QP-PQ
Quantum Probability and White Noise Analysis
Volume XVII

Proceedings of the Japan–Italy Joint Workshop on Quantum
Open Systems, Quantum Chaos and Quantum Measurement

*F*undamental *A*spects
of
*Q*uantum *P*hysics

Waseda University, Tokyo, Japan   27 – 29 September 2001

Edited by

Luigi Accardi
University of Roma II, Italy

Shuichi Tasaki
Waseda University, Japan

FUNDAMENTAL ASPECTS OF QUANTUM PHYSICS
QP–PQ: Quantum Probability and White Noise Analysis — Vol. 17

QUANTUM ZENO EFFECT, ADIABATICITY
AND DYNAMICAL SUPERSELECTION RULES

PAOLO FACCHI

*Dipartimento di Fisica, Università di Bari I-70126 Bari, Italy
and Istituto Nazionale di Fisica Nucleare, Sezione di Bari, I-70126 Bari, Italy*

The evolution of a quantum system undergoing very frequent measurements takes place in a proper subspace of the total Hilbert space (quantum Zeno effect). When the measuring apparatus is included in the quantum description, the Zeno effect becomes a pure consequence of the dynamics. We show that for continuous measurement processes the quantum Zeno evolution derives from an adiabatic theorem. The system is forced to evolve in a set of orthogonal subspaces of the total Hilbert space and a dynamical superselection rule arises. The dynamical properties of this evolution are investigated and several examples are considered.

1 Introduction

The quantum Zeno effect[1,2] is a direct consequence of general features of the Schrödinger equation that yield quadratic behavior of the survival probability at short times.[3,4] It consists in the hindrance of the evolution of a quantum system when very frequent measurements are performed, in order to ascertain whether it is still in its initial state.

Following an interesting idea by Cook,[5] both the experimental and theoretical investigations of the last decade have dealt with oscillating (mainly, two-level) systems.[6] However, a few years ago, the presence of a short-time quadratic region was experimentally confirmed for an unstable quantum mechanical system[7] and the existence of the Zeno effect (as well as its inverse[8,9]) has been recently proved.[10]

Interestingly, the quantum Zeno effect (QZE) does not necessarily freeze everything. On the contrary, for frequent projections onto a multi-dimensional subspace, the system can evolve away from its initial state, although it remains in the subspace defined by the "measurement." This continuing time evolution within the projected subspace has been recently investigated and called *quantum Zeno dynamics*.[11] It involves interesting, yet unsettled, physical and mathematical[12,13,2] issues.

The above-mentioned investigations deal with "pulsed" measurements, according to von Neumann's projection postulate.[14] However, from a physical point of view, a "measurement" is nothing but an interaction with an external system (another quantum object, or a field, or simply a different degree of freedom of the very system investigated), playing the role of apparatus. In

this sense von Neumann's postulate can be considered as a useful shorthand notation, summarizing the effect of the quantum measurement.

By including the apparatus in the quantum description, several authors,[6,16,17] during the last two decades, have demonstrated the QZE without making use of projection operators (and non-unitary dynamics). In particular, the QZE has been reformulated in terms of "continuous" measurements,[4,15] obtaining the same physical effects (as well as a quantitative comparison with the "pulsed" situation[18]) in terms of a continuous (eventually strong) coupling to an external agent.

The studies of the last few years pave the way to interesting possible applications of the QZE. Indeed, we have a physical and mathematical framework that enable us to analyze the modification of the evolution of a quantum system and possibly to tailor the interaction in order to slow the evolution down (or eventually accelerate it). The potential importance of such a scheme cannot be underestimated.

It is therefore important to understand in more details which features of the coupling between the "observed" system and the "measuring" apparatus are needed to obtain a QZE. In other words, one wants to know when an external quantum system can be considered a good apparatus and why. The purpose of the present article is to clarify these issues and cast the quantum Zeno evolution in terms of an adiabatic theorem. We will show that the evolution of a quantum system under the action of a continuous measurement process can be profoundly modified: the system is forced to evolve in a set of orthogonal subspaces of the total Hilbert space and an effective superselection rule arises in the strong coupling limit. These *quantum Zeno subspaces*[19] are just the eigenspaces (belonging to different eigenvalues) of the Hamiltonian describing the interaction between the system and the apparatus: they are subspaces that the measurement process is able to distinguish.

The general ideas will be applied to some relevant examples. Some interesting issues and possible applications will be discussed in details.

2   Pulsed measurements

Let Q be a quantum system, whose states belong to the Hilbert space $\mathcal{H}$ and whose evolution is described by the unitary operator $U(t) = \exp(-iHt)$, where H is a time-independent lower-bounded Hamiltonian. Let P be a projection operator and $\text{Ran}P = \mathcal{H}_P$ its range. We assume that the initial density matrix ρ_0 of system Q belongs to $\mathcal{H}_P$:

$$\rho_0 = P\rho_0 P, \qquad \text{Tr}[\rho_0 P] = 1. \tag{1}$$

Under the action of the Hamiltonian H (i.e., if no measurements are performed in order to get information about the quantum state), the state at time t reads

$$\rho(t) = U(t)\rho_0 U^\dagger(t) \tag{2}$$

and the *survival probability*, namely the probability that the system is still in $\mathcal{H}_P$ at time t, is

$$p(t) = \text{Tr}\left[U(t)\rho_0 U^\dagger(t)P\right]. \tag{3}$$

No distinction is made between one- and many-dimensional projections.

The above evolution is "undisturbed," in the sense that the quantum systems evolves only under the action of its Hamiltonian for a time t, without undergoing any measurement process. Assume, on the other hand, that we do perform a selective measurement at time τ, in order to check whether Q has survived inside $\mathcal{H}_P$. By this, we mean that we select the survived component and stop the other one. The state of Q changes (up to a normalization constant) into

$$\rho_0 \to \rho(\tau) = PU(\tau)\rho_0 U^\dagger(\tau)P \tag{4}$$

and the survival probability in $\mathcal{H}_P$ is

$$p(\tau) = \text{Tr}\left[U(\tau)\rho_0 U^\dagger(\tau)P\right] = \text{Tr}\left[V(\tau)\rho_0 V^\dagger(\tau)\right], \qquad V(\tau) \equiv PU(\tau)P. \tag{5}$$

We stress that the measurement occurs instantaneously (this is the essence of von Neumann's projection postulate).[14]

The QZE is the following. We prepare Q in the initial state ρ_0 at time 0 and perform a series of P-observations at time intervals $\tau = t/N$. The state of Q at time t reads (up to a normalization constant)

$$\rho^{(N)}(t) = V_N(t)\rho_0 V_N^\dagger(t), \qquad V_N(t) \equiv [PU(t/N)P]^N \tag{6}$$

and the survival probability in $\mathcal{H}_P$ is given by

$$p^{(N)}(t) = \text{Tr}\left[V_N(t)\rho_0 V_N^\dagger(t)\right]. \tag{7}$$

Equations (6)-(7) are the formal statement of the QZE, according to which very frequent observations modify the dynamics of the quantum system: under general conditions, if N is sufficiently large, all transitions outside $\mathcal{H}_P$ are inhibited. Notice that the dynamics (6)-(7) is *not reversible.*

We emphasize that close scrutiny of the features of the survival probability has clarified that if N is not too large the system can display an inverse Zeno effect,[8,9,20] by which decay is accelerated. Both effects have recently been seen in the same experimental setup.[10] We will not elaborate on this here.

2.1 Misra and Sudarshan's theorem

In order to consider the $N \to \infty$ limit ("continuous observation"), one needs some mathematical requirements: assume that the limit operator

$$\mathcal{V}(t) \equiv \lim_{N \to \infty} V_N(t) \tag{8}$$

exists (in the strong sense) for $t > 0$. The final state of Q is then

$$\rho(t) = \lim_{N \to \infty} \rho^{(N)}(t) = \mathcal{V}(t)\rho_0\mathcal{V}^{\dagger}(t) \tag{9}$$

and the probability to find the system in $\mathcal{H}_P$ is

$$\mathcal{P}(t) \equiv \lim_{N \to \infty} p^{(N)}(t) = \text{Tr}\left[\mathcal{V}(t)\rho_0\mathcal{V}^{\dagger}(t)\right]. \tag{10}$$

By assuming the strong continuity of $\mathcal{V}(t)$ at $t = 0$

$$\lim_{t \to 0+} \mathcal{V}(t) = P, \tag{11}$$

one can prove that under general conditions the operators

$$\mathcal{V}(t) \quad \text{exist for all real } t \text{ and form a semigroup.} \tag{12}$$

Moreover, by time-reversal invariance

$$\mathcal{V}^{\dagger}(t) = \mathcal{V}(-t), \tag{13}$$

so that $\mathcal{V}^{\dagger}(t)\mathcal{V}(t) = P$. This implies, by (1), that

$$\mathcal{P}(t) = \text{Tr}\left[\rho_0\mathcal{V}^{\dagger}(t)\mathcal{V}(t)\right] = \text{Tr}\left[\rho_0 P\right] = 1. \tag{14}$$

If the particle is "continuously" observed, in order to check whether it has survived inside $\mathcal{H}_P$, it will never make a transition to $\mathcal{H}_P^{\perp}$ (QZE).

Two important remarks are now in order: first, it is not clear whether the dynamics in the $N \to \infty$ limit is time reversible. Although one ends up, in general, with a semigroup, there are concrete elements of reversibility in the above equations.[11] Second, the theorem just summarized *does not* state that the system *remains* in its initial state, after the series of very frequent measurements. Rather, the system evolves in the subspace $\mathcal{H}_P$, instead of evolving "naturally" in the total Hilbert space $\mathcal{H}$.

2.2   Complete measurements

Let us first consider the particular case of complete selective measurements. $\mathcal{H}_P$ has dimension 1 and the initial state is a pure (normalized) state $|a\rangle$:

$$P_a = |a\rangle\langle a|, \qquad \rho_0 = |a\rangle\langle a|. \tag{15}$$

The time evolution operator (6) after N measurements in a time interval t reads

$$V_N(t) = [P_a U(t/N) P_a]^N = [P_a \langle a|U(t/N)|a\rangle]^N = P_a \mathcal{A}(t/N)^N, \tag{16}$$

where $\mathcal{A}(t)$ is the (undisturbed) *survival amplitude* in state $|a\rangle$ at time t

$$\mathcal{A}(t) = \langle a|e^{-iHt}|a\rangle. \tag{17}$$

Therefore, in this case the problem of the existence of the limit operator $\mathcal{V}(t)$ is reduced to the existence of the limit function $\lim_N \mathcal{A}(t/N)^N$. Let $\tau = t/N$ be the time interval between two successive measurements. We can write

$$\mathcal{A}\left(\frac{t}{N}\right)^N = \mathcal{A}(\tau)^N = \exp\left[N \log \mathcal{A}(\tau)\right] = \exp\left[t\,\frac{\log \mathcal{A}(\tau)}{\tau}\right]$$

$$= \exp\left[-t\left(\frac{\gamma(\tau)}{2} + i\omega(\tau)\right)\right], \tag{18}$$

where

$$\gamma(\tau) = -\frac{1}{\tau}\log|\mathcal{A}(\tau)|^2, \qquad \omega(\tau) = -\frac{1}{\tau}\arg \mathcal{A}(\tau), \tag{19}$$

and the "observed" survival probability has a purely exponential decay with an effective rate $\gamma(\tau)$:

$$p_a^{(N)}(t) = |\mathcal{A}(\tau)|^{2N} = \exp[-\gamma(\tau)t]. \tag{20}$$

From (16) and (18) one sees that $\mathcal{V}(t)$ in (8) exists (in the strong sense), for $t > 0$, if and only if $\gamma(0^+)$ and $\omega(0^+)$ exist and are finite [if $\gamma(0^+) = +\infty$, the existence of the limit $\omega(0^+)$ is irrelevant], and it reads

$$\mathcal{V}(t) = P_a \exp\left[-\left(\frac{\gamma(0^+)}{2} + i\omega(0^+)\right)t\right]. \tag{21}$$

Notice that, for one-dimensional projections, when $\mathcal{V}(t)$ exists [and $\gamma(0^+) < \infty$ so that $\mathcal{V}(t) \neq 0$], the strong continuity in the origin $\mathcal{V}(0^+) = P_a$ [see Eq. (11)] follows from the very existence of (21) and need not be assumed as an independent hypothesis.

A sufficient condition for the existence of $\mathcal{V}(t)$ is that the initial state belongs to the domain of the Hamiltonian, $|a\rangle \in D(H)$. Indeed, in such a case, the first and second moment of the Hamiltonian exist,

$$\langle a|H^2|a\rangle = \|H|a\rangle\|^2 < \infty, \qquad \langle a|H|a\rangle = E_a < \infty, \qquad (22)$$

and the survival amplitude has the following asymptotic behavior at short times:[a]

$$\mathcal{A}(\tau) \sim 1 - iE_a\tau - \langle a|H^2|a\rangle\frac{\tau^2}{2}, \qquad \tau \to 0, \qquad (23)$$

i.e. the survival probability exhibits a short-time quadratic behavior,

$$p_a(\tau) = |\mathcal{A}(\tau)|^2 \sim 1 - \frac{\tau^2}{\tau_Z^2}, \qquad \tau \to 0, \qquad (24)$$

where

$$\tau_Z \equiv \frac{1}{\sqrt{\langle a|H^2|a\rangle - \langle a|H|a\rangle^2}} \qquad (25)$$

is called the *Zeno time*. Therefore, by plugging (23) into (19),

$$\gamma(\tau) \sim \frac{\tau}{\tau_Z^2}, \qquad \omega(\tau) \sim E_a, \qquad \tau \to 0, \qquad (26)$$

and one gets

$$\mathcal{V}(t) = P_a \exp\left(-iE_a t\right), \qquad \rho(t) = \rho_0, \qquad \mathcal{P}_a(t) = 1. \qquad (27)$$

More and more frequent measurements hinder the evolution and eventually freeze it. This is the quantum Zeno paradox: "A watched pot never boils".

Notice that $\mathcal{V}(t)$ in (27) form a strongly continuous one-parameter unitary group within $\mathcal{H}_P$. Therefore, starting from the dynamics (16), which is irreversible and probability-nonconserving, one ends up with a fully reversible evolution. Reversibility is recovered in the limit.

For one-dimensional projections, interesting behaviors can be obtained (in the very limit $N \to \infty$) only by relaxing the hypothesis $|a\rangle \in D(H)$. In such a case the survival probability, at variance with (24), can be no longer quadratic at short times and one can obtain different results depending on its short-time behavior.[21] Assume, for example, that

$$p_a(\tau) \sim 1 - \left(\frac{\tau}{\tau_c}\right)^\alpha, \qquad \text{for} \quad \tau \to 0, \qquad (28)$$

---

[a]More precisely, without further information on the third moment, $\mathcal{A}(\tau) = 1 - iE_a\tau - \langle a|H^2|a\rangle\tau^2/2 + o(\tau^2)$, whence $\gamma(\tau) = \tau/\tau_Z^2 + o(\tau)$.

for some $\alpha > 0$. The effective decay rate (19) becomes

$$\gamma(\tau) \sim \frac{\tau^{\alpha-1}}{\tau_c^\alpha}, \qquad \text{for} \quad \tau \to 0. \tag{29}$$

When $\alpha > 1$, $\gamma(0^+) = 0$ and the limit (27) is recovered again. On the other hand, for $0 < \alpha < 1$, $\gamma(0^+) = +\infty$, and the limit operator $\mathcal{V}(t)$ in Eq. (21) vanishes. In such a case irreversibility is still present in the limit and probability is *immediately* lost. Notice that in this case the limiting operator is not continuous at $t = 0$, $\mathcal{V}(0^+) = 0 \neq \mathcal{V}(0) = P_a$, whence Misra and Sudarshan's theorem does not apply. [Note: from (16), $\mathcal{V}(0) = \lim_N V_N(0) = P_a$.]

An interesting case arises at the threshold value $\alpha = 1$, when $\gamma(0^+) = 1/\tau_c \neq 0$. Irreversibility is still present in the limit, but it manifests itself in a gentler way, through the limit operator

$$\mathcal{V}(t) = P_a \exp(-t/2\tau_c), \qquad \text{for} \quad t \geq 0: \tag{30}$$

probability is not conserved and the Zeno paradox does not arise. Notice that in Eq. (30) we had to assume $\omega(0^+) < \infty$, in order to assure the existence of $\mathcal{V}(t)$.

2.3 Incomplete measurements

In the case of incomplete measurements some outcomes are lumped together, for example, because the experimental equipment has insufficient resolution.[22] Therefore, the projection operator P, which selects a particular lump, is many-dimensional. Let us first consider a finite dimensional $\mathcal{H}_P = \text{Ran}P$,

$$\text{Tr}P = s, \qquad \text{with} \quad s < \infty. \tag{31}$$

The resulting time evolution operator is just a generalization of (16) to a finite dimensional matrix and has the explicit form [see (27)]

$$\mathcal{V}(t) = \lim_{N \to \infty} V_N(t) = \lim_{N \to \infty} [PU(t/N)P]^N = P\exp(-iPHPt). \tag{32}$$

As shown in Sec. 2.2, if $\mathcal{H}_P \subset D(H)$, then $\mathcal{V}(t)$ in (32) is unitary within $\mathcal{H}_P$ and is generated by a resulting self-adjoint Hamiltonian PHP.[23] Again, reversibility is recovered in the limit.

For infinite dimensional projection, $s = \infty$, one can always formally write the limiting evolution in the form (32) but has to define the meaning of PHP. In such a case the time evolution operator $\mathcal{V}(t)$ may be not unitary and interesting phenomena can arise, related to the self-adjointness of the limiting

Hamiltonian PHP. This an interesting problem[11,12] that will not be discussed here.

In general, for incomplete selective measurements, the system Q does not remain in its initial state, after the series of very frequent measurements. Rather, the system is confined (and evolves) in the subspace $\mathcal{H}_P$, instead of evolving "naturally" in the total Hilbert space $\mathcal{H}$.

2.4 Nonselective measurements

We will now consider the case of nonselective measurements and extend Misra and Sudarshan's theorem[2] in order to accomodate multiple projectors and to build a bridge for our subsequent discussion. For nonselective measurements the measuring apparatus functions, but no selection of outcomes is performed,[24] and all "beams" go through the whole Zeno dynamics. Let $\{P_n\}_n$,

$$P_n P_m = \delta_{mn} P_n, \qquad \sum_n P_n = 1, \tag{33}$$

be a (countable) collection of projection operators and $\mathrm{Ran} P_n = \mathcal{H}_{P_n}$ the relative subspaces. This induces a partition on the total Hilbert space

$$\mathcal{H} = \bigoplus_n \mathcal{H}_{P_n}. \tag{34}$$

Consider the associated nonselective measurement described by the superoperator

$$\hat{P}\rho = \sum_n P_n \rho P_n. \tag{35}$$

The free evolution reads

$$\hat{U}_t \rho_0 = U(t) \rho_0 U^\dagger(t), \qquad U(t) = \exp(-iHt) \tag{36}$$

and the Zeno evolution after N measurements in a time t is governed by the superoperator

$$\hat{V}_t^{(N)} = \hat{P}\left(\hat{U}\left(t/N\right)\hat{P}\right)^{N-1}, \tag{37}$$

which yields

$$\rho(t) = \hat{V}_t^{(N)} \rho_0 = \sum_{n_1,\ldots,n_N} V_{n_1\ldots n_N}^{(N)}(t)\, \rho_0\, V_{n_1\ldots n_N}^{(N)\dagger}(t), \tag{38}$$

where

$$V_{n_1\ldots n_N}^{(N)}(t) = P_{n_N} U\left(t/N\right) P_{n_{N-1}} \cdots P_{n_2} U\left(t/N\right) P_{n_1}. \tag{39}$$

We follow Misra and Sudarshan[2] and assume, as in Sec. 2.1, the existence of the strong limits ($t > 0$)

$$\mathcal{V}_n(t) = \lim_{N \to \infty} V_{n\ldots n}^{(N)}(t), \qquad \lim_{t \to 0^+} \mathcal{V}_n(t) = P_n, \quad \forall n . \tag{40}$$

Then $\mathcal{V}_n(t)$ exist for all real t and form a semigroup,[2] and

$$\mathcal{V}_n^\dagger(t)\mathcal{V}_n(t) = P_n. \tag{41}$$

Moreover, it is easy to show that

$$\lim_{N \to \infty} V_{n\ldots n'\ldots}^{(N)}(t) = 0, \qquad \text{for} \quad n' \neq n. \tag{42}$$

Therefore the final state is

$$\rho(t) = \hat{\mathcal{V}}_t \rho_0 = \sum_n \mathcal{V}_n(t)\rho_0 \mathcal{V}_n^\dagger(t), \quad \text{with} \quad \sum_n \mathcal{V}_n^\dagger(t)\mathcal{V}_n(t) = \sum_n P_n = 1. \tag{43}$$

The components $\mathcal{V}_n(t)\rho_0\mathcal{V}_n^\dagger(t)$ make up a block diagonal matrix: the initial density matrix is reduced to a mixture and any interference between different subspaces $\mathcal{H}_{P_n}$ is destroyed (complete decoherence). In conclusion,

$$p_n(t) = \text{Tr}\left[\rho(t)P_n\right] = \text{Tr}\left[\rho_0 P_n\right] = p_n(0), \quad \forall n. \tag{44}$$

In words, probability is conserved in each subspace and no probability "leakage" between any two subspaces is possible. The total Hilbert space splits into invariant subspaces and the different components of the wave function (or density matrix) evolve independently within each sector. One can think of the total Hilbert space as the shell of a tortoise, each invariant subspace being one of the scales. Motion among different scales is impossible. (See Fig. 1 in the following.)

If $\text{Tr} P_n = s_n < \infty$, then the limiting evolution operator $\mathcal{V}_n(t)$ (40) within the subspace $\mathcal{H}_{P_n}$ has the form (32),

$$\mathcal{V}_n(t) = P_n \exp(-iP_n H P_n t), \tag{45}$$

is unitary in $\mathcal{H}_{P_n}$ and the resulting Hamiltonian $P_n H P_n$ is self-adjoint, provided that $\mathcal{H}_{P_n} \subset D(H)$.

The original limit result (14) is reobtained when $p_n = 1$ for some n, in (44): the initial state is then in one of the invariant subspaces and the survival probability in that subspace remains unity. However, even if the limits are the same, notice that the setup described here is conceptually different from that of Sec. 2.1. Indeed, the dynamics (39) allows transitions among different subspaces $\mathcal{H}_{P_n} \to \mathcal{H}_{P_m}$, while the dynamics (6) completely forbids them. Therefore, for finite N, (39) takes into account the possibility that

one subspace $\mathcal{H}_{P_n}$ gets *repopulated*[25,4] after the system has made transitions to other subspaces, while in (6) the system must be found in $\mathcal{H}_{P_n}$ at every measurement.

3   Dynamical quantum Zeno effect

Our whole discussion has dealt so far with "pulsed" measurements, according to von Neumann's projection postulate.[14] However, from a physical point of view, a "measurement" is nothing but an interaction with an external system (another quantum object, or a field, or simply another degree of freedom of the very system investigated), playing the role of apparatus. We emphasize that in such a case the QZE is a consequence of the dynamical features (i.e. the form factors) of the coupling between the system investigated and the external system, and no use is made of projection operators (and non-unitary dynamics). The idea of "continuous" measurement in a QZE context has been proposed several times during the last two decades,[6,16,17] although the first quantitative comparison with the "pulsed" situation is rather recent.[18]

We consider therefore a purely dynamical evolution, by including the detector in the quantum description. In general,[4,15] one can consider the Hamiltonian

$$H_K = H + K H_{\text{meas}}, \tag{46}$$

where H is the Hamiltonian of the system under observation (and can include the free Hamiltonian of the apparatus, $H = H_{\text{sys}} + H_{\text{det}}$) and H_{meas} is the interaction Hamiltonian between the system and the apparatus, K representing the strength of the measurement or, equivalently, the inverse response time of the apparatus (see examples in Sec. 4).

3.1   A theorem

We now state a theorem,[26,19] which is the exact analog of Misra and Sudarshan's theorem for a dynamical evolution of the type (46). Consider the time evolution operator

$$U_K(t) = \exp(-iH_K t). \tag{47}$$

We will prove that in the "infinitely strong measurement" ("infinitely quick detector") limit $K \to \infty$ the evolution operator

$$\mathcal{U}(t) = \lim_{K \to \infty} U_K(t), \tag{48}$$

becomes diagonal with respect to H_{meas}:

$$[\mathcal{U}(t), P_n] = 0, \qquad \text{where} \quad H_{\text{meas}} P_n = \eta_n P_n, \tag{49}$$

P_n being the orthogonal projection onto $\mathcal{H}_{P_n}$, the eigenspace of H_{meas} belonging to the eigenvalue η_n. Note that in Eq. (49) one has to consider distinct eigenvalues, i.e., $\eta_n \neq \eta_m$ for $n \neq m$, whence the $\mathcal{H}_{P_n}$'s are in general multidimensional.

Moreover, the limiting evolution operator has the explicit form

$$\mathcal{U}(t) = \exp[-i(H^{\text{diag}} + K H_{\text{meas}})t], \tag{50}$$

where

$$H^{\text{diag}} = \sum_n P_n H P_n \tag{51}$$

is the diagonal part of the system Hamiltonian H with respect to the interaction Hamiltonian H_{meas}.

It is worth noticing that the limiting evolution (48) is understood in the sense of the intertwining relations (49), that is

$$\lim_{K \to \infty} \left(U_K P_n - P_n U_K \right) = 0, \tag{52}$$

while, strictly speaking, each single addend has no limit, due to a fast oscillating phase. In other words, one should read Eq. (50) as

$$U_K(t) = \exp[-i(H^{\text{diag}} + K H_{\text{meas}})t] + o(1), \qquad \text{for} \quad K \to \infty. \tag{53}$$

3.2 Dynamical superselection rules

Before proving the theorem of Sec. 3.1 let us briefly consider its physical implications. In the $K \to \infty$ limit, due to (49), the time evolution operator becomes diagonal with respect to H_{meas}, namely

$$[\mathcal{U}(t), H_{\text{meas}}] = 0, \tag{54}$$

a superselection rule arises and the total Hilbert space is split into subspaces $\mathcal{H}_{P_n}$ which are invariant under the evolution. These subspaces are simply defined by the P_n's, i.e., they are eigenspaces belonging to distinct eigenvalues η_n: in other words, *subspaces that the apparatus is able to distinguish*. On the other hand, due to (51), the dynamics within each Zeno subspace $\mathcal{H}_{P_n}$ is governed by the diagonal part $P_n H P_n$ of the system Hamiltonian H. The evolution reads

$$\rho(t) = \mathcal{U}(t) \rho_0 \mathcal{U}^\dagger(t) = e^{-i(H^{\text{diag}} + K H_{\text{meas}})t} \rho_0 e^{i(H^{\text{diag}} + K H_{\text{meas}})t} \tag{55}$$

and the probability to find the system in $\mathcal{H}_{P_n}$

$$
\begin{aligned}
p_n(t) &= \mathrm{Tr}\left[\rho(t)P_n\right] = \mathrm{Tr}\left[\mathcal{U}(t)\rho_0\mathcal{U}^\dagger(t)P_n\right] = \mathrm{Tr}\left[\mathcal{U}(t)\rho_0 P_n\mathcal{U}^\dagger(t)\right] \\
&= \mathrm{Tr}\left[\rho_0 P_n\right] = p_n(0)
\end{aligned}
\tag{56}
$$

is constant. As a consequence, if the initial state of the system belongs to a specific sector, it will be forced to remain there forever (QZE):

$$
\psi_0 \in \mathcal{H}_{P_n} \to \psi(t) \in \mathcal{H}_{P_n}.
\tag{57}
$$

More generally, if the initial state is an incoherent superposition of the form $\rho_0 = \hat{P}\rho_0$, with $\hat{P}$ defined in (35), then each component will evolve separately, according to

$$
\begin{aligned}
\rho(t) &= \mathcal{U}(t)\rho_0\mathcal{U}^\dagger(t) = \sum_n e^{-i(H^{\mathrm{diag}}+KH_{\mathrm{meas}})t}P_n\rho_0 P_n e^{i(H^{\mathrm{diag}}+KH_{\mathrm{meas}})t} \\
&= \sum_n e^{-iP_n HP_n t}P_n\rho_0 P_n e^{iP_n HP_n t} = \sum_n \mathcal{V}_n(t)\rho_0\mathcal{V}_n^\dagger(t),
\end{aligned}
\tag{58}
$$

with $\mathcal{V}_n(t) = P_n \exp(-iP_n HP_n t)$, which is exactly the same result (43)-(45) found for the case of nonselective pulsed measurements. This bridges the gap with the description of Sec. 2.4 and clarifies the role of the detection apparatus. In Fig. 1 we endeavored to give a pictorial representation of the decomposition of the Hilbert space as K is increased.

3.3 Proof of the theorem

We will now use perturbation theory and prove[26] that the limiting evolution operator has the form (50). From that, property (49) follows. In the next section we will give a more direct proof of (49), which relies on the adiabatic theorem.

Rewrite the time evolution operator in the form

$$
U_K(t) = \exp(-iH_K t) = \exp(-iH_\lambda \tau) = U_\lambda(\tau)
\tag{59}
$$

where

$$
\lambda = 1/K, \qquad \tau = Kt = t/\lambda, \qquad H_\lambda = \lambda H_K = H_{\mathrm{meas}} + \lambda H,
\tag{60}
$$

and apply perturbation theory to the Hamiltonian H_λ for small λ. To this end, choose the unperturbed degenerate projections $P_{n\alpha}$

$$
H_{\mathrm{meas}}P_{n\alpha} = \eta_n P_{n\alpha}, \qquad P_n = \sum_\alpha P_{n\alpha},
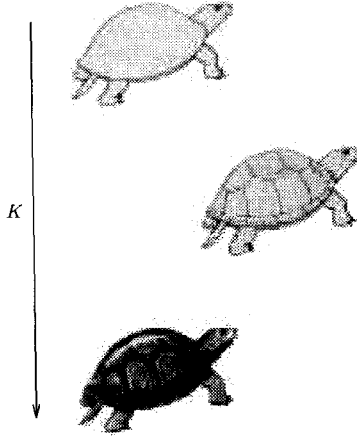\tag{61}
$$

Figure 1. The Hilbert space of the system: a superselection rule appears as the coupling K to the apparatus is increased.

whose degeneration α is resolved at some order in the coupling constant λ. This means that by denoting with $\widetilde{\eta}_{n\alpha}$ and $\widetilde{P}_{n\alpha}$ the eigenvalues and the orthogonal projections of the total Hamiltonian H_λ

$$H_\lambda \widetilde{P}_{n\alpha} = \widetilde{\eta}_{n\alpha} \widetilde{P}_{n\alpha}, \tag{62}$$

they reduce to the unperturbed ones when the perturbation vanishes

$$\widetilde{P}_{n\alpha} \xrightarrow{\lambda \to 0} P_{n\alpha}, \qquad \widetilde{\eta}_{n\alpha} \xrightarrow{\lambda \to 0} \eta_n. \tag{63}$$

Therefore, by applying standard perturbation theory,[27] we get the eigenprojections

$$\begin{aligned} \widetilde{P}_{n\alpha} &= P_{n\alpha} + \lambda P_{n\alpha}^{(1)} + \mathrm{O}(\lambda^2) \\ &= P_{n\alpha} + \lambda \left(\frac{Q_n}{a_n} H P_{n\alpha} + P_{n\alpha} H \frac{Q_n}{a_n} \right) + \mathrm{O}(\lambda^2), \end{aligned} \tag{64}$$

where

$$Q_n = 1 - P_n = \sum_{m \neq n} P_m, \qquad \frac{Q_n}{a_n} = \frac{Q_n}{\eta_n - H_{\mathrm{meas}}} = \sum_{m \neq n} \frac{P_m}{\eta_n - \eta_m}. \tag{65}$$

The perturbative expansion of the eigenvalues reads

$$\widetilde{\eta}_{n\alpha} = \eta_n + \lambda \eta_{n\alpha}^{(1)} + \lambda^2 \eta_{n\alpha}^{(2)} + \mathrm{O}(\lambda^3) \tag{66}$$

where

$$\eta_{n\alpha}^{(1)} P_{n\alpha} = P_{n\alpha} H P_{n\alpha}, \qquad \eta_{n\alpha}^{(2)} P_{n\alpha} = P_{n\alpha} H \frac{Q_n}{a_n} H P_{n\alpha}, \tag{67}$$

with the choice

$$P_{n\alpha} H P_{n\beta} = P_{n\alpha} H \frac{Q_n}{a_n} H P_{n\beta} = 0, \qquad \alpha \neq \beta. \tag{68}$$

Write now the spectral decomposition of the evolution operator (59) in terms of the projections $\widetilde{P}_{n\alpha}$

$$U_\lambda(\tau) = \exp(-iH_\lambda \tau) \sum_{n,\alpha} \widetilde{P}_{n\alpha} = \sum_{n,\alpha} \exp(-i\widetilde{\eta}_{n\alpha}\tau) \widetilde{P}_{n\alpha} \tag{69}$$

and plug in the perturbation expansions (64), to obtain

$$U_\lambda(\tau) = \sum_{n,\alpha} e^{-i\widetilde{\eta}_{n\alpha}\tau} P_{n\alpha}$$

$$+\lambda \sum_{n,\alpha} \left(\frac{Q_n}{a_n} H P_{n\alpha} e^{-i\widetilde{\eta}_{n\alpha}\tau} + e^{-i\widetilde{\eta}_{n\alpha}\tau} P_{n\alpha} H \frac{Q_n}{a_n} \right) + O(\lambda^2). \tag{70}$$

Let us define a new operator $\widetilde{H}_\lambda$ as

$$\widetilde{H}_\lambda = \sum_{n,\alpha} \widetilde{\eta}_{n\alpha} P_{n\alpha}$$

$$= H_{\text{meas}} + \lambda \sum_n P_n H P_n + \lambda^2 \sum_n P_n H \frac{Q_n}{a_n} H P_n + O(\lambda^3), \tag{71}$$

where Eqs. (66)-(68) were used. By plugging Eq. (71) into Eq. (70) and making use of the property

$$\sum_n P_n H \frac{Q_n}{a_n} = -\sum_n \frac{Q_n}{a_n} H P_n, \tag{72}$$

we finally obtain

$$U_\lambda(\tau) = \exp(-i\widetilde{H}_\lambda \tau) + \lambda \left[\sum_n \frac{Q_n}{a_n} H P_n, \ \exp(-i\widetilde{H}_\lambda \tau) \right] + O(\lambda^2), \tag{73}$$

Now, by recalling the definition (60), we can write the time evolution operator $U_K(t)$ as the sum of two terms

$$U_K(t) = U_{\text{ad},K}(t) + \frac{1}{K} U_{\text{na},K}(t), \tag{74}$$

where

$$U_{\mathrm{ad},K}(t) = e^{-i(KH_{\mathrm{meas}}+\sum_n P_n H P_n+\frac{1}{K}\sum_n P_n H \frac{Q_n}{a_n} H P_n +\mathrm{O}(K^{-2}))t} \tag{75}$$

is a diagonal, *adiabatic* evolution and

$$U_{\mathrm{na},K}(t) = \left[\sum_n \frac{Q_n}{a_n} H P_n,\ U_{\mathrm{ad},K}(t)\right] + \mathrm{O}\left(K^{-1}\right) \tag{76}$$

is the off-diagonal, *nonadiabatic* correction. In the $K \to \infty$ limit only the adiabatic term survives and one obtains

$$U_K(t) = U_{\mathrm{ad},K}(t) + \mathrm{O}\left(K^{-1}\right) = e^{-i\left(KH_{\mathrm{meas}}+\sum_n P_n H P_n\right)t} + \mathrm{O}\left(K^{-1}\right), \tag{77}$$

which is formula (50) [and implies also (49)]. The proof is complete. As a byproduct we get the corrections to the exact limit, valid for large, but finite, values of K.

3.4 Zeno evolution from an adiabatic theorem

We now give an alternative proof [and a generalization to time-dependent Hamiltonians $H(t)$] of Eq. (49). The adiabatic theorem deals with the time evolution operator $U(t)$ when the Hamiltonian $H(t)$ depends slowly on time. The traditional formulation[27] replaces the physical time t by the scaled time $s = t/T$ and considers the solution of the scaled Schrödinger equation

$$i\frac{d}{ds}U_T(s) = TH(s)U_T(s) \tag{78}$$

in the $T \to \infty$ limit.

Given a family $P(s)$ of smooth spectral projections of $\dot{H}(s)$

$$H(s)P(s) = E(s)P(s), \tag{79}$$

the adiabatic time evolution $U_{\mathrm{A}}(s) = \lim_{T\to\infty} U_T(s)$ has the intertwining property[28,27]

$$U_{\mathrm{A}}(s)P(0) = P(s)U_{\mathrm{A}}(s), \tag{80}$$

that is, $U_{\mathrm{A}}(s)$ maps $\mathcal{H}_{P(0)}$ onto $\mathcal{H}_{P(s)}$.

Theorem (49) and its generalization,

$$\mathcal{U}(t)P_n(0) = P_n(t)\mathcal{U}(t), \tag{81}$$

valid for generic time dependent Hamiltonians,

$$H_K(t) = H(t) + KH_{\mathrm{meas}}(t), \tag{82}$$

are easily proven by recasting them in the form of an adiabatic theorem.[19] In the H interaction picture, given by

$$i\frac{d}{dt}U_S(t) = HU_S(t), \qquad H^I_{meas}(t) = U^\dagger_S(t)H_{meas}U_S(t), \qquad (83)$$

the Schrödinger equation reads

$$i\frac{d}{dt}U^I_K(t) = KH^I_{meas}(t)\, U^I_K(t). \qquad (84)$$

The Zeno evolution pertains to the $K \to \infty$ limit. And in such a limit Eq. (84) has exactly the same form of the adiabatic evolution (78): the large coupling K limit corresponds to the large time T limit and the physical time t to the scaled time $s = t/T$. Therefore, let us consider a spectral projection of $H^I_{meas}(t)$,

$$P^I_n(t) = U^\dagger_S(t)P_n(t)U_S(t), \qquad (85)$$

such that

$$H^I_{meas}(t)P^I_n(t) = \eta_n(t)P^I_n(t), \qquad H_{meas}(t)P_n(t) = \eta_n(t)P_n(t). \qquad (86)$$

The limiting operator

$$\mathcal{U}^I(t) = \lim_{K\to\infty} U^I_K(t) \qquad (87)$$

has the intertwining property (80)

$$\mathcal{U}^I(t)P^I_n(0) = P^I_n(t)\mathcal{U}^I(t), \qquad (88)$$

i.e. maps $\mathcal{H}_{P^I_n(0)}$ onto $\mathcal{H}_{P^I_n(t)}$:

$$\psi^I_0 \in \mathcal{H}_{P^I_n(0)} \to \psi^I(t) \in \mathcal{H}_{P^I_n(t)}. \qquad (89)$$

In the Schrödinger picture the limiting operator

$$\mathcal{U}(t) = \lim_{K\to\infty} U_S(t)U^I_K(t) = U_S(t)\mathcal{U}^I(t) \qquad (90)$$

satisfies the intertwining property (81) [see (85)]

$$\mathcal{U}(t)P_n(0) = U_S(t)\mathcal{U}^I(t)P_n(0) = U_S(t)\mathcal{U}^I(t)P^I_n(0)$$
$$= U_S(t)P^I_n(t)\mathcal{U}^I(t) = P_n(t)U_S(t)\mathcal{U}^I(t) = P_n(t)\mathcal{U}(t), \qquad (91)$$

and maps $\mathcal{H}_{P_n(0)}$ onto $\mathcal{H}_{P_n(t)}$:

$$\psi_0 \in \mathcal{H}_{P_n(0)} \to \psi(t) \in \mathcal{H}_{P_n(t)}. \qquad (92)$$

The probability to find the system in $\mathcal{H}_{P_n(t)}$,

$$p_n(t) = \text{Tr}\left[P_n(t)\mathcal{U}(t)\rho_0\mathcal{U}^\dagger(t)\right] = \text{Tr}\left[\mathcal{U}(t)P_n(0)\rho_0\mathcal{U}^\dagger(t)\right]$$
$$= \text{Tr}\left[P_n(0)\rho_0\right] = p_n(0), \tag{93}$$

is constant: if the initial state of the system belongs to a given sector, it will be forced to remain there forever (QZE).

For a time independent Hamiltonian $H_{\text{meas}}(t) = H_{\text{meas}}$, the projections are constant, $P_n(t) = P_n$, hence Eq. (81) reduces to (49) and the above property holds *a fortiori* and reduces to (56).

3.5 Generalizations

The formulation of a Zeno dynamics in terms of an adiabatic theorem is powerful. Indeed one can use all the machinery of adiabatic theorems in order to get results in this context. An interesting extension would be to consider time-dependent measurements

$$H_{\text{meas}} = H_{\text{meas}}(t), \tag{94}$$

whose spectral projections $P_n = P_n(t)$ have a nontrivial time evolution. In this case, instead of confining the quantum state to a fixed sector, one can transport it along a given path (subspace) $\mathcal{H}_{P_n(t)}$, according to Eqs. (92)-(93). One then obtains a dynamical generalization of the process pioneered by Von Neumann in terms of projection operators.[14,29]

4 Applications

As a first example, consider the Hamiltonian

$$H_{3\text{lev}} = H + KH_{\text{meas}} = \begin{pmatrix} 0 & \Omega & 0 \\ \Omega & 0 & K \\ 0 & K & 0 \end{pmatrix}, \tag{95}$$

describing a two-level system, with Hamiltonian

$$H = \Omega(|1\rangle\langle 2| + |2\rangle\langle 1|) = \Omega\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{96}$$

coupled to a third one, that plays the role of measuring apparatus:

$$H_{\text{meas}} = |2\rangle\langle 3| + |3\rangle\langle 2| = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{97}$$

This example was considered by Peres.[16] One expects the third level to perform better as a measuring apparatus when the coupling K becomes larger. Indeed, if initially the system is in state $|1\rangle$, the survival probability reads[4]

$$p_0(t) = \frac{1}{K_1^4}\left[K^2 + \Omega^2 \cos(K_1 t)\right]^2 \xrightarrow{K\to\infty} 1, \qquad K_1 = \sqrt{K^2 + \Omega^2}\,. \qquad (98)$$

In spite of its simplicity, this model clarifies the physical meaning of a "continuous" measurement performed by an "external apparatus" (which can even be another degree of freedom of the system investigated). Also, it captures many interesting features of a Zeno dynamics. Indeed, as K is increased, the Hilbert space is split into three invariant subspaces $\mathcal{H} = \bigoplus \mathcal{H}_{P_n}$, the three eigenspaces of H_{meas}:

$$\mathcal{H}_{P_0} = \{|1\rangle\}, \quad \mathcal{H}_{P_1} = \{(|2\rangle + |3\rangle)/\sqrt{2}\}, \quad \mathcal{H}_{P_{-1}} = \{(|2\rangle - |3\rangle)/\sqrt{2}\}, \quad (99)$$

corresponding to projections

$$P_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_1 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad P_{-1} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}, \quad (100)$$

with eigenvalues $\eta_0 = 0$ and $\eta_{\pm 1} = \pm 1$. Therefore the diagonal part of the system Hamiltonian H vanishes, $H^{\text{diag}} = \sum P_n H P_n = 0$, the Zeno evolution is governed by

$$H^{\text{diag}} + K H_{\text{meas}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & K \\ 0 & K & 0 \end{pmatrix} \qquad (101)$$

and any transition between $|1\rangle$ and $|2\rangle$ is inhibited: a watched pot never boils.

Second example: consider

$$H_{4\text{lev}} = \Omega\sigma_1 + K\tau_1 + K'\tau_1' = \begin{pmatrix} 0 & \Omega & 0 & 0 \\ \Omega & 0 & K & 0 \\ 0 & K & 0 & K' \\ 0 & 0 & K' & 0 \end{pmatrix}, \qquad (102)$$

where states $|1\rangle$ and $|2\rangle$ make Rabi oscillations,

$$\Omega\sigma_1 = \Omega(|2\rangle\langle 1| + |1\rangle\langle 2|) = \Omega\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad (103)$$

while state $|3\rangle$ "observes" them,

$$K\tau_1 = K(|3\rangle\langle 2| + |2\rangle\langle 3|) = K \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{104}$$

and state $|4\rangle$ "observes" whether level $|3\rangle$ is populated,

$$K'\tau_1' = K'(|4\rangle\langle 3| + |3\rangle\langle 4|) = K' \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{105}$$

If $K \gg \Omega$ and K', then the total Hilbert space is split into the three eigenspaces of τ_1 [compare with (99)]:

$$\mathcal{H}_{P_0} = \{|1\rangle, |4\rangle\}, \quad \mathcal{H}_{P_1} = \{(|2\rangle + |3\rangle)/\sqrt{2}\}, \quad \mathcal{H}_{P_{-1}} = \{(|2\rangle - |3\rangle)/\sqrt{2}\}, \tag{106}$$

the Zeno evolution is governed by

$$H_{4\text{lev}}^{\text{diag}} = K\tau_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & K & 0 \\ 0 & K & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{107}$$

and the Rabi oscillations between states $|1\rangle$ and $|2\rangle$ are hindered. On the other hand, if $K' \gg K \gg \Omega$, the total Hilbert space is instead divided into the three eigenspaces of τ_1' [notice the differences with (106)]:

$$\mathcal{H}_{P_0'} = \{|1\rangle, |2\rangle\}, \quad \mathcal{H}_{P_1'} = \{(|3\rangle + |4\rangle)/\sqrt{2}\}, \quad \mathcal{H}_{P_{-1}'} = \{(|3\rangle - |4\rangle)/\sqrt{2}\}, \tag{108}$$

the Zeno Hamiltonian reads

$$H_{4\text{lev}}^{\text{diag}'} = \Omega\sigma_1 + K'\tau_1' = \begin{pmatrix} 0 & \Omega & 0 & 0 \\ \Omega & 0 & 0 & 0 \\ 0 & 0 & 0 & K' \\ 0 & 0 & K' & 0 \end{pmatrix} \tag{109}$$

and the Ω oscillations are fully *restored* (in spite of $K \gg \Omega$).[30] A watched cook can freely watch a boiling pot.

Third example (decoherence-free subspaces[31] in quantum computation). The Hamiltonian[32]

$$H_{\text{meas}} = ig \sum_{i=1}^{2} \left(b \, |2\rangle_{ii}\langle 1| - b^\dagger \, |1\rangle_{ii}\langle 2| \right) - i\kappa b^\dagger b \tag{110}$$

describes a system of two $(i = 1, 2)$ three-level atoms in a cavity. The atoms are in a Λ configuration with split ground states $|0\rangle_i$ and $|1\rangle_i$ and excited state $|2\rangle_i$, while the cavity has a single resonator mode b in resonance with the atomic transition 1-2. Spontaneous emission inside the cavity is neglected, but a photon leaks out through the nonideal mirrors with a rate κ.

The excitation number $\mathcal{N}$,

$$\mathcal{N} = \sum_{i=1,2} |2\rangle_{ii}\langle 2| + b^\dagger b, \tag{111}$$

commutes with the Hamiltonian,

$$[H_{\text{meas}}, \mathcal{N}] = 0. \tag{112}$$

Therefore we can solve the eigenvalue equation inside each eigenspace of $\mathcal{N}$.

A comment is now in order. Strictly speaking, the Hamiltonian (110) is non-Hermitian and we can not apply directly the theorem of Sec. 3.1. (Notice that the proof of the theorem heavily hinges upon Hermiticity of Hamiltonians and unitarity of evolutions.) However, we can enlarge our Hilbert space $\mathcal{H}$, by including the photon modes outside the cavity a_ω and their coupling with the cavity mode b. The enlarged dynamics is generated by the *Hermitian* Hamiltonian

$$\tilde{H}_{\text{meas}} = ig \sum_{i=1}^{2} \left(b \, |2\rangle_{ii}\langle 1| - b^\dagger \, |1\rangle_{ii}\langle 2| \right)$$

$$+ \int d\omega \, \omega a_\omega^\dagger a_\omega + \sqrt{\frac{\kappa}{\pi}} \int d\omega \left[a_\omega^\dagger b + a_\omega b^\dagger \right]. \tag{113}$$

It is easy to show that the evolution engendered by $\tilde{H}_{\text{meas}}$, when projected back to $\mathcal{H}$, is given by the effective non-Hermitian Hamiltonian (110), provided the field outside the cavity is initially in the vacuum state. Notice that any complex eigenvalue of H_{meas} engenders a dissipation of $\mathcal{H}$ into the enlarged Hilbert space embedding it. On the other hand, any real eigenvalue of H_{meas} generates a unitary dynamics which preserves probability within $\mathcal{H}$. Hence it is also an eigenvalue of $\tilde{H}_{\text{meas}}$ and its eigenvectors are the eigenvectors of the restriction $\tilde{H}_{\text{meas}}|_{\mathcal{H}}$. Therefore, as a general rule, the theorem of Sec. 3.1 can be applied also to non-Hermitian measurement Hamiltonians $\mathcal{H}_{\text{meas}}$, provided one restricts one's attention only to their real eigenvalues.

The eigenspace $\mathcal{S}_0$ corresponding to $\mathcal{N} = 0$ is spanned by four vectors

$$\mathcal{S}_0 = \{|000\rangle, |001\rangle, |010\rangle, |011\rangle\}, \tag{114}$$

where $|0j_1j_2\rangle$ denotes a state with no photons in the cavity and the atoms in state $|j_1\rangle_1|j_2\rangle_2$. The restriction of H_{meas} to $\mathcal{S}_0$ is the null operator

$$H_{\text{meas}}|_{\mathcal{S}_0} = 0, \tag{115}$$

hence $\mathcal{S}_0$ is a subspace of the eigenspace $\mathcal{H}_{P_0}$ of H_{meas} belonging to the eigenvalue $\eta_0 = 0$

$$\mathcal{S}_0 \subset \mathcal{H}_{P_0}, \qquad H_{\text{meas}}P_0 = 0. \tag{116}$$

The eigenspace $\mathcal{S}_1$ corresponding to $\mathcal{N} = 1$ is spanned by eight vectors

$$\mathcal{S}_1 = \{|020\rangle, |002\rangle, |100\rangle, |110\rangle, |101\rangle, |021\rangle, |012\rangle, |111\rangle\}, \tag{117}$$

and the restriction of H_{meas} to $\mathcal{S}_1$ is represented by the 8-dimensional matrix

$$H_{\text{meas}}|_{\mathcal{S}_1} = \begin{pmatrix} 0 & 0 & 0 & ig & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & ig & 0 & 0 & 0 \\ 0 & 0 & -i\kappa & 0 & 0 & 0 & 0 & 0 \\ -ig & 0 & 0 & -i\kappa & 0 & 0 & 0 & 0 \\ 0 & -ig & 0 & 0 & -i\kappa & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & ig \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & ig \\ 0 & 0 & 0 & 0 & 0 & -ig & -ig & -i\kappa \end{pmatrix}. \tag{118}$$

The eigenvector $(|021\rangle - |012\rangle)/\sqrt{2}$ has eigenvalue $\eta_0 = 0$ and all the others have eigenvalues with negative imaginary parts. Moreover, all restrictions $H_{\text{meas}}|_{\mathcal{S}_n}$ with $n > 1$ have eigenvalues with negative imaginary parts. Indeed they are spanned by states containing at least one photon, which dissipates through nonideal mirrors, according to $-i\kappa b^\dagger b$ in (110). The only exception is state $|0, 2, 2\rangle$ of $\mathcal{S}_2$, but also in this case it easy to prove that all eigenstates of $H_{\text{meas}}|_{\mathcal{S}_2}$ dissipate. Therefore the eigenspace $\mathcal{H}_{P_0}$ of H_{meas} belonging to the eigenvalue $\eta_0 = 0$ is 5-dimensional and is spanned by

$$\mathcal{H}_{P_0} = \{|000\rangle, |001\rangle, |010\rangle, |011\rangle, (|021\rangle - |012\rangle)/\sqrt{2}\}, \tag{119}$$

If the coupling g and the cavity loss κ are sufficiently strong, any other weak Hamiltonian H added to (110) reduces to P_0HP_0 and changes the state of the system only *within* the decoherence-free subspace (119).

Fourth example. Let

$$H_{\text{decay}} = H + KH_{\text{meas}} = \begin{pmatrix} 0 & \tau_Z^{-1} & 0 \\ \tau_Z^{-1} & -i2/\tau_Z^2\gamma & K \\ 0 & K & 0 \end{pmatrix}. \tag{120}$$

This describes the spontaneous emission $|1\rangle \to |2\rangle$ of a system into a (structured) continuum, while level $|2\rangle$ is resonantly coupled to a third level $|3\rangle$.[4]

This case is also relevant for quantum computation, if one is interested in protecting a given subspace (level $|1\rangle$) from decoherence[31,32] by inhibiting spontaneous emission.[33] Here γ represents the decay rate to the continuum and τ_Z is the Zeno time (convexity of the initial quadratic region).

Notice that, in a certain sense, this situation is complementary to that in (110); here the measurement Hamiltonian H_{meas} is Hermitian, while the system Hamiltonian H is not. Again, we have to enlarge our Hilbert space, apply the theorem to the dilation and project back the Zeno evolution. As a result one can simply apply the theorem to the original Hamiltonian, for, in this case, H_{meas} has a complete set of orthogonal projections that univocally defines a partition of $\mathcal{H}$ into quantum Zeno subspaces. We shall elaborate further on this interesting aspect in a future work.

As the Rabi frequency K is increased one is able to hinder spontaneous emission from level $|1\rangle$ (to be protected) to level $|2\rangle$. However, in order to get an effective "protection" of level $|1\rangle$, one needs $K > 1/\tau_Z$. More to this, when the presence of the inverse Zeno effect is taken into account, this requirement becomes even more stringent[9] and yields $K > 1/\tau_Z^2\gamma$. Both these conditions can be very demanding for a real system subject to dissipation.[4,15,9] For instance, typical values for spontaneous decay in vacuum are $\gamma \simeq 10^9\text{s}^{-1}$, $\tau_Z^2 \simeq 10^{-29}\text{s}^2$ and $1/\tau_Z^2\gamma \simeq 10^{20}\text{s}^{-1}$.[34]

5   Conclusions

If very frequent measurements are performed on a quantum system, in order to ascertain whether it is still in its initial state, transitions to other states are hindered and the QZE takes place. This formulation of the QZE hinges upon the notion of pulsed measurements, according to von Neumann's projection postulate. However, as we have seen by means of several examples, a "measurement" is nothing but an interaction with an external system (another quantum object, or a field, or simply another degree of freedom of the very system investigated), playing the role of apparatus. This enables one to reformulate the QZE in terms of a (strong) coupling to an external agent and to cast the quantum Zeno evolution in terms of an adiabatic theorem. There are many interesting examples, varying from quantum computation to decoherence-free subspaces to "protection" from decoherence. Additional work is in progress, also in view of possible practical applications.

Acknowledgments

References

1. A. Beskow and J. Nilsson, *Arkiv für Fysik* **34**, 561 (1967); L. A. Khalfin, *JETP Letters* **8**, 65 (1968).
2. B. Misra and E. C. G. Sudarshan, *J. Math. Phys.* **18**, 756 (1977).
3. H. Nakazato, M. Namiki, and S. Pascazio, *Int. J. Mod. Phys.* B **10**, 247 (1996); D. Home and M. A. B. Whitaker, *Ann. Phys.* **258**, 237 (1997).
4. P. Facchi and S. Pascazio, *Progress in Optics*, edited by E. Wolf (Elsevier, Amsterdam, 2001), Vol. 42, Ch. 3, p. 147.
5. R. J. Cook, *Phys. Scr. T* **21**, 49 (1988).
6. W. H. Itano, D. J. Heinzen, J. J. Bollinger, and D. J. Wineland, *Phys. Rev.* A **41**, 2295 (1990); T. Petrosky, S. Tasaki, and I. Prigogine, *Phys. Lett.* A **151**, 109 (1990); *Physica* A **170**, 306 (1991); A. Peres and A. Ron, *Phys. Rev.* A **42**, 5720 (1990); S. Pascazio, M. Namiki, G. Badurek, and H. Rauch, *Phys. Lett.* A **179**, 155 (1993); T. P. Altenmüller and A. Schenzle, *Phys. Rev.* A **49**, 2016 (1994); J. I. Cirac, A. Schenzle, and P. Zoller, *Europhys. Lett.* **27**, 123 (1994); S. Pascazio and M. Namiki, *Phys. Rev.* A **50**, 4582 (1994); P. Kwiat, H. Weinfurter, T. Herzog, A. Zeilinger, and M. Kasevich, *Phys. Rev. Lett.* **74**, 4763 (1995); A. Beige and G. Hegerfeldt, *Phys. Rev.* A **53**, 53 (1996); A. Luis and J. Periña, *Phys. Rev. Lett.* **76**, 4340 (1996).
7. S. R. Wilkinson, C. F. Bharucha, M. C. Fischer, K. W. Madison, P. R. Morrow, Q. Niu, B. Sundaram, and M. G. Raizen, *Nature* **387**, 575 (1997).
8. A. M. Lane, *Phys. Lett.* A **99**, 359 (1983); W. C. Schieve, L. P. Horwitz, and J. Levitan, *Phys. Lett.* A **136**, 264 (1989); A. G. Kofman and G. Kurizki, *Nature* **405**, 546 (2000).
9. P. Facchi, H. Nakazato, and S. Pascazio, *Phys. Rev. Lett.* **86**, 2699 (2001).
10. M.C. Fischer, B. Gutiérrez-Medina, and M.G. Raizen, *Phys. Rev. Lett.* **87**, 040402 (2001).
11. P. Facchi, V. Gorini, G. Marmo, S. Pascazio, and E. C. G. Sudarshan, *Phys. Lett.* A **275**, 12 (2000); P. Facchi, S. Pascazio, A. Scardicchio, and L. S. Schulman, *Phys. Rev.* A **65**, 012108 (2002).
12. C. N. Friedman, *Indiana Univ. Math. J.* **21**, 1001 (1972).
13. K. Gustafson, *Irreversibility questions in chemistry, quantum-counting, and time-delay*, in *Energy storage and redistribution in molecules*, edited by J. Hinze (Plenum, 1983), and refs. [10,12] therein. See also K. Gustafson and B. Misra, *Lett. Math. Phys.* **1**, 275 (1976).
14. J. von Neumann, *Mathematical Foundation of Quantum Mechanics*

(Princeton University Press, Princeton, 1955).

15. P. Facchi and S. Pascazio, *Quantum Zeno effects with "pulsed" and "continuous" measurements*, in *Time's arrows, quantum measurements and superluminal behavior*, edited by D. Mugnai, A. Ranfagni, and L. S. Schulman (CNR, Rome, 2001) p. 139; *Fortschr. Phys.* **49**, 941 (2001).

16. A. Peres, *Am. J. Phys.* **48**, 931 (1980).

17. K. Kraus, *Found. Phys.* **11**, 547 (1981); A. Sudbery, *Ann. Phys.* **157**, 512 (1984); A. Venugopalan and R. Ghosh, *Phys. Lett.* A **204**, 11 (1995); M. P. Plenio, P. L. Knight, and R. C. Thompson, *Opt. Comm.* **123**, 278 (1996); M. V. Berry and S. Klein, *J. Mod. Opt.* **43**, 165 (1996). E. Mihokova, S. Pascazio, and L. S. Schulman, *Phys. Rev.* A **56**, 25 (1997); A. D. Panov, *Phys. Lett.* A **260**, 441 (1999); J. Řeháček, J. Peřina, P. Facchi, S. Pascazio, and L. Mišta, *Phys. Rev.* A **62**, 013804 (2000); P. Facchi and S. Pascazio, *Phys. Rev.* A **62**, 023804 (2000); B. Militello, A. Messina, and A. Napoli, *Phys. Lett.* A **286**, 369 (2001).

18. L. S. Schulman, *Phys. Rev.* A **57**, 1509 (1998).

19. P. Facchi and S. Pascazio, *Phys. Rev. Lett.* **89**, 080401 (2002).

20. P. Facchi and S. Pascazio, *Quantum Zeno phenomena in quantum field theory*, these proceedings (2002).

21. See, for example, J. G. Muga, G. W. Wei, and R. F. Snider, *Europhys. Lett.* **35**, 247 (1996); H. Nakazato, *To Decay Or Not To Decay? Temporal behavior of a quantum system –Analysis based on a solvable model–*, these proceedings (2002).

22. A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer Academic Publishers, Dordrecht, 1998).

23. For an interesting example, see K. Machida, H. Nakazato, S. Pascazio, H. Rauch, and S. Yu, *Phys. Rev.* A **60**, 3448 (1999).

24. J. Schwinger, *Proc. Nat. Acad. Sc.* **45**, 1552 (1959), reproduced in J. Schwinger, *Quantum kinematics and dynamics* (Perseus Publishing, New York, 1991).

25. H. Nakazato, M. Namiki, S. Pascazio, and H. Rauch, *Phys. Lett.* A **217**, 203 (1996).

26. P. Facchi, *Quantum Time Evolution: Free and Controlled Dynamics*, PhD Thesis (October 2000) Bari.

27. A. Messiah, *Quantum mechanics* (Interscience, New York, 1961).

28. M. Born and V. Fock, *Z. Phys.* **51**, 165 (1928); T. Kato, *J. Phys. Soc. Jap.* **5**, 435 (1950).

29. Y. Aharonov and M. Vardi, *Phys. Rev.* D **21**, 2235 (1980).

30. B. Militello, A. Messina, and A. Napoli, *Fortschr. Phys.* **49**, 1041 (2001).

31. G. M. Palma, K. A. Suominen, A. K. Ekert, *Proc. R. Soc. Lond.* A

452, 567 (1996); L. M. Duan and G. C. Guo, *Phys. Rev. Lett.* **79**, 1953 (1997); P. Zanardi and M. Rasetti, *Phys. Rev. Lett.* **79**, 3306 (1997); *Mod. Phys. Lett.* B **11**, 1085 (1997); P. Zanardi, *Phys. Rev.* A **57**, 3276 (1998); D. A. Lidar, I. L. Chuang, and K. B. Whaley, *Phys. Rev. Lett.* **81**, 2594 (1998); L. Viola, E. Knill, and S. Lloyd, *Phys. Rev. Lett.* **82**, 2417 (1999).

32. A. Beige, D. Braun, B. Tregenna, and P. L. Knight, *Phys. Rev. Lett.* **85**, 1762 (2000).

33. G. S. Agarwal, M. O. Scully, and H. Walther, *Phys. Rev. Lett.* **86**, 4271 (2001).

34. P. Facchi and S. Pascazio, *Phys. Lett.* A **241**, 139 (1998).