Control of decoherence via quantum Zeno subspaces

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Abstract. We discuss three control strategies, whose objective is to counter the effects of decoherence: the first strategy hinges upon the quantum Zeno effect, the second makes use of frequent unitary interruptions ("bang-bang" pulses), and the third of a strong, continuous coupling. Decoherence can be suppressed only if the frequency τ^{-1} of the measurements/pulses is large enough or if the coupling K is sufficiently strong. Otherwise, if τ^{-1} or K are large, but not sufficiently large, all these control procedures accelerate decoherence.

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1. Introduction

The control of decoherence [1] is an important problem with many practical applications. Decoherence hinders the preservation of quantum superpositions and entanglement over long periods of time, and this is clearly very detrimental for many physical applications, e.g. when one is interested in quantum computation [2]. In this article we will briefly analyze three schemes whose objective is to counter the effects of decoherence. The first is based on the quantum Zeno effect [3], the second on "bang-bang" (BB) pulses and their generalization, quantum dynamical decoupling [4] and the third on a strong, continuous coupling [5]. These methods are seemingly different, but a systematic study shows that they are in fact related to each other, within the unifying framework of the quantum Zeno subspaces [6, 7], and it is interesting to understand under which circumstances and physical conditions these controls may *accelerate*, rather than hinder decoherence [8]. The method we propose is general and can be applied to diverse situations of practical interest, such as atoms and ions in cavities, organic molecules, quantum dots and Josephson junctions.

2. Generalities and notation

We introduce notation and set up a general framework. Let the total system consist of a target system and a reservoir and its Hilbert space $\mathcal{H}_{\rm tot} = \mathcal{H}_S \otimes \mathcal{H}_B$ be expressed as the tensor product of the system Hilbert space $\mathcal{H}_S$ and the reservoir Hilbert space $\mathcal{H}_B$. The total Hamiltonian

$$H_{\rm tot} = H_0 + H_{SB} = H_S \otimes \mathbf{1}_B + \mathbf{1}_S \otimes H_B + H_{SB} \tag{1}$$

is the sum of the system Hamiltonian $H_S \otimes \mathbf{1}_B$, the reservoir Hamiltonian $\mathbf{1}_S \otimes H_B$ and their interaction H_{SB}, which is responsible for decoherence; the operators $\mathbf{1}_S$ and $\mathbf{1}_B$ are the identity operators in the Hilbert spaces $\mathcal{H}_S$ and $\mathcal{H}_B$, respectively, and the operators H_S and H_B act on $\mathcal{H}_S$ and $\mathcal{H}_B$, respectively.

The dynamics of the total system is conveniently expressed in terms of the Liouvillian

$$\mathcal{L}_{\rm tot}\rho \equiv -i[H_{\rm tot}, \rho] = -i\left(H_{\rm tot}\rho - \rho H_{\rm tot}\right) \ , \tag{2}$$

where ρ is the density matrix of the total system. Starting from the Hamiltonian (1), the Liouvillian is decomposed into

$$\mathcal{L}_{\rm tot} = \mathcal{L}_0 + \mathcal{L}_{SB} = \mathcal{L}_S + \mathcal{L}_B + \mathcal{L}_{SB}, \tag{3}$$

where the meaning of the symbols is obvious.

We assume that the interaction Hamiltonian H_{SB} in (1) can be written as [9]

$$H_{SB} = \sum_m \left(X_m \otimes A_m^\dagger + X_m^\dagger \otimes A_m\right), \tag{4}$$

where X_m are the eigenoperators of the system Liouvillian, satisfying

$$\mathcal{L}_S X_m = i\omega_m X_m \qquad (\omega_m \neq \omega_n, \quad \text{for} \quad m \neq n), \tag{5}$$

and A_m are the destruction operators of the bath

$$A_m = A(g_m) = \int d^3k \; g_m^*(\boldsymbol{k}) \; a(\boldsymbol{k}) \ , \tag{6}$$

expressed in terms of bosonic operators $a(\boldsymbol{k})$, with form factors $g_m(\boldsymbol{k})$. A particular case of the above is the qubit Hamiltonian

$$H_S = \frac{\Omega}{2}\sigma_z, \qquad H_B = \int d^3k \; \omega(k) \; a^\dagger(\boldsymbol{k}) a(\boldsymbol{k}), \tag{7}$$

$$H_{SB} = \sigma_z \otimes \left[A(g_0) + A^\dagger(g_0)\right] + \sigma_x \otimes \left[A(g_1) + A^\dagger(g_1)\right], \tag{8}$$

where the states of the qubit, $|\downarrow\rangle$ and $|\uparrow\rangle$, are the eigenstates of H_S, and $\omega(k) \geq 0$ the energy of the boson with wavenumber $\boldsymbol{k}$. Let us introduce the bare spectral density functions (form factors)

$$\kappa_m(\omega) = \int d^3k \; |g_m(\boldsymbol{k})|^2 \delta(\omega(k) - \omega) \qquad (\kappa_m(\omega) = 0 \ \text{ for } \ \omega < 0) \tag{9}$$

and the thermal spectral density functions at the inverse temperature β

$$\kappa_m^\beta(\omega) = \frac{1}{1 - e^{-\beta\omega}}\left[\kappa_m(\omega) - \kappa_m(-\omega)\right] \ , \tag{10}$$

which extend along the whole real axis due to the counter-rotating terms and satisfy the KMS symmetry [10]

$$\kappa_m^\beta(-\omega) = \frac{N(\omega)}{N(\omega)+1}\,\kappa_m^\beta(\omega) = \exp(-\beta\omega)\,\kappa_m^\beta(\omega), \quad N(\omega) = \frac{1}{e^{\beta\omega} - 1}. \tag{11}$$

Let us focus, for the sake of clarity, on two particular Ohmic cases: exponential and polynomial form factors

$$\kappa_m^{(E)}(\omega) = g^2\omega \exp(-\omega/\Lambda_E)\theta(\omega), \tag{12}$$

$$\kappa_m^{(P)}(\omega) = g^2 \frac{\omega}{[1+(\omega/\Lambda_P)^2]^n}\theta(\omega), \tag{13}$$

respectively, where g is a coupling constant, Λ a cutoff and θ the unit step function. In order to properly compare these two cases, we will require that the bandwidth be the same:

$$W \equiv \frac{\int d\omega\ \omega\ \kappa_m^{(E)}(\omega)}{\int d\omega\ \kappa_m^{(E)}(\omega)} = \frac{\int d\omega\ \omega\ \kappa_m^{(P)}(\omega)}{\int d\omega\ \kappa_m^{(P)}(\omega)}. \tag{14}$$

We focus on a proper subspace $\mathcal{H}_{\text{comp}} \subset \mathcal{H}_S$, in which quantum computation is to be performed

$$\mathcal{H}_S = \mathcal{H}_{\text{comp}} \oplus \mathcal{H}_{\text{orth}}. \tag{15}$$

The initial state of the total system $\rho(0)$ is set to be the tensor product of the system and reservoir initial states

$$\rho(0) = \sigma(0) \otimes \rho_B\ , \tag{16}$$

where the reservoir equilibrium state has an inverse temperature β

$$\rho_B = \frac{1}{Z}\exp(-\beta H_B), \qquad Z = \text{tr}_B e^{-\beta H_B} \qquad (\mathcal{L}_B \rho_B = 0). \tag{17}$$

The system state $\sigma(t)$ at time t is given by the partial trace of the state $\rho(t)$ of the whole system with respect to the reservoir degrees of freedom:

$$\sigma(t) \equiv \text{tr}_B \rho(t). \tag{18}$$

There is decoherence when $\sigma(t)$ is not unitarily equivalent to $\sigma(0)$ for a given class of initial states: the purpose of the control is to suppress such decoherence.

Under the assumption that the bath is in a thermal state (17), and in the Markov approximation, the reduced state of the system (18) satisfies the master equation

$$\dot\sigma(t) = (\mathcal{L}_S + \mathcal{L})\,\sigma(t)\ , \tag{19}$$

where, up to a renormalization of the free Liouvillian $\mathcal{L}_S$ by Lamb and Stark shift terms, $\mathcal{L}$ engenders the dissipation due to the interaction with the bath,

$$\mathcal{L}\sigma = \sum_n \gamma_n \left(X_n \sigma X_n^\dagger - \frac{1}{2}\left\{X_n^\dagger X_n, \sigma\right\}\right), \tag{20}$$

where $X_{-n} = X_n^\dagger$ and

$$\gamma_n = 2\pi\kappa_n^\beta(\omega_n) \tag{21}$$

are the dissipation rates.

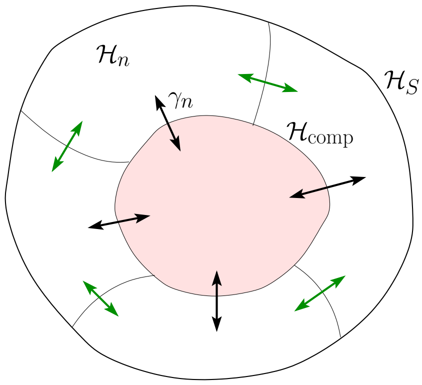

Figure 1. The Zeno subspaces are formed when the frequency τ^{-1} of interruptions (measurements or BB pulses, see later) or the strength K of the coupling tend to ∞. The shaded region represents the computational subspace $\mathcal{H}_{\rm comp} \subset \mathcal{H}_S$ defined in Eq. (15). The decay rates γ_n depend on τ or K.

3. Control and quantum Zeno subspaces

Before analyzing in some detail the different control procedures, it is useful to sketch the main ideas that motivate this article. If the interaction between the system and some external agent (that performs the "control") is described in terms of some relevant parameters (such as a frequency of interruptions τ^{-1} or the strength K of a relevant coupling), the (effective) decay rates (21) will be in general functions of these parameters. Decoherence and/or dissipation can be controlled if

$$\gamma_n = \gamma_n(\tau \text{ or } K) \stackrel{1/\tau \text{ or } K\to\infty}{\longrightarrow} 0. \qquad (22)$$

In such a case, the Hilbert space of the system splits into invariant sectors, that we will call *quantum Zeno subspaces* [6]: see Fig. 1. If one of these invariant subspaces is the "computational" subspace $\mathcal{H}_{\rm comp}$ introduced in Eq. (15), one can inhibit decoherence and/or dissipation within this subspace.

There is, however, a very important issue, relevant for applications: the limit (22) is mathematical, and the *physical* meaning of the expressions $\tau^{-1}, K \to \infty$ must be scrutinized with great care. The principal objective of our study is to understand *how* the limit is attained and analyze the deviations from the ideal situation. We will see that in general the functional dependence of the decay rates in (22) can be complicated, and yields an *enhancement* of decoherence in some cases and a *suppression* in other cases.

4. Control procedures

We can now compare the three control methods both with exponential (12) and polynomial form factors (13). We will focus on the transition between a regime in which decoherence is reduced ("controlled") and a regime in which it is enhanced. We will work in a high-temperature case, which is rather critical from an experimental point of view, because of temperature-induced transitions. In the two-level system (7) we set $\Omega = 0.01W$ and $\beta = 50W^{-1}$, so that the temperature $\beta^{-1} = 2\Omega$.

4.1. Quantum Zeno control

The Zeno control is obtained by performing frequent measurements on the system. The measurement is described by a projection superoperator $\hat{P}$ acting on the density matrix

$$\rho \to \hat{P}\rho \equiv \sum_n P_n \rho P_n, \tag{23}$$

where $\{P_n\}$ is a set of orthogonal projection operators acting on $\mathcal{H}_S$. In the following, we restrict our analysis to a measuring apparatus that does not "select" the different outcomes (nonselective measurement) [11], with a complete set of projection operators $\sum_n P_n = \mathbf{1}_S$. The measurement is designed so that

$$\hat{P} H_{SB} = \sum_n P_n H_{SB} P_n = 0. \tag{24}$$

In terms of the Liouvillian, this condition reads

$$\hat{P} \mathcal{L}_{SB} \hat{P} = 0. \tag{25}$$

(In the next subsection a similar condition will be required for the BB control and for the control via a continuous coupling.) The Zeno control consists in performing repeated nonselective measurements at times $t_k = k\tau$ $(k = 0, 1, 2, \ldots)$ (we include an initial "state preparation" at $t = 0$). Between successive measurements, the system evolves via H_{tot}. The density matrix after $N+1$ measurements, in the limit $\tau \to 0$ while keeping $t = N\tau$ constant, with an initial state $\rho(0)$, is given by

$$\begin{aligned} \rho(t) = \rho(N\tau) &= \left(\hat{P} e^{\mathcal{L}_{\mathrm{tot}}\tau} \hat{P}\right)^N \rho(0) \\ &= \hat{P}\left[1 + \hat{P}\mathcal{L}_{\mathrm{tot}}\hat{P}\tau + \mathrm{O}\left(\tau^2\right)\right]^{\frac{t}{\tau}} \rho(0) \stackrel{\tau\to 0}{\longrightarrow} \hat{P} e^{\mathcal{L}'_{\mathrm{tot}} t} \rho(0) \ , \end{aligned} \tag{26}$$

where the controlled Liouvillian $\mathcal{L}'_{\mathrm{tot}}$ reads

$$\mathcal{L}'_{\mathrm{tot}} = \hat{P}\mathcal{L}_{\mathrm{tot}}\hat{P} = \hat{P}\mathcal{L}_S\hat{P} + \mathcal{L}_B\hat{P} = \mathcal{L}'_S + \mathcal{L}_B\hat{P} \ . \tag{27}$$

Hence, as a result of infinitely frequent measurements, the system-reservoir coupling is eliminated and, thus, decoherence is halted. We notice the formation of the invariant Zeno subspaces [6]: in the limit of very frequent measurements, the evolution is given by (27) and transitions among different sectors of the Hilbert space become forbidden, yielding a superselection rule. The subspaces are defined by the superoperator $\hat{P}$ defining the measurement and, owing to the condition (25), they are all "decoherence-free". The computational subspace can be any one of these Zeno subspaces.

We will assume for simplicity that $\hat{P}$ commutes with the system Liouvillian

$$\hat{P}\mathcal{L}_S = \mathcal{L}_S\hat{P}, \tag{28}$$

so that

$$\mathcal{L}'_{\mathrm{tot}} = (\mathcal{L}_S + \mathcal{L}_B)\hat{P} \ . \tag{29}$$

Let us now look at the quantum Zeno dynamics with a *finite* time interval $\tau = t/N$ between measurements,

$$\rho(t) = \left[\hat{P} e^{\mathcal{L}_{\rm tot}\tau} \hat{P}\right]^{\frac{t}{\tau}} \rho(0) \ , \tag{30}$$

where $\mathcal{L}_{\rm tot}$ and $\hat{P}$ are given by (3) and (23), respectively. We will consider the subtle effects on the decay rate arising from the presence of the short-time quadratic (Zeno) region. Therefore the standard method [12] is not applicable to the present situation and the limit must be evaluated by a different technique. We only sketch the main results, more details can be found in [8].

The evolution of the density matrix is governed by

$$\dot{\sigma}(t) = \left[\mathcal{L}_S + \mathcal{L}_{\rm Z}(\tau)\right] \sigma(t) \ , \tag{31}$$

where the dissipative part is found to have the explicit form [analogous to Eq. (20)]

$$\mathcal{L}_{\rm Z}(\tau)\sigma = \sum_m \gamma_m^{\rm Z}(\tau) \hat{P} \left(X_m \hat{P}\sigma X_m^\dagger - \frac{1}{2}\left\{ X_m^\dagger X_m, \hat{P}\sigma \right\} \right) \tag{32}$$

and the controlled decay rates read

$$\gamma_m^{\rm Z}(\tau) = \tau \int_{-\infty}^{\infty} d\omega \ \kappa_m^\beta(\omega) \ \mathrm{sinc}^2 \left(\frac{\omega - \omega_m}{2}\tau \right) \ , \tag{33}$$

with $\mathrm{sinc}(x) = \sin(x)/x$. We focus for simplicity on the two-level case (7) (with energy gap Ω) and drop the suffix m. Notice that, in the $\tau \to 0$ limit, the dissipative part disappears, $\gamma^{\rm Z}(\tau) \to 0$, and decoherence is suppressed, as expected. On the other hand, $\gamma^{\rm Z}(\tau) \to \gamma = 2\pi\kappa^\beta(\Omega)$, when $\tau \to \infty$ [uncontrolled evolution, see (21)]. The ratio $\gamma^{\rm Z}(\tau)/\gamma$ is shown in Fig. 2 as a function of τ [in units of W–the bandwidth defined in Eq. (14)]. In general, (33) yields both Zeno (suppression/control of decoherence) and inverse Zeno effects [13, 14] (enhancement of decoherence) as τ is changed. The transition between the two regimes takes place at $\tau = \tau^*$, where τ^* is defined by the equation [14]

$$\gamma^{\rm Z}(\tau^*) = \gamma. \tag{34}$$

4.2. Control via "Bang-Bang" Pulses

We now turn our attention to the so-called quantum dynamical decoupling [4], and in particular to a kicked control. In this case, one applies after each time interval τ an *instantaneous* unitary operators $U_{\rm k}$ with spectral decomposition

$$U_{\rm k} = \sum_n e^{-i\lambda_n} P_n, \qquad (\lambda_n \neq \lambda_m \bmod 2\pi, \quad \text{for } n \neq m) \tag{35}$$

and considers the $\tau \to 0$ limit [7]. As in the case discussed in the previous subsection, one observes the formation of invariant Zeno subspaces: transitions among different subspaces vanish in the $\tau \to 0$ limit, yielding a superselection rule. In this case, the subspaces are defined by the eigenprojections in (35) and are nothing but the ergodic

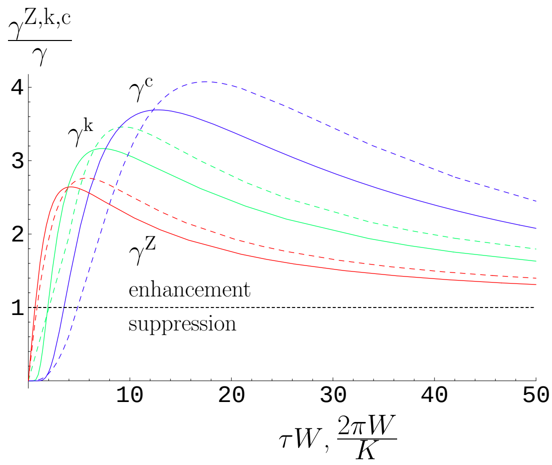

Figure 2. Comparison among the three control methods. Full lines: exponential form factor (12). Dashed lines: polynomial form factor (13) with $n = 2$.

sectors of $U_{\rm k}$. Indeed, by assuming again, as in (25) and (28), that $[\hat{P}, \mathcal{L}_S] = 0$ and that $\hat{P}\mathcal{L}_{SB}\hat{P} = 0$ we get the controlled evolution

$$\rho(t) = \left[e^{\mathcal{L}_{\rm k}} e^{\mathcal{L}_{\rm tot}\tau}\right]^{\frac{t}{\tau}} \hat{P}\rho(0) \stackrel{\tau\to 0}{\longrightarrow} e^{\mathcal{L}'_{\rm tot}t}\hat{P}\rho(0), \tag{36}$$

where $\mathcal{L}_{\rm k}$ is the Liouvillian corresponding to the kick (35): $e^{\mathcal{L}_k}\rho \equiv U_k \rho U_k^\dagger$, and

$$\mathcal{L}'_{\rm tot} = \hat{P}\mathcal{L}_{\rm tot}\hat{P} = (\mathcal{L}_S + \mathcal{L}_B)\hat{P}, \tag{37}$$

exactly as in (29). Note that the controlled Liouvillians for bang bang pulses, (37), and for the Zeno control, (29), coincide when the set of orthogonal projections (23) is equal to the set (35) of eigenprojections of $U_{\rm k}$, namely

$$\mathcal{L}_{\rm k}\hat{P} = 0, \qquad (\hat{P}\mathbf{1}) = \mathbf{1}. \tag{38}$$

Therefore, the two controls are equivalent in the ideal (limiting) case [7].

We now investigate the nonideal bang-bang control of decoherence (for finite τ), so that the effects on the decay rate arising from the presence of a short-time quadratic (Zeno) region play a fundamental role. Once again, we only give the main results.

We focus for simplicity on the two level system (8) with $g_0 = 0$ (spin-flip decoherence). We include an additional third level—that performs the control—and add the following Hamiltonian (acting on $\mathcal{H}_S \oplus {\rm span}\{|M\rangle\}$) to (7)-(8)

$$H_M = -\frac{\Omega}{2}|M\rangle\langle M|, \tag{39}$$

so that $|M\rangle$ is degenerate with $|{\downarrow}\rangle$. The control consists of a sequence of 2π pulses between $|{\downarrow}\rangle$ and $|M\rangle$, given by

$$U_{\rm k} = \exp\left[-i\pi\left(|{\downarrow}\rangle\langle M| + |M\rangle\langle{\downarrow}|\right)\right] = P_\uparrow - P_{-1}\ , \tag{40}$$

where

$$P_\uparrow = |\uparrow\rangle\langle\uparrow|, \qquad P_{-1} = P_\downarrow + P_M = |\downarrow\rangle\langle\downarrow| + |M\rangle\langle M|, \tag{41}$$

are the eigenprojections of $U_{\rm k}$ (belonging respectively to $e^{-i\lambda_\uparrow}=1$ and $e^{-i\lambda_{-1}}=-1$) which define two Zeno subspaces. One gets for the decay rate out of state $|\uparrow\rangle$

$$\gamma^{\rm k}(\tau) = \frac{2}{\pi}\sum_{j=0}^{\infty}\frac{1}{\left(j+\frac{1}{2}\right)^2}\left[\kappa^\beta\left(\Omega+\frac{\pi}{\tau}(2j+1)\right)+\kappa^\beta\left(\Omega-\frac{\pi}{\tau}(2j+1)\right)\right]$$
$$\stackrel{\tau\to 0}{\sim} \frac{2}{\pi}\sum_{j=0}^{\infty}\frac{1}{\left(j+\frac{1}{2}\right)^2}\,\kappa^\beta\left(\frac{\pi}{\tau}(2j+1)\right), \qquad (42)$$

where we assumed that β is not too small (as compared to τ). In contrast to the Zeno case, the mechanism of decoherence suppression is not fully determined by $\mathcal{L}_{\rm tot}$ and $\hat{P}$, but depends also on the details of the Liouvillian $\mathcal{L}_{\rm k}$. Again, in the $\tau\to 0$ limit, the dissipative part disappears (with a law that depends on the form factor), $\gamma^{\rm k}(\tau)\to 0$, and decoherence is suppressed, as expected. On the other hand, $\gamma^{\rm k}(\tau)\to\gamma=2\pi\kappa^\beta(\Omega)$, when $\tau\to\infty$ [uncontrolled evolution, see (21)]. The ratio $\gamma^{\rm k}(\tau)/\gamma$ is shown in Fig. 2 as a function of τ. Once again, the transition between the two regimes takes place at $\tau=\tau^*$, where τ^* is defined by the equation

$$\gamma^{\rm k}(\tau^*) = \gamma. \qquad (43)$$

4.3. Control via a strong continuous coupling

The formulation in the preceding sections hinges upon instantaneous processes, that can be unitary or nonunitary. However, the main results can also be obtained by making use of a continuous coupling, when the external system takes a sort of steady "gaze" at the system of interest. The mathematical formulation of this idea is contained in a theorem [6] on the (large-K) dynamical evolution governed by a *generic* Liouvillian of the type

$$\mathcal{L}_K = \mathcal{L}_{\rm tot} + K\mathcal{L}_{\rm c}, \qquad (44)$$

where $\mathcal{L}_{\rm c}$ can be viewed as an "additional" interaction Hamiltonian performing the "measurement" and K is a coupling constant. The evolution reads [see (36)]

$$\rho(t) = e^{(K\mathcal{L}_c+\mathcal{L}_{\rm tot})t}\hat{P}\rho(0) \stackrel{K\to\infty}{\longrightarrow} e^{\mathcal{L}'_{\rm tot}t}\hat{P}\rho(0), \qquad (45)$$

where $\mathcal{L}'_{\rm tot}$ is again given by (27) under the assumption (25) and [see (38)]

$$\mathcal{L}_c\hat{P}=0, \qquad (\hat{P}\mathbf{1})=\mathbf{1}. \qquad (46)$$

The above statements can be proved by making use of the adiabatic theorem [15]. One observes again the formation of invariant Zeno subspaces, that are in this case the eigenspaces of the interaction (46). The links between the quantum Zeno effect and the notion of "continuous coupling" to an external apparatus or environment has often been proposed in the literature of the last 25 years [5]. The novelty here lies in its generalization to *any* interaction and in the gradual formation of the Zeno subspaces as K becomes increasingly large. In this case, they are nothing but the adiabatic subspaces of the interaction.

In general, as in the BB control but in contrast to the Zeno case, the mechanism of decoherence suppression is not fully determined by H_S and depends on the details of the Hamiltonians H_S *and* $H_{\rm c}$. We clarify this point by considering again a specific example: consider the two-level system (8) with $g_0 = 0$ (spin flip decoherence). We again add to (8) the Hamiltonian (39), acting on $\mathcal{H}_S \oplus \mathrm{span}\{|M\rangle\}$. However, the control consists now in the Hamiltonian

$$KH_{\rm c} = K(|{\downarrow}\rangle\langle M| + |M\rangle\langle{\downarrow}|), \tag{47}$$

which "continuously" couples the third state $|M\rangle$ to state $|{\downarrow}\rangle$, $K \in \mathbb{R}$ being the strength of the coupling. As K is increased, state $|M\rangle$ performs a better "continuous observation" of $|{\downarrow}\rangle$, yielding the Zeno subspaces. In terms of its eigenprojections

$$P_\uparrow = |{\uparrow}\rangle\langle{\uparrow}|, \qquad P_\pm = \frac{(|{\downarrow}\rangle \pm |M\rangle)(\langle{\downarrow}| \pm \langle M|)}{2}, \tag{48}$$

$H_{\rm c}$ reads

$$H_{\rm c} = \eta_\uparrow P_\uparrow + \eta_- P_- + \eta_+ P_+, \tag{49}$$

with $\eta_\uparrow = 0$ and $\eta_\pm = \pm 1$. In the Zeno limit ($K \to \infty$) the subspaces $\mathcal{H}_\uparrow$, $\mathcal{H}_+$ and $\mathcal{H}_-$ decouple due to wildly oscillating phases $\mathrm{O}(K)$ and

$$\hat{P} H_{SB} = P_\uparrow H_{SB} P_\uparrow + P_- H_{SB} P_- + P_+ H_{SB} P_+ = 0. \tag{50}$$

The decay rate out of state $|{\uparrow}\rangle$ reads

$$\gamma^{\rm c}(K) = \pi \left(\kappa^\beta(\Omega - K) + \kappa^\beta(\Omega + K)\right) \stackrel{K\,\mathrm{large}}{\sim} \pi\kappa^\beta(K). \tag{51}$$

Hence, in the $K \to \infty$ limit, the dissipative part disappears $\gamma^{\rm c}(K) \to 0$ (with a law that depends on the form factor), and decoherence is suppressed, as expected. On the other hand, $\gamma^{\rm c}(K) \to \gamma$, when $K \to 0$ [uncontrolled evolution, see (21)]. Notice that the role of K in this subsection and the role of $1/\tau$ in the previous ones are equivalent. This yields a natural comparison [7] between different timescales (τ for measurements and kicks, $1/K$ for continuous coupling).

The ratio $\gamma^{\rm c}(K)/\gamma$ is shown in Fig. 2 as a function of $2\pi/K$. The transition between these two regimes takes now place at $K = K^*$ where K^* is defined by the equation

$$\gamma^{\rm c}(K^*) = \gamma. \tag{52}$$

5. SUMMARY AND CONCLUDING REMARKS

We have analyzed and compared three control methods for combating decoherence. The first is based on repeated quantum measurements (projection operators) and involves a description in terms of nonunitary processes. The second and third methods are both dynamical, as they can be described in terms of unitary evolutions. In all cases, decoherence can be halted by very rapidly/strongly driving or very frequently measuring the system state. However, if the frequency is not high enough or the coupling not strong enough, the controls may accelerate the decoherence process and deteriorate the performance of the quantum state manipulation. The acceleration of decoherence is analogous to the inverse Zeno effect, namely the acceleration of the decay of an unstable state due to frequent measurements [13, 14].

As a general rule, when one endeavors to control decoherence by suitably tailoring the coupling of the system of interest to another system (such as an external field, or a measuring apparatus), one should carefully look at the relevant timescales, as it is not true that repeated measurements/interruptions always lead to a suppression of decoherence.

It is convenient to summarize the main results obtained for the two-level system (7)-(8) (qubit) with energy difference Ω. If the frequency τ^{-1} of measurements or BB kicks, or the strength K of the coupling tend to ∞, the computational (Zeno) subspace becomes isolated and decoherence is completely suppressed. However, if τ^{-1} and K are large, but not sufficiently large, the transition (decay) rates between the computational subspace and the remaining sectors of the Hilbert space display a complicated dependence on τ^{-1} and K, and decoherence can be suppressed or enhanced, depending on the situation.

At low temperatures $\beta^{-1} \ll \Omega \ll W$, where W is the bandwidth of the form factor of the interaction, the decay rates read

$$\begin{cases} \gamma^{\mathrm{Z}}(\tau) \sim \frac{\tau}{\tau_Z^2}, & \tau \to 0 \ , \\ \gamma^{\mathrm{k}}(\tau) \sim \frac{8}{\pi}\kappa\left(\frac{\pi}{\tau}\right), & \tau \to 0 \ , \\ \gamma^{\mathrm{c}}(K) \sim \pi\kappa(K), & K \to \infty \ , \end{cases} \tag{53}$$

where the superscripts Z, k and c denote (Zeno) measurements, (BB) kicks and continuous coupling, respectively, κ is the form factor and $1/\tau_Z^2 \simeq \int d\omega\kappa(\omega)$ the Zeno time. As we have shown, there is a characteristic transition time τ^* [coupling K^*], such that one obtains:

$$\begin{aligned} &\text{for } \tau < \tau^* \text{ or } K > K^* \ \Rightarrow \ \text{decoherence suppression: } \gamma(\tau \text{ or } K) < \gamma, \\ &\text{for } \tau > \tau^* \text{ or } K < K^* \ \Rightarrow \ \text{decoherence enhancement: } \gamma(\tau \text{ or } K) > \gamma. \end{aligned}$$

Therefore, in order to obtain a suppression of decoherence, the interruptions/coupling must be *very* frequent/strong. Notice, in this context, that both τ^* and $2\pi/K^*$ are not simply related to the inverse bandwidth $2\pi W^{-1}$: they can be in general (much) shorter. For instance, in the Ohmic polynomial case (13), one gets

$$\begin{cases} \tau_Z^*/(2\pi W^{-1}) \simeq 2(n-1)\alpha_n^2 \frac{\Omega}{W} \ll 1 \ , \\ \tau_{\mathrm{k}}^*/(2\pi W^{-1}) \simeq \frac{\alpha_n}{2}\left(\frac{\alpha_n \pi^2}{4}\frac{\Omega}{W}\right)^{\frac{1}{2n-1}} \ll 1 \ , \\ K^*/W \simeq \alpha_n^{-1}\left(\frac{2}{\alpha_n}\frac{W}{\Omega}\right)^{\frac{1}{2n-1}} \gg 1 \ , \end{cases} \tag{54}$$

where $\alpha_n = (\sqrt{\pi}/2)\Gamma(n-3/2)/\Gamma(n-1) \leq \pi/2$ is a coefficient of order 1 and n characterizes the polynomial fall off of the form factor (13). The above times/coupling may be (very) difficult to achieve in practice. In fact, we see here that the relevant timescale is not simply the inverse bandwidth $2\pi W^{-1}$, but can be much shorter if $\Omega \ll W$, as is typically the case. These conclusions, summarized here for the simple case of a qubit, are valid *in general*, when one aims at protecting from decoherence a higher dimensional subspace.

The results obtained in this paper are of general validity and bring to light the different features of the control procedures as well as the crucial role played by the form factor of the interaction. We do not expect any drastic change for different decoherence mechanisms and/or different physical systems. The only somewhat delicate issue, in our opinion, is to understand whether the system investigated can be consistently described by means of a set of discrete levels.

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