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Qualitative Theory of Dynamical Systems

Frame Independent Dynamics in the Newtonian Space-Time

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In honour of Ernesto Lacomba, our friend and collaborator

Abstract. A procedure based on a version of the Erlangen programme is used to construct the absolute Minkowski space-time of special relativity and the Newtonian space-time starting from groupoids of transformations. The passage from special relativity to Newtonian geometry is performed at the level of groupoids. A frame independent Lagrangian (and Hamiltonian) formulation of dynamics is formulated.

Keywords. Groupoids of transformations, frame-independent dynamics.

1. Introduction

In the first part of this paper the absolute Minkowski space-time of special relativity will be constructed starting from the transformation laws of the observed data of an event recorded by different relativistic observers. In particular, we will consider a class of "inertial" observers, each one equipped with its own "private" affine space-time where the time and the relative position of a physical event are recorded. All observers in this class are assumed to be equivalent and their private space-times are all assumed to have a Minkowski metric. The information exchanges between different observers are represented by affine isomorphisms that preserve the metrics. From the above physical assumptions this family of isomorphisms turns out to be an affine groupoid. By applying a procedure based on a version of the Erlangen programme, the absolute Minkowsi space-time is constructed starting from the groupoid of Poincaré mappings.

The same procedure is xeroxed starting from Newtonian observers whose observations are connected by the affine Galilei groupoid that preserves simultaneity

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and the spatial Euclidean metrics. The resulting abstract space is the absolute Newtonian space-time.

Finally, we will show how the reduction from special relativity to Newtonian geometry can be performed at the level of the above-mentioned groupoids.

The second part of the paper will be devoted to particle dynamics in Newtonian space-time. In particular, we will consider a homogeneous formulation whose dynamical solutions are in the space-time-energy-momentum phase space.

Starting from the observation that the Lagrangian function of any observer is not invariant under Galileian transformations, we will conceive the intrinsic Lagrangian as a section of a suitable affine bundle, whose structure will turn out to be related to the Bargmann group of transformations of the quantum phase of the wave function under the Galilei group [1].

The general theory of variational principles developed in [3,4] will be specialized to our affine setting. Again, the spaces and the pairings needed to compute the variational principle will be constructed starting from observers' groupoids and their compatibilities. The resulting variational principle will produce a frame independent Lagrangian (and Hamiltonian) formulation of dynamics in the space-time-energy-momentum phase space [2].

2. Preliminary constructions

We will present a number of constructions obtained by applying procedures based on a version of the Erlangen programme of Felix Klein. In preparation for these constructions we provide an outline of the procedures that will be used throughout the paper for constructing all the spaces needed and we establish notational conventions

2.1. A version of the Erlangen programme

We consider an indexed family $\{A_i\}_{i\in I}$ of vector spaces. With each pair (i,j) of indices we associate an affine isomorphism

$$\alpha_{ij} : A_j \mapsto A_i\,. \tag{1}$$

The *groupoid properties*

$$\alpha_{ii} = 1_{A_i} \tag{2}$$

and

$$\alpha_{ij} \circ \alpha_{jk} = \alpha_{ik} \tag{3}$$

are assumed to hold.

We introduce an equivalence relation in the set

$$\mathbb{A} = \bigcup_{i\in I} \big(\{i\} \times A_i\big) \tag{4}$$

of pairs (i,a_i) such that $a_i \in A_i$. Two pairs (i,a_i) and (j,a_j) are equivalent if $a_i = \alpha_{ij}(a_j)$. Let A be the quotient set. For each $i \in I$ there is the *chart*

$$\alpha_i : A \to A_i : [(j,a_j)] \mapsto a_i = \alpha_{ij}(a_j)\,, \tag{5}$$

whose inverse is the canonical projection

$$\alpha_i^{-1} : A_i \to A : a_i \mapsto a = [(i, a_i)] \,. \tag{6}$$

Relations

$$\alpha_i \circ \alpha_j^{-1} = \alpha_{ij} \tag{7}$$

hold and are regarded as coordinate transitions functions on A which will then be proved to be an affine space. The model vector space, its dual space and the corresponding pairing will be constructed by using groupoids associated with the given groupoid (1).

Linear parts

$$\overline{\alpha}_{ij} : A_j \mapsto A_i \tag{8}$$

of the affine mappings (1) form a groupoid of linear isomorphisms. Then, applying the same procedure as above, we introduce a new equivalence relation in the set (4) and obtain a quotient set $\overline{A}$ and charts

$$\overline{\alpha}_i : \overline{A} \to A_i \tag{9}$$

compatible with the linear groupoid (8).

We show that the set $\overline{A}$ is a vector space and each chart $\overline{\alpha}_i$ is a linear mapping. The sum $a^1 + a^2$ is defined by

$$a^1 + a^2 = \overline{\alpha}_i^{-1}\big(\overline{\alpha}_i(a^1) + \overline{\alpha}_i(a^2)\big) \tag{10}$$

and multiplication by a number is defined by

$$\lambda a = \overline{\alpha}_i^{-1}\big(\lambda \overline{\alpha}_i(a)\big) \,. \tag{11}$$

These definitions are correct since

$$\begin{aligned} \overline{\alpha}_j^{-1}\big(\overline{\alpha}_j(a^1) + \overline{\alpha}_j(a^2)\big) &= \overline{\alpha}_i^{-1}\Big(\overline{\alpha}_{ji}^{-1}\Big(\overline{\alpha}_{ji}\big(\overline{\alpha}_i(a^1)\big) + \overline{\alpha}_{ji}\big(\overline{\alpha}_i(a^2)\big)\Big)\Big) \\ &= \overline{\alpha}_i^{-1}\Big(\overline{\alpha}_{ji}^{-1}\Big(\overline{\alpha}_{ji}\big(\overline{\alpha}_i(a^1) + \overline{\alpha}_i(a^1)\big)\Big)\Big) \\ &= \overline{\alpha}_i^{-1}\big(\overline{\alpha}_i(a^1) + \overline{\alpha}_i(a^2)\big) \end{aligned} \tag{12}$$

and

$$\begin{aligned} \overline{\alpha}_j^{-1}\big(\lambda \overline{\alpha}_j(a)\big) &= \overline{\alpha}_i^{-1}\Big(\overline{\alpha}_{ji}^{-1}\Big(\lambda \overline{\alpha}_{ji}\big(\overline{\alpha}_i(a)\big)\Big)\Big) \\ &= \overline{\alpha}_i^{-1}\Big(\overline{\alpha}_{ji}^{-1}\Big(\overline{\alpha}_{ji}\big(\lambda \overline{\alpha}_i(a)\big)\Big)\Big) \\ &= \overline{\alpha}_i^{-1}\big(\lambda \overline{\alpha}_i(a)\big) \end{aligned} \tag{13}$$

for each pair (i, j) of indices. The linear operations in $\overline{A}$ have all the required properties since they are copies of the corresponding linear operations in the vector spaces A_i. A chart $\overline{\alpha}_i$ is a linear mapping since

$$\overline{\alpha}_i(a^1 + a^2) = \overline{\alpha}_i\Big(\overline{\alpha}_i^{-1}\big(\overline{\alpha}_i(a^1) + \overline{\alpha}_i(a^2)\big)\Big) = \overline{\alpha}_i(a^1) + \overline{\alpha}_i(a^2) \tag{14}$$

and

$$\overline{\alpha}_i(\lambda a) = \overline{\alpha}_i\Big(\overline{\alpha}_i^{-1}\big(\lambda\overline{\alpha}_i(a)\big)\Big) = \lambda\overline{\alpha}_i(a)\,. \tag{15}$$

We show that the set A is an affine space modelled on the vector space $\overline{A}$. The 'difference' $a^2 - a^1 \in \overline{A}$ of two elements of A is defined by

$$a^2 - a^1 = \overline{\alpha}_i^{-1}\big(\alpha_i(a^2) - \alpha_i(a^1)\big)\,. \tag{16}$$

This is again a correct definition since

$$\begin{aligned}\overline{\alpha}_j^{-1}\big(\alpha_j(a^2) - \alpha_j(a^1)\big) &= \overline{\alpha}_i^{-1}\Big(\overline{\alpha}_i j\Big(\alpha_{ji}\big(\alpha_i(a^2)\big) - \alpha_{ji}\big(\alpha_i(a^1)\big)\Big)\Big)\\ &= \overline{\alpha}_i^{-1}\Big(\overline{\alpha}_{ij}\Big(\overline{\alpha}_{ji}\big(\alpha_i(a^2) - \alpha_i(a^1)\big)\Big)\Big)\\ &= \overline{\alpha}_i^{-1}\big(\alpha_i(a^2) - \alpha_i(a^1)\big)\,. \end{aligned}\tag{17}$$

It follows from

$$\overline{\alpha}_i(a^2 - a^1) = \overline{\alpha}_i\Big(\overline{\alpha}_i^{-1}\big(\alpha_i(a^2) - \alpha_i(a^1)\big)\Big) = \alpha_i(a^2) - \alpha_i(a^1)\,. \tag{18}$$

that each chart α_i is an affine mapping and the corresponding chart $\overline{\alpha}_i$ is its linear part.

The mappings

$$\overline{\alpha}_{ij}^* : A_i^* \mapsto A_j^* : h_i \mapsto \overline{\alpha}_{ij}^*(h_i) = h_i \circ \overline{\alpha}_{ij} \tag{19}$$

dual to the mappings $\overline{\alpha}_{ij}$ form a linear groupoid. Following the established procedure we introduce an equivalence relation in the set

$$\mathbb{A}^* = \bigcup_{i\in I}\big(\{i\} \times A_i^*\big)\,. \tag{20}$$

A quotient set $\overline{A}^*$ and charts

$$\alpha_i^{*-1} : \overline{A}^* \to A_i^* : [(j, h_j)] \mapsto h_i = \alpha_{ji}^*(h_j) \tag{21}$$

are obtained. Results derived for the linear groupoid (8) are valid for any linear groupoid. It follows that the set $\overline{A}^*$ is a vector space and charts are linear mappings. It will be shown that the mapping

$$\alpha_i^* : A_i^* \to \overline{A}^* : h_i \mapsto [(i, h_i)] \tag{22}$$

is the dual of $\overline{\alpha}_i$ as is already implied by the adopted symbol.

The vector space $\overline{A}^*$ is the dual of $\overline{A}$. The pairing

$$\langle\ ,\ \rangle : \overline{A}^* \times \overline{A} \to \mathbb{R} \tag{23}$$

is defined by

$$\langle h, a\rangle = \big\langle \alpha_i^{*-1}(h), \overline{\alpha}_i(a)\big\rangle\,. \tag{24}$$

This definition is correct since

$$\begin{aligned}\big\langle \alpha_j^{*-1}(h), \overline{\alpha}_j(a)\big\rangle &= \Big\langle \alpha_{ji}^{*\,-1}\big(\alpha_i^{*-1}(h)\big), \overline{\alpha}_{ji}\big(\overline{\alpha}_i(a)\big)\Big\rangle\\ &= \big\langle \alpha_i^{*-1}(h), \overline{\alpha}_i(a)\big\rangle\,. \end{aligned}\tag{25}$$

It follows from

$$\left\langle \alpha_i^*(h_i), a \right\rangle = \left\langle \alpha_i^{*-1}\big(\alpha_i^*(h_i)\big), \overline{\alpha}_i(a) \right\rangle = \left\langle h_i, \overline{\alpha}_i(a) \right\rangle \tag{26}$$

that α_i^* is the dual of $\overline{\alpha}_i$.

Let there be two vector spaces A_i and A_i' associated with each index $i \in I$ and let

$$\alpha_{ij} : A_j \mapsto A_i \tag{27}$$

and

$$\alpha_{ij}' : A_j' \mapsto A_i' \tag{28}$$

be two affine groupoids. If for each index i there is an affine mapping

$$\varphi_i : A_i \to A_i' \tag{29}$$

and if the family $\{\varphi_i\}_{i\in I}$ is compatible with the groupoids in the sense that

$$\varphi_i = \alpha_{ij}' \circ \varphi_j \circ \alpha_{ji} \tag{30}$$

for each pair (i, j) of indices, then a mapping

$$\varphi : A \to A' \tag{31}$$

is defined by

$$\varphi = \alpha_i'^{-1} \circ \varphi_i \circ \alpha_i \,. \tag{32}$$

The definition is correct since

$$\alpha_j'^{-1} \circ \varphi_j \circ \alpha_j = \alpha_i'^{-1} \circ \alpha_{ij}' \circ \varphi_j \circ \alpha_{ji} \circ \alpha_i = \alpha_i'^{-1} \circ \varphi_i \circ \alpha_i \,. \tag{33}$$

The family $\{\overline{\varphi}_i\}_{i\in I}$ of linear parts is compatible with the linear groupoids

$$\overline{\alpha}_{ij} : A_j \mapsto A_i \tag{34}$$

and

$$\overline{\alpha}_{ij}' : A_j' \mapsto A_i' \,. \tag{35}$$

A mapping

$$\overline{\varphi} : \overline{A} \to \overline{A}' \tag{36}$$

is correctly defined by

$$\overline{\varphi} = \overline{\alpha}_i'^{-1} \circ \overline{\varphi}_i \circ \overline{\alpha}_i \,. \tag{37}$$

The mapping $\overline{\varphi}$ is linear since it is the composition of linear mappings. It is the linear part of the mapping φ since the composition

$$\overline{\alpha}_i'^{-1} \circ \overline{\varphi}_i \circ \overline{\alpha}_i \tag{38}$$

is the linear part of the composition

$$\alpha_i'^{-1} \circ \varphi_i \circ \alpha_i \,. \tag{39}$$

2.2. Notational conventions

We will be working with Euclidean vector spaces. Let Q be an Euclidean space with a metric tensor $h : Q \to Q^*$. The symbol $\mathbf{a}\,\vec{v}$ will be used to denote the value of a covector $\mathbf{a} : Q \to \mathbb{R}$ applied to a vector $\vec{v} \in Q$. The covector $h(\vec{v}) \in Q^*$ associated with a vector $\vec{v} \in Q$ will be usually denoted by $\vec{v}\,\cdot$. The evaluation of the covector $\vec{v}\,\cdot$ on a vector $\vec{w}$ is the scalar product $\vec{v} \cdot \vec{w}$. The norm of a vector $\vec{v}$ is defined by

$$\|\vec{v}\,\| = \sqrt{\vec{v} \cdot \vec{v}}\ . \tag{40}$$

Matrix notation will be used in the vector space $\mathbb{R} \times Q$ denoted by

$$\begin{pmatrix} \mathbb{R} \\ Q \end{pmatrix}. \tag{41}$$

An element of this space will be written as a 2×1 matrix (a column vector)

$$\begin{pmatrix} t \\ \vec{q} \end{pmatrix}. \tag{42}$$

The dual space of covectors in $\mathbb{R} \times Q$ is the space $\mathbb{R} \times Q^*$ denoted by

$$\begin{pmatrix} \mathbb{R} & Q^* \end{pmatrix}. \tag{43}$$

A covector will be written as a 1×2 matrix (a row vector)

$$\begin{pmatrix} a & \mathbf{b} \end{pmatrix} \tag{44}$$

and the evaluation of the covector (44) on the vector (42) is the matrix product

$$\begin{pmatrix} a & \mathbf{b} \end{pmatrix} \begin{pmatrix} t \\ \vec{q} \end{pmatrix} = ax + \mathbf{b}\,\vec{y}\,. \tag{45}$$

Let Q and Q' be two vector spaces. A linear mapping from $\mathbb{R} \times Q$ to $\mathbb{R} \times Q'$ is represented by a 2×2 matrix

$$\begin{pmatrix} a & \mathbf{b} \\ \vec{v} & \alpha \end{pmatrix}, \tag{46}$$

with $a \in \mathbb{R}$, $\mathbf{b} \in Q^*$, $\vec{v} \in Q'$, and α a linear mapping from Q to Q'. Products of matrices with linear mappings as components are interpreted as compositions. The same interpretation is applied to 'products' of linear mappings. The convention

$$U : \begin{pmatrix} \mathbb{R} \\ Q \end{pmatrix} \to \mathbb{R} : \begin{pmatrix} t \\ \vec{q} \end{pmatrix} \mapsto U \begin{pmatrix} t \\ \vec{q} \end{pmatrix} \tag{47}$$

will be used for values of a differentiable functions.

3. The Minkowski space-time

The outlined procedure will be used to construct the absolute Minkowski space-time of special relativity.

3.1. Lorentz transformations

Let Q be a Euclidean vector space of dimension 3 with a metric tensor $h : Q \to Q^*$. A Minkowski metric of signature $(1,3)$ in the space $\mathbb{R} \times Q$ is represented by the metric tensor

$$g : \begin{pmatrix} \mathbb{R} \\ Q \end{pmatrix} \to \begin{pmatrix} \mathbb{R} & Q^* \end{pmatrix} : \begin{pmatrix} \delta t \\ \delta \vec{q} \end{pmatrix} \mapsto \begin{pmatrix} c^2 \delta t & -\delta \vec{q} \cdot \end{pmatrix}. \tag{48}$$

A *Lorentz transformation* is a linear mapping

$$\overline{\mu} : \begin{pmatrix} \mathbb{R} \\ Q \end{pmatrix} \to \begin{pmatrix} \mathbb{R} \\ Q \end{pmatrix} \tag{49}$$

preserving the Minkowski metric relations in the space $\mathbb{R} \times Q$. This property is expressed by the equality

$$\overline{\mu}^* \circ g \circ \overline{\mu} = g\,. \tag{50}$$

Equivalently, if

$$\begin{pmatrix} \delta t' \\ \delta \vec{q}\,' \end{pmatrix} = \overline{\mu} \begin{pmatrix} \delta t \\ \delta \vec{q} \end{pmatrix}, \tag{51}$$

then

$$c^2 (\delta t')^2 - \|\delta \vec{q}\,'\|^2 = c^2 (\delta t)^2 - \|\delta \vec{q}\|^2\,. \tag{52}$$

An affine mapping

$$\mu : \begin{pmatrix} \mathbb{R} \\ Q \end{pmatrix} \to \begin{pmatrix} \mathbb{R} \\ Q \end{pmatrix} \tag{53}$$

is called a *Poincaré* transformation if its linear part $\overline{\mu}$ is a Lorentz transformation.

Let $\overline{\mu}$ be a Lorentz transformation and let

$$\overline{\mu} \begin{pmatrix} 1 \\ \vec{0} \end{pmatrix} = \begin{pmatrix} x \\ \vec{y} \end{pmatrix}. \tag{54}$$

It follows from

$$c^2 x^2 - \|\vec{y}\|^2 = c^2 \tag{55}$$

that

$$x^2 = 1 + c^{-2} \|\vec{y}\|^2 \geq 1\,. \tag{56}$$

The vector $\vec{v} = x^{-1} \vec{y}$ can be introduced since $x \neq 0$. If x is positive, then $\overline{\mu}$ is said to be an *orthochronous*. The general relation

$$x = \pm \left(\sqrt{1 - c^{-2} \|\vec{v}\|^2} \right)^{-1}, \tag{57}$$

follows from (55) with $\vec{y} = x \vec{v}$. We have obtained a useful relation

$$\overline{\mu} \begin{pmatrix} 1 \\ \vec{0} \end{pmatrix} = \pm \left(\sqrt{1 - c^{-2} \|\vec{v}\|^2} \right)^{-1} \begin{pmatrix} 1 \\ \vec{v} \end{pmatrix}. \tag{58}$$

We will find the general form of the Lorentz transformation $\overline{\mu}$ in terms of the vector $\vec{v}$ and an isometry $\kappa : Q \to Q$. The mapping $\overline{\mu}$ preserves orthogonality. If $\vec{v} = 0$, then vectors

$$\overline{\mu}\begin{pmatrix} 1 \\ \vec{0} \end{pmatrix} = \pm \begin{pmatrix} 1 \\ \vec{0} \end{pmatrix} \quad \text{and} \quad \overline{\mu}\begin{pmatrix} 0 \\ \delta\vec{q} \end{pmatrix} = \begin{pmatrix} \delta t' \\ \delta\vec{q}\,' \end{pmatrix} \tag{59}$$

are orthogonal. It follows that

$$\delta t' = c^{-2} \begin{pmatrix} c^2 & -\vec{0}\,\cdot \end{pmatrix} \begin{pmatrix} \delta t' \\ \delta\vec{q}\,' \end{pmatrix} = 0\,. \tag{60}$$

Hence,

$$\overline{\mu}\begin{pmatrix} 0 \\ \delta\vec{q} \end{pmatrix} = \begin{pmatrix} 0 \\ \delta\vec{q}\,' \end{pmatrix}. \tag{61}$$

The matrix form

$$\overline{\mu} = \pm \begin{pmatrix} 1 & \vec{0}\,\cdot \\ \vec{0} & \kappa \end{pmatrix} \tag{62}$$

of $\overline{\mu}$ follows from (59) and (61). The component $\kappa : Q \to Q$ is an isometry.

If $\vec{v} \neq \vec{0}$, then we introduce orthogonal projection operators

$$Pr^{\|} : Q \to Q : \delta\vec{q} \mapsto \|\vec{v}\|^{-2}(\vec{v}\cdot\delta\vec{q})\vec{v} \tag{63}$$

and

$$Pr^{\perp} = 1_Q - Pr^{\|} \tag{64}$$

in Q, and orthogonal projection operators

$$\mathbf{M} = \begin{pmatrix} 1 & \vec{0}\,\cdot \\ \vec{0} & Pr^{\|} \end{pmatrix} \tag{65}$$

and

$$\mathbf{E} = \begin{pmatrix} 0 & \vec{0}\,\cdot \\ \vec{0} & Pr^{\perp} \end{pmatrix} = \begin{pmatrix} 1 & \vec{0}\,\cdot \\ \vec{0} & 1_Q \end{pmatrix} - \begin{pmatrix} 1 & \vec{0}\,\cdot \\ \vec{0} & Pr^{\|} \end{pmatrix} \tag{66}$$

in $\mathbb{R} \times Q$. The image of $\mathbf{M}$ is a Minkowski plane with a metric of signature $(1,1)$ and the image of $\mathbf{E}$ is a Euclidean plane. We introduce the mapping

$$\chi = \pm\left(\sqrt{1 - c^{-2}\|\vec{v}\|^2}\right)^{-1} \begin{pmatrix} 1 & c^{-2}\vec{v}\,\cdot \\ \vec{v} & Pr^{\|} \end{pmatrix} \pm \begin{pmatrix} 0 & \vec{0}\,\cdot \\ \vec{0} & Pr^{\perp} \end{pmatrix}. \tag{67}$$

Vectors

$$\begin{pmatrix} 1 \\ \vec{0} \end{pmatrix} \quad \text{and} \quad \|\vec{v}\|^{-1}\begin{pmatrix} 0 \\ \vec{v} \end{pmatrix} \tag{68}$$

form an orthonormal basis in the image of $\mathbf{M}$ since

$$\begin{pmatrix} c^2 & -\vec{0}\,\cdot \end{pmatrix} \begin{pmatrix} 1 \\ \vec{0} \end{pmatrix} = c^2\,, \tag{69}$$

$$\|\vec{v}\|^{-2} \begin{pmatrix} 0 & -\vec{v}\,\cdot \end{pmatrix} \begin{pmatrix} 0 \\ \vec{v} \end{pmatrix} = -1\,, \tag{70}$$

and

$$\|\vec{v}\|^{-1} \begin{pmatrix} 0 & -\vec{v}\,\cdot \end{pmatrix} \begin{pmatrix} 1 \\ \vec{0} \end{pmatrix} = 0\,. \tag{71}$$

The mapping χ applied to these vectors produces vectors

$$\chi \begin{pmatrix} 1 \\ \vec{0} \end{pmatrix} = \pm \left(\sqrt{1 - c^{-2}\|\vec{v}\|^2}\right)^{-1} \begin{pmatrix} 1 \\ \vec{v} \end{pmatrix} \tag{72}$$

$$\|\vec{v}\|^{-1}\chi \begin{pmatrix} 0 \\ \vec{v} \end{pmatrix} = \pm\|\vec{v}\|^{-1} \left(\sqrt{1 - c^{-2}\|\vec{v}\|^2}\right)^{-1} \begin{pmatrix} c^{-2}\|\vec{v}\|^2 \\ \vec{v} \end{pmatrix} \tag{73}$$

satisfying the same orthonormality conditions. In addition, if $\delta\vec{q}$ is orthogonal to $\vec{v}$, then

$$\chi \begin{pmatrix} 0 \\ \delta\vec{q} \end{pmatrix} = \begin{pmatrix} 0 \\ \delta\vec{q} \end{pmatrix}. \tag{74}$$

We conclude that χ is a Lorentz transformation. It follows from

$$\left(\chi^{-1} \circ \overline{\mu}\right) \begin{pmatrix} 1 \\ \vec{0} \end{pmatrix} = \begin{pmatrix} 1 \\ \vec{0} \end{pmatrix} \tag{75}$$

that

$$\overline{\mu} = \left[\pm \left(\sqrt{1 - c^{-2}\|\vec{v}\|^2}\right)^{-1} \begin{pmatrix} 1 & c^{-2}\vec{v}\,\cdot \\ \vec{v} & Pr^{\parallel} \end{pmatrix} \pm \begin{pmatrix} 0 & \vec{0}\,\cdot \\ \vec{0} & Pr^{\perp} \end{pmatrix}\right] \begin{pmatrix} 1 & \vec{0}\,\cdot \\ \vec{0} & \kappa \end{pmatrix}. \tag{76}$$

where κ is an isometry in Q. This expression is valid also in the case of $\vec{v} = \vec{0}$ with an arbitrary vector $\vec{v} \neq \vec{0}$ used to defined the projection operators.

3.2. Relativistic observers

We consider a class of observers indexed over a set I. To each physical event an observer O_i assigns a time $t_i \in \mathbb{R}$ and a relative position $\vec{q}_i$ in a Euclidean vector space Q_i of dimension 3 with a metric tensor $h_i : Q_i \to Q_i^*$. A motion of a material point is seen by an observer as a one dimensional submanifold in the observer's space-time $\mathbb{R} \times Q_i$. This submanifold is the image of a differentiable embedding

$$\gamma : \mathbb{R} \to \begin{pmatrix} \mathbb{R} \\ Q_i \end{pmatrix} : s \mapsto \begin{pmatrix} \tau(s) \\ \vec{\sigma}(s) \end{pmatrix}. \tag{77}$$

The motion of a point distant from any possible sources of interaction is said to be *inertial*. An observer is said to be *inertial* if any inertial motion is seen by the observer as a straight line presentable as the image of an affine mapping

$$\gamma : \mathbb{R} \to \begin{pmatrix} \mathbb{R} \\ Q_i \end{pmatrix} : s \mapsto \begin{pmatrix} t_i^0 + s\,x \\ \vec{q}_i^{\,0} + s\,\vec{y} \end{pmatrix}. \tag{78}$$

Only inertial observers will be considered.

A Minkowski metric of signature $(1,3)$ in a observer's space-time is represented by the metric tensor

$$g_i : \begin{pmatrix} \mathbb{R} \\ Q_i \end{pmatrix} \to \begin{pmatrix} \mathbb{R} & Q_i^* \end{pmatrix} : \begin{pmatrix} \delta t_i \\ \delta\vec{q}_i \end{pmatrix} \mapsto \begin{pmatrix} c^2\delta t_i & -\delta\vec{q}_i\,\cdot \end{pmatrix}. \tag{79}$$

Vectors

$$\begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} t_j \\ \vec{q}_j \end{pmatrix} \tag{80}$$

assigned to the same physical event by two observers O_i and O_j are in the relation

$$\begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix} = \mu_{ij} \begin{pmatrix} t_j \\ \vec{q}_j \end{pmatrix}, \tag{81}$$

where μ_{ij} is an affine mapping. The observers find that the linear part $\overline{\mu}_{ij}$ of μ_{ij} preserves the Minkowski metric relations in the two space-times $\mathbb{R}\times Q_i$ and $\mathbb{R}\times Q_j$. This property is expressed by the equality

$$\overline{\mu}^*_{ij} \circ g_i \circ \overline{\mu}_{ij} = g_j \, . \tag{82}$$

We use an arbitrary isometry $\varphi : Q_j \to Q_i$ to obtain a convenient factorization

$$\overline{\mu}_{ij} = \overline{\mu} \circ \begin{pmatrix} 1 & \vec{0}_j \, \cdot \\ \vec{0}_i & \varphi \end{pmatrix}. \tag{83}$$

The linear mapping

$$\overline{\mu} : \begin{pmatrix} \mathbb{R} \\ Q_i \end{pmatrix} \to \begin{pmatrix} \mathbb{R} \\ Q_i \end{pmatrix} \tag{84}$$

is a Lorentz transformation. The expression

$$\begin{aligned} \mu_{ij} = \Bigg[\pm \left(\sqrt{1 - c^{-2} \|\vec{v}_{ij}\|^2} \right)^{-1} \begin{pmatrix} 1 & c^{-2}\vec{v}_{ij} \, \cdot \\ \vec{v}_{ij} & Pr^{\|} \end{pmatrix} \\ \pm \begin{pmatrix} 0 & \vec{0}_i \, \cdot \\ \vec{0}_i & Pr^{\perp} \end{pmatrix} \Bigg] \begin{pmatrix} 1 & \vec{0}_j \, \cdot \\ \vec{0}_i & \kappa_{ij} \end{pmatrix} + \begin{pmatrix} \Delta t_{ij} \\ \Delta \vec{q}_{ij} \end{pmatrix} \end{aligned} \tag{85}$$

for the Poincaré mapping μ_{ij} follows.

The world line of the observer O_j in the space-time of this observer is the image of the mapping

$$\gamma_j : \mathbb{R} \to \begin{pmatrix} \mathbb{R} \\ Q_j \end{pmatrix} : t_j \mapsto \begin{pmatrix} t_j \\ \vec{0}_j \end{pmatrix}. \tag{86}$$

Seen by the observer O_i this world line is the image of

$$\mu_{ij} \circ \gamma_j : \mathbb{R} \to \begin{pmatrix} \mathbb{R} \\ Q_i \end{pmatrix} : t_j \mapsto t_j \begin{pmatrix} x \\ \vec{y} \end{pmatrix} + \begin{pmatrix} \Delta t_{ij} \\ \Delta \vec{q}_{ij} \end{pmatrix}. \tag{87}$$

This mapping is converted to

$$\gamma_{ij} : \mathbb{R} \to \begin{pmatrix} \mathbb{R} \\ Q_i \end{pmatrix} : t_i \mapsto \begin{pmatrix} t_i \\ \Delta \vec{q}_{ij} + (t_i - \Delta t_{ij}) \vec{v}_{ij} \end{pmatrix} \tag{88}$$

by setting $t_j = x^{-1}(t_i - \Delta t_{ij})$ and $\vec{y} = x^{-1}\vec{v}_{ij}$. The parameter t_i in (81) is the time of the observer O_i. The vector $\vec{v}_{ij}$ is the velocity of the observer O_j in Q_i and

$$\begin{pmatrix} \Delta t_{ij} \\ \Delta \vec{q}_{ij} \end{pmatrix} \tag{89}$$

is the initial position in $\mathbb{R} \times Q_i$ of the observer O_j at $t_i = \Delta t_{ij}$.

3.3. The absolute Minkowski space-time

A Poincaré mapping

$$\mu_{ij} : M_j \to M_i \tag{90}$$

is assigned to each pair of observers O_i and O_j. The spaces

$$M_i = \begin{pmatrix} \mathbb{R} \\ Q_i \end{pmatrix} \quad \text{and} \quad M_j = \begin{pmatrix} \mathbb{R} \\ Q_j \end{pmatrix} \tag{91}$$

are used. The family $\{\mu_{ij}\}_{(i,j)\in I\times I}$ is a groupoid. The family $\{\overline{\mu}_{ij}\}_{(i,j)\in I\times I}$ of Lorentz mappings

$$\overline{\mu}_{ij} : M_j \to M_i \tag{92}$$

and the family $\{\overline{\mu}^*_{ij}\}_{(i,j)\in I\times I}$ of dual mappings

$$\overline{\mu}^*_{ij} : M^*_i \mapsto M^*_j : h_i \mapsto \overline{\mu}^*_{ij}(h_i) = h_i \circ \overline{\mu}_{ij} \tag{93}$$

with

$$M^*_i = \begin{pmatrix} \mathbb{R} & Q^*_i \end{pmatrix} \quad \text{and} \quad M^*_j = \begin{pmatrix} \mathbb{R} & Q^*_j \end{pmatrix} \tag{94}$$

are again groupoids.

The affine absolute Minkowski space-time M is obtained by applying the procedure described in Part A to the *Poincaré groupoid* $\{\mu_{ij}\}_{(i,j)\in I\times I}$. When applied to the *Lorentz groupoid* $\{\overline{\mu}_{ij}\}_{(i,j)\in I\times I}$ the procedure produces the model vector space $\overline{M}$. The dual $\overline{M}^*$ of the model space is generated from the dual groupoid $\{\overline{\mu}^*_{ij}\}_{(i,j)\in I\times I}$.

There is a Minkowski metric tensor

$$g_i : \begin{pmatrix} \mathbb{R} \\ Q_i \end{pmatrix} \to \begin{pmatrix} \mathbb{R} & Q^*_i \end{pmatrix} : \begin{pmatrix} \delta t_i \\ \delta \vec{q}_i \end{pmatrix} \mapsto \begin{pmatrix} c^2 \delta t_i & -\delta \vec{q}_i \,\cdot \end{pmatrix}. \tag{95}$$

for each index i. The equality (75) expresses compatibility of the family $\{g_i\}_{i\in I}$ with the groupoids (83) and (82). A metric tensor

$$g : \overline{M} \to \overline{M}^* . \tag{96}$$

of signature $(1, 3)$ is defined by

$$g = \overline{\mu}^*_i \circ g_i \circ \overline{\mu}_i . \tag{97}$$

If

$$\overline{\mu}_i(v) = \begin{pmatrix} \delta t_i \\ \delta \vec{q}_i \end{pmatrix}, \tag{98}$$

then

$$\langle g(v), v \rangle = c^2 |\delta t_i|^2 - \|\delta \vec{q}_i\|^2 . \tag{99}$$

We impose certain saturation requirements on the Poincaré groupoid. Two observers O_i and O_j are considered equivalent if

$$\mu_{ij} = \begin{pmatrix} 1 & \vec{0}_j \,\cdot \\ \vec{0}_i & \kappa_{ij} \end{pmatrix}, \tag{100}$$

where $\kappa_{ij} : Q_j \to Q_i$ is an isometry with $\det(\kappa_{ij}) = 1$. Let

$$\mu_{i\star} : M_i \to M_i \tag{101}$$

be a Poincaré transformation. Be setting $Q_\star = Q_i$, $\mu_{j\star} = \mu_{ji} \circ \mu_{i\star}$ and $\mu_{\star j} = \mu_{j\star}^{-1}$ for each index $j \in I$ a vitual observer $O_\star$ is constructed. The Poincaré groupoid $\{\mu_{ij}\}_{(i,j)\in I\times I}$ is *saturated* if for each such construction there is an index $k \in I$ such that O_k and $O_\star$ are equivalent.

There is a Minkowski metric in M. If Poincaré groupoid is saturated, then no other structure is present in the affine space M.

Additional structures can be introduced in the space-time M if the groupoid is restricted. We give an example. The groupoid is composed of only orthochronous Poincaré mappings. The saturation requirement is modified by using only orthochronous mappings $\mu_{i\star}$ and an unchainged definition of equivalence. The resulting space-time M has a temporal orientation. Examples of other additional structures can be easily produced.

4. The Newtonian space-time

The outlined procedure will be used to construct the absolute Newtonian space-time. The passage from special relativity to Newtonian geometry is performed at the level of groupoids.

4.1. Galilei transformations

Let Q be a Euclidean vector space of dimension 3 with a metric tensor $h : Q \to Q^*$. The metric tensor and the canonical projection

$$\tau : \begin{pmatrix} \mathbb{R} \\ Q \end{pmatrix} \to \mathbb{R} : \begin{pmatrix} t \\ \vec{q} \end{pmatrix} \mapsto t \tag{102}$$

establish the structure of a Newtonian space-time in the space $\left(\begin{smallmatrix} \mathbb{R} \\ Q \end{smallmatrix}\right)$.

Let

$$\nu : \begin{pmatrix} \mathbb{R} \\ Q \end{pmatrix} \to \begin{pmatrix} \mathbb{R} \\ Q \end{pmatrix} \tag{103}$$

be an affine mapping, let the linear part $\overline{\nu}$ of this mapping conserve the projection τ in the sense that

$$\tau \circ \overline{\nu} = \tau \tag{104}$$

and let the linear mapping $\lambda : Q \to Q$ characterized by

$$\overline{\nu} \begin{pmatrix} 0 \\ \delta\vec{q} \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda\delta\vec{q}_j \end{pmatrix} \tag{105}$$

be an isometry. If all these conditions are satisfied, then $\overline{\nu}$ is said to be a *linear Galilei transformation* and the mapping ν is an *affine Galilei transformation.*

The projection τ is a covector represented by the matrix

$$\begin{pmatrix} 1 & \vec{0}\cdot \end{pmatrix}. \tag{106}$$

Let

$$\overline{\nu} = \begin{pmatrix} a & \mathbf{b} \\ \vec{v} & \lambda \end{pmatrix} \tag{107}$$

be a linear Galilei transformation. Condition (104) in its matrix form

$$\begin{pmatrix} 1 & \vec{0}\cdot \end{pmatrix} \begin{pmatrix} a & \mathbf{b} \\ \vec{v} & \lambda \end{pmatrix} = \begin{pmatrix} 1 & \vec{0}\cdot \end{pmatrix} \tag{108}$$

implies that $a = 1$ and $\mathbf{b} = \vec{0}\cdot$. The mapping λ in (107) is an isometry since it is the mapping characterized by (105). We have obtained the general form

$$\overline{\nu} = \begin{pmatrix} 1 & \vec{0}\cdot \\ \vec{v} & \lambda \end{pmatrix} \tag{109}$$

of a linear Galilei transformation with λ an isometry and $\vec{v}$ an arbitrary vector. The mapping

$$\overline{\nu} = \begin{pmatrix} 1 & \vec{0}\cdot \\ \vec{v} & \lambda \end{pmatrix} + \begin{pmatrix} \Delta t \\ \Delta \vec{q} \end{pmatrix} \tag{110}$$

is a general affine Galilei transformation.

4.2. Newtonian observers

We consider a class of observers indexed over a set J. To each physical event an observer O_i assigns a time $t \in \mathbb{R}$ and a relative position $\vec{q}$ in a Euclidean vector space Q_i of dimension 3 with a metric tensor $h_i : Q_i \to Q_i^*$.

Vectors

$$\begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix} \in N_i = \begin{pmatrix} \mathbb{R} \\ Q_i \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} t_j \\ \vec{q}_j \end{pmatrix} \in N_j = \begin{pmatrix} \mathbb{R} \\ Q_j \end{pmatrix} \tag{111}$$

assigned to the same physical event by two observers O_i and O_j are in the relation

$$\begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix} = \nu_{ij} \begin{pmatrix} t_j \\ \vec{q}_j \end{pmatrix}, \tag{112}$$

where ν_{ij} is an affine mapping. For each observer O_i there is the projection

$$\tau_i : N_i \to \mathbb{R} : \begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix} \mapsto \begin{pmatrix} 1 & \vec{0}_i\cdot \end{pmatrix} \begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix} = t_i\,. \tag{113}$$

The observers find that the linear part $\overline{\nu}_{ij}$ of ν_{ij} preserves the projections τ_i and τ_j in the sense that

$$\tau_i \circ \overline{\nu}_{ij} = \tau_j \tag{114}$$

and that the mapping $\lambda_{ij} : Q_j \to Q_i$ characterized by

$$\overline{\nu}_{ij} \begin{pmatrix} 0 \\ \delta\vec{q}_j \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda_{ij}\delta\vec{q}_j \end{pmatrix} \tag{115}$$

is an isometry. It follows that the linear mapping $\overline{\nu}_{ij}$ is represented by the matrix

$$\begin{pmatrix} 1 & \vec{0}_j\cdot \\ \vec{v}_{ij} & \lambda_{ij} \end{pmatrix} \tag{116}$$

and the affine mapping ν_{ij} is the mapping

$$\nu_{ij} : N_j \to N_i : \begin{pmatrix} t_j \\ \vec{q}_j \end{pmatrix} \mapsto \begin{pmatrix} 1 & \vec{0}_j\, \cdot \\ \vec{v}_{ij} & \lambda_{ij} \end{pmatrix} \begin{pmatrix} t_j \\ \vec{q}_j \end{pmatrix} + \begin{pmatrix} \Delta t_{ij} \\ \Delta \vec{q}_{ij} \end{pmatrix}$$
$$= \begin{pmatrix} t_j \\ t_j \vec{v}_{ij} + \lambda_{ij} \vec{q}_j \end{pmatrix} + \begin{pmatrix} \Delta t_{ij} \\ \Delta \vec{q}_{ij} \end{pmatrix}. \quad (117)$$

We say that ν_{ij} is an *affine Galilei mapping* and its linear part $\overline{\nu}_{ij}$ is a *linear Galilei mapping*.

The world line of the observer O_j in its own space-time is the image of the mapping

$$\gamma_j : \mathbb{R} \to N_j : t_j \mapsto t_j \begin{pmatrix} 1 \\ \vec{0}_j \end{pmatrix}. \quad (118)$$

Seen by the observer O_i this world line is the image of

$$\gamma_{ij} = \nu_{ij} \circ \gamma_j : \mathbb{R} \to N_i : t_j \mapsto t_j \overline{\nu}_{ij} \begin{pmatrix} 1 \\ \vec{v}_{ij} \end{pmatrix} + \begin{pmatrix} \Delta t_{ij} \\ \Delta \vec{q}_{ij} \end{pmatrix} \quad (119)$$

converted to

$$\gamma_{ij} : \mathbb{R} \to N_i : t_i \mapsto \begin{pmatrix} t_i \\ t_i \vec{v}_{ij} \end{pmatrix} + \begin{pmatrix} \Delta t_{ij} \\ \Delta \vec{q}_{ij} \end{pmatrix}. \quad (120)$$

The vector $\vec{v}_{ij}$ is the velocity of the observer O_j in Q_i and

$$\begin{pmatrix} \Delta t_{ij} \\ \Delta \vec{q}_{ij} \end{pmatrix} \quad (121)$$

is the initial position in $\mathbb{R} \times Q_i$ of the observer O_j at $t_i = \Delta t_{ij}$.

4.3. The absolute Newtonian space-time

Galilei mappings ν_{ij} and $\overline{\nu}_{ij}$ are assigned to each pair of observers O_i and O_j. The families $\{\nu_{ij}\}_{(i,j)\in J\times J}$ and $\{\overline{\nu}_{ij}\}_{(i,j)\in J\times J}$ are groupoids.

The absolute Newtonian space-time N is obtained by applying the procedure described in Section 2 to the *affine Galilei groupoid* ν_{ij}. The affine charts

$$\nu_i : N \to N_i \quad (122)$$

are introduced. The space N is an affine space modelled on the vector space $\overline{N}$ constructed by applying the procedure to the *linear Galilei groupoid* $\overline{\nu}_{ij}$. The linear charts

$$\overline{\nu}_i : \overline{N} \to N_i \quad (123)$$

are created.

There is a linear mapping

$$\tau : \overline{N} \to \mathbb{R}\,. \quad (124)$$

If

$$\overline{\nu}_i(\delta x) = \begin{pmatrix} \delta t_i \\ \delta \vec{q}_i \end{pmatrix}, \quad (125)$$

then

$$\tau(\delta x) = \delta t_i\,. \quad (126)$$

The number $\tau(x_2-x_1)$ is the time elapsed between events x_1 and x_2 in N. Events x_1 and x_2 are *simultaneous* if $\tau(x_2-x_1)=0$.

In the space

$$T_0 = \tau^{-1}(0) \subset W \tag{127}$$

there is a Euclidean metric represented by a metric tensor

$$h : T_0 \to T_0^* \,. \tag{128}$$

If

$$\overline{\nu}_i(\delta x) = \begin{pmatrix} 0 \\ \delta\vec{q}_i \end{pmatrix}, \tag{129}$$

then

$$\langle h(\delta x), \delta x \rangle = \|\delta\vec{q}_i\|^2 \,. \tag{130}$$

The metric h expresses geometric relations between simultaneous events. The norm

$$\|x_2 - x_1\| = \sqrt{\langle h(x_2-x_1), x_2-x_1 \rangle} \tag{131}$$

represents the distance between simultaneous events x_1 and x_2.

The concept of saturation introduced for the Poincaré groupoid is applicable to the Galilei groupoid with the Poincaré isomorphisms replaced with Galilei isomorphisms. We will see from the discussion in the subsequent section that requiring a complete saturation of the Galilei groupoid is not necessarily the best choice.

4.4. A derivation of the Galilei groupoid from the Poincaré groupoid

We single out a family of observers $\{O_i\}_{i\in J}$ with $J \subset I$. Observations made by the observers within the family are related by orthochronous Poincaré mappings

$$\mu_{ij} = \left[\left(\sqrt{1 - c^{-2}\|\vec{v}_{ij}\|^2}\right)^{-1} \begin{pmatrix} 1 & c^{-2}\vec{v}_{ij}\,\cdot \\ \vec{v}_{ij} & Pr^{\|} \end{pmatrix} + \begin{pmatrix} 0 & \vec{0}_i\,\cdot \\ \vec{0}_i & Pr^{\perp} \end{pmatrix} \right] \begin{pmatrix} 1 & \vec{0}_j\,\cdot \\ \vec{0}_i & \kappa_{ij} \end{pmatrix} + \begin{pmatrix} \Delta t_{ij} \\ \Delta\vec{q}_{ij} \end{pmatrix}. \tag{132}$$

The observers are prepared to tolerate errors below a certain proportion Δ of the quantity being measured and errors in absolute time measurements below a threshold T. If the relative velocities of observers within the family are limited by the inequality

$$c^{-1}\|v_{ij}\| < \Delta \,, \tag{133}$$

then the matrices

$$\begin{pmatrix} 1 & c^{-2}\vec{v}_{ij}\,\cdot \\ \vec{v}_{ij} & 1_{Q_i} \end{pmatrix} \begin{pmatrix} 1 & \vec{0}_j\,\cdot \\ \vec{0}_i & \kappa_{ij} \end{pmatrix} \tag{134}$$

are valid approximations of the linear parts of the mappings μ_{ij}. If an approximate linear part is applied to vectors

$$\begin{pmatrix} \delta t_j \\ \delta\vec{q}_j \end{pmatrix} \tag{135}$$

such that

$$c^{-1}\left\|\frac{\delta\vec{q}_j}{\delta t_j}\right\| < \Delta\,, \tag{136}$$

then

$$\begin{pmatrix} \delta t_j + c^{-2}\vec{v}_{ij}\cdot\kappa_{ij}\delta\vec{q}_j \\ \delta t_j\vec{v}_{ij} + \kappa_{ij}\delta\vec{q}_j \end{pmatrix} = \begin{pmatrix} 1 & c^{-2}\vec{v}_{ij}\,\cdot \\ \vec{v}_{ij} & 1_{Q_i} \end{pmatrix}\begin{pmatrix} 1 & \vec{0}_j\,\cdot \\ \vec{0}_i & \kappa_{ij} \end{pmatrix}\begin{pmatrix} \delta t_j \\ \delta\vec{q}_j \end{pmatrix} \tag{137}$$

is correctly approximated by

$$\begin{pmatrix} \delta t_j \\ \delta t_j\vec{v}_{ij} + \kappa_{ij}\delta\vec{q}_j \end{pmatrix} = \begin{pmatrix} 1 & \vec{0}_j\,\cdot \\ \vec{v}_{ij} & \kappa_{ij} \end{pmatrix}\begin{pmatrix} \delta t_j \\ \delta\vec{q}_j \end{pmatrix}. \tag{138}$$

If $\delta t_j < T$, then (138) is a correct approximation even if the inequality (136) does not hold.

We consider the Minkowski geometry the correct structure of space-time. We have identified limits of applicability of the Newtonian structure of space-time. Observers can use this structure if the observed velocities, observed distances, and observed time periods are small. There is no precise correspondence between events observed by these observers and the Minkowskian events. Different families of Newtonian observers can be considered. Observations made by observers belonging to different families are not related. It is customary not to impose the limitations explicitly. Saturated Galilei groupoids can be used only if the limitations are disregarded.

5. Particle mechanics in Newtonian space-time

The Lagrange equations for a particle of mass m are derived by an observer O_i by applying a variational principle to the homogeneous function

$$\widetilde{L}_i : N_i \times N_i \to \mathbb{R} : \left(\begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix}, \begin{pmatrix} t'_i \\ \vec{q}'_i \end{pmatrix}\right) \mapsto \frac{m}{2}\left\|\frac{\vec{q}'_i}{t'_i}\right\|^2 t'_i - U_i\begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix} t'_i\,. \tag{139}$$

The family $\{\widetilde{L}_i\}_{i\in J}$, however, is not compatible with the groupoid $\nu_{ij}\times\bar{\nu}_{ij}$ and then it does not produce an intrinsic Lagrangian function on $N\times\bar{N}$.

In order to give an intrinsic definition of a Lagrangian, we will conceive it as a section of a suitable affine bundle and, as a consequence, we will introduce all the required definitions to write the variational principle.

5.1. Geometric constructions

This section is devoted to the construction of the spaces needed. In particular, the first space Y we will introduce is the above-mentioned bundle, where not only the Lagrangian, but also all the pairing used will take their values.

The second space S, related to the differential of the Lagrangian, will play the role of a dual space of the Newtonian $\overline{N}$.

The last one is the energy-momentum space P. It is another dual space of $\overline{N}$, but with respect to a noncanonical pairing [2].

5.1.1. The Bargman affine bundle. Let us consider the linear Galilei groupoid $\{\overline{\nu}_{ij}\}_{(i,j)\in J\times J}$ and put, for any $i \in J$,

$$Y_i = \begin{pmatrix} \mathbb{R} \\ Q_i \end{pmatrix} \times \mathbb{R}\,. \tag{140}$$

Then define the affine groupoid

$$\begin{aligned} \eta_{ij} &: Y_j \to Y_i \\ &: \left(\begin{pmatrix} t'_j \\ \vec{q}'_j \end{pmatrix}, y_j \right) \mapsto \left(\overline{\nu}_{ij} \begin{pmatrix} t'_j \\ \vec{q}'_j \end{pmatrix}, y_j + \frac{m}{2} \left\|\vec{v}_{ij}\right\|^2 t'_j - m\vec{v}_{ji} \cdot \vec{q}'_j \right). \end{aligned} \tag{141}$$

The method outlined in Section 2 produces an abstract affine space Y and affine charts

$$\eta_i : Y \to Y_i\,. \tag{142}$$

Moreover, the groupoid

$$\overline{\eta}_{ij} : Y_j \to Y_i : \left(\begin{pmatrix} t'_j \\ \vec{q}'_j \end{pmatrix}, y_j \right) \mapsto \left(\overline{\nu}_{ij} \begin{pmatrix} t'_j \\ \vec{q}'_j \end{pmatrix}, y_j \right), \tag{143}$$

composed of all the linear parts of η_{ij} produces the model vector space $\overline{Y}$ and linear charts

$$\overline{\eta}_i : \overline{Y} \to Y_i\,. \tag{144}$$

It follows from $\overline{\eta}_{ij} = \overline{\nu}_{ij} \times 1_{\mathbb{R}}$ that $\overline{Y} = \overline{N} \times \mathbb{R}$.

The space Y is the total space of an affine bundle over $\overline{N}$. To prove it, consider the canonical projections

$$\vartheta_i : Y_i = N_i \times \mathbb{R} \to N_i\,. \tag{145}$$

Since

$$\begin{aligned} (\overline{\nu}_{ij} \circ \vartheta_j \circ \eta_{ji}) &\left(\begin{pmatrix} t'_i \\ \vec{q}'_i \end{pmatrix}, y_i \right) \\ &= (\overline{\nu}_{ij} \circ \vartheta_j) \left(\overline{\nu}_{ji} \begin{pmatrix} t'_i \\ \vec{q}'_i \end{pmatrix}, y_i + \frac{m}{2} \left\|\vec{v}_{ij}\right\|^2 t'_i - m\vec{v}_{ij} \cdot \vec{q}'_i \right) \\ &= \overline{\nu}_{ij} \left(\overline{\nu}_{ji} \begin{pmatrix} t'_i \\ \vec{q}'_i \end{pmatrix} \right) \\ &= \begin{pmatrix} t'_i \\ \vec{q}'_i \end{pmatrix} \\ &= \vartheta_i \left(\begin{pmatrix} t'_i \\ \vec{q}'_i \end{pmatrix}, y_i \right), \end{aligned} \tag{146}$$

the family $\{\vartheta_i\}_{i\in J}$ is compatible, in the sense of (30), with the groupoids η_{ij} and $\overline{\nu}_{ij}$. As a consequence, a mapping

$$\vartheta : Y \to \overline{N} \tag{147}$$

is defined by

$$\vartheta = \overline{\nu}_i^{-1} \circ \vartheta_i \circ \eta_i \,. \tag{148}$$

On the other hand, the canonical projections (145) are linear mappings compatible also with the groupoids $\overline{\eta}_{ij}$ and $\overline{\nu}_{ij}$. It follows that the linear part of (147), defined by

$$\overline{\vartheta} = \overline{\nu}_i^{-1} \circ \vartheta_i \circ \overline{\eta}_i \,, \tag{149}$$

is the canonical projection

$$\overline{\vartheta} : \overline{Y} = \overline{N} \times \mathbb{R} \to \overline{N} \,. \tag{150}$$

Finally, let y^1 and y^2 be elements of Y with $\vartheta(y^2) = \vartheta(y^1)$ and put

$$\eta_i(y^1) = \left(\begin{pmatrix} t_i' \\ \vec{q}_i' \end{pmatrix}, y_i^1 \right), \tag{151}$$

and

$$\eta_i(y^2) = \left(\begin{pmatrix} t_i' \\ \vec{q}_i' \end{pmatrix}, y_i^2 \right). \tag{152}$$

It is

$$y^2 - y^1 = \overline{\eta}_i^{-1} \left(\begin{pmatrix} t_i' \\ \vec{q}_i' \end{pmatrix}, y_i^2 - y_i^1 \right) = \left(\overline{\nu}_i^{-1} \begin{pmatrix} t_i' \\ \vec{q}_i' \end{pmatrix}, y_i^2 - y_i^1 \right) \in \overline{Y} \,. \tag{153}$$

We conclude that the mapping ϑ is an affine fibration and (150) is the model vector fibration. The Lagrangian will be a section of ϑ.

5.1.2. An evaluation space. The next construction will be related to the differential of the Lagrangian. We start again by defining an affine groupoid

$$\sigma_{ij} : S_j \to S_i : \begin{pmatrix} a_j & \mathbf{b}_j \end{pmatrix} \mapsto \begin{pmatrix} a_j & \mathbf{b}_j \end{pmatrix} \overline{\nu}_{ji} + \begin{pmatrix} -\frac{m}{2} \|\vec{v}_{ij}\|^2 & m\vec{v}_{ij} \,\cdot \end{pmatrix}, \tag{154}$$

where, for any $i \in J$,

$$S_i = \begin{pmatrix} \mathbb{R} & Q_i^* \end{pmatrix}. \tag{155}$$

In correspondence to σ_{ij} we have an abstract affine space S and affine charts

$$\sigma_i : S \to S_i \,. \tag{156}$$

The model vector space $\overline{S}$ and linear charts

$$\overline{\sigma}_i : \overline{S} \to S_i \tag{157}$$

are then constructed from the linear groupoid

$$\overline{\sigma}_{ij} : \begin{pmatrix} \mathbb{R} & Q_j^* \end{pmatrix} \to \begin{pmatrix} \mathbb{R} & Q_i^* \end{pmatrix} : \begin{pmatrix} a_j' & \mathbf{b}_j' \end{pmatrix} \mapsto \begin{pmatrix} a_j' & \mathbf{b}_j' \end{pmatrix} \overline{\nu}_{ji} \,. \tag{158}$$

We will introduce a paring in $S \times \overline{N}$ and its "linearization" in $\overline{S} \times \overline{N}$. To this end, let us consider for any index $i \in J$ the mapping

$$\begin{aligned} \langle \ , \ \rangle_i^\natural &: S_i \times N_i \to Y_i \\ &: \left(\begin{pmatrix} a_i & \mathbf{b}_i \end{pmatrix}, \begin{pmatrix} \delta t_i \\ \delta \vec{q}_i \end{pmatrix} \right) \mapsto \left(\begin{pmatrix} \delta t_i \\ \delta \vec{q}_i \end{pmatrix}, \begin{pmatrix} a_i & \mathbf{b}_i \end{pmatrix} \begin{pmatrix} \delta t_i \\ \delta \vec{q}_i \end{pmatrix} \right). \end{aligned} \tag{159}$$

The family (159) is compatible with the groupoids

$$\sigma_{ij} \times \overline{\nu}_{ij} : S_j \times N_j \to S_i \times N_i \tag{160}$$

and

$$\eta_{ij} : Y_j \to Y_i \,, \tag{161}$$

as established by

$$\begin{aligned}
&\eta_{ij} \left\langle \sigma_{ji} \left(\begin{array}{cc} a_i \mathbf{b}_i \end{array} \right), \overline{\nu}_{ji} \left(\begin{array}{c} \delta t_i \\ \delta \vec{q}_i \end{array} \right) \right\rangle_j^\natural \\
&= \eta_{ij} \left\langle \left(\begin{array}{cc} a_i \mathbf{b}_i \end{array} \right) \overline{\nu}_{ij} + \left(\begin{array}{c} -\frac{m}{2} \|\vec{v}_{ij}\|^2 m\vec{v}_{ji} \,\cdot \end{array} \right), \overline{\nu}_{ji} \left(\begin{array}{c} \delta t_i \\ \delta \vec{q}_i \end{array} \right) \right\rangle_j^\natural \\
&= \eta_{ij} \left(\overline{\nu}_{ji} \left(\begin{array}{c} \delta t_i \\ \delta \vec{q}_i \end{array} \right), \left(\begin{array}{cc} a_i & \mathbf{b}_i \end{array} \right) \left(\begin{array}{c} \delta t_i \\ \delta \vec{q}_i \end{array} \right) \right. \\
&\quad \left. + \left(\begin{array}{c} -\frac{m}{2} \|\vec{v}_{ij}\|^2 m\vec{v}_{ji} \,\cdot \end{array} \right) \overline{\nu}_{ji} \left(\begin{array}{c} \delta t_i \\ \delta \vec{q}_i \end{array} \right) \right) \\
&= \left(\left(\begin{array}{c} \delta t_i \\ \delta \vec{q}_i \end{array} \right), \left(\begin{array}{cc} a_i \mathbf{b}_i \end{array} \right) \left(\begin{array}{c} \delta t_i \\ \delta \vec{q}_i \end{array} \right) \right) \\
&= \left\langle \left(\begin{array}{cc} a_i \mathbf{b}_i \end{array} \right), \left(\begin{array}{c} \delta t_i \\ \delta \vec{q}_i \end{array} \right) \right\rangle_i^\natural .
\end{aligned} \tag{162}$$

Consequently, the pairing

$$\langle \;,\; \rangle^\natural : S \times \overline{N} \to Y : (\mathbf{b}, \delta x) \mapsto \eta_i^{-1} \left(\langle \sigma_i(\mathbf{b}), \overline{\nu}_i(\delta x) \rangle_i^\natural \right) \tag{163}$$

is well defined.

Analogously, the family

$$\begin{aligned}
&\overline{\langle \;,\; \rangle}_i^\natural : S_i \times N_i \to Y_i \\
&\quad : \left(\left(\begin{array}{cc} a_i' & \mathbf{b}_i' \end{array} \right), \left(\begin{array}{c} \delta t_i \\ \delta \vec{q}_i \end{array} \right) \right) \mapsto \left(\left(\begin{array}{c} \delta t_i \\ \delta \vec{q}_i \end{array} \right), \left(\begin{array}{cc} a_i' & \mathbf{b}_i' \end{array} \right) \left(\begin{array}{c} \delta t_i \\ \delta \vec{q}_i \end{array} \right) \right)
\end{aligned} \tag{164}$$

is compatible with the groupoids

$$\overline{\sigma}_{ij} \times \overline{\nu}_{ij} : S_j \times N_j \to S_i \times N_i \tag{165}$$

and

$$\overline{\eta}_{ij} : Y_j \to Y_i \,, \tag{166}$$

as established by

$$\overline{\eta}_{ij}\overline{\left\langle\overline{\sigma}_{ji}\left(\begin{array}{cc} a'_i & \mathbf{b}'_i \end{array}\right),\ \overline{\nu}_{ji}\left(\begin{array}{c}\delta t_i\\ \delta\vec{q}_i\end{array}\right)\right\rangle}^{\natural}_j = \overline{\eta}_{ij}\overline{\left\langle\left(\begin{array}{cc} a'_i & \mathbf{b}'_i \end{array}\right)\overline{\nu}_{ij}\ ,\overline{\nu}_{ji}\left(\begin{array}{c}\delta t_i\\ \delta\vec{q}_i\end{array}\right)\right\rangle}^{\natural}_j$$
$$= \overline{\eta}_{ij}\left(\overline{\nu}_{ji}\left(\begin{array}{c}\delta t_i\\ \delta\vec{q}_i\end{array}\right),\left(\begin{array}{cc} a'_i & \mathbf{b}'_i \end{array}\right)\left(\begin{array}{c}\delta t_i\\ \delta\vec{q}_i\end{array}\right)\right)$$
$$= \left(\left(\begin{array}{c}\delta t_i\\ \delta\vec{q}_i\end{array}\right),\left(\begin{array}{cc} a'_i & \mathbf{b}'_i \end{array}\right)\left(\begin{array}{c}\delta t_i\\ \delta\vec{q}_i\end{array}\right)\right)$$
$$= \overline{\left\langle\left(\begin{array}{cc} a'_i & \mathbf{b}'_i \end{array}\right),\left(\begin{array}{c}\delta t_i\\ \delta\vec{q}_i\end{array}\right)\right\rangle}^{\natural}_i . \tag{167}$$

Therefore, the pairing

$$\overline{\langle\ ,\ \rangle}^{\natural} : \overline{S}\times\overline{N}\to\overline{Y} : (\mathbf{b}',\delta x)\mapsto \overline{\eta}_i^{-1}\left(\overline{\langle\overline{\sigma}_i(\mathbf{b}'),\overline{\nu}_i(\delta x)\rangle}^{\natural}_i\right) \tag{168}$$

is well defined.

Notice that for each $\delta x\in\overline{N}$ the mapping

$$\langle\ ,\delta x\rangle^{\natural} : S\to Y : \mathbf{b}\mapsto\langle\mathbf{b},\delta x\rangle^{\natural} \tag{169}$$

is an affine mapping and the mapping

$$\overline{\langle\ ,\delta x\rangle}^{\natural} : \overline{S}\to\overline{Y} : \mathbf{b}'\mapsto\overline{\langle\mathbf{b}',\delta x\rangle}^{\natural} \tag{170}$$

is its linear part. The pairing (168) is bilinear.

The groupoid $\overline{\sigma}_{ij}$ conforms to the description of the groupoid $\overline{\nu}^*_{ij}$ dual to the groupoid $\overline{\nu}_{ij}$. It follows that the space $\overline{S}$ is the dual $\overline{N}^*$ of $\overline{N}$. Pairings

$$\langle\ ,\ \rangle_i : S_i\times N_i\to\mathbb{R} : \left(\left(\begin{array}{cc} a'_i & \mathbf{b}'_i \end{array}\right),\left(\begin{array}{c}\delta t_i\\ \delta\vec{q}_i\end{array}\right)\right)\mapsto\left(\begin{array}{cc} a'_i & \mathbf{b}'_i \end{array}\right)\left(\begin{array}{c}\delta t_i\\ \delta\vec{q}_i\end{array}\right), \tag{171}$$

are compatible with the groupoid

$$\overline{\sigma}_{ij}\times\overline{\nu}_{ij} : S_j\times N_j\to S_i\times N_i\,. \tag{172}$$

The compatibility follows from

$$\left\langle\overline{\sigma}_{ji}\left(\begin{array}{cc} a'_i & \mathbf{b}'_i \end{array}\right),\overline{\nu}_{ji}\left(\begin{array}{c}\delta t_i\\ \delta\vec{q}_i\end{array}\right)\right\rangle_j = \left\langle\left(\begin{array}{cc} a'_i & \mathbf{b}'_i \end{array}\right)\overline{\nu}_{ij},\overline{\nu}_{ji}\left(\begin{array}{c}\delta t_i\\ \delta\vec{q}_i\end{array}\right)\right\rangle_j$$
$$= \left(\begin{array}{cc} a'_i & \mathbf{b}'_i \end{array}\right)\left(\begin{array}{c}\delta t_i\\ \delta\vec{q}_i\end{array}\right)$$
$$= \left\langle\left(\begin{array}{cc} a'_i & \mathbf{b}'_i \end{array}\right),\left(\begin{array}{c}\delta t_i\\ \delta\vec{q}_i\end{array}\right)\right\rangle_i . \tag{173}$$

The pairing

$$\langle\ ,\ \rangle : \overline{S}\times\overline{N}\to\mathbb{R} : (\mathbf{b}',\delta x)\mapsto\left\langle\overline{\sigma}_i(\mathbf{b}'),\overline{\nu}_i(\delta x)\right\rangle_i \tag{174}$$

is the canonical bilinear pairing. It can be defined by

$$\langle\ ,\ \rangle = pr_{\mathbb{R}}\circ\overline{\langle\ ,\ \rangle}^{\natural} \tag{175}$$

in terms of the canonical projection

$$pr_{\mathbb{R}} : \overline{Y} = \overline{N} \times \mathbb{R} \to \mathbb{R} \,. \tag{176}$$

5.1.3. The energy-momentum phase space. Let us define the affine groupoid

$$\begin{aligned} \pi_{ij} &: P_j \to P_i \\ &: \begin{pmatrix} e_j & \mathbf{p}_j \end{pmatrix} \mapsto \Big(\begin{pmatrix} e_j & \mathbf{p}_j \end{pmatrix} \alpha_j \overline{\nu}_{ji} + \begin{pmatrix} -\frac{m}{2} \|\vec{v}_{ij}\|^2 & m\vec{v}_{ij} \, \cdot \end{pmatrix} \Big) \alpha_i^{-1} \,, \end{aligned} \tag{177}$$

where, for any $i \in J$,

$$P_i = \begin{pmatrix} \mathbb{R} & Q_i^* \end{pmatrix} \tag{178}$$

and

$$\alpha_i = \begin{pmatrix} -1 & \vec{0}_i \, \cdot \\ \vec{0}_i & 1_{Q_i} \end{pmatrix} . \tag{179}$$

An abstract affine space P and affine charts

$$\pi_i : P \to P_i \tag{180}$$

are constructed. The model vector space $\overline{P}$ and linear charts

$$\overline{\pi}_i : \overline{P} \to P_i \tag{181}$$

are constructed from the groupoid

$$\overline{\pi}_{ij} : P_j \to P_i : \begin{pmatrix} e'_j & \mathbf{p}'_j \end{pmatrix} \mapsto \begin{pmatrix} e'_j & \mathbf{p}'_j \end{pmatrix} \alpha_j \overline{\nu}_{ji} \alpha_i^{-1} \,. \tag{182}$$

For each index i there are mappings

$$\begin{aligned} \langle \ , \ \rangle_i^\sharp &: P_i \times N_i \to Y_i \\ &: \left(\begin{pmatrix} e_i & \mathbf{p}_i \end{pmatrix}, \begin{pmatrix} \delta t_i \\ \delta \vec{q}_i \end{pmatrix} \right) \mapsto \left(\begin{pmatrix} \delta t_i \\ \delta \vec{q}_i \end{pmatrix}, \begin{pmatrix} e_i & \mathbf{p}_i \end{pmatrix} \alpha_i \begin{pmatrix} \delta t_i \\ \delta \vec{q}_i \end{pmatrix} \right) \end{aligned} \tag{183}$$

and

$$\begin{aligned} \overline{\langle \ , \ \rangle}_i^\sharp &: P_i \times N_i \to Y_i \\ &: \left(\begin{pmatrix} e'_i & \mathbf{p}'_i \end{pmatrix}, \begin{pmatrix} \delta t_i \\ \delta \vec{q}_i \end{pmatrix} \right) \mapsto \left(\begin{pmatrix} \delta t_i \\ \delta \vec{q}_i \end{pmatrix}, \begin{pmatrix} e'_i & \mathbf{p}'_i \end{pmatrix} \alpha_i \begin{pmatrix} \delta t_i \\ \delta \vec{q}_i \end{pmatrix} \right). \end{aligned} \tag{184}$$

The family (183) is compatible with the groupoids

$$\pi_{ij} \times \overline{\nu}_{ij} : P_j \times N_j \to P_i \times N_i \tag{185}$$

and

$$\eta_{ij} : Y_j \to Y_i \,. \tag{186}$$

The family (184) is compatible with the groupoids

$$\overline{\pi}_{ij} \times \overline{\nu}_{ij} : P_j \times N_j \to P_i \times N_i \tag{187}$$

and

$$\overline{\eta}_{ij} : Y_j \to Y_i \,. \tag{188}$$

Compatibilities follow from

$$\eta_{ij}\left\langle \pi_{ji}\left(\begin{array}{cc} e_i & \mathbf{p}_i \end{array}\right), \overline{\nu}_{ji}\left(\begin{array}{c} \delta t_i \\ \delta\vec{q}_i \end{array}\right)\right\rangle_j^\sharp$$

$$= \eta_{ij}\left\langle \left(\left(\begin{array}{cc} e_i & \mathbf{p}_i \end{array}\right)\alpha_i\overline{\nu}_{ij} + \left(\begin{array}{cc} -\frac{m}{2}\|\vec{v}_{ij}\|^2 & m\vec{v}_{ji}\,\cdot \end{array}\right)\right)\alpha_j^{-1}, \overline{\nu}_{ji}\left(\begin{array}{c} \delta t_i \\ \delta\vec{q}_i \end{array}\right)\right\rangle_j^\sharp$$

$$= \eta_{ij}\Bigg(\overline{\nu}_{ji}\left(\begin{array}{c} \delta t_i \\ \delta\vec{q}_i \end{array}\right), \left(\begin{array}{cc} e_i & \mathbf{p}_i \end{array}\right)\alpha_i\left(\begin{array}{c} \delta t_i \\ \delta\vec{q}_i \end{array}\right)$$

$$+ \left(\begin{array}{cc} -\frac{m}{2}\|\vec{v}_{ij}\|^2 & m\vec{v}_{ji}\,\cdot \end{array}\right)\overline{\nu}_{ji}\left(\begin{array}{c} \delta t_i \\ \delta\vec{q}_i \end{array}\right)\Bigg)$$

$$= \left(\left(\begin{array}{c} \delta t_i \\ \delta\vec{q}_i \end{array}\right), \left(\begin{array}{cc} e_i & \mathbf{p}_i \end{array}\right)\alpha_i\left(\begin{array}{c} \delta t_i \\ \delta\vec{q}_i \end{array}\right)\right)$$

$$= \left\langle \left(\begin{array}{cc} e_i & \mathbf{p}_i \end{array}\right), \left(\begin{array}{c} \delta t_i \\ \delta\vec{q}_i \end{array}\right)\right\rangle_i^\sharp \tag{189}$$

and

$$\overline{\eta}_{ij}\overline{\left\langle \overline{\pi}_{ji}\left(\begin{array}{cc} e_i' & \mathbf{p}_i' \end{array}\right),\right.}\, \overline{\nu}_{ji}\left(\begin{array}{c} \delta t_i \\ \delta\vec{q}_i \end{array}\right)\Bigg\rangle_j^\sharp = \overline{\eta}_{ij}\overline{\left\langle \left(\begin{array}{cc} e_i' & \mathbf{p}_i' \end{array}\right)\alpha_i\overline{\nu}_{ij}\alpha_i^{-1}\right.}, \overline{\nu}_{ji}\left(\begin{array}{c} \delta t_i \\ \delta\vec{q}_i \end{array}\right)\Bigg\rangle_j^\sharp$$

$$= \overline{\eta}_{ij}\left(\overline{\nu}_{ji}\left(\begin{array}{c} \delta t_i \\ \delta\vec{q}_i \end{array}\right), \left(\begin{array}{cc} e_i' & \mathbf{p}_i' \end{array}\right)\alpha_i\left(\begin{array}{c} \delta t_i \\ \delta\vec{q}_i \end{array}\right)\right)$$

$$= \left(\left(\begin{array}{c} \delta t_i \\ \delta\vec{q}_i \end{array}\right), \left(\begin{array}{cc} e_i' & \mathbf{p}_i' \end{array}\right)\alpha_i\left(\begin{array}{c} \delta t_i \\ \delta\vec{q}_i \end{array}\right)\right)$$

$$= \overline{\left\langle \left(\begin{array}{cc} e_i' & \mathbf{p}_i' \end{array}\right),\right.}\left(\begin{array}{c} \delta t_i \\ \delta\vec{q}_i \end{array}\right)\Bigg\rangle_i^\sharp. \tag{190}$$

Consequently there are pairings

$$\langle\ ,\ \rangle^\sharp : P \times \overline{N} \to Y : (\mathbf{p}, \delta x) \mapsto \eta_i^{-1}\left(\langle \pi_i(\mathbf{p}), \overline{\nu}_i(\delta x)\rangle_i^\sharp\right) \tag{191}$$

and

$$\overline{\langle\ ,}\ \rangle^\sharp : \overline{P} \times \overline{N} \to \overline{Y} : (\mathbf{p}', \delta x) \mapsto \overline{\eta}_i^{-1}\left(\overline{\langle \overline{\pi}_i(\mathbf{p}'),}\overline{\nu}_i(\delta x)\rangle_i^\sharp\right). \tag{192}$$

For each $\delta x \in \overline{N}$ the mapping

$$\langle\ , \delta x\rangle^\sharp : P \to Y : \mathbf{p} \mapsto \langle \mathbf{p}, \delta x\rangle^\sharp \tag{193}$$

is an affine mapping and the mapping

$$\overline{\langle\ ,}\delta x\rangle^\sharp : \overline{P} \to \overline{Y} : \mathbf{p}' \mapsto \overline{\langle \mathbf{p}',}\delta x\rangle^\sharp \tag{194}$$

is its linear part. The pairing (192) is bilinear.

Pairings

$$\langle\ ,\ \rangle_i^\flat : P_i \times N_i \to \mathbb{R} : \left(\left(\begin{array}{cc} e_i' & \mathbf{p}_i' \end{array}\right), \left(\begin{array}{c} \delta t_i \\ \delta\vec{q}_i \end{array}\right)\right) \mapsto \left(\begin{array}{cc} e_i' & \mathbf{p}_i' \end{array}\right)\alpha_i\left(\begin{array}{c} \delta t_i \\ \delta\vec{q}_i \end{array}\right) \tag{195}$$

are compatible with the groupoid

$$\overline{\pi}_{ij} \times \overline{\nu}_{ij} : P_j \times N_j \to P_i \times N_i \,. \tag{196}$$

Compatibility is established by

$$\begin{aligned}\left\langle \overline{\pi}_{ji} \begin{pmatrix} e'_i & \mathbf{p}'_i \end{pmatrix}, \overline{\nu}_{ji} \begin{pmatrix} \delta t_i \\ \delta \vec{q}_i \end{pmatrix} \right\rangle_j^\flat &= \left\langle \begin{pmatrix} e'_i & \mathbf{p}'_i \end{pmatrix} \overline{\nu}_{ij}, \overline{\nu}_{ji} \begin{pmatrix} \delta t_i \\ \delta \vec{q}_i \end{pmatrix} \right\rangle_j^\flat \\ &= \begin{pmatrix} e'_i & \mathbf{p}'_i \end{pmatrix} \begin{pmatrix} \delta t_i \\ \delta \vec{q}_i \end{pmatrix} \\ &= \left\langle \begin{pmatrix} e'_i & \mathbf{p}'_i \end{pmatrix}, \begin{pmatrix} \delta t_i \\ \delta \vec{q}_i \end{pmatrix} \right\rangle_i^\flat . \end{aligned} \tag{197}$$

A pairing

$$\langle \ , \ \rangle^\flat : \overline{P} \times \overline{N} \to \mathbb{R} : (\mathbf{p}', \delta x) \mapsto \left\langle \overline{\pi}_i(\mathbf{p}'), \overline{\nu}_i(\delta x) \right\rangle_i^\flat \tag{198}$$

is defined. The formula

$$\langle \ , \ \rangle^\flat = pr_{\mathbb{R}} \circ \overline{\langle \ , \ \rangle^\sharp} \tag{199}$$

is an alternative definition.

The family $\{\alpha_i\}_{i \in I}$ of matrices α_i introduced in (179) interpreted as mappings

$$P_i \to S_i : \begin{pmatrix} e_i & \mathbf{p}_i \end{pmatrix} \mapsto \begin{pmatrix} e_i & \mathbf{p}_i \end{pmatrix} \alpha_i \tag{200}$$

is compatible with the affine groupoids

$$\pi_{ij} : P_j \to P_i \tag{201}$$

and

$$\sigma_{ij} : S_j \to S_i \,, \tag{202}$$

and also with the linear groupoids

$$\overline{\pi}_{ij} : P_j \to P_i \tag{203}$$

and

$$\overline{\sigma}_{ij} : S_j \to S_i \,. \tag{204}$$

Compatibilities follow from

$$\begin{aligned}\sigma_{ij}\Big(\Big(\pi_{ji} \begin{pmatrix} e_i & \mathbf{p}_i \end{pmatrix}\Big)\alpha_j\Big) &= \sigma_{ij}\Big(\begin{pmatrix} e'_i & \mathbf{p}'_i \end{pmatrix} \alpha_i \overline{\nu}_{ij} + \begin{pmatrix} -\frac{m}{2}\|\vec{v}_{ij}\|^2 & m\vec{v}_{ij} \,\cdot \end{pmatrix}\Big) \\ &= \begin{pmatrix} e_i & \mathbf{p}_i \end{pmatrix} \alpha_i \end{aligned} \tag{205}$$

and

$$\overline{\sigma}_{ij}\Big(\Big(\overline{\pi}_{ji} \begin{pmatrix} e'_i & \mathbf{p}'_i \end{pmatrix}\Big)\alpha_j\Big) = \overline{\sigma}_{ij}\Big(\begin{pmatrix} e_i & \mathbf{p}_i \end{pmatrix} \alpha_i \overline{\nu}_{ij}\Big) = \begin{pmatrix} e'_i & \mathbf{p}'_i \end{pmatrix} \alpha_i \,. \tag{206}$$

We have mappings

$$\alpha : P \to S : \mathbf{p} \mapsto \sigma_i^{-1}\big(\pi_i(\mathbf{p})\alpha_i\big) \tag{207}$$

and

$$\overline{\alpha} : \overline{P} \to \overline{S} : \mathbf{p}' \mapsto \overline{\sigma}_i^{-1}\big(\overline{\pi}_i(\mathbf{p})\alpha_i\big) \,. \tag{208}$$

Relations

$$\langle \mathbf{p}, \delta x\rangle^{\sharp} = \left\langle \alpha(\mathbf{p}), \delta x\right\rangle^{\natural}, \tag{209}$$

$$\overline{\langle \mathbf{p}', \delta x\rangle^{\sharp}} = \overline{\left\langle \overline{\alpha}(\mathbf{p}'), \delta x\right\rangle^{\natural}}, \tag{210}$$

and

$$\langle \mathbf{p}', \delta x\rangle^{\flat} = \left\langle \overline{\alpha}(\mathbf{p}'), \delta x\right\rangle \tag{211}$$

are established.

5.2. The Lagrangian

The mapping

$$\begin{aligned} L_i \colon N_i \times N_i &\to Y_i \\ &: \left(\begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix}, \begin{pmatrix} t_i' \\ \vec{q}_i' \end{pmatrix} \right) \mapsto \left(\begin{pmatrix} t_i' \\ \vec{q}_i' \end{pmatrix}, \frac{m}{2} \left\| \frac{\vec{q}_i'}{t_i'} \right\|^2 t_i' - U_i \begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix} t_i' \right) \end{aligned} \tag{212}$$

is a version of the usual Lagrangian

$$\widetilde{L}_i : N_i \times N_i \to \mathbb{R} : \left(\begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix}, \begin{pmatrix} t_i' \\ \vec{q}_i' \end{pmatrix} \right) \mapsto \frac{m}{2} \left\| \frac{\vec{q}_i'}{t_i'} \right\|^2 t_i' - U_i \begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix} t_i'. \tag{213}$$

The argument t_i' in the Lagrangian is always positive. Compatibility of this mapping with the groupoids

$$\nu_{ij} \times \overline{\nu}_{ij} : N_j \times N_j \to N_i \times N_i \tag{214}$$

and

$$\eta_{ij} : Y_j \to Y_i \tag{215}$$

follows from

$$\begin{aligned} &\eta_{ij} \left(L_j \left(\nu_{ji} \begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix}, \overline{\nu}_{ji} \begin{pmatrix} t_i' \\ \vec{q}_i' \end{pmatrix} \right) \right) \\ &\quad = \eta_{ij} \left(\overline{\nu}_{ji} \begin{pmatrix} t_i' \\ \vec{q}_i' \end{pmatrix}, \frac{m}{2} \left\| \vec{v}_{ji} + \lambda_{ji} \left(\frac{\vec{q}_i'}{t_i'} \right) \right\|^2 t_j' - U_j \left(\nu_{ji} \begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix} \right) t_j' \right) \\ &\quad = \left(\begin{pmatrix} t_i' \\ \vec{q}_i' \end{pmatrix}, \frac{m}{2} \left\| \frac{\vec{q}_i'}{t_i'} \right\|^2 t_i' - U_i \begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix} t_i' \right) \\ &\quad = L_i \left(\begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix}, \begin{pmatrix} t_i' \\ \vec{q}_i' \end{pmatrix} \right). \end{aligned} \tag{216}$$

The absolute Lagrangian is the mapping

$$L : N \times \overline{N} \to Y \tag{217}$$

defined by

$$L = \eta_i^{-1} \circ L_i \circ (\nu_i \times \overline{\nu}_i). \tag{218}$$

5.3. The differential of the Lagrangian

The differential of the Lagrangian L_i is represented by mappings

$$\begin{aligned}\left(\begin{array}{cc}\frac{\partial L_i}{\partial t_i} & \frac{\partial L_i}{\partial \vec{q}_i}\end{array}\right) : N_i \times N_i \to S_i : & \left(\left(\begin{array}{c} t_i \\ \vec{q}_i \end{array}\right), \left(\begin{array}{c} t'_i \\ \vec{q}'_i \end{array}\right)\right) \\ & \mapsto \left(\begin{array}{cc} -\frac{\partial U_i}{\partial t_i}\left(\begin{array}{c} t_i \\ \vec{q}_i \end{array}\right) t'_i & -\frac{\partial U_i}{\partial \vec{q}_i}\left(\begin{array}{c} t_i \\ \vec{q}_i \end{array}\right) t'_i \end{array}\right)\end{aligned} \tag{219}$$

and

$$\begin{aligned}\left(\begin{array}{cc}\frac{\partial L_i}{\partial t'_i} & \frac{\partial L_i}{\partial \vec{q}'_i}\end{array}\right) : N_i \times N_i \to S_i : & \left(\left(\begin{array}{c} t_i \\ \vec{q}_i \end{array}\right), \left(\begin{array}{c} t'_i \\ \vec{q}'_i \end{array}\right)\right) \\ & \mapsto \left(\begin{array}{cc} -\frac{m}{2}\left\|\frac{\vec{q}'_i}{t'_i}\right\|^2 - U_i\left(\begin{array}{c} t_i \\ \vec{q}_i \end{array}\right) & m\frac{\vec{q}'_i\,\cdot}{t'_i} \end{array}\right).\end{aligned} \tag{220}$$

The first of these mappings is compatible with the groupoids

$$\nu_{ij} \times \overline{\nu}_{ij} : N_j \times N_j \to N_i \times N_i \tag{221}$$

and

$$\overline{\sigma}_{ij} : S_j \to S_i\,. \tag{222}$$

The second is compatible with the groupoids

$$\nu_{ij} \times \overline{\nu}_{ij} : N_j \times N_j \to N_i \times N_i \tag{223}$$

and

$$\sigma_{ij} : S_j \to S_i\,. \tag{224}$$

Compatibilities are established by

$$\begin{aligned}&\overline{\sigma}_{ij}\left(\left(\begin{array}{cc}\frac{\partial L_j}{\partial t_j} & \frac{\partial L_j}{\partial \vec{q}_j}\end{array}\right)\left(\nu_{ji}\left(\begin{array}{c} t_i \\ \vec{q}_i \end{array}\right), \overline{\nu}_{ji}\left(\begin{array}{c} t'_i \\ \vec{q}'_i \end{array}\right)\right)\right) \\ &\quad = \overline{\sigma}_{ij}\left(\begin{array}{cc} -\frac{\partial U_j}{\partial t_j}\left(\nu_{ji}\left(\begin{array}{c} t_i \\ \vec{q}_i \end{array}\right)\right) t'_i & -\frac{\partial U_j}{\partial \vec{q}_j}\left(\nu_{ji}\left(\begin{array}{c} t_i \\ \vec{q}_i \end{array}\right)\right) t'_i \end{array}\right) \\ &\quad = \left(\begin{array}{cc} -\frac{\partial U_j}{\partial t_j}\left(\nu_{ji}\left(\begin{array}{c} t_i \\ \vec{q}_i \end{array}\right)\right) t'_i & -\frac{\partial U_j}{\partial \vec{q}_j}\left(\nu_{ji}\left(\begin{array}{c} t_i \\ \vec{q}_i \end{array}\right)\right) t'_i \end{array}\right)\overline{\nu}_{ji} \\ &\quad = \left(\begin{array}{cc} -\frac{\partial U_i}{\partial t_i}\left(\begin{array}{c} t_i \\ \vec{q}_i \end{array}\right) t'_i & -\frac{\partial U_i}{\partial \vec{q}_i}\left(\begin{array}{c} t_i \\ \vec{q}_i \end{array}\right) t'_i \end{array}\right) \\ &\quad = \left(\begin{array}{cc}\frac{\partial L_i}{\partial t_i} & \frac{\partial L_i}{\partial \vec{q}_i}\end{array}\right)\left(\left(\begin{array}{c} t_i \\ \vec{q}_i \end{array}\right), \left(\begin{array}{c} t'_i \\ \vec{q}'_i \end{array}\right)\right)\end{aligned} \tag{225}$$

and

$$\begin{aligned}
&\sigma_{ij}\left(\left(\begin{array}{cc} \frac{\partial L_j}{\partial t'_j} & \frac{\partial L_j}{\partial \vec{q}'_j} \end{array}\right)\left(\nu_{ji}\left(\begin{array}{c} t_i \\ \vec{q}_i \end{array}\right), \overline{\nu}_{ji}\left(\begin{array}{c} t'_i \\ \vec{q}'_i \end{array}\right)\right)\right) \\
&\quad = \sigma_{ij}\left(\begin{array}{cc} -\frac{m}{2}\left\|\vec{v}_{ji} + \lambda_{ji}\left(\frac{\vec{q}'_i}{t'_i}\right)\right\|^2 - U_j\left(\nu_{ji}\left(\begin{array}{c} t_i \\ \vec{q}_i \end{array}\right)\right) & m\left(\vec{v}_{ji}\cdot + \frac{\vec{q}'_i\cdot}{t'_i}\lambda_{ij}\right) \end{array}\right) \\
&\quad = \left(\begin{array}{cc} -\frac{\partial U_j}{\partial t_j}\left(\nu_{ji}\left(\begin{array}{c} t_i \\ \vec{q}_i \end{array}\right)\right)t'_i & -\frac{\partial U_j}{\partial \vec{q}_j}\left(\nu_{ji}\left(\begin{array}{c} t_i \\ \vec{q}_i \end{array}\right)\right)t'_i \end{array}\right)\overline{\nu}_{ji} \\
&\quad = \left(\begin{array}{cc} -\frac{m}{2}\left\|\frac{\vec{q}'_i}{t'_i}\right\|^2 - U_i\left(\begin{array}{c} t_i \\ \vec{q}_i \end{array}\right) & m\frac{\vec{q}'_i\cdot}{t'_i} \end{array}\right) \\
&\quad = \left(\begin{array}{cc} \frac{\partial L_i}{\partial t'_l} & \frac{\partial L_i}{\partial \vec{q}'_i} \end{array}\right)\left(\left(\begin{array}{c} t_i \\ \vec{q}_i \end{array}\right), \left(\begin{array}{c} t'_i \\ \vec{q}'_i \end{array}\right)\right).
\end{aligned} \tag{226}$$

An absolute version of the differential is represented by mappings

$$\frac{\partial L}{\partial x} = \overline{\sigma}_i^{-1} \circ \left(\begin{array}{cc} \frac{\partial L_i}{\partial t_i} & \frac{\partial L_i}{\partial \vec{q}_i} \end{array}\right) \circ (\nu_i \times \overline{\nu}_i) : N \times \overline{N} \to \overline{S} \tag{227}$$

and

$$\frac{\partial L}{\partial x'} = \sigma_i^{-1} \circ \left(\begin{array}{cc} \frac{\partial L_i}{\partial t'_i} & \frac{\partial L_i}{\partial \vec{q}'_i} \end{array}\right) \circ (\nu_i \times \overline{\nu}_i) : N \times \overline{N} \to S\,. \tag{228}$$

5.4. Observed dynamics of a particle

We will formulate the *Lagrange equations* for a trajectory

$$\left(\left(\begin{array}{c} \tau_i \\ \xi_i \end{array}\right), \left(\begin{array}{cc} \varepsilon_i & \pi_i \end{array}\right)\right) : \mathbb{R} \to N_i \times P_i \tag{229}$$

of a particle of mass m in the space-time-energy-momentum phase space of an observer O_i. External forces will not be considered. The equations are first order differential equations. The derivatives

$$\left(\left(\begin{array}{c} \tau'_i \\ \xi'_i \end{array}\right), \left(\begin{array}{cc} \varepsilon'_i & \pi'_i \end{array}\right)\right) : \mathbb{R} \to N_i \times P_i \tag{230}$$

of the trajectory are involved. A first order differential equation in the space-time-energy-momentum phase space is a set

$$D \subset N_i \times P_i \times N_i \times P_i\,. \tag{231}$$

The trajectory (229) is a solution of the equation if

$$\left(\left(\begin{array}{c} \tau_i(s) \\ \xi_i(s) \end{array}\right), \left(\begin{array}{cc} \varepsilon_i(s) & \pi_i(s) \end{array}\right), \left(\begin{array}{c} \tau'_i(s) \\ \xi'_i(s) \end{array}\right), \left(\begin{array}{cc} \varepsilon'_i(s) & \pi'_i(s) \end{array}\right)\right) \in D \tag{232}$$

for each $s \in \mathbb{R}$. The dynamics of the particle is represented by the differential equation

$$D_i = \Bigg\{ \bigg(\begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix}, \begin{pmatrix} e_i & \mathbf{p}_i \end{pmatrix}, \begin{pmatrix} t_i' \\ \vec{q}_i' \end{pmatrix}, \begin{pmatrix} e_i' & \mathbf{p}_i' \end{pmatrix} \bigg) \in N_i \times P_i \times N_i \times P_i \, ;$$
$$t_i' > 0, \quad \begin{pmatrix} e_i & \mathbf{p}_i \end{pmatrix} \alpha_i = \begin{pmatrix} -\frac{m}{2} \left\| \frac{\vec{q}_i'}{t_i'} \right\|^2 - U_i \begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix} & m \frac{\vec{q}_i' \cdot}{t_i'} \end{pmatrix},$$
$$\begin{pmatrix} e_i' & \mathbf{p}_i' \end{pmatrix} \alpha_i = \begin{pmatrix} -\frac{\partial U_i}{\partial t_i} \begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix} & -\frac{\partial U_i}{\partial \vec{q}_i} \begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix} \end{pmatrix} t_i' \Bigg\} . \tag{233}$$

This equation is derived from the variational equalities

$$\begin{pmatrix} \frac{\partial L_i}{\partial t_i} \left(\begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix}, \begin{pmatrix} t_i' \\ \vec{q}_i' \end{pmatrix} \right) & \frac{\partial L_i}{\partial \vec{q}_i} \left(\begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix}, \begin{pmatrix} t_i' \\ \vec{q}_i' \end{pmatrix} \right) \end{pmatrix} \begin{pmatrix} \delta t_i \\ \delta \vec{q}_i \end{pmatrix}$$
$$= \begin{pmatrix} e_i' & \mathbf{p}_i' \end{pmatrix} \alpha_i \begin{pmatrix} \delta t_i \\ \delta \vec{q}_i \end{pmatrix} \tag{234}$$

and

$$\begin{pmatrix} \frac{\partial L_i}{\partial t_i'} \left(\begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix}, \begin{pmatrix} t_i' \\ \vec{q}_i' \end{pmatrix} \right) & \frac{\partial L_i}{\partial \vec{q}_i'} \left(\begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix}, \begin{pmatrix} t_i' \\ \vec{q}_i' \end{pmatrix} \right) \end{pmatrix} \begin{pmatrix} \delta t_i' \\ \delta \vec{q}_i' \end{pmatrix}$$
$$= \begin{pmatrix} e_i & \mathbf{p}_i \end{pmatrix} \alpha_i \begin{pmatrix} \delta t_i \\ \delta \vec{q}_i \end{pmatrix} . \tag{235}$$

5.5. Lagrangian formulation of dynamics in the absolute space-time-energy-momentum

A space-time-energy-momentum phase space trajectory consists of mappings

$$\gamma : \mathbb{R} \to N \tag{236}$$

and

$$\pi : \mathbb{R} \to P \, . \tag{237}$$

Derivatives are mappings

$$\gamma' : \mathbb{R} \to \overline{N} \tag{238}$$

and

$$\pi' : \mathbb{R} \to \overline{P} \, . \tag{239}$$

Dynamics is governed by the differential equation

$$D = \Bigg\{ (x, \mathbf{p}, x', \mathbf{p}') \in N \times P \times \overline{N} \times \overline{P};$$
$$\tau(x') > 0 \, , \; \alpha(\mathbf{p}) = \frac{\partial L}{\partial x'}(x, x') \, , \; \overline{\alpha}(\mathbf{p}') = \frac{\partial L}{\partial x}(x, x') \Bigg\} \tag{240}$$

derived from the variational equalities

$$\overline{\left\langle \frac{\partial L}{\partial x}(x, x'), \delta x \right\rangle}^{\natural} = \overline{\langle \mathbf{p}', \delta x \rangle}^{\sharp} = \overline{\langle \overline{\alpha}(\mathbf{p}'), \delta x \rangle}^{\natural} \tag{241}$$

and

$$\left\langle \frac{\partial L}{\partial x'}(x,x'), \delta x' \right\rangle^{\natural} = \langle \mathbf{p}, \delta x' \rangle^{\sharp} = \langle \alpha(\mathbf{p}), \delta x' \rangle^{\natural} . \tag{242}$$

5.6. The Legendre transformation

For each observer O_i there is the function

$$\begin{aligned} E_i &\colon N_i \times N_i \times P_i \to \mathbb{R} \\ &: \left(\begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix}, \begin{pmatrix} t'_i \\ \vec{q}'_i \end{pmatrix}, \begin{pmatrix} e_i & \mathbf{p}_i \end{pmatrix} \right) \mapsto \begin{pmatrix} e_i & \mathbf{p}_i \end{pmatrix} \alpha_i \begin{pmatrix} t'_i \\ \vec{q}'_i \end{pmatrix} \\ &\qquad - \frac{m}{2} \left\| \frac{\vec{q}'_i}{t'_i} \right\|^2 t'_i + U_i \begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix} t'_i \end{aligned} \tag{243}$$

with a positive argument t'_i.

Compatibility of the family of such functions with the groupoids

$$\nu_{ij} \times \overline{\nu}_{ij} \times \pi_{ij} : N_j \times N_j \times P_j \to N_i \times N_i \times P_i \tag{244}$$

follows from

$$\begin{aligned} &E_j \left(\nu_{ji} \begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix}, \overline{\nu}_{ji} \begin{pmatrix} t'_i \\ \vec{q}'_i \end{pmatrix}, \pi_{ji} \begin{pmatrix} e_i & \mathbf{p}_i \end{pmatrix} \right) \\ &= \pi_{ji} \begin{pmatrix} e_i & \mathbf{p}_i \end{pmatrix} \alpha_j \overline{\nu}_{ji} \begin{pmatrix} t'_i \\ \vec{q}'_i \end{pmatrix} - \frac{m}{2} \left\| \vec{v}_{ji} + \lambda_{ji} \left(\frac{\vec{q}'_i}{t'_i} \right) \right\|^2 t'_i \\ &\quad + U_j \left(\nu_{ji} \begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix} \right) t'_i \\ &= \left(\begin{pmatrix} e_i & \mathbf{p}_i \end{pmatrix} \alpha_i \overline{\nu}_{ij} + \begin{pmatrix} -\frac{m}{2} \|\vec{v}_{ij}\|^2 & m\vec{v}_{ji} \cdot \end{pmatrix} \right) \overline{\nu}_{ji} \begin{pmatrix} t'_i \\ \vec{q}'_i \end{pmatrix} \\ &\quad - \frac{m}{2} \left\| \vec{v}_{ji} + \lambda_{ji} \left(\frac{\vec{q}'_i}{t'_i} \right) \right\|^2 t'_i + U_j \left(\nu_{ji} \begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix} \right) t'_i \\ &= \begin{pmatrix} e_i & \mathbf{p}_i \end{pmatrix} \alpha_i \begin{pmatrix} t'_i \\ \vec{q}'_i \end{pmatrix} + \begin{pmatrix} -\frac{m}{2} \|\vec{v}_{ij}\|^2 & m\vec{v}_{ji} \cdot \end{pmatrix} \begin{pmatrix} t'_i \\ t'_i \vec{v}_{ji} + \lambda_{ji} \vec{q}'_i \end{pmatrix} \\ &\quad - \frac{m}{2} \|\vec{v}_{ij}\|^2 t'_i - \frac{m}{2} \left\| \frac{\vec{q}'_i}{t'_i} \right\|^2 t'_i + m\vec{v}_{ij} \cdot \vec{q}'_i + U_i \begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix} t'_i \\ &= \begin{pmatrix} e_i & \mathbf{p}_i \end{pmatrix} \alpha_i \begin{pmatrix} t'_i \\ \vec{q}'_i \end{pmatrix} - \frac{m}{2} \left\| \frac{\vec{q}'_i}{t'_i} \right\|^2 t'_i + U_i \begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix} t'_i \\ &= E_i \left(\begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix}, \begin{pmatrix} t'_i \\ \vec{q}'_i \end{pmatrix}, \begin{pmatrix} e_i & \mathbf{p}_i \end{pmatrix} \right). \end{aligned} \tag{245}$$

A function

$$E : N \times \overline{N} \times P \to \mathbb{R} \tag{246}$$

is defined by

$$E = E_i \circ (\nu_i \times \overline{\nu}_i \times \pi_i) . \tag{247}$$

The formula

$$E = pr_{\mathbb{R}} \circ \widetilde{E} \tag{248}$$

with

$$\widetilde{E} : N \times \overline{N} \times P \to \overline{Y} : (x, x', \mathbf{p}) \mapsto \langle \mathbf{p}, x' \rangle^{\sharp} - L(x, x') \tag{249}$$

gives an alternative definition of the function E.

The differential of the function E_i is represented by mappings

$$\begin{aligned} \begin{pmatrix} \frac{\partial E_i}{\partial t_i} & \frac{\partial E_i}{\partial \vec{q}_i} \end{pmatrix} &: N_i \times N_i \times Pi \to S_i : \left(\begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix}, \begin{pmatrix} t'_i \\ \vec{q}'_i \end{pmatrix}, \begin{pmatrix} e_i & \mathbf{p}_i \end{pmatrix} \right) \\ &\mapsto \begin{pmatrix} \frac{\partial U_i}{\partial t_i} \begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix} t'_i & \frac{\partial U_i}{\partial \vec{q}_i} \begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix} t'_i \end{pmatrix}, \end{aligned} \tag{250}$$

$$\begin{aligned} \begin{pmatrix} \frac{\partial E_i}{\partial t'_i} & \frac{\partial E_i}{\partial \vec{q}'_i} \end{pmatrix} &: N_i \times N_i \times P_i \to S_i : \left(\begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix}, \begin{pmatrix} t'_i \\ \vec{q}'_i \end{pmatrix}, \begin{pmatrix} e_i & \mathbf{p}_i \end{pmatrix} \right) \\ &\mapsto \begin{pmatrix} e_i & \mathbf{p}_i \end{pmatrix} \alpha_i + \begin{pmatrix} \frac{m}{2} \left\| \frac{\vec{q}'_i}{t'_i} \right\|^2 + U_i \begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix} & -m \frac{\vec{q}'_i \cdot}{t'_i} \end{pmatrix}, \end{aligned} \tag{251}$$

and

$$\begin{aligned} \begin{pmatrix} \frac{\partial E_i}{\partial e_i} \\ \frac{\partial E_i}{\partial \mathbf{p}_i} \end{pmatrix} &: N_i \times N_i \times P_i \to N_i \\ &: \left(\begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix}, \begin{pmatrix} t'_i \\ \vec{q}'_i \end{pmatrix}, \begin{pmatrix} e_i & \mathbf{p}_i \end{pmatrix} \right) \mapsto \alpha_i \begin{pmatrix} t'_i \\ \vec{q}'_i \end{pmatrix}. \end{aligned} \tag{252}$$

The first two of these mappings are compatible with the groupoids

$$\nu_{ij} \times \overline{\nu}_{ij} \times \pi_{ij} : N_j \times N_j \times P_j \to N_i \times N_i \times P_i \tag{253}$$

and

$$\overline{\sigma}_{ij} : S_j \to S_i \,. \tag{254}$$

The third is compatible with the groupoids

$$\nu_{ij} \times \overline{\nu}_{ij} \times \pi_{ij} : N_j \times N_j \times P_j \to N_i \times N_i \times P_i \tag{255}$$

and

$$\overline{\nu}_{ij} : N_j \to N_i \,. \tag{256}$$

Compatibilities are established by

$$
\begin{aligned}
&\overline{\sigma}_{ij}\left(\left(\begin{array}{cc}\frac{\partial E_j}{\partial t_j} & \frac{\partial E_j}{\partial \vec{q}_j}\end{array}\right)\left(\nu_{ji}\left(\begin{array}{c}t_i\\ \vec{q}_i\end{array}\right),\overline{\nu}_{ji}\left(\begin{array}{c}t_i'\\ \vec{q}_i'\end{array}\right)\right),\pi_{ji}\left(\begin{array}{cc}e_i & \mathbf{p}_i\end{array}\right)\right)\\
&\quad=\overline{\sigma}_{ij}\left(\begin{array}{cc}\frac{\partial U_j}{\partial t_j}\left(\nu_{ji}\left(\begin{array}{c}t_i\\ \vec{q}_i\end{array}\right)\right)t_i' & \frac{\partial U_j}{\partial \vec{q}_j}\left(\nu_{ji}\left(\begin{array}{c}t_i\\ \vec{q}_i\end{array}\right)\right)t_i'\end{array}\right)\\
&\quad=\left(\begin{array}{cc}\frac{\partial U_j}{\partial t_j}\left(\nu_{ji}\left(\begin{array}{c}t_i\\ \vec{q}_i\end{array}\right)\right)t_i' & \frac{\partial U_j}{\partial \vec{q}_j}\left(\nu_{ji}\left(\begin{array}{c}t_i\\ \vec{q}_i\end{array}\right)\right)t_i'\end{array}\right)\overline{\nu}_{ji}\\
&\quad=\left(\begin{array}{cc}\frac{\partial U_i}{\partial t_i}\left(\begin{array}{c}t_i\\ \vec{q}_i\end{array}\right)t_i' & \frac{\partial U_i}{\partial \vec{q}_i}\left(\begin{array}{c}t_i\\ \vec{q}_i\end{array}\right)t_i'\end{array}\right)\\
&\quad=\left(\begin{array}{cc}\frac{\partial E_i}{\partial t_i} & \frac{\partial E_i}{\partial \vec{q}_i}\end{array}\right)\left(\left(\begin{array}{c}t_i\\ \vec{q}_i\end{array}\right),\left(\begin{array}{c}t_i'\\ \vec{q}_i'\end{array}\right),\left(\begin{array}{cc}e_i & \mathbf{p}_i\end{array}\right)\right),
\end{aligned}
\tag{257}
$$

$$
\begin{aligned}
&\overline{\sigma}_{ij}\left(\left(\begin{array}{cc}\frac{\partial E_j}{\partial t_j'} & \frac{\partial E_j}{\partial \vec{q}_j'}\end{array}\right)\left(\nu_{ji}\left(\begin{array}{c}t_i\\ \vec{q}_i\end{array}\right),\overline{\nu}_{ji}\left(\begin{array}{c}t_i'\\ \vec{q}_i'\end{array}\right),\pi_{ji}\left(\begin{array}{cc}e_i & \mathbf{p}_i\end{array}\right)\right)\right)\\
&\quad=\overline{\sigma}_{ij}\Big(\pi_{ji}\left(\begin{array}{cc}e_i & \mathbf{p}_i\end{array}\right)\alpha_j\Big)\\
&\qquad+\overline{\sigma}_{ij}\left(\begin{array}{cc}\frac{m}{2}\left\|\vec{v}_{ji}+\lambda_{ji}\left(\frac{\vec{q}_i'}{t_i'}\right)\right\|^2+U_j\left(\nu_{ji}\left(\begin{array}{c}t_i\\ \vec{q}_i\end{array}\right)\right) & -m\left(\vec{v}_{ji}\cdot+\frac{\vec{q}_i'\cdot}{t_i'}\lambda_{ij}\right)\end{array}\right)\\
&\quad=\left(\begin{array}{cc}e_i & \mathbf{p}_i\end{array}\right)\alpha_i+\left(\begin{array}{cc}\frac{m}{2}\left\|\frac{\vec{q}_i'}{t_i'}\right\|^2+U_i\left(\begin{array}{c}t_i\\ \vec{q}_i\end{array}\right) & -m\frac{\vec{q}_i'\cdot}{t_i'}\end{array}\right)\\
&\quad=\left(\begin{array}{cc}\frac{\partial E_i}{\partial t_i'} & \frac{\partial E_i}{\partial \vec{q}_i'}\end{array}\right)\left(\left(\begin{array}{c}t_i\\ \vec{q}_i\end{array}\right),\left(\begin{array}{c}t_i'\\ \vec{q}_i'\end{array}\right),\left(\begin{array}{cc}e_i & \mathbf{p}_i\end{array}\right)\right)
\end{aligned}
\tag{258}
$$

and

$$
\begin{aligned}
\overline{\nu}_{ij}\left(\left(\begin{array}{c}\frac{\partial E_i}{\partial e_i}\\ \frac{\partial E_i}{\partial \mathbf{p}_i}\end{array}\right)\left(\nu_{ji}\left(\begin{array}{c}t_i\\ \vec{q}_i\end{array}\right),\overline{\nu}_{ji}\left(\begin{array}{c}t_i'\\ \vec{q}_i'\end{array}\right),\pi_{ji}\left(\begin{array}{cc}e_i & \mathbf{p}_i\end{array}\right)\right)\right)&=\overline{\nu}_{ij}\alpha_j\overline{\nu}_{ji}\left(\begin{array}{c}t_i'\\ \vec{q}_i'\end{array}\right)\\
&=\alpha_i\left(\begin{array}{c}t_i'\\ \vec{q}_i'\end{array}\right).
\end{aligned}
\tag{259}
$$

An absolute version of the differential is represented by mappings

$$
\frac{\partial E}{\partial x}=\overline{\sigma}_i^{-1}\circ\left(\begin{array}{cc}\frac{\partial E_i}{\partial t_i} & \frac{\partial E_i}{\partial \vec{q}_i}\end{array}\right)\circ(\nu_i\times\overline{\nu}_i\times\pi_i):N\times\overline{N}\times P\to\overline{S}\,,
\tag{260}
$$

$$
\frac{\partial E}{\partial x'}=\overline{\sigma}_i^{-1}\circ\left(\begin{array}{cc}\frac{\partial E_i}{\partial t_i'} & \frac{\partial E_i}{\partial \vec{q}_i'}\end{array}\right)\circ(\nu_i\times\overline{\nu}_i\times\pi_i):N\times\overline{N}\times P\to\overline{S}\,,
\tag{261}
$$

and

$$
\frac{\partial E}{\partial \mathbf{p}}=\overline{\nu}_i^{-1}\circ\left(\begin{array}{c}\frac{\partial E_i}{\partial e_i}\\ \frac{\partial E_i}{\partial \mathbf{p}_i}\end{array}\right)\circ(\nu_i\times\overline{\nu}_i\times\pi_i):N\times\overline{N}\times P\to\overline{N}\,.
\tag{262}
$$

Alternative definitions

$$\frac{\partial E}{\partial x}(x, x', \mathbf{p}) = \frac{\partial U}{\partial x}(x) \in \overline{S}\,, \tag{263}$$

$$\frac{\partial E}{\partial x'}(x, x', \mathbf{p}) = \alpha(\mathbf{p}) - \frac{\partial L}{\partial x'}(x, x') \in \overline{S}\,, \tag{264}$$

and

$$\frac{\partial E}{\partial \mathbf{p}}(x, x', \mathbf{p}) = \overline{\alpha}^*(x') \in \overline{N} \tag{265}$$

are possible.

There is a family of projections

$$\begin{aligned}\psi_i\colon & N_i \times N_i \times P_i \to N_i \times \mathbb{R} \times P_i \\ : & \left(\begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix}, \begin{pmatrix} t'_i \\ \vec{q}'_i \end{pmatrix}, \begin{pmatrix} e_i & \mathbf{p}_i \end{pmatrix}\right) \mapsto \left(\begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix}, t'_i, \begin{pmatrix} e_i & \mathbf{p}_i \end{pmatrix}\right).\end{aligned} \tag{266}$$

The mapping

$$\psi : N \times \overline{N} \times P \to N \times \mathbb{R} \times P : (x, x', \mathbf{p}) \mapsto \big(x, \tau(x'), \mathbf{p}\big) \tag{267}$$

is the corresponding abstract version. Each function E_i gives rise to a family (E_i, ψ_i) of functions defined on fibres of the projection ψ_i and the function E is a family of functions on fibres of the projection ψ. The set

$$\begin{aligned}Cr(E_i, \psi_i) &= \Bigg\{\left(\begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix}, \begin{pmatrix} t'_i \\ \vec{q}'_i \end{pmatrix}, \begin{pmatrix} e_i & \mathbf{p}_i \end{pmatrix}\right) \in N_i \times N_i \times P_i; \\ &\qquad \forall_{\delta\vec{q}'_i} \begin{pmatrix} \frac{\partial E_i}{\partial t'_i} & \frac{\partial E_i}{\partial \vec{q}'_i} \end{pmatrix} \begin{pmatrix} 0 \\ \delta\vec{q}'_i \end{pmatrix} = 0\Bigg\} \\ &= \Bigg\{\left(\begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix}, \begin{pmatrix} t'_i \\ \vec{q}'_i \end{pmatrix}, \begin{pmatrix} e_i & \mathbf{p}_i \end{pmatrix}\right) \in N_i \times N_i \times P_i; \\ &\qquad \mathbf{p}_i = m\frac{\vec{q}'_i\,\cdot}{t'_i}\Bigg\}\end{aligned} \tag{268}$$

is the critical set of the family (E_i, ψ_i). The definition of this set is compatible with the groupoids

$$\pi_{ij} : P_j \to P_i \tag{269}$$

and

$$\overline{\nu}_{ij} : N_j \to N_i \tag{270}$$

since

$$\mathbf{p}_j = \mathbf{p}_i\lambda_{ij} + m\vec{v}_{ji}\,\cdot \tag{271}$$

from

$$\begin{pmatrix} e_j & \mathbf{p}_j \end{pmatrix} = \Big(\begin{pmatrix} e_i & \mathbf{p}_i \end{pmatrix}\alpha_i\overline{\nu}_{ij} + \begin{pmatrix} -\frac{m}{2}\,\|\vec{v}_{ij}\|^2 & m\vec{v}_{ji}\,\cdot \end{pmatrix}\Big)\alpha_j^{-1} \tag{272}$$

and

$$m\frac{\vec{q}'_j\,\cdot}{t'_j} = m\vec{v}_{ji} + m\frac{\vec{q}'_i\,\cdot}{t'_i}\lambda_{ij} \tag{273}$$

from

$$\begin{pmatrix} t'_j \\ \vec{q}'_j \end{pmatrix} = \begin{pmatrix} t'_i \\ t'_i\vec{v}_{ji} + \lambda_{ji}\vec{q}'_i \end{pmatrix}. \tag{274}$$

The set

$$\begin{aligned} Cr(E,\psi) &= \left(\nu_i^{-1} \times \overline{\nu}_i^{-1} \times \pi_i^{-1}\right)\left(Cr(E_i,\psi_i)\right) \\ &= \Big\{ (x,x',\mathbf{p}) \in N \times \overline{N} \times P;\ \forall_{\delta x' \in \overline{N}} \quad \text{if} \quad \tau(\delta x') = 0, \\ &\qquad\qquad \text{then } \left\langle \frac{\partial E}{\partial x'}(x,x',\mathbf{p}), \delta x' \right\rangle = 0 \Big\} \end{aligned} \tag{275}$$

is the critical set of the family (E,ψ).

The critical set $Cr(E_i,\psi_i)$ is the image of the section

$$\begin{aligned} \rho_i &: N_i \times \mathbb{R} \times P_i \to N_i \times N_i \times P_i \\ &: \left(\begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix}, t'_i, \begin{pmatrix} e_i & \mathbf{p}_i \end{pmatrix} \right) \mapsto \left(\begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix}, t'_i \begin{pmatrix} 1 \\ \frac{1}{m}\vec{p}_i \end{pmatrix}, \begin{pmatrix} e_i & \mathbf{p}_i \end{pmatrix} \right) \end{aligned} \tag{276}$$

of the fibration ψ_i. We are denoting by $\vec{p}_i$ the vector $h_i^{-1}(\mathbf{p}_i) \in Q_i$. The function

$$\begin{aligned} H_i = E_i \circ \rho_i &: N_i \times \mathbb{R}_+ \times P_i \to \mathbb{R} : \left(\begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix}, \lambda_i, \begin{pmatrix} e_i & \mathbf{p}_i \end{pmatrix} \right) \\ &\mapsto \left(\frac{1}{2m}\left\|\mathbf{p}_i\right\|^2 + U_i \begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix} - e_i \right)\lambda_i \end{aligned} \tag{277}$$

interpreted as fhe family (H_i,ζ_i) of functions defined on fibres of the projection

$$\begin{aligned} \zeta_i &: N_i \times \mathbb{R}_+ \times P_i \to N_i \times P_i \\ &: \left(\begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix}, \lambda_i, \begin{pmatrix} e_i & \mathbf{p}_i \end{pmatrix} \right) \mapsto \left(\begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix}, \begin{pmatrix} e_i & \mathbf{p}_i \end{pmatrix} \right) \end{aligned} \tag{278}$$

is the Hamiltonian family.

It follows from

$$\begin{aligned} & H_j\left(\nu_{ji}\begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix}, \lambda_i, \pi_{ji}\begin{pmatrix} e_i & \mathbf{p}_i \end{pmatrix} \right) \\ &= \left(\frac{1}{2m}\left\|\mathbf{p}_i\lambda_{ij} + m\vec{v}_{ji}\cdot\right\|^2 + U_j\left(\nu_{ji}\begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix}\right) - e_i + \mathbf{p}_i\vec{v}_{ij} - \frac{m}{2}\left\|\vec{v}_{ij}\right\|^2 \right)\lambda_i \\ &= \left(\frac{1}{2m}\left\|\mathbf{p}_i\right\|^2 + U_i\begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix} - e_i \right)\lambda_i \\ &= H_i\left(\begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix}, \lambda_i, \begin{pmatrix} e_i & \mathbf{p}_i \end{pmatrix} \right) \end{aligned} \tag{279}$$

that the functions H_i are compatible with the groupoid

$$\nu_{ij} \times 1_{\mathbb{R}_+} \times \pi_{ij} : N_j \times \mathbb{R}_+ \times P_j \to N_i \times \mathbb{R}_+ \times P_i\,. \tag{280}$$

The function

$$H : N \times \mathbb{R}_+ \times P \to \mathbb{R} : (x, \lambda, \mathbf{p}) \mapsto H_i\big(\nu_i(x), t', \pi_i(\mathbf{p})\big) \tag{281}$$

defines the Hamiltonian family (H, ζ) of functions on fibres of the projection

$$\zeta : N \times \mathbb{R}_+ \times P \to N \times P : \ (x, \lambda, \mathbf{p}) \mapsto (x, \mathbf{p}) \, . \tag{282}$$

The passage from the Lagrangian to the Hamiltonian family is the Legendre transformation.

5.7. Hamiltonian formulation of mechanics

The set

$$\begin{aligned} D_i = \Bigg\{ \Bigg(\begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix}, \begin{pmatrix} e_i & \mathbf{p}_i \end{pmatrix}, \begin{pmatrix} t'_i \\ \vec{q}'_i \end{pmatrix}, \begin{pmatrix} e'_i & \mathbf{p}'_i \end{pmatrix} \Bigg) \in N_i \times P_i \times N_i \times P_i \, ; \\ e_i = \frac{1}{2m} \|\mathbf{p}_i\|^2 + U_i \begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix}, \ t'_i > 0 \, , \ \vec{q}'_i = \frac{1}{m} \vec{p}_i t'_i \, , \\ \begin{pmatrix} e'_i & \mathbf{p}'_i \end{pmatrix} \alpha_i = \begin{pmatrix} -\frac{\partial U_i}{\partial t_i} \begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix} & -\frac{\partial U_i}{\partial \vec{q}_i} \begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix} \end{pmatrix} t'_i \Bigg\} \end{aligned} \tag{283}$$

represents the dynamics reformulated in Hamiltonian terms. It is derived from variational equalities

$$\begin{aligned} \begin{pmatrix} \frac{\partial H_i}{\partial t_i} \Bigg(\begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix}, \lambda_i, \begin{pmatrix} e_i & \mathbf{p}_i \end{pmatrix} \Bigg) & \frac{\partial H_i}{\partial \vec{q}_i} \Bigg(\begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix}, \lambda_i, \begin{pmatrix} e_i & \mathbf{p}_i \end{pmatrix} \Bigg) \end{pmatrix} \begin{pmatrix} \delta t_i \\ \delta \vec{q}_i \end{pmatrix} \\ = - \begin{pmatrix} e'_i & \mathbf{p}'_i \end{pmatrix} \alpha_i \begin{pmatrix} \delta t_i \\ \delta \vec{q}_i \end{pmatrix}, \end{aligned} \tag{284}$$

$$\begin{pmatrix} \delta e_i & \delta \mathbf{p}_i \end{pmatrix} \begin{pmatrix} \frac{\partial H_i}{\partial e_i} \Bigg(\begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix}, \lambda_i, \begin{pmatrix} e_i & \mathbf{p}_i \end{pmatrix} \Bigg) \\ \frac{\partial H_i}{\partial \mathbf{p}_i} \Bigg(\begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix}, \lambda_i, \begin{pmatrix} e_i & \mathbf{p}_i \end{pmatrix} \Bigg) \end{pmatrix} = \begin{pmatrix} \delta e_i & \delta \mathbf{p}_i \end{pmatrix} \alpha_i \begin{pmatrix} t'_i \\ \vec{q}'_i \end{pmatrix}, \tag{285}$$

and

$$\frac{\partial H_i}{\partial \lambda_i} \Bigg(\begin{pmatrix} t_i \\ \vec{q}_i \end{pmatrix}, \lambda_i, \begin{pmatrix} e_i & \mathbf{p}_i \end{pmatrix} \Bigg) \delta \lambda_i = 0 \tag{286}$$

to be satisfied for arbitrary variations

$$\begin{pmatrix} \delta t_i \\ \delta \vec{q}_i \end{pmatrix}, \quad \begin{pmatrix} \delta e_i & \delta \mathbf{p}_i \end{pmatrix}, \quad \delta \lambda_i \tag{287}$$

and some value of the parameter λ_i.

The frame independent version of dynamics is the set

$$\begin{aligned} D = \Big\{ (x, \mathbf{p}, x', \mathbf{p}') \in N \times P \times \overline{N} \times \overline{P} \, ; \ \tau(x') > 0 \, , \ \exists_{\lambda > 0}, \\ \overline{\alpha}^*(x') = \frac{\partial H}{\partial \mathbf{p}}(x, \lambda, \mathbf{p}) \, , \ \overline{\alpha}(\mathbf{p}') = -\frac{\partial H}{\partial x}(x, \lambda, \mathbf{p}), \ \frac{\partial H}{\partial \lambda}(x, \lambda, \mathbf{p}) = 0 \Big\} \end{aligned} \tag{288}$$

derived from the equalities

$$\left\langle \frac{\partial H}{\partial x}(x,\lambda,\mathbf{p}),\delta x \right\rangle = -\left\langle \mathbf{p}',\delta x\right\rangle^\flat = -\left\langle \overline{\alpha}(\mathbf{p}'),\delta x\right\rangle, \tag{289}$$

$$\left\langle \delta\mathbf{p},\frac{\partial H}{\partial \mathbf{p}}(x,\lambda,\mathbf{p}) \right\rangle = \left\langle \delta\mathbf{p},x'\right\rangle^\flat = \left\langle \overline{\alpha}(\delta\mathbf{p}),x'\right\rangle, \tag{290}$$

and

$$\frac{\partial H}{\partial \lambda}\,(x,\lambda,\mathbf{p})\,\delta\lambda = 0\,. \tag{291}$$

to be satisfied for arbitrary variations δx, $\delta\mathbf{p}$, $\delta\lambda$ and some value of the parameter λ.

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