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ScienceDirect
Annals of Physics 322 (2007) 657–676
ANNALS of PHYSICS
www.elsevier.com/locate/aop

On the assumption of initial factorization in the master equation for weakly coupled systems II: Solvable models

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Received 23 February 2006; accepted 28 June 2006
Available online 9 August 2006

Abstract

We analyze some solvable models of a quantum mechanical system in interaction with a reservoir when the initial state is not factorized. We apply Nakajima–Zwanzig's projection method by choosing a reference state of the reservoir endowed with the mixing property. In van Hove's limit, the dynamics is described in terms of a master equation. We observe that Markovianity becomes a valid approximation for timescales that depend both on the form factors of the interaction and on the observables of the reservoir that can be measured.

PACS: 03.65.Yz; 05.30.−d

Keywords: Master equation; van Hove's limit; Dissipation; Nakajima–Zwanzig's projection method; Correlations

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0003-4916/$ - see front matter
doi:10.1016/j.aop.2006.06.003

1. Introduction

The dissipative dynamics of a small quantum system weakly coupled to a large reservoir is described in terms of a master equation [1–4]. In the standard approach to this problem, one usually takes for granted that there are no initial correlations between the system and the reservoir. In the preceding article, hereafter referred to as Article I, we reconsidered this hypothesis in the framework of Nakajima–Zwanzig's projection method [2,4–6] and proved that, in order to get a consistent description, the reference state of the reservoir should be endowed with the mixing property. In such a case, the initial correlations disappear in the Markovian (van Hove) limit and the system behaves as if it started from a factorized initial condition. Interestingly, one arrives at the same conclusions also for uncorrelated initial conditions. The mixing property is therefore crucial, and a "wrong" choice of the reservoir state provokes the appearance of secular terms.

In this article, we shall focus on the hypotheses that are necessary for the derivation of the theorem proved in Article I. These will be scrutinized in terms of two exactly solvable models, in which an oscillator is coupled to a bosonic reservoir. This will enable us to describe the onset to Markovianity and the timescales at which Markovianity becomes a valid approximation.

This article is organized as follows. We introduce notation and summarize previous results in Section 2. The first exactly solvable model is introduced in Section 3 and solved in Sections 4–6. The second model is briefly discussed in Section 7. Section 8 is devoted to a discussion and some concluding remarks. Two Appendices contain the details of the derivations.

2. Summary of previous results

2.1. Notation

We start by briefly summarizing the main ideas of Article I and introduce notation. Let the total system consist of a "large" reservoir B and a "small" (sub)system S, so that the total Hilbert space can be expressed as the tensor product of the Hilbert spaces of the reservoir $\mathcal{H}_{\mathrm{B}}$ and of the system $\mathcal{H}_{\mathrm{S}}$,

$$\mathcal{H}_{\mathrm{tot}} = \mathcal{H}_{\mathrm{S}} \otimes \mathcal{H}_{\mathrm{B}}. \tag{2.1}$$

The Hamiltonian and the corresponding Liouvillian of the total system read

$$H = H_0 + \lambda H_{\mathrm{SB}} = H_{\mathrm{S}} + H_{\mathrm{B}} + \lambda H_{\mathrm{SB}}, \tag{2.2}$$

$$\mathcal{L} = \mathcal{L}_0 + \lambda \mathcal{L}_{\mathrm{SB}} = \mathcal{L}_{\mathrm{S}} + \mathcal{L}_{\mathrm{B}} + \lambda \mathcal{L}_{\mathrm{SB}}, \tag{2.3}$$

respectively, where λ is the coupling constant. Clearly,

$$[H_{\mathrm{S}}, H_{\mathrm{B}}] = 0, \quad [\mathcal{L}_{\mathrm{S}}, \mathcal{L}_{\mathrm{B}}] = 0. \tag{2.4}$$

We assume that the system Hamiltonian H_{S} admits a pure point spectrum, and the system Liouvillian $\mathcal{L}_{\mathrm{S}}$ is resolved in terms of its eigenprojections $\tilde{Q}_m$,

$$\mathcal{L}_{\mathrm{S}} = -\mathrm{i}\sum_m \omega_m \tilde{Q}_m, \quad \sum_m \tilde{Q}_m = 1, \quad \tilde{Q}_m \tilde{Q}_n = \delta_{mn}\tilde{Q}_m. \tag{2.5}$$

2.2. Nakajima–Zwanzig's projection method

Let $\rho(t)$ be the density operator of the total system at time t, which has evolved from the initial state ρ_0

$$\rho(t) = \mathrm{e}^{\mathcal{L}t}\rho_0 \tag{2.6}$$

and is the solution of the von Neumann equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = \mathcal{L}\rho(t), \quad \rho(0) = \rho_0. \tag{2.7}$$

We are interested in the reduced dynamics of system S, which is described by the density operator of S,

$$\rho_{\mathrm{S}}(t) = \mathrm{tr}_{\mathrm{B}}\rho(t). \tag{2.8}$$

In order to derive a master equation for $\rho_{\mathrm{S}}(t)$, Nakajima–Zwanzig's procedure makes use of the projection operators [2,4–6]

$$\mathcal{P}\rho = \mathrm{tr}_{\mathrm{B}}\{\rho\} \otimes \Omega_{\mathrm{B}} = \sigma \otimes \Omega_{\mathrm{B}}, \quad \mathcal{Q} = 1 - \mathcal{P}, \tag{2.9}$$

where Ω_{B} is a certain *reference state* of the reservoir. Due to normalization $\mathrm{tr}_{\mathrm{B}}\Omega_{\mathrm{B}} = 1$, it follows that $\mathcal{P}^2 = \mathcal{P}$ and $\mathcal{Q}^2 = \mathcal{Q}$. In particular,

$$\mathcal{P}\rho(t) = \rho_{\mathrm{S}}(t) \otimes \Omega_{\mathrm{B}}, \quad \mathcal{Q}\rho(t) = \rho(t) - \rho_{\mathrm{S}}(t) \otimes \Omega_{\mathrm{B}}, \tag{2.10}$$

where we used the definition (2.8).

In the standard derivation of a master equation, the initial state of the total system, ρ_0, is taken to be the tensor product of a system initial state ρ_{S} and a reservoir state ρ_{B},

$$\rho_0 = \rho_{\mathrm{S}} \otimes \rho_{\mathrm{B}}. \tag{2.11}$$

This is an *uncorrelated* initial state. The reservoir is assumed to be at equilibrium (with respect to the reservoir free evolution $\mathcal{L}_{\mathrm{B}}$)

$$\mathcal{L}_{\mathrm{B}}\rho_{\mathrm{B}} = 0, \tag{2.12}$$

and in most applications $\rho_{\mathrm{B}} = Z_\beta^{-1}\mathrm{e}^{-\beta H_{\mathrm{B}}}$ is a thermal state at the inverse temperature $\beta = (k_{\mathrm{B}}T)^{-1}$ with the normalization constant Z_β. Then, the reservoir state ρ_{B} in the uncorrelated initial state (2.11) is usually taken as the reference state Ω_{B}.

When the assumption of a factorized initial state is not justified, however, an ambiguity arises regarding the choice of the reference state Ω_{B}. Indeed, if

$$\rho_0 = \rho_{\mathrm{S}} \otimes \rho_{\mathrm{B}} + \delta\rho_0, \tag{2.13}$$

where

$$\rho_{\mathrm{S}} = \mathrm{tr}_{\mathrm{B}}\rho_0, \quad \rho_{\mathrm{B}} = \mathrm{tr}_{\mathrm{S}}\rho_0, \tag{2.14}$$

and the term $\delta\rho_0$ represents the correlation between system S and reservoir B, the relation between ρ_{B} and Ω_{B} is by no means obvious. We discussed this point in Article I and proved the following theorem.

2.3. Theorem

Given a correlated initial state ρ_0, if

(i) 0 is the unique simple eigenvalue of the reservoir Liouvillian $\mathcal{L}_{\mathrm{B}}$ corresponding to the eigenvector Ω_{B} and the remaining part of the spectrum of $\mathcal{L}_{\mathrm{B}}$ is absolutely continuous (strictly speaking, the spectrum of $\mathcal{L}_{\mathrm{B}}$ can be defined only once the sector has been specified: in our case, the relevant sector is that containing the state Ω_{B});

(ii) the initial (correlated) state of the total system is given in the form

$$\rho_0 = \Lambda(1_{\mathrm{S}} \otimes \Omega_{\mathrm{B}}) = \sum_i L_i(1_{\mathrm{S}} \otimes \Omega_{\mathrm{B}})L_i^{\dagger}, \tag{2.15}$$

where Λ is a bounded superoperator (i.e., L_i's are bounded operators) satisfying the normalization condition $\mathrm{tr}\rho_0 = 1$, namely, the initial state ρ_0 belongs to the sector specified by $1_{\mathrm{S}} \otimes \Omega_{\mathrm{B}}$,

then van Hove's "$\lambda^2 t$" limit [1,7,8] of the $\mathcal{P}$-projected density operator in the interaction picture,

$$\rho_{\mathrm{I}}(\tau) = \lim_{\lambda\to 0} \rho_{\mathrm{I}}^{(\lambda)}(\tau) = \lim_{\lambda\to 0} \mathrm{e}^{-\mathcal{L}_{\mathrm{S}}\tau/\lambda^2}\mathcal{P}\rho(\tau/\lambda^2), \tag{2.16}$$

is the solution of

$$\rho_{\mathrm{I}}(\tau) = \mathcal{P}\rho_0 + \int_0^{\tau} \mathrm{d}\tau' \mathcal{K}\rho_{\mathrm{I}}(\tau') \tag{2.17}$$

with

$$\mathcal{K} = -\sum_m \mathcal{P}\tilde{Q}_m\mathcal{L}_{\mathrm{SB}}\frac{\mathcal{Q}}{\mathcal{L}_0 + \mathrm{i}\omega_m - 0^+}\mathcal{L}_{\mathrm{SB}}\tilde{Q}_m\mathcal{P}, \tag{2.18}$$

or equivalently,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\rho_{\mathrm{I}}(\tau) = \mathcal{K}\rho_{\mathrm{I}}(\tau), \quad \rho_{\mathrm{I}}(0) = \mathcal{P}\rho_0 = \mathrm{tr}_{\mathrm{B}}\{\rho_0\} \otimes \Omega_{\mathrm{B}}. \tag{2.19}$$

That is, even if the initial state ρ_0 is not in a factorized form, all correlations disappear in van Hove's limit and system S behaves as if the total system started from the factorized initial state in (2.19) with the reservoir state Ω_{B}.

In addition, we showed that

$$\lim_{\lambda\to 0} \mathcal{Q}\rho(\tau/\lambda^2) = 0, \tag{2.20}$$

which makes the dynamics consistent, for no spurious term will develop in the master equation and no correlations can appear at later times: not only the initial state, but also the state at any moment t is factorized in van Hove's limit. This supports the validity of the assumption of the factorized state, that is frequently applied in literature in order to derive a master equation [2–4]. The state of system S evolves according to the master equation (2.19), while the reservoir B remains in the state Ω_{B}.

It is important to note that, in van Hove's limit, the reservoir state immediately relaxes into Ω_{B}, which is the eigenstate of the reservoir Liouvillian $\mathcal{L}_{\mathrm{B}}$ belonging to its unique

simple eigenvalue 0, and the spectral properties required in hypothesis (i) imply that it is a *mixing* state. The right choice for the reference state of the projection (2.10) is this mixing state Ω_{B}, and such a projection is nothing but the eigenprojection of the reservoir Liouvillian $\mathcal{L}_{\mathrm{B}}$ belonging to the simple eigenvalue 0. This is the criterion for the reference state, that covers both equilibrium states and nonequilibrium steady states. Furthermore, as clarified in Article I, the interaction between system S and reservoir B is not essential to the factorization or the mixing; the total system is factorized and the reservoir relaxes into the mixing state through its own free evolution.

The purpose of the present article is to scrutinize these issues in some explicit examples. In particular, we shall focus on: (a) the disappearance of the initial correlation, (b) the factorization of the total system, and (c) the relaxation of the reservoir into the mixing state, in van Hove's limit. This will also enable us to discuss the relevant timescales for the factorization and the mixing.

3. An exactly solvable model

Let us corroborate the above general arguments by scrutinizing an exactly solvable model. We consider an oscillator a coupled to a reservoir b_ω, whose Hamiltonian is given by (2.2) with

$$H_{\mathrm{S}} = \omega_{\mathrm{S}} a^\dagger a, \quad H_{\mathrm{B}} = \int_0^\infty \mathrm{d}\omega\, \omega b_\omega^\dagger b_\omega, \quad H_{\mathrm{SB}} = \mathrm{i}\int_0^\infty \mathrm{d}\omega (g_\omega^* a^\dagger b_\omega - g_\omega a b_\omega^\dagger), \tag{3.1}$$

where $a(a^\dagger)$ and $b_\omega(b_\omega^\dagger)$ are annihilation (creation) operators satisfying the canonical commutation relations

$$[a, a^\dagger] = 1, \quad [b_\omega, b_{\omega'}^\dagger] = \delta(\omega - \omega'), \tag{3.2}$$

and g_ω is the form factor of the interaction. Even though system S has an infinite number of levels, and does not fulfill the conditions of the main theorem proved in Article I, the following explicit calculation will show that all the conclusions are still valid and therefore the theorem has a wider applicability.

The above model is exactly solvable [9–11]. Indeed, the Heisenberg equations of motion for $a(t) = \mathrm{e}^{\mathrm{i}Ht} a \mathrm{e}^{-\mathrm{i}Ht}$ and $b_\omega(t) = \mathrm{e}^{\mathrm{i}Ht} b_\omega \mathrm{e}^{-\mathrm{i}Ht}$ read

$$\dot{a}(t) = -\mathrm{i}\omega_{\mathrm{S}} a(t) + \lambda \int_0^\infty \mathrm{d}\omega\, g_\omega^* b_\omega(t), \tag{3.3a}$$

$$\dot{b}_\omega(t) = -\mathrm{i}\omega b_\omega(t) - \lambda g_\omega a(t), \tag{3.3b}$$

and by integrating the second equation and inserting it into the first, one obtains an integro-differential equation for $a(t)$,

$$\dot{a}(t) = -\mathrm{i}\omega_{\mathrm{S}} a(t) - \lambda^2 \int_0^t \mathrm{d}t' K(t - t') a(t') + \lambda B(t), \tag{3.4}$$

with

$$K(t) = \int_0^\infty \mathrm{d}\omega\, |g_\omega|^2 \mathrm{e}^{-\mathrm{i}\omega t}, \quad B(t) = \int_0^\infty \mathrm{d}\omega\, g_\omega^* \mathrm{e}^{-\mathrm{i}\omega t} b_\omega, \tag{3.5}$$

which is solved via Laplace transform to yield

$$a(t) = G(t)a + \lambda \int_0^t \mathrm{d}t' G(t-t')B(t'), \tag{3.6a}$$

$$b_\omega(t) = \mathrm{e}^{-\mathrm{i}\omega t} b_\omega - \lambda \int_0^t \mathrm{d}t' \mathrm{e}^{-\mathrm{i}\omega(t-t')} g_\omega a(t'), \tag{3.6b}$$

where

$$G(t) = \int_{C_\mathrm{B}} \frac{\mathrm{d}s}{2\pi\mathrm{i}} \frac{\mathrm{e}^{st}}{s + \mathrm{i}\omega_\mathrm{S} + \lambda^2 \hat{K}(s)}, \quad \hat{K}(s) = \int_0^\infty \mathrm{d}\omega \frac{|g_\omega|^2}{s + \mathrm{i}\omega}, \tag{3.7}$$

C_B being the Bromwich path on the complex s-plane. Note that $G(0^+) = 1$ and $\dot{G}(0^+) = -\mathrm{i}\omega_\mathrm{S}$.

4. A correlated initial state

Any physical preparation of a quantum state is based on concrete physical procedures that cannot be controlled with complete accuracy. The real initial state is therefore unknown to some extent and in general has certainly some correlations built in. As an example of a correlated initial state, that has the advantage of being solvable, we take

$$\rho_0 = \frac{1}{Z_0} \mathrm{e}^{a^\dagger \xi^\dagger b} (\sigma_\mathrm{S} \otimes \rho_\mathcal{W}) \mathrm{e}^{b^\dagger \xi a}, \tag{4.1}$$

with any positive operator σ_S of system S and a reservoir state

$$\rho_\mathcal{W} = \frac{1}{Z_\mathcal{W}} \mathrm{e}^{-b^\dagger \mathcal{W} b}, \tag{4.2}$$

where the summations over the reservoir modes ω are implicit (and so henceforth as long as no confusion can arise)

$$\xi^\dagger b = \int_0^\infty \mathrm{d}\omega\, \xi_\omega^* b_\omega, \quad b^\dagger \mathcal{W} b = \int_0^\infty \mathrm{d}\omega \int_0^\infty \mathrm{d}\omega' b_\omega^\dagger \mathcal{W}_{\omega\omega'} b_{\omega'}. \tag{4.3}$$

$\mathcal{W}_{\omega\omega'}$ is Hermitian ($\mathcal{W}_{\omega\omega'} = \mathcal{W}_{\omega'\omega}^*$) and consists of $\mathcal{W}^{(0)}_{\omega\omega'}$, that is proportional to $\delta(\omega - \omega')$, and the remaining square integrable part $\tilde{\mathcal{W}}_{\omega\omega'}$,

$$\mathcal{W}_{\omega\omega'} = \mathcal{W}^{(0)}_{\omega\omega'} + \tilde{\mathcal{W}}_{\omega\omega'}, \quad \mathcal{W}^{(0)}_{\omega\omega'} = W(\omega)\delta(\omega - \omega'). \tag{4.4}$$

The states ρ_0 and $\rho_\mathcal{W}$ are normalized with the normalization constants Z_0 and $Z_\mathcal{W}$, and ξ_ω is the relevant parameter to the initial correlation between system S and reservoir B.

For $\xi_\omega = 0$, the state (4.1) is obviously factorized, while it becomes a tightly correlated state for any $\xi_\omega \neq 0$, with correlations proportional to ξ_ω, as will be shown later in (5.7). Actually, the operator $\mathrm{e}^{a^\dagger \xi^\dagger b}$ appearing in the initial state (4.1) generates a correlation between S and B: it changes the n-particle states $b^\dagger\eta_1 \cdots b^\dagger\eta_n|\,\mathrm{vac}\rangle$ of the reservoir into $\mathrm{e}^{a^\dagger\xi^\dagger b} b^\dagger\eta_1 \cdots b^\dagger\eta_n|\mathrm{vac}\rangle = (b^\dagger\eta_1 + a^\dagger\xi^\dagger\eta_1)\cdots(b^\dagger\eta_n + a^\dagger\xi^\dagger\eta_n)|\mathrm{vac}\rangle$, so that S and B are entangled for any nonvanishing value of ξ_ω. It is also possible to explicitly compute the correlation functions in the initial state (4.1): see the generating functional (5.4) and the correlation function (5.7) below. The choice of this particular form for the initial state ρ_0 is mainly due to the fact that it allows us to solve the dynamics of the total system exactly and to discuss the correlation between system S and reservoir B. One can think of the

correlations in (4.1) as engendered by a linear interaction of the form $H_{\rm prep} \propto a^\dagger \xi^\dagger b + \text{h.c.}$ in a rotating-wave-like approximation.

As shown in Appendix A of Article I, the (normalized) reservoir state

$$\Omega_{\rm B} = \frac{1}{Z_{\mathcal{W}_0}} \mathrm{e}^{-b^\dagger \mathcal{W}^{(0)} b} \tag{4.5}$$

is mixing with respect to the reservoir dynamics driven by the Hamiltonian $H_{\rm B}$ in (3.1), and the initial state ρ_0 in (4.1) belongs to the sector specified by $1_{\rm S} \otimes \Omega_{\rm B}$ in the sense of (2.15). Indeed, ρ_0 is the state perturbed from $1_{\rm S} \otimes \Omega_{\rm B}$ by a local operator L,

$$\rho_0 = L(1_{\rm S} \otimes \Omega_{\rm B})L^\dagger, \quad L = \frac{1}{\sqrt{Z_0}} \mathrm{e}^{a^\dagger \xi^\dagger b} (\sqrt{\sigma_{\rm S}} \otimes L_{\rm B}), \tag{4.6}$$

where

$$L_{\rm B} = \rho_{\mathcal{W}}^{1/2} \Omega_{\rm B}^{-1/2} = \sqrt{\frac{Z_{\mathcal{W}_0}}{Z_{\mathcal{W}}}} \bar{\mathrm{T}} \exp\left(- \int_0^{1/2} \mathrm{d}\beta\, b^\dagger \mathrm{e}^{-\beta \mathcal{W}^{(0)}} \tilde{\mathcal{W}} \mathrm{e}^{\beta \mathcal{W}^{(0)}} b \right) \tag{4.7}$$

is a local perturbation such that

$$\rho_{\mathcal{W}} = L_{\rm B} \Omega_{\rm B} L_{\rm B}^\dagger, \tag{4.8}$$

$\bar{\mathrm{T}}$ denoting the anti-chronologically ordered product and

$$b^\dagger \mathrm{e}^{-\beta \mathcal{W}^{(0)}} \tilde{\mathcal{W}} \mathrm{e}^{\beta \mathcal{W}^{(0)}} b = \int_0^\infty \mathrm{d}\omega \int_0^\infty \mathrm{d}\omega' b_\omega^\dagger \mathrm{e}^{-\beta W(\omega)} \tilde{\mathcal{W}}_{\omega\omega'} \mathrm{e}^{\beta W(\omega')} b_{\omega'}. \tag{4.9}$$

Even though the initial state ρ_0 does not satisfy the hypotheses of the theorem proved in Article I, the following analysis extends the general results valid for a bounded perturbation.

Note that the reservoir Gaussian state $\rho_{\mathcal{W}}$ in (4.2) is fully characterized by the two-point function

$$\mathcal{N}_{\omega\omega'} = \langle b_{\omega'}^\dagger b_\omega \rangle_{\rho_{\mathcal{W}}} = \mathrm{tr}_{\rm B}\{ b_{\omega'}^\dagger b_\omega \rho_{\mathcal{W}} \} \tag{4.10}$$

and, as shown in Appendix A in Article I, it is also composed of two parts like $\mathcal{W}_{\omega\omega'}$ in (4.4),

$$\mathcal{N}_{\omega\omega'} = \mathcal{N}^{(0)}_{\omega\omega'} + \tilde{\mathcal{N}}_{\omega\omega'}. \tag{4.11}$$

The first term is the two-point function in the mixing state $\Omega_{\rm B}$,

$$\mathcal{N}^{(0)}_{\omega\omega'} = \langle b_{\omega'}^\dagger b_\omega \rangle_{\Omega_{\rm B}} = N(\omega)\delta(\omega - \omega'), \quad N(\omega) = \frac{1}{\mathrm{e}^{W(\omega)} - 1}, \tag{4.12}$$

which is the Bose distribution function when $W(\omega) = \beta\omega$, while the second one is a local function representing the effect of the local perturbation $L_{\rm B}$ in (4.8).

5. Dynamics of the total system

Since we are interested in the correlation between system S and reservoir B, we need to look at the state of the *total* system, $\rho(t)$. In order to treat the reservoir degrees of freedom rigorously, we should restrict ourselves to reservoir observables whose expectation values are finite and discuss the state of the total system through a set of such expectation values.

The relevant quantity for our discussion is therefore a characteristic functional of the state $\rho(t)$, e.g.,

$$\mathcal{G}[J_a, J_a^*, J_b, J_b^\dagger; t] = \mathrm{tr}\{\mathrm{e}^{J_a^* a}\mathrm{e}^{J_b^\dagger b}\mathrm{e}^{-b^\dagger J_b}\mathrm{e}^{-a^\dagger J_a}\rho(t)\}, \tag{5.1}$$

where $J_b^\dagger b = \int_0^\infty \mathrm{d}\omega J_{b,\omega}^* b_\omega$, which is the generating functional of the expectation values of any anti-normally ordered products of a, $a^\dagger$, b_ω, and $b_\omega^\dagger$ and characterizes the state of the total system, $\rho(t)$. It is important to note that we are not interested in infinitely extended objects, such as the Hamiltonian of the reservoir H_B, since their expectation values are infinite: our targets are *locally* distributed objects. Such a formalization of the problem is reasonable, since we cannot observe infinitely extended objects in practice, and this is nothing but the starting point of the C^*-algebraic approach to quantum statistical mechanics [12]. In the characteristic functional (5.1), the bandwidth of $J_{b,\omega}$ represents the locality of the observables.

Let us begin with the characteristic functional of the initial state ρ_0 in (4.1),

$$\mathcal{G}_0[J_a, J_a^*, J_b, J_b^\dagger] = \mathcal{G}[J_a, J_a^*, J_b, J_b^\dagger; 0] = \mathrm{tr}\{\mathrm{e}^{J_a^* a}\mathrm{e}^{J_b^\dagger b}\mathrm{e}^{-b^\dagger J_b}\mathrm{e}^{-a^\dagger J_a}\rho_0\}, \tag{5.2}$$

which, in the coherent-state representation (Q-representation [4])

$$a|\alpha\rangle = \alpha|\alpha\rangle, \quad \langle\alpha|\alpha'\rangle = \mathrm{e}^{-|\alpha|^2/2-|\alpha'|^2/2+\alpha^*\alpha'}, \quad \int \frac{\mathrm{d}^2\alpha}{\pi}|\alpha\rangle\langle\alpha| = 1_\mathrm{S}, \tag{5.3}$$

is evaluated as

$$\begin{aligned}
&\mathcal{G}_0[J_a, J_a^*, J_b, J_b^\dagger] \\
&= \frac{1}{Z_0}\mathrm{e}^{-J_b^\dagger J_b}\int \frac{\mathrm{d}^2\alpha}{\pi}\langle\alpha|\sigma_\mathrm{S}|\alpha\rangle\mathrm{e}^{J_a^*\alpha-\alpha^* J_a}\left\langle \mathrm{e}^{b^\dagger(\xi\alpha-J_b)}\mathrm{e}^{(\alpha^*\xi^\dagger+J_b^\dagger)b}\right\rangle_\mathcal{W} \\
&= \frac{1}{Z_0}\mathrm{e}^{-J_b^\dagger(1+\mathcal{N})J_b}\int \frac{\mathrm{d}^2\alpha}{\pi}\langle\alpha|\sigma_\mathrm{S}|\alpha\rangle\mathrm{e}^{\alpha^*\xi^\dagger\mathcal{N}\xi\alpha}\mathrm{e}^{(J_a^*+J_b^\dagger\mathcal{N}\xi)\alpha}\mathrm{e}^{-\alpha^*(J_a+\xi^\dagger\mathcal{N}J_b)} \\
&= \mathrm{e}^{-J_b^\dagger(1+\mathcal{N})J_b}\mathcal{G}_\mathrm{S}(J_a+\xi^\dagger\mathcal{N}J_b, J_a^*+J_b^\dagger\mathcal{N}\xi),
\end{aligned} \tag{5.4}$$

where

$$\mathcal{G}_\mathrm{S}(J_a, J_a^*) = \mathcal{G}_0[J_a, J_a^*, 0, 0] = \mathrm{tr}_\mathrm{S}\{\mathrm{e}^{J_a^* a}\mathrm{e}^{-a^\dagger J_a}\rho_\mathrm{S}\} \tag{5.5}$$

is the characteristic function of the initial state of system S and

$$\rho_\mathrm{S} = \mathrm{tr}_\mathrm{B}\rho_0. \tag{5.6}$$

One can see from this characteristic functional how the parameter ξ_ω embodies the initial correlation. For example,

$$\begin{aligned}
\langle ab_\omega^\dagger\rangle_{\rho_0} &= -\frac{\partial}{\partial J_a^*}\frac{\delta}{\delta J_{b,\omega}}\mathcal{G}_0[J_a, J_a^*, J_b, J_b^\dagger]\Big|_{J_a, J_a^*, J_b, J_b^\dagger=0} \\
&= -\int_0^\infty \mathrm{d}\omega'\xi_{\omega'}^*\mathcal{N}_{\omega'\omega}\frac{\partial^2\mathcal{G}_\mathrm{S}(J_a, J_a^*)}{\partial J_a\partial J_a^*}\Big|_{J_a, J_a^*=0} \\
&= \int_0^\infty \mathrm{d}\omega'\xi_{\omega'}^*\mathcal{N}_{\omega'\omega}\langle aa^\dagger\rangle_{\rho_\mathrm{S}}.
\end{aligned} \tag{5.7}$$

Let us discuss the evolution of the state of the total system

$$\rho(t) = \mathrm{e}^{-\mathrm{i}Ht}\rho_0\mathrm{e}^{\mathrm{i}Ht}. \tag{5.8}$$

The characteristic functional (5.1) of the state $\rho(t)$ is easily computed in the Heisenberg picture

$$\begin{aligned}\mathcal{G}[J_a, J_a^*, J_b, J_b^\dagger; t] &= \mathrm{tr}\{\mathrm{e}^{J_a^* a(t)}\mathrm{e}^{J_b^\dagger b(t)}\mathrm{e}^{-b^\dagger(t)J_b}\mathrm{e}^{-a^\dagger(t)J_a}\rho_0\}\\ &= \mathrm{tr}\{\mathrm{e}^{J_a^* a(t)}\mathrm{e}^{J_b^\dagger b(t)}\mathrm{e}^{-b^\dagger J_b(t)}\mathrm{e}^{-a^\dagger J_a(t)}\rho_0\}\\ &= \mathcal{G}_0[J_a(t), J_a^*(t), J_b(t), J_b^\dagger(t)],\end{aligned} \tag{5.9}$$

where $J_a(t)$ and $J_b(t)$ are functionals of J_a and J_b, defined via $a^\dagger(t)J_a + b^\dagger(t)J_b = a^\dagger J_a(t) + b^\dagger J_b(t)$. Note that the solutions (3.6) for $a(t)$ and $b_\omega(t)$ are linear in a and b_ω, but do not contain $a^\dagger$ or $b_\omega^\dagger$. The characteristic functional of the initial state ρ_0 is given in (5.4) and Eq. (5.9) is further reduced to

$$\mathcal{G}[J_a, J_a^*, J_b, J_b^\dagger; t] = \mathrm{e}^{-J_b^\dagger(t)(1+\mathcal{N})J_b(t)}\mathcal{G}_\mathrm{S}(J_a(t) + \xi^\dagger\mathcal{N}J_b(t), J_a^*(t) + J_b^\dagger(t)\mathcal{N}\xi). \tag{5.10}$$

We thus obtain the exact characteristic functional of the state of the *total* system, $\rho(t)$,

$$\mathcal{G}[J_a, J_a^*, J_b, J_b^\dagger; t] = \mathrm{e}^{-\mathcal{J}^\dagger\mathcal{A}(t)\mathcal{J}}\mathcal{G}_\mathrm{S}(h^\dagger(t)\mathcal{J}, \mathcal{J}^\dagger h(t)), \tag{5.11}$$

where

$$\mathcal{J}^\dagger\mathcal{A}(t)\mathcal{J} = \begin{pmatrix}J_a^* & J_b^\dagger\end{pmatrix}\begin{pmatrix}\mathcal{A}_{aa}(t) & \mathcal{A}_{ab}(t)\\ \mathcal{A}_{ba}(t) & \mathcal{A}_{bb}(t)\end{pmatrix}\begin{pmatrix}J_a\\ J_b\end{pmatrix}, \quad h^\dagger(t)\mathcal{J} = \begin{pmatrix}h_a^*(t) & h_b^\dagger(t)\end{pmatrix}\begin{pmatrix}J_a\\ J_b\end{pmatrix} \tag{5.12}$$

with

$$\mathcal{A}_{aa}(t) = \lambda^2\int_0^t \mathrm{d}t'\int_0^t \mathrm{d}t'' G(t-t')\Phi_{gg}(t',t'')G^*(t-t''), \tag{5.13a}$$

$$\begin{aligned}J_b^\dagger\mathcal{A}_{bb}(t)J_b = \Phi_{J_bJ_b}(t,t) &- 2\lambda^2\mathrm{Re}\int_0^t \mathrm{d}t'(K_{J_bg} * G)(t-t')\Phi_{gJ_b}(t',t)\\ &+ \lambda^4\int_0^t \mathrm{d}t'\int_0^t \mathrm{d}t''(K_{J_bg} * G)(t-t')\Phi_{gg}(t',t'')(G^* * K_{J_bg}^*)(t-t''),\end{aligned} \tag{5.13b}$$

$$\begin{aligned}J_b^\dagger\mathcal{A}_{ba}(t) = \lambda&\int_0^t \mathrm{d}t'\Phi_{J_bg}(t,t')G^*(t-t')\\ &- \lambda^3\int_0^t \mathrm{d}t'\int_0^t \mathrm{d}t''(K_{J_bg} * G)(t-t')\Phi_{gg}(t',t'')G^*(t-t'')\end{aligned} \tag{5.13c}$$

and

$$h_a(t) = G(t) + \lambda(G * K_{g(\mathcal{N}\xi)})(t), \tag{5.13d}$$

$$J_b^\dagger h_b(t) = K_{J_b(\mathcal{N}\xi)}(t) - \lambda(K_{J_bg} * G)(t) - \lambda^2(K_{J_bg} * G * K_{g(\mathcal{N}\xi)})(t). \tag{5.13e}$$

We have introduced

$$K_{fg}(t) = \int_0^\infty \mathrm{d}\omega f_\omega^* \mathrm{e}^{-\mathrm{i}\omega t} g_\omega,$$
$$\Phi_{fg}(t,t') = \int_0^\infty \mathrm{d}\omega \int_0^\infty \mathrm{d}\omega' f_\omega^* \mathrm{e}^{-\mathrm{i}\omega t}(1+\mathcal{N})_{\omega\omega'} \mathrm{e}^{\mathrm{i}\omega' t'} g_{\omega'}, \tag{5.14}$$

where $1_{\omega\omega'} = \delta(\omega - \omega')$, and the convolution

$$(F * G)(t) = \int_0^t \mathrm{d}t' F(t-t')G(t'). \tag{5.15}$$

The characteristic functional of the total system (5.11) is exact and valid for any time t.

The functions $\lambda(G * K_{g(\mathcal{N}\xi)})(t)$ in $h_a(t)$ and $K_{J_b(\mathcal{N}\xi)}(t)$ in $h_b(t)$ describe how the initial correlation propagates, while $\mathcal{A}_{ba}(t)$ and $\lambda(K_{J_{bg}} * G)(t)$ in $h_b(t)$ describe the correlation established through the interaction between system S and reservoir B. System S forgets its initial state through the decay of $G(t)$ and approaches an equilibrium state via the action of $\mathcal{A}_{aa}(t)$, while $\mathcal{A}_{bb}(t)$ governs the relaxation of reservoir B into its equilibrium, i.e., the mixing state Ω_{B}, as explained in the following.

6. The van Hove limit of the characteristic functional and discussion

We are now in a position to discuss the van Hove limit of the evolution of the total system and demonstrate the validity of the general theorem proved in Article I: (a) the disappearance of the initial correlation, (b) the factorization of the total system, and (c) the relaxation into the mixing state, in van Hove's limit.

In order to discuss van Hove's limit, let us remove the (rapid) oscillation of system S. That is, let us look at the characteristic functional of the density operator $\mathrm{e}^{\mathrm{i}H_{\mathrm{S}}t}\rho(t)\mathrm{e}^{-\mathrm{i}H_{\mathrm{S}}t}$ in the scaled time $\tau = \lambda^2 t$,

$$\mathcal{G}_{\mathrm{I}}^{(\lambda)}[J_a, J_a^*, J_b, J_b^\dagger; \tau] = \mathcal{G}[J_a \mathrm{e}^{-\mathrm{i}\omega_{\mathrm{S}}\tau/\lambda^2}, J_a^* \mathrm{e}^{\mathrm{i}\omega_{\mathrm{S}}\tau/\lambda^2}, J_b, J_b^\dagger; \tau/\lambda^2]. \tag{6.1}$$

Then, the van Hove limits of the constituent functions (5.13) (Appendix A),

$$\lim_{\lambda\to 0} \mathcal{A}_{aa}(\tau/\lambda^2) = [1+N(\omega_{\mathrm{S}})](1-\mathrm{e}^{-\Gamma(\omega_{\mathrm{S}})\tau}), \tag{6.2a}$$

$$\lim_{\lambda\to 0} J_b^\dagger \mathcal{A}_{bb}(\tau/\lambda^2) J_b = \int_0^\infty \mathrm{d}\omega J_{b,\omega}^* [1+N(\omega)] J_{b,\omega}, \tag{6.2b}$$

$$\lim_{\lambda\to 0} \mathrm{e}^{\mathrm{i}\omega_{\mathrm{S}}\tau/\lambda^2} \mathcal{A}_{ab}(\tau/\lambda^2) J_b = \lim_{\lambda\to 0} J_b^\dagger \mathcal{A}_{ba}(\tau/\lambda^2) \mathrm{e}^{-\mathrm{i}\omega_{\mathrm{S}}\tau/\lambda^2} = 0, \tag{6.2c}$$

$$\lim_{\lambda\to 0} h_a(\tau/\lambda^2)\mathrm{e}^{\mathrm{i}\omega_{\mathrm{S}}\tau/\lambda^2} = \mathrm{e}^{-\Gamma(\omega_{\mathrm{S}})\tau/2}\mathrm{e}^{-\mathrm{i}\Delta(\omega_{\mathrm{S}})\tau}, \quad \lim_{\lambda\to 0} J_b^\dagger h_b(\tau/\lambda^2) = 0, \tag{6.2d}$$

lead us to the van Hove limit of the characteristic functional (5.11)

$$\begin{aligned}\mathcal{G}_{\mathrm{I}}[J_a, J_a^*, J_b, J_b^\dagger; \tau] &= \lim_{\lambda\to 0} \mathcal{G}_{\mathrm{I}}^{(\lambda)}[J_a, J_a^*, J_b, J_b^\dagger; \tau] \\ &= \mathrm{e}^{-J_a^* J_a [1+N(\omega_{\mathrm{S}})](1-\mathrm{e}^{-\Gamma(\omega_{\mathrm{S}})\tau})} \mathrm{e}^{-\int_0^\infty \mathrm{d}\omega J_{b,\omega}^* [1+N(\omega)] J_{b,\omega}} \\ &\quad \times \mathcal{G}_{\mathrm{S}}(J_a \mathrm{e}^{-\Gamma(\omega_{\mathrm{S}})\tau/2}\mathrm{e}^{\mathrm{i}\Delta(\omega_{\mathrm{S}})\tau}, J_a^* \mathrm{e}^{-\Gamma(\omega_{\mathrm{S}})\tau/2}\mathrm{e}^{-\mathrm{i}\Delta(\omega_{\mathrm{S}})\tau}),\end{aligned} \tag{6.3}$$

where

$$\Gamma(\omega) = 2\pi|g_\omega|^2, \quad \Delta(\omega) = \mathrm{P}\int_0^\infty \frac{\mathrm{d}\omega'}{2\pi}\frac{\Gamma(\omega')}{\omega-\omega'}. \tag{6.4}$$

It is clear from (6.3) that (a) the initial correlation (or, equivalently, ξ_ω) disappears and (b) the state of the total system is factorized at all times in van Hove's limit. Furthermore, (c) the local perturbation in the initial state ρ_0, i.e., L in (4.6) (especially, the contribution of $\tilde{\mathcal{W}}_{\omega\omega'}$, which appears in the characteristic functional through $\tilde{\mathcal{N}}_{\omega\omega'}$), decays out and the reservoir relaxes into the mixing state Ω_B given in (4.5). The dynamics of the system in van Hove's limit is exactly the same as that derived from the *uncorrelated* initial state $\mathrm{tr}\{\rho_0\}\otimes\Omega_\mathrm{B}$ with the mixing state Ω_B and it is actually possible to show that the density operator $\rho_\mathrm{I}(\tau)$ characterized by the characteristic functional (6.3) obeys the master equation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\rho_\mathrm{I}(\tau) = &-\mathrm{i}[\Delta(\omega_\mathrm{S})a^\dagger a, \rho_\mathrm{I}(\tau)]\\ &-\tfrac{1}{2}[1+N(\omega_\mathrm{S})]\Gamma(\omega_\mathrm{S})[a^\dagger a\rho_\mathrm{I}(\tau)+\rho_\mathrm{I}(\tau)a^\dagger a-2a\rho_\mathrm{I}(\tau)a^\dagger]\\ &-\tfrac{1}{2}N(\omega_\mathrm{S})\Gamma(\omega_\mathrm{S})[aa^\dagger\rho_\mathrm{I}(\tau)+\rho_\mathrm{I}(\tau)aa^\dagger-2a^\dagger\rho_\mathrm{I}(\tau)a].\end{aligned} \tag{6.5}$$

This is nothing but the familiar master equation derived from the factorized initial condition with the reservoir in the thermal equilibrium state at a finite temperature, $\rho_0 \sim \rho_\mathrm{S}\otimes \mathrm{e}^{-\beta H_\mathrm{B}}$ [3,4].

These points corroborate the theorem in Article I, suggesting that the mixing state Ω_B, which is contained in the initial state ρ_0, should be selected as the reference state of Nakajima–Zwanzig's projection $\mathcal{P}$. Note that the characteristic functional in van Hove's limit, Eq. (6.3), approaches

$$\mathcal{G}_\mathrm{I}[J_a, J_a^*, J_b, J_b^\dagger; \tau] \xrightarrow{\tau\to\infty} \mathrm{e}^{-J_a^* J_a[1+N(\omega_\mathrm{S})]}\mathrm{e}^{-\int_0^\infty \mathrm{d}\omega J_{b,\omega}^*[1+N(\omega)]J_{b,\omega}}, \tag{6.6}$$

which means that the equilibrium state (in van Hove's limit) is

$$\rho_\mathrm{eq} = \frac{1}{Z_\mathrm{eq}}\mathrm{e}^{-W(\omega_\mathrm{S})a^\dagger a}\otimes\Omega_\mathrm{B}, \quad Z_\mathrm{eq}^{-1} = 1-\mathrm{e}^{-W(\omega_\mathrm{S})}, \tag{6.7}$$

i.e., system S relaxes into the equilibrium state with the same structure as that of the mixing state Ω_B.

As discussed in Article I, the state of the total system is factorized through its free evolution and the interaction between system S and reservoir B is not essential, which is also confirmed by the present exact solution. In the absence of the interaction, the exact characteristic functional (5.11) reads

$$\mathcal{G}[J_a, J_a^*, J_b, J_b^\dagger; t] = \mathrm{e}^{-\Phi_{J_b J_b}(t,t)}\mathcal{G}_\mathrm{S}(J_a\mathrm{e}^{\mathrm{i}\omega_\mathrm{S}t}+K_{J_b(\mathcal{N}\xi)}^*(t), J_a^*\mathrm{e}^{-\mathrm{i}\omega_\mathrm{S}t}+K_{J_b(\mathcal{N}\xi)}(t)) \tag{6.8}$$

and approaches

$$\mathcal{G}[J_a, J_a^*, J_b, J_b^\dagger; t] \xrightarrow{t\to\infty} \mathrm{e}^{-\int_0^\infty \mathrm{d}\omega J_{b,\omega}^*[1+N(\omega)]J_{b,\omega}}\mathcal{G}_\mathrm{S}(J_a\mathrm{e}^{\mathrm{i}\omega_\mathrm{S}t}, J_a^*\mathrm{e}^{-\mathrm{i}\omega_\mathrm{S}t}) \tag{6.9}$$

by Riemann–Lebesgue's lemma [see the discussion below and Eq. (A.1b)]. The state is thus factorized into

$$\rho(t) \xrightarrow{t\to\infty} \rho_\mathrm{S}(t)\otimes\Omega_\mathrm{B}, \quad \rho_\mathrm{S}(t) = \mathrm{e}^{\mathcal{L}_\mathrm{S}t}\mathrm{tr}_\mathrm{B}\rho_0 \tag{6.10}$$

through the free evolution, which confirms the second part of the theorem in Article I.

The timescales of the factorization and the relaxation into the mixing state are clear from (6.8): the former is governed by the function

$$K_{J_b(\mathcal{N}\xi)}(t) = \int_0^\infty \mathrm{d}\omega \left(\int_0^\infty \mathrm{d}\omega' J^*_{b,\omega} \mathcal{N}_{\omega\omega'} \xi_{\omega'} \right) \mathrm{e}^{-\mathrm{i}\omega t} \tag{6.11}$$

contained in $J_b^\dagger h_b(t)$ in (5.13e), and the latter by the leading term of $\mathcal{A}_{bb}(t)$ in (5.13b),

$$\Phi_{J_b J_b}(t,t) = \Phi^{(0)}_{J_b J_b}(0) + 2\mathrm{Re} \int_0^\infty \mathrm{d}\omega \left(\int_{\omega/2}^\infty \mathrm{d}\bar{\omega} J^*_{b,\bar{\omega}+\omega/2} \tilde{\mathcal{N}}_{(\bar{\omega}+\omega/2)(\bar{\omega}-\omega/2)} J_{b,\bar{\omega}-\omega/2} \right) \mathrm{e}^{-\mathrm{i}\omega t}, \tag{6.12}$$

where

$$\Phi^{(0)}_{fg}(t) = \int_0^\infty \mathrm{d}\omega f^*_\omega [1 + N(\omega)] g_\omega \mathrm{e}^{-\mathrm{i}\omega t}. \tag{6.13}$$

The timescales of the decay of these functions are determined by the bandwidths of their Fourier transforms,

$$\tilde{K}_{J_b(\mathcal{N}\xi)}(\omega) = 2\pi \int_0^\infty \mathrm{d}\omega' J^*_{b,\omega} \mathcal{N}_{\omega\omega'} \xi_{\omega'} \tag{6.14}$$

for the former, and

$$\tilde{\Phi}_{J_b J_b}(\omega) = 2\pi \int_{\omega/2}^\infty \mathrm{d}\bar{\omega} J^*_{b,\bar{\omega}+\omega/2} \tilde{\mathcal{N}}_{(\bar{\omega}+\omega/2)(\bar{\omega}-\omega/2)} J_{b,\bar{\omega}-\omega/2} \tag{6.15}$$

for the latter. Therefore, besides the spread of the initial correlation (ξ_ω) and of the perturbation from the mixing state ($\tilde{\mathcal{N}}_{\omega\omega'}$), *the size of the relevant reservoir observables* ($J_{b,\omega}$) influences the timescales of the factorization and the mixing. In the weak-coupling regime they are very short compared with the timescale of the dissipative dynamics of system S, which is of order $1/\lambda^2$ in the original time t.

It is interesting to discuss what happens from a physical point of view. The initial correlations and the local perturbations propagate outwards from the region of interest (defined by the "size" of the relevant local observables) and never come back. What remains is the "unperturbed" state, that is the mixing state Ω_B and is the stable "ground state" within the sector it specifies. The relaxation time of such a process is the time necessary for the disturbance to pass through the range of the interaction, that of the initial correlation, and the extension of the observable. It should be noted that, when we work in the interaction picture $\mathrm{e}^{\mathrm{i}H_0 t}\rho(t)\mathrm{e}^{-\mathrm{i}H_0 t}$, instead of $\mathrm{e}^{\mathrm{i}H_\mathrm{S} t}\rho(t)\mathrm{e}^{-\mathrm{i}H_\mathrm{S} t}$ considered above in (6.1), we should duly take into account the time dependence of the observables in such a picture, $X(t) = \mathrm{e}^{\mathrm{i}H_0 t} X \mathrm{e}^{-\mathrm{i}H_0 t}$, otherwise mixing is not observed.

7. A solvable model with counter-rotating interaction

Let us look at another solvable example: the same model as the previous one (3.1) but with a different interaction Hamiltonian

$$H_\mathrm{SB} = \mathrm{i}(a + a^\dagger) \int_0^\infty \mathrm{d}\omega (g^*_\omega b_\omega - g_\omega b^\dagger_\omega), \tag{7.1}$$

containing *counter-rotating* terms. This model is also exactly solvable [9,11,13–15]. Let us only briefly sketch the main results. More details are given in Appendix B. The exact characteristic functional of the state of the total system, $\rho(t)$, reads

$$\mathcal{G}[J_a, J_a^*, J_b, J_b^\dagger; t] = \mathrm{e}^{-\mathcal{J}^\dagger \mathcal{A}(t)\mathcal{J}} \mathrm{e}^{-\mathcal{J}^\dagger \bar{\mathcal{A}}(t)\mathcal{J}^*} \mathrm{e}^{-\mathcal{J}^{\mathrm{T}} \bar{\mathcal{A}}^\dagger(t)\mathcal{J}} \times \mathcal{G}_{\mathrm{S}}(h^\dagger(t)\mathcal{J} + \mathcal{J}^\dagger \bar{h}(t), \mathcal{J}^\dagger h(t) + \bar{h}^\dagger(t)\mathcal{J}) \tag{7.2}$$

for the same correlated initial state as before, ρ_0 in (4.1), where T denotes the transpose matrix, and

$$\mathcal{J}^\dagger \bar{\mathcal{A}}(t)\mathcal{J}^* = \begin{pmatrix} J_a^* & J_b^\dagger \end{pmatrix} \begin{pmatrix} \bar{\mathcal{A}}_{aa}(t) & \bar{\mathcal{A}}_{ab}(t) \\ \bar{\mathcal{A}}_{ba}(t) & \bar{\mathcal{A}}_{bb}(t) \end{pmatrix} \begin{pmatrix} J_a^* \\ J_b^* \end{pmatrix}, \quad \bar{h}^\dagger(t)\mathcal{J} = \begin{pmatrix} \bar{h}_a^*(t) & \bar{h}_b^\dagger(t) \end{pmatrix} \begin{pmatrix} J_a \\ J_b \end{pmatrix}. \tag{7.3}$$

The details of these functions are given in Appendix B.

This characteristic functional contains different types of terms from those in the previous example (5.11): the counter-rotating interaction provokes "squeezing." In van Hove's limit, however, these contributions disappear. Indeed, the van Hove limits of the constituent functions (B.4)–(B.6) of the characteristic functional (7.2) are

$$\lim_{\lambda\to 0} \mathcal{A}_{aa}(\tau/\lambda^2) = [1 + N(\omega_{\mathrm{S}})](1 - \mathrm{e}^{-\Gamma(\omega_{\mathrm{S}})\tau}), \tag{7.4a}$$

$$\lim_{\lambda\to 0} J_b^\dagger \mathcal{A}_{bb}(\tau/\lambda^2) J_b = \int_0^\infty \mathrm{d}\omega J_{b,\omega}^* [1 + N(\omega)] J_{b,\omega}, \tag{7.4b}$$

$$\lim_{\lambda\to 0} h_a(\tau/\lambda^2)\mathrm{e}^{\mathrm{i}\omega_{\mathrm{S}}\tau/\lambda^2} = \mathrm{e}^{-\Gamma(\omega_{\mathrm{S}})\tau/2} \mathrm{e}^{-\mathrm{i}[\Delta(\omega_{\mathrm{S}}) - \bar{\Delta}(\omega_{\mathrm{S}})]\tau}, \tag{7.4c}$$

while all other limits vanish (see Appendix B.3), and one ends up with the same dynamics as the previous one (6.3) except for the frequency shift; $\Delta(\omega_{\mathrm{S}})$ must be substituted with $\Delta(\omega_{\mathrm{S}}) - \bar{\Delta}(\omega_{\mathrm{S}})$, where $\bar{\Delta}(\omega)$ is defined in (B.10). The present example again supports the validity of the theorem proved in Article I: (a) the initial correlation disappears, (b) the state of the total system is factorized at all times, and (c) the reservoir remains in the mixing state, in van Hove's limit. The effect of the counter-rotating interaction manifests itself only in the frequency shift; no other differences in the resultant dynamics from the previous example with the rotating-wave interaction [8].

Furthermore, the timescales of the factorization and of the mixing are governed by the functions $K_{J_b}(\mathcal{N}\xi)(t)$ and $\Phi_{J_bJ_b}(t,t)$, respectively (see Appendix B.2); they are the same as those in the previous example [Eqs. (6.11) and (6.12)]. This also supports the general conclusion that the free evolution of the reservoir plays an essential role for the factorization and the mixing, but the interaction does not. The counter-rotating interaction gives rise to no significant effect on the factorization or the mixing.

8. Concluding remarks

We have investigated two solvable models in the light of the general theorem proved in Article I. In both cases, we confirmed that when the initial state of the quantum system and the reservoir is not factorized, a correct application of Nakajima–Zwanzig's projection method requires that the reference state of the latter be mixing. In addition, close scrutiny of the solvable models enabled us to focus on the relevant timescales. It turns

out that an effective factorization of the state of the total system depends on the *free* dynamics of the reservoir (responsible for mixing) as well as on the interaction. Indeed, the free dynamics itself is sufficient to drive a complete factorization. Moreover, the timescales for mixing (that in turn govern the very applicability of the projection method in terms of the "reference" state of the reservoir) depend on the "size" of local observables of the reservoir: clearly, if one has access to information that is distributed over *larger* portion of the reservoir, one can in general detect *finer* deviations from mixing. The timescales at which Markovianity can be considered a good approximation depend on the structure of the local observables that one can measure, that is on the dimension of the (sub)system whose evolution one wants to describe. This conclusion, physically sound, is in some sense a strict consequence of the philosophy at the basis of the C^*-algebraic approach to infinite systems (in the case at hand, the reservoir, whose observables one can measure).

There are other very interesting problems that we have not analyzed and that are related to the general features of the evolutions when it is not permissible to consider a factorized initial state [11,13,15–19]. Among others, the problems related to the (complete) positivity of the evolution requires additional investigations [20–27]. Another interesting issue would be to discuss the applicability of this method to more articulated (and intriguing) thermodynamical situations, such as those of nonequilibrium steady states [28], shortly discussed in Article I (see Fig. 1 in Article I). It is indeed possible to apply the method we propose to discuss the relaxation of a system driven by a reservoir at a nonequilibrium steady state and this aspect will be discussed elsewhere.

Acknowledgments

This work is partly supported by the bilateral Italian–Japanese Projects II04C1AF4E on "Quantum Information, Computation and Communication" of the Italian Ministry of Instruction, University and Research, and 15C1 on "Quantum Information and Computation" of the Italian Ministry for Foreign Affairs, by the European Community through the Integrated Project EuroSQIP, by the Grant for The 21st Century COE Program "Holistic Research and Education Center for Physics of Self-Organization Systems" at Waseda University, the Grant-in-Aid for the COE Research "Establishment of Molecular Nano-Engineering by Utilizing Nanostructure Arrays and Its Development into Micro-Systems" at Waseda University (No. 13CE2003), and the Grants-in-Aid for Scientific Research on Priority Areas "Control of Molecules in Intense Laser Fields" (No. 14077219), "Dynamics of Strings and Fields" (No. 13135221), and for Young Scientists (B) (No. 18740250) from the Ministry of Education, Culture, Sports, Science and Technology, Japan, and by Grants-in-Aid for Scientific Research (C) (Nos. 14540280, 17540365, and 18540292) from the Japan Society for the Promotion of Science.

Appendix A. Prototypes of the van Hove limits

The characteristic functional $\mathcal{G}[J_a, J_a^*, J_b, J_b^\dagger; t]$ in (5.11) is expressed in terms of the functions given in (5.13). The van Hove limits of these functions fall into the following types: by taking the weak-coupling limit $\lambda \to 0$ keeping $\tau = \lambda^2 t$ finite, one obtains

$$\text{(i)}\ G(t)e^{i\omega_S t} \to e^{-\Gamma(\omega_S)\tau/2} e^{-i\Delta(\omega_S)\tau}, \tag{A.1a}$$

$$\text{(ii)}\ K_{fg}(t) \to 0, \quad \Phi_{fg}(t,t) \to \int_0^\infty \mathrm{d}\omega f_\omega^* [1+N(\omega)] g_\omega, \tag{A.1b}$$

$$\text{(iii)}\ (K_{fg} * G * K_{gf'})(t) \mathrm{e}^{\mathrm{i}\omega_S t} \to \hat{K}_{fg}(-\mathrm{i}\omega_S + 0^+)\hat{K}_{gf'}(-\mathrm{i}\omega_S + 0^+)\mathrm{e}^{-\Gamma(\omega_S)\tau/2}\mathrm{e}^{-\mathrm{i}\Delta(\omega_S)\tau}, \tag{A.1c}$$

$$\text{(iv)}\ \lambda^2 \int_0^t \mathrm{d}t' \int_0^t \mathrm{d}t''(K_{fg} * G)(t-t')\Phi_{gg}(t',t'')(G^* * K_{f'g}^*)(t-t'') \to \hat{K}_{fg}(-\mathrm{i}\omega_S + 0^+)[\hat{K}_{f'g}(-\mathrm{i}\omega_S + 0^+)]^*[1+N(\omega_S)](1-\mathrm{e}^{-\Gamma(\omega_S)\tau}), \tag{A.1d}$$

$$\text{(v)}\ \int_0^t \mathrm{d}t' \Phi_{J_b g}(t,t')(G^* * K_{fg}^*)(t-t') \to \int_0^\infty \mathrm{d}\omega J_{b,\omega}^* [1+N(\omega)] g_\omega \frac{[\hat{K}_{fg}(-\mathrm{i}\omega + 0^+)]^*}{\mathrm{i}(\omega-\omega_S)+0^+}, \tag{A.1e}$$

where $\Gamma(\omega)$ and $\Delta(\omega)$ are defined in (6.4), and $\hat{K}_{fg}(s)$ is the Laplace transform of $K_{fg}(t)$ in (5.14). Let us prove these results.

(i) The van Hove limit of $G(t)$, which is defined in (3.7), is the ordinary one [8]:

$$G(t)\mathrm{e}^{\mathrm{i}\omega_S t} = \int_{C_B} \frac{\mathrm{d}\tilde{s}}{2\pi\mathrm{i}} \frac{\mathrm{e}^{\tilde{s}\tau}}{\tilde{s} + \hat{K}(\lambda^2\tilde{s} - \mathrm{i}\omega_S)} \overset{\lambda\to 0}{\to} \int_{C_B} \frac{\mathrm{d}\tilde{s}}{2\pi\mathrm{i}} \frac{\mathrm{e}^{\tilde{s}\tau}}{\tilde{s} + \hat{K}(-\mathrm{i}\omega_S + 0^+)}, \tag{A.2}$$

which results in (A.1a), by noting the formula for $\hat{K}(s)$ in (3.7),

$$\hat{K}(-\mathrm{i}\omega_S + 0^+) = \tfrac{1}{2}\Gamma(\omega_S) + \mathrm{i}\Delta(\omega_S), \tag{A.3}$$

with $\Gamma(\omega)$ and $\Delta(\omega)$ defined in (6.4).

(ii) The van Hove limits in (A.1b) are just the long-time limits and are due to Riemann–Lebesgue's lemma. The timescales of the decays are determined by the band widths of their Fourier transforms. See Eqs. (5.14) and (6.12).

(iii) In terms of the inverse Laplace transform, the convolution in (A.1c) is written as

$$(K_{fg} * G * K_{gf'})(t)\mathrm{e}^{\mathrm{i}\omega_S t} = \int_{C_B} \frac{\mathrm{d}s}{2\pi\mathrm{i}} \frac{\hat{K}_{fg}(s)\hat{K}_{gf'}(s)}{s + \mathrm{i}\omega_S + \lambda^2\hat{K}(s)} \mathrm{e}^{(s+\mathrm{i}\omega_S)t}, \tag{A.4}$$

whose van Hove limit proceeds like in (A.2).

(iv) Notice first that the contribution of $\tilde{\mathcal{N}}_{\omega\omega'}$ to Eq. (A.1d) through the function $\Phi_{fg}(t,t')$, which represents the effect of the local perturbation L_B for $\rho_{\mathcal{W}}$ in (4.8), decays out in van Hove's limit, since the van Hove limit of this contribution is a generalization of (iii) but with a vanishing prefactor λ^2 in (A.1d). Therefore, the main contribution comes from the mixing state through $\Phi_{fg}^{(0)}(t-t')$ defined in (6.13): by noting that

$$\begin{aligned}
&\frac{1}{\lambda^2}\Phi_{fg}^{(0)}(t-t')\mathrm{e}^{\mathrm{i}\omega_S(t-t')} \\
&\quad = \frac{1}{\lambda^2}\int_0^\infty \mathrm{d}\omega f_\omega^* g_\omega [1+N(\omega)]\mathrm{e}^{-\mathrm{i}(\omega-\omega_S)(t-t')} \\
&\quad = \int_{-\omega_S/\lambda^2}^\infty \mathrm{d}\tilde{\omega} f_{\lambda^2\tilde{\omega}+\omega_S}^* g_{\lambda^2\tilde{\omega}+\omega_S}[1+N(\lambda^2\tilde{\omega}+\omega_S)]\mathrm{e}^{-\mathrm{i}\tilde{\omega}(\tau-\tau')}
\end{aligned}$$

$$\overset{\lambda\to 0}{\to} \int_{-\infty}^{\infty} \mathrm{d}\tilde{\omega} f^*_{\omega_\mathrm{S}} g_{\omega_\mathrm{S}} [1 + N(\omega_\mathrm{S})] \mathrm{e}^{-\mathrm{i}\tilde{\omega}(\tau-\tau')} = 2\pi f^*_{\omega_\mathrm{S}} g_{\omega_\mathrm{S}} [1 + N(\omega_\mathrm{S})] \delta(\tau - \tau'), \tag{A.5}$$

Eq. (A.1d) is deduced via

$$\frac{1}{\lambda^2} \int_0^\tau \mathrm{d}\tau' \int_0^\tau d\tau'' (K_{fg} * G)((\tau - \tau')/\lambda^2) \mathrm{e}^{\mathrm{i}\omega_\mathrm{S}(\tau-\tau')/\lambda^2} \Phi^{(0)}_{gg}((\tau' - \tau'')/\lambda^2) \mathrm{e}^{\mathrm{i}\omega_\mathrm{S}(\tau'-\tau'')/\lambda^2} \times (G^* * K^*_{f'g})((\tau - \tau'')/\lambda^2) \mathrm{e}^{-\mathrm{i}\omega_\mathrm{S}(\tau-\tau'')/\lambda^2} \overset{\lambda\to 0}{\to} \hat{K}_{fg}(-\mathrm{i}\omega_\mathrm{S} + 0^+) [\hat{K}_{f'g}(-\mathrm{i}\omega_\mathrm{S} + 0^+)]^* \Gamma(\omega_\mathrm{S}) [1 + N(\omega_\mathrm{S})] \int_0^\tau \mathrm{d}\tau' \mathrm{e}^{-\Gamma(\omega_\mathrm{S})\tau'}. \tag{A.6}$$

(v) While the contribution of $\tilde{\mathcal{N}}_{\omega\omega'}$ decays out in van Hove's limit, which is shown by generalizing (ii) and (iii), that of $\mathcal{N}^{(0)}_{\omega\omega'}$,

$$\int_0^\infty \mathrm{d}\omega J^*_{b,\omega} [1 + N(\omega)] g_\omega \left(\int_{C_\mathrm{B}} \frac{\mathrm{d}s}{2\pi\mathrm{i}} \frac{\hat{K}_{fg}(s)}{s + \mathrm{i}\omega_\mathrm{S} + \lambda^2 \hat{K}(s)} \frac{\mathrm{e}^{(s+\mathrm{i}\omega)\tau/\lambda^2}}{s + \mathrm{i}\omega} \right)^* = \int_0^\infty \mathrm{d}\omega J^*_{b,\omega} [1 + N(\omega)] g_\omega \left(\int_{C_\mathrm{B}} \frac{\mathrm{d}\tilde{s}}{2\pi\mathrm{i}} \frac{\hat{K}_{fg}(\lambda^2\tilde{s} - \mathrm{i}\omega)}{\lambda^2\tilde{s} - \mathrm{i}(\omega - \omega_\mathrm{S}) + \lambda^2 \hat{K}(\lambda^2\tilde{s} - \mathrm{i}\omega)} \frac{\mathrm{e}^{\tilde{s}\tau}}{\tilde{s}} \right)^*, \tag{A.7}$$

yields (A.1e).

The prototypes (i)–(v) lead to the van Hove limits of the components (6.2).

Appendix B. Solution to the model with the counter-rotating interaction

We summarize the exact solution to the model with the counter-rotating interaction (7.2).

B.1. Heisenberg operators

The exact solution to the Heisenberg equations of motion for $a(t) = \mathrm{e}^{\mathrm{i}Ht} a \mathrm{e}^{-\mathrm{i}Ht}$ and $b_\omega(t) = \mathrm{e}^{\mathrm{i}Ht} b_\omega \mathrm{e}^{-\mathrm{i}Ht}$ reads

$$a(t) = [F(t) + \lambda^2 \bar{F}(t)] a + \lambda^2 \bar{F}(t) a^\dagger + \lambda \int_0^t \mathrm{d}t' F(t - t') [B(t') - B^\dagger(t')], \tag{B.1a}$$

$$b_\omega(t) = \mathrm{e}^{-\mathrm{i}\omega t} b_\omega - \lambda \int_0^t \mathrm{d}t' \mathrm{e}^{-\mathrm{i}\omega(t-t')} g_\omega [a(t') + a^\dagger(t')], \tag{B.1b}$$

where $B(t)$ is defined in (3.5) and

$$F(t) = \int_{C_\mathrm{B}} \frac{\mathrm{d}s}{2\pi\mathrm{i}} \frac{s - \mathrm{i}\omega_\mathrm{S}}{s^2 + \omega_\mathrm{S}^2 + 2\lambda^2 \omega_\mathrm{S} \hat{L}(s)} \mathrm{e}^{st}, \tag{B.2a}$$

$$\bar{F}(t) = -\int_{C_B} \frac{ds}{2\pi i} \frac{i\hat{L}(s)}{s^2 + \omega_S^2 + 2\lambda^2 \omega_S \hat{L}(s)} e^{st} \tag{B.2b}$$

with

$$\hat{L}(s) = -\int_0^\infty d\omega |g_\omega|^2 \frac{2\omega}{s^2 + \omega^2}. \tag{B.3}$$

Note that $F(0^+) = 1$, $\dot{F}(0^+) = -i\omega_S$ and $\bar{F}(0^+) = 0$, $\dot{\bar{F}}(0^+) = 0$.

B.2. Characteristic functional

The characteristic functional of the state of the total system, $\rho(t)$, is given by (7.2), which is composed of the functions

$$\begin{aligned}\mathcal{A}_{aa}(t) = {} & \frac{1}{2}[1 - |F(t) + \lambda^2 \bar{F}(t)|^2 - \lambda^4 |\bar{F}(t)|^2] \\ & + \lambda^2 \int_0^t dt' \int_0^t dt'' F(t - t') \mathrm{Re}\Phi_{gg}^{\beta}(t', t'') F^*(t - t''),\end{aligned} \tag{B.4a}$$

$$\begin{aligned}J_b^\dagger \mathcal{A}_{bb}(t) J_b = {} & \Phi_{J_b J_b}(t, t) - \frac{1}{2}\lambda^2 [|(K_{J_b g} * F)(t)|^2 + |(K_{J_b g}^* * F)(t)|^2] \\ & + 2\lambda^2 \mathrm{Im} \int_0^t dt' (K_{J_b g} * \mathrm{Im} F)(t - t') \Phi_{g J_b}^{\beta}(t', t) \\ & + 4\lambda^4 \int_0^t dt' \int_0^t dt'' (K_{J_b g} * \mathrm{Im} F)(t - t') \\ & \times \mathrm{Re}\Phi_{gg}^{\beta}(t', t'') (\mathrm{Im} F * K_{J_b g}^*)(t - t''),\end{aligned} \tag{B.4b}$$

$$\begin{aligned}J_b^\dagger \mathcal{A}_{ba}(t) = {} & \frac{1}{2}\lambda (K_{J_b g} * F)(t) F^*(t) + i\lambda^3 (K_{J_b g} * \mathrm{Im} F)(t) \bar{F}^*(t) \\ & + \frac{1}{2}\lambda \int_0^t dt' \Phi_{J_b g}^{\beta}(t, t') F^*(t - t') \\ & + 2\lambda^3 \int_0^t dt' \int_0^t dt'' (K_{J_b g} * \mathrm{Im} F)(t - t') \mathrm{Im}\Phi_{gg}^{\beta}(t', t'') F^*(t - t''),\end{aligned} \tag{B.4c}$$

$$\bar{\mathcal{A}}_{aa}(t) = \frac{1}{2}\lambda^2 [F(t) + \lambda^2 \bar{F}(t)] \bar{F}^*(t) - \frac{1}{2}\lambda^2 \int_0^t dt' \int_0^t dt'' F^*(t - t') \mathrm{Re}\Phi_{gg}^{\beta}(t', t'') F^*(t - t''), \tag{B.5a}$$

$$\begin{aligned}J_b^\dagger \bar{\mathcal{A}}_{bb}(t) J_b^* = {} & -\frac{1}{2}\lambda^2 (K_{J_b g} * F)(t) (K_{J_b g} * F^*)(t) + i\lambda^2 \int_0^t dt' \Phi_{J_b g}^{\beta}(t, t') (\mathrm{Im} F * K_{J_b g})(t - t') \\ & + 2\lambda^4 \int_0^t dt' \int_0^t dt'' (K_{J_b g} * \mathrm{Im} F)(t - t') \mathrm{Re}\Phi_{gg}^{\beta}(t', t'') (\mathrm{Im} F * K_{J_b g})(t - t''),\end{aligned} \tag{B.5b}$$

$$\begin{aligned}J_b^\dagger \bar{\mathcal{A}}_{ba}(t) = {} & -\frac{1}{2}\lambda^3 (K_{J_b g} * F)(t) \bar{F}^*(t) - \frac{1}{2}\lambda \int_0^t dt' \Phi_{J_b g}^{\beta}(t, t') F(t - t') \\ & + i\lambda^3 \int_0^t dt' \int_0^t dt'' (K_{J_b g} * \mathrm{Im} F)(t - t') \Phi_{gg}^{\beta}(t', t'') F(t - t''),\end{aligned} \tag{B.5c}$$

$$\bar{\mathcal{A}}_{ab}(t)J_b^* = \frac{1}{2}\lambda[F(t) + \lambda^2\bar{F}(t)](K_{J_{bg}} * F^*)(t) + \mathrm{i}\lambda^3\int_0^t \mathrm{d}t'\int_0^t \mathrm{d}t'' F(t-t')\Phi_{gg}^{\beta}(t',t'')(K_{J_{bg}} * \mathrm{Im}F)(t-t''), \tag{B.5d}$$

and

$$h_a(t) = F(t) + \lambda(F * K_{g(\mathcal{N}\xi)})(t) + \lambda^2\bar{F}(t), \tag{B.6a}$$

$$J_b^\dagger h_b(t) = K_{J_b(\mathcal{N}\xi)}(t) - \lambda(K_{J_{bg}} * F)(t) - 2\mathrm{i}\lambda^2(K_{J_{bg}} * \mathrm{Im}F * K_{g(\mathcal{N}\xi)})(t), \tag{B.6b}$$

$$\bar{h}_a(t) = -\lambda(F * K^*_{g(\mathcal{N}\xi)})(t) - \lambda^2\bar{F}^*(t), \tag{B.6c}$$

$$J_b^\dagger \bar{h}_b(t) = -\lambda(K_{J_{bg}} * F^*)(t) + 2\mathrm{i}\lambda^2(K_{J_{bg}} * \mathrm{Im}F * K^*_{g(\mathcal{N}\xi)})(t), \tag{B.6d}$$

where

$$\Phi_{fg}^{\beta}(t,t') = \int_0^\infty \mathrm{d}\omega\int_0^\infty \mathrm{d}\omega' f_\omega^* \mathrm{e}^{-\mathrm{i}\omega t}(1+2\mathcal{N})_{\omega\omega'}\mathrm{e}^{\mathrm{i}\omega' t'} g_{\omega'}. \tag{B.7}$$

B.3. van Hove's limit

In addition to the prototypes (A.1), the following limits are necessary for the van Hove limit of the characteristic functional (7.2): by taking the weak-coupling limit $\lambda \to 0$ keeping $\tau = \lambda^2 t$ finite, we have

$$\begin{aligned} F(t)\mathrm{e}^{\mathrm{i}\omega_{\mathrm{S}}t} &= \int_{C_{\mathrm{B}}} \frac{\mathrm{d}\tilde{s}}{2\pi\mathrm{i}} \frac{\lambda^2\tilde{s} - 2\mathrm{i}\omega_{\mathrm{S}}}{\lambda^2\tilde{s}^2 - 2\mathrm{i}\omega_{\mathrm{S}}\tilde{s} + 2\omega_{\mathrm{S}}\hat{L}(\lambda^2\tilde{s} - \mathrm{i}\omega_{\mathrm{S}})}\mathrm{e}^{\tilde{s}\tau} \\ &\xrightarrow{\lambda\to 0} \int_{C_{\mathrm{B}}} \frac{\mathrm{d}\tilde{s}}{2\pi\mathrm{i}} \frac{1}{\tilde{s} + \mathrm{i}\hat{L}(-\mathrm{i}\omega_{\mathrm{S}} + 0^+)}\mathrm{e}^{\tilde{s}\tau} = \mathrm{e}^{-\Gamma(\omega_{\mathrm{S}})\tau/2}\mathrm{e}^{-\mathrm{i}[\Delta(\omega_{\mathrm{S}}) - \bar{\Delta}(\omega_{\mathrm{S}})]\tau}, \end{aligned} \tag{B.8a}$$

$$F(t)\mathrm{e}^{-\mathrm{i}\omega_{\mathrm{S}}t} = \int_{C_{\mathrm{B}}} \frac{\mathrm{d}\tilde{s}}{2\pi\mathrm{i}} \frac{\lambda^2\tilde{s}}{\lambda^2\tilde{s}^2 + 2\mathrm{i}\omega_{\mathrm{S}}\tilde{s} + 2\omega_{\mathrm{S}}\hat{L}(\lambda^2\tilde{s} + \mathrm{i}\omega_{\mathrm{S}})}\mathrm{e}^{\tilde{s}\tau} \xrightarrow{\lambda\to 0} 0, \tag{B.8b}$$

$$\lambda^2\bar{F}(t)\mathrm{e}^{\pm\mathrm{i}\omega_{\mathrm{S}}t} = -\lambda^2\int_{C_{\mathrm{B}}} \frac{\mathrm{d}\tilde{s}}{2\pi\mathrm{i}} \frac{\mathrm{i}\hat{L}(\lambda^2\tilde{s} \mp \mathrm{i}\omega_{\mathrm{S}})}{\lambda^2\tilde{s}^2 \mp 2\mathrm{i}\omega_{\mathrm{S}}\tilde{s} + 2\omega_{\mathrm{S}}\hat{L}(\lambda^2\tilde{s} \mp \mathrm{i}\omega_{\mathrm{S}})}\mathrm{e}^{\tilde{s}\tau} \xrightarrow{\lambda\to 0} 0, \tag{B.8c}$$

where

$$\pm\mathrm{i}\hat{L}(\mp\mathrm{i}\omega_{\mathrm{S}} + 0^+) = \tfrac{1}{2}\Gamma(\omega_{\mathrm{S}}) \pm \mathrm{i}[\Delta(\omega_{\mathrm{S}}) - \bar{\Delta}(\omega_{\mathrm{S}})] \tag{B.9}$$

with $\Gamma(\omega)$ and $\Delta(\omega)$ in (6.4), and

$$\bar{\Delta}(\omega) = \int_0^\infty \frac{\mathrm{d}\omega'}{2\pi}\frac{\Gamma(\omega')}{\omega + \omega'}. \tag{B.10}$$

Then, the van Hove limits of the components (B.4)–(B.6) of the characteristic functional (7.2) yield (7.4).

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