in (10) implies that the only functions of the parameter $\alpha(\boldsymbol{\varphi})$ which admit unbiased estimators with finite variance are of the form $\alpha(\boldsymbol{\varphi}) \equiv g(f(\boldsymbol{\varphi}))$, with $g(\cdot)$ a smooth function.

Since $\mathcal{I}$ in (C6) is a rank-one matrix, it has a single nonzero eigenvalue λ. By direct calculation, the nonzero eigenvalue associated with the (normalized) eigenvector

$$\boldsymbol{v} = \frac{\nabla f(\boldsymbol{\varphi})}{|\nabla f(\boldsymbol{\varphi})|} \tag{D1}$$

reads

$$\lambda = 8\varrho(k,\ell)|\nabla f(\boldsymbol{\varphi})|^2N^2. \tag{D2}$$

In order for a given function $\alpha(\boldsymbol{\varphi})$ of the parameters to admit an unbiased estimator $\tilde{\alpha}$ with finite variance, it must happen that $\nabla_{\varphi}\alpha(\boldsymbol{\varphi})$ belongs to the support of $\mathcal{I}$ [28,41], which in our case is one dimensional and spanned by $\boldsymbol{v}$, whence

$$\nabla_{\varphi}\alpha(\boldsymbol{\varphi}) \propto \nabla_{\varphi}f(\boldsymbol{\varphi}), \tag{D3}$$

which is verified for all $\boldsymbol{\varphi}$, only if $\alpha(\boldsymbol{\varphi}) = g(f(\boldsymbol{\varphi}))$, with $g(\cdot)$ a differentiable real-valued function. For every unbiased estimator $\tilde{\alpha} \equiv \tilde{g}$ of such functions, the Cramér-Rao bound reads

$$\mathrm{Var}[\tilde{g}] \geqslant \left(\frac{dg}{df}\right)^2\frac{1}{8\varrho(k,\ell)N^2}, \tag{D4}$$

and, for the particular case $g(f(\boldsymbol{\varphi})) = f(\boldsymbol{\varphi})$,

$$\mathrm{Var}[\tilde{f}] \geqslant \frac{1}{8\varrho(k,\ell)N^2}. \tag{D5}$$

APPENDIX E: MAXIMUM-LIKELIHOOD ESTIMATOR

In this Appendix we find the maximum-likelihood estimator $\tilde{f}_{\mathrm{MLE}}$ that saturates the Cramér-Rao bound in Eq. (12). Let us assume that, after ν independent iterations, the homodyne measurement of the quadrature field $\hat{x}_{\theta}$ yields the outcomes $\vec{x} = (x_1, \ldots, x_{\nu})$. Due to the independence of the measurements, the likelihood $\mathcal{L}(\boldsymbol{\varphi}|\vec{x})$, namely the probability that the set of outcomes $\vec{x}$ are observed as a function of the unknown parameters $\boldsymbol{\varphi}$, is given by

$$\mathcal{L}(\boldsymbol{\varphi}|\vec{x}) = \prod_{j=1}^{\nu}p(x_j|\boldsymbol{\varphi}), \tag{E1}$$

with $p(x|\boldsymbol{\varphi})$ given in Eq. (4). The estimator $\tilde{f}_{\mathrm{MLE}} \equiv \tilde{f}_{\mathrm{MLE}}(\vec{x})$ is defined as the value of $f(\boldsymbol{\varphi})$ that most likely yields the observed outcomes $\vec{x}$. To find that, we maximize the log-likelihood function, and thus obtain

$$\begin{aligned}0 &= \frac{\partial}{\partial f}\ln\mathcal{L}(\boldsymbol{\varphi}|\vec{x})\Big|_{f=\tilde{f}_{\mathrm{MLE}}} \\ &= \frac{\partial}{\partial f}\sum_{j=1}^{\nu}\ln p(x_j|\boldsymbol{\varphi})\Big|_{f=\tilde{f}_{\mathrm{MLE}}} \\ &= \frac{\partial}{\partial f}\sum_{j=1}^{\nu}\left(-\frac{1}{2}\ln\sigma_{\varphi}^2 - \frac{x_j^2}{2\sigma_{\varphi}^2}\right)\Bigg|_{f=\tilde{f}_{\mathrm{MLE}}}\end{aligned}$$

$$= \left[\frac{\partial \sigma_{\varphi}^2}{\partial f} \sum_{j=1}^{\nu} \left(-\frac{1}{2\sigma_{\varphi}^2} + \frac{x_j^2}{2\sigma_{\varphi}^4} \right) \right] \Bigg|_{f=\tilde{f}_{\rm MLE}}. \tag{E2}$$

By assuming that $\partial\sigma_{\varphi}^2/\partial f \neq 0$ the solution is given by the value of $f(\boldsymbol{\varphi})$ that solves

$$\sigma_{\varphi}^2 = \sigma^2(\vec{x}) \equiv \frac{1}{\nu} \sum_{j=1}^{\nu} x_j^2, \tag{E3}$$

where σ_{φ}^2 is given by (5), with P_{φ} given by (8) and θ being phase of the local oscillator. Thus the estimator is ultimately given by

$$\tilde{f}_{\rm MLE}(\vec{x}) = \theta + \frac{1}{2} \arccos \left(\frac{[2\sigma^2(\vec{x}) - 1] - 2P_{\varphi} \sinh^2 r}{2P_{\varphi} \sinh r \cosh r} \right). \tag{E4}$$

APPENDIX F: GENERALIZATION OF THE SETUP IN FIG. 3

In this Appendix we show that employing as local networks inside $\hat{U}_{\varphi}$ of Fig. 3, the network shown in Fig. 2, with conditions (15) satisfied, still yields a setup which allows Heisenberg-scaling sensitivity, this time for the estimation of a linear combination of *functions* of parameters. In fact, even though these local networks do not exactly behave as single-mode phase shifts, since (15) holds locally, they still satisfy some local conditions

$$P_i \sim 1 - \frac{\ell_i}{N}, \quad \ell_i \geqslant 0, \quad i = 1, \dots, m_2, \tag{F1}$$

similar to the global condition (8), where P_i in this case is the probability that a photon, injected in the first channel of the ith local network, comes out from its upper channel. Thus, the probability amplitude (28) generalizes to

$$\begin{aligned} \chi_{\varphi} &= e^{iL(\varphi_{\rm cl})} \sum_{i=1}^{M} \omega_i \sqrt{1 - \frac{\ell_i}{N}} e^{i\delta\varphi_i} \\ &= e^{iL[\varphi_{\rm cl})} \left[1 + \sum_{i=1}^{M} i\omega_i \delta\varphi_i - \frac{1}{2} \sum_{i=1}^{M} \omega_i \left(\delta\varphi_i^2 + \frac{\ell_i}{N} \right) \right] \\ &\quad + O(N^{-3/2}), \end{aligned} \tag{F2}$$

where we made use of condition (F1) to write the transition amplitudes associated with each channel of $\hat{U}_{\varphi}$. Exploiting once again the requirement that $\delta\varphi_i = O(N^{-1/2})$, for $i = 1, \dots, M$, we notice that the one-photon probability,

$$\begin{aligned} P_{\varphi} &= \left| e^{iL(\varphi_{\rm cl})} \sum_{i=1}^{M} \omega_i \sqrt{1 - \frac{\ell_i}{N}} e^{i\delta\varphi_i} \right|^2 \\ &= 1 + \left(\sum_{i=1}^{M} \omega_i \delta\varphi_i \right)^2 - \sum_{i=1}^{M} \omega_i \left(\delta\varphi_i^2 + \frac{\ell_i}{N} \right) + O(N^{-3/2}) \\ &\equiv 1 - \frac{\ell}{N} + O(N^{-3/2}) \end{aligned} \tag{F3}$$

still satisfies condition (8), so that Heisenberg-scaling sensitivity in the estimation of the total acquired phase shown in (30) is achieved.

[1] V. Giovannetti, S. Lloyd, and L. Maccone, Quantum-enhanced measurements: Beating the standard quantum limit, Science **306**, 1330 (2004).

[2] V. Giovannetti, S. Lloyd, and L. Maccone, Quantum Metrology, Phys. Rev. Lett. **96**, 010401 (2006).

[3] J. P. Dowling, Quantum optical metrology—the lowdown on high-N00N states, Contemp. Phys. **49**, 125 (2008).

[4] V. Giovannetti, S. Lloyd, and L. Maccone, Advances in quantum metrology, Nat. Photonics **5**, 222 (2011).

[5] J. P. Dowling and K. P. Seshadreesan, Quantum optical technologies for metrology, sensing, and imaging, J. Lightwave Technol. **33**, 2359 (2015).

[6] A. De Pasquale, P. Facchi, G. Florio, V. Giovannetti, K. Matsuoka, and K. Yuasa, Two-mode bosonic quantum metrology with number fluctuations, Phys. Rev. A **92**, 042115 (2015).

[7] S. Zhou, M. Zhang, J. Preskill, and L. Jiang, Achieving the Heisenberg limit in quantum metrology using quantum error correction, Nat. Commun. **9**, 78 (2018).

[8] W. Ge, K. Jacobs, Z. Eldredge, A. V. Gorshkov, and M. Foss-Feig, Distributed Quantum Metrology with Linear Networks and Separable Inputs, Phys. Rev. Lett. **121**, 043604 (2018).

[9] K. Qian, Z. Eldredge, W. Ge, G. Pagano, C. Monroe, J. V. Porto, and A. V. Gorshkov, Heisenberg-scaling measurement protocol for analytic functions with quantum sensor networks, Phys. Rev. A **100**, 042304 (2019).

[10] R. McConnell, G. H. Low, T. J. Yoder, C. D. Bruzewicz, I. L. Chuang, J. Chiaverini, and J. M. Sage, Heisenberg scaling of imaging resolution by coherent enhancement, Phys. Rev. A **96**, 051801(R) (2017).

[11] M. Unternährer, B. Bessire, L. Gasparini, M. Perenzoni, and A. Stefanov, Super-resolution quantum imaging at the Heisenberg limit, Optica **5**, 1150 (2018).

[12] A. De Pasquale and T. M. Stace, Quantum thermometry, *Thermodynamics in the Quantum Regime*, Fundamental Theories of Physics, Vol. 195, edited by F. Binder, L. Correa, C. Gogolin, J. Anders, and G. Adesso (Springer, Cham, 2018), pp. 503–527.

[13] S. Seah, S. Nimmrichter, D. Grimmer, J. P. Santos, V. Scarani, and G. T. Landi, Collisional Quantum Thermometry, Phys. Rev. Lett. **123**, 180602 (2019).

[14] L. Razzoli, L. Ghirardi, I. Siloi, P. Bordone, and M. G. A. Paris, Lattice quantum magnetometry, Phys. Rev. A **99**, 062330 (2019).

[15] S. Bhattacharjee, U. Bhattacharya, W. Niedenzu, V. Mukherjee, and A. Dutta, Quantum magnetometry using two-stroke thermal machines, New J. Phys. **22**, 013024 (2020).

[16] J. Aasi, J. Abadie, B. P. Abbott, R. Abbott, T. D. Abbott, M. R. Abernathy, C. Adams, T. Adams, P. Addesso, R. X. Adhikari, *et al.*, Enhanced sensitivity of the LIGO gravitational wave detector by using squeezed states of light, Nat. Photonics **7**, 613 (2013).

[17] A. Monras, Optimal phase measurements with pure Gaussian states, Phys. Rev. A **73**, 033821 (2006).

[18] L. Pezzé and A. Smerzi, Mach-Zehnder Interferometry at the Heisenberg Limit with Coherent and

Squeezed-Vacuum Light, Phys. Rev. Lett. **100**, 073601 (2008).

[19] M. Aspachs, J. Calsamiglia, R. Muñoz-Tapia, and E. Bagan, Phase estimation for thermal Gaussian states, Phys. Rev. A **79**, 033834 (2009).

[20] M. D. Lang and C. M. Caves, Optimal Quantum-Enhanced Interferometry Using a Laser Power Source, Phys. Rev. Lett. **111**, 173601 (2013).

[21] L. Maccone and A. Riccardi, Squeezing metrology, Quantum **4**, 292 (2020).

[22] T. Matsubara, P. Facchi, V. Giovannetti, and K. Yuasa, Optimal Gaussian metrology for generic multimode interferometric circuit, New J. Phys. **21**, 033014 (2019).

[23] C. Oh, C. Lee, C. Rockstuhl, H. Jeong, J. Kim, H. Nha, and S.-Y. Lee, Optimal Gaussian measurements for phase estimation in single-mode Gaussian metrology, npj Quantum Inf. **5**, 10 (2019).

[24] D. Gatto, P. Facchi, F. A. Narducci, and V. Tamma, Distributed quantum metrology with a single squeezed-vacuum source, Phys. Rev. Res. **1**, 032024(R) (2019).

[25] D. Gatto, P. Facchi, and V. Tamma, Phase space Heisenberg-limited estimation of the average phase shift in a Mach–Zehnder interferometer, Int. J. Quantum Inform. **18**, 1941019 (2020).

[26] G. Gramegna, D. Triggiani, P. Facchi, F. A. Narducci, and V. Tamma, Typicality of Heisenberg scaling precision in multimode quantum metrology, Phys. Rev. Res. **3**, 013152 (2021).

[27] G. Gramegna, D. Triggiani, P. Facchi, F. A. Narducci, and V. Tamma, Heisenberg scaling precision in multi-mode distributed quantum metrology, New J. Phys. **23**, 053002 (2021).

[28] J. A. Gross and C. M. Caves, One from many: Estimating a function of many parameters, J. Phys. A: Math. Theor. **54**, 014001 (2021).

[29] P. C. Humphreys, M. Barbieri, A. Datta, and I. A. Walmsley, Quantum Enhanced Multiple Phase Estimation, Phys. Rev. Lett. **111**, 070403 (2013).

[30] T. Baumgratz and A. Datta, Quantum Enhanced Estimation of a Multidimensional Field, Phys. Rev. Lett. **116**, 030801 (2016).

[31] M. Szczykulska, T. Baumgratz, and A. Datta, Multi-parameter quantum metrology, Adv. Phys.: X **1**, 621 (2016).

[32] M. Gessner, L. Pezzè, and A. Smerzi, Sensitivity Bounds for Multiparameter Quantum Metrology, Phys. Rev. Lett. **121**, 130503 (2018).

[33] Q. Zhuang, Z. Zhang, and J. H. Shapiro, Distributed quantum sensing using continuous-variable multipartite entanglement, Phys. Rev. A **97**, 032329 (2018).

[34] Z. Eldredge, M. Foss-Feig, J. A. Gross, S. L. Rolston, and A. V. Gorshkov, Optimal and secure measurement protocols for quantum sensor networks, Phys. Rev. A **97**, 042337 (2018).

[35] Y. Xia, W. Li, W. Clark, D. Hart, Q. Zhuang, and Z. Zhang, Demonstration of a Reconfigurable Entangled Radio-Frequency Photonic Sensor Network, Phys. Rev. Lett. **124**, 150502 (2020).

[36] X. Guo, C. R. Breum, J. Borregaard, S. Izumi, M. V. Larsen, T. Gehring, M. Christandl, J. S. Neergaard-Nielsen, and U. L. Andersen, Distributed quantum sensing in a continuous-variable entangled network, Nat. Phys. **16**, 281 (2020).

[37] H. Cramér, *Mathematical Methods of Statistics* (Princeton University Press, 1999), Vol. 9.

[38] S. Olivares and M. G. Paris, Bayesian estimation in homodyne interferometry, J. Phys. B: At., Mol. Opt. Phys. **42**, 055506 (2009).

[39] A. A. Berni, T. Gehring, B. M. Nielsen, V. Händchen, M. G. Paris, and U. L. Andersen, Ab initio quantum-enhanced optical phase estimation using real-time feedback control, Nat. Photonics **9**, 577 (2015).

[40] L. Pezzè and A. Smerzi, Quantum phase estimation algorithm with Gaussian spin states, PRX Quantum **2**, 040301 (2021).

[41] P. Stoica and T. L. Marzetta, Parameter estimation problems with singular information matrices, IEEE Trans. Signal Process. **49**, 87 (2001).