Entanglement of two blocks of spins in the critical Ising model

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We compute the entropy of entanglement of two blocks of L spins at a distance d in the ground state of an Ising chain in an external transverse magnetic field. We numerically study the von Neumann entropy for different values of the transverse field. At the critical point we obtain analytical results for blocks of size $L=1$ and 2. In the general case, the critical entropy is shown to be additive when $d\to\infty$. Finally, based on simple arguments, we derive an expression for the entropy at the critical point as a function of both L and d. This formula is in excellent agreement with numerical results.

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I. INTRODUCTION

A comprehension of the features of entanglement in systems with many degrees of freedom, such as quantum spin chains, is currently one of the most challenging problems, at the borderline of quantum-information science [1] and statistical physics. In the last few years several measures of entanglement have been proposed [2–5] and calculated (analytically in the simplest cases, otherwise numerically) for the ground states of many-body systems [6]. States out of equilibrium have also been investigated, see [7].

Despite accurate investigations and different proposals, there is still no consensus on the correct characterization of the *multipartite* entanglement of the ground state of a many-body system. We will consider here the entanglement entropy, a measure that can sometimes be tackled by analytic investigations and for which quantum field theoretical methods can be employed. The entanglement entropy is just the von Neumann entropy associated with the reduced density matrix, that is the entropy of a subsystem of the chain, and was explicitly evaluated for quantum spin chains [4,8–12].

One of the most striking features of the entanglement entropy is its universal behavior at and close to a quantum phase transition. Indeed, it is found that entropy in noncritical systems generally tends to saturate towards a finite value as the size of the subsystem increases, but this value (logarithmically) diverges with the size of the subsystem as the system approaches a quantum critical point. Close to a quantum critical point, where the correlation length ξ is much larger than the lattice spacing, correlations are described by a $(1+1)$-dimensional quantum field theory and at the critical point, where ξ diverges, the field theory is also a *conformal* field theory [13]. In the latter case, the behavior of entropy calculated by analytical and numerical techniques for several spin systems is confirmed by the predictions of the corresponding field theory.

In this work we extend the characterization of the entanglement entropy to a more general subsystem, in which the correlations between two disjoint blocks of spins and the rest of the chain are studied as a function of the distance d between the blocks and their common size L. This entropy of entanglement will be denoted $S(L,d)$ and will be analyzed by analytical and numerical methods.

The physical system we shall consider is the Ising model in a transverse magnetic field, since it fulfills a convenient combination of requirements. It is solvable, its ground state can be computed by using well-known analytical and numerical techniques [8] and, at the same time, it successfully describes a rich spectrum of physical phenomena that include the ordered and disordered magnetic phases, connected by a quantum phase transition [14].

We will analytically compute the entropy of entanglement at the quantum critical point (QCP) for blocks of $L=1$ and 2 spins and will tackle the problem numerically for larger values of L. We will first study the behavior of the entropy as a function of the magnetic field λ, then at the QCP, $\lambda=1$, as a function of the distance d between the blocks and their size L. We will investigate the limits $d\to 0$ and $d\to\infty$. Our results will include as a particular case $(d=0)$ the logarithmic behavior of the entropy of a single block of L spins at criticality $S_L=\frac{1}{6}\log L+\mathcal{K}$, where $\mathcal{K}$ is a constant [4,8–12,15–17]. We will also show the additivity of $S(L,d)$ at the critical point as $d\to\infty$. Finally, we will plot $S(L,d)$ as a function of both L and d, getting an accurate idea of the features of the entropy at the QCP.

This paper is divided in six sections. In Sec. II we review previous works on spin chains, following [4,8,18–21], where the ground state of the Ising model is computed: the explicit expressions obtained will be used in the following sections in order to obtain the reduced density matrix ϱ_L and the Von Neumann entropy S_L of L contiguous spins. In Sec. III we extend the definition of the correlation matrix given in [4,8] to a bipartition of two blocks of L spins separated by a generic distance d. We define here the entropy $S(L,d)$, describing the entanglement of the two blocks with the rest of the chain. By making use of the newly defined reduced density matrix $\varrho_{L,d}$, in Sec. IV, we analytically compute the entropy for blocks of one and two spins at the critical point. In Sec. V we carry out numerical computations of $S(L,d)$ for several sizes of the blocks. Finally, we plot the entropy as a function of the size L of the blocks and their reciprocal distance d, in order to get a general idea of the features of the entropy of

entanglement at the critical point. Our results are summarized and discussed in Sec. VI. In the Appendixes we included, for self-consistency, additional material and explicit calculations.

II. GROUND STATE OF THE ISING MODEL

The Ising chain in a tranverse field consists of $2N+1$ spins with nearest-neighbor interactions and an external magnetic field, described by the Hamiltonian

$$\mathcal{H}_I = -J \sum_{-N\leqslant i\leqslant N} (\lambda \sigma_i^z + \sigma_i^x \sigma_{i+1}^x). \tag{1}$$

Here i labels the spins (we take an odd number of spins for simplicity), $J>0$, and we consider open boundary conditions, $\sigma_{N+1}^x=0$. σ_i^μ $(\mu=x,y,z)$ are the Pauli matrices acting on spin i. The determination of the ground state proceeds with the Jordan-Wigner transformation in terms of Dirac or Majorana fermionic operators [18–21]. Here it will be convenient to consider Majorana fermions, whose operators are defined by

$$\check{a}_{2l-1} \equiv \left(\prod_{m<l} \sigma_m^z\right)\sigma_l^x, \quad \check{a}_{2l} \equiv \left(\prod_{m<l} \sigma_m^z\right)\sigma_l^y, \tag{2}$$

with $-N\leqslant l\leqslant N$. They are Hermitian and obey anticommutation relations,

$$\check{a}_m^\dagger = \check{a}_m, \quad \{\check{a}_m, \check{a}_n\} = 2\delta_{mn}, \tag{3}$$

and their expectation values in the ground state $|\psi_0\rangle$

$$\langle\psi_0|\check{a}_m\check{a}_n|\psi_0\rangle = \langle\check{a}_m\check{a}_n\rangle = \delta_{mn} + i\Gamma_{mn}^A, \tag{4}$$

with $-2N-1\leqslant m, n\leqslant 2N$, completely characterize $|\psi_0\rangle$. Consider now a block of L contiguos spins labeled by i with

$$k \leqslant i \leqslant k+L-1, \tag{5}$$

with $k>-N$ and $k+L-1<N$. The expectation values of the Majorana operators of the block are encoded in the $2L\times 2L$ submatrix

$$(\Gamma_L^A)_{mn} = -i(\langle\check{a}_m\check{a}_n\rangle - \delta_{mn}), \tag{6}$$

with $2k-1\leqslant m$, $n\leqslant 2k+2L-2$.

We are interested in the thermodynamic limit of an infinite chain, $N\to\infty$. In such a limit, the ground state becomes translation invariant, and all correlations inherit such an invariance: $\langle\check{a}_{2m}\check{a}_{2n}\rangle=\langle\check{a}_{2m-1}\check{a}_{2n-1}\rangle=0$, $\forall m, n$ with $m\neq n$, while $\langle\check{a}_{2m-1}\check{a}_{2n}\rangle=ig_{m-n}$ depend only on the difference $m-n$. Therefore the block correlation matrix Γ_L^A becomes independent of k and reads

$$\Gamma_L^A = \begin{pmatrix} \Pi_0 & \Pi_{-1} & \cdots & \Pi_{-L+1} \\ \Pi_1 & \Pi_0 & & \vdots \\ \vdots & & \ddots & \vdots \\ \Pi_{L-1} & \cdots & \cdots & \Pi_0 \end{pmatrix}, \tag{7}$$

with

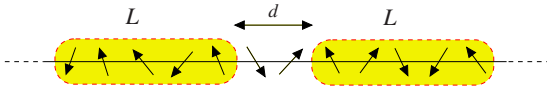

FIG. 1. (Color online) Two blocks of L adjacent spins at a distance d. The state $\varrho_{L,d}$ is obtained from the ground state $|\psi_0\rangle$ of the spin chain by tracing out the spins that do not belong to the blocks.

$$\Pi_l = \begin{pmatrix} 0 & -i\langle\check{a}_{2l-1}\check{a}_0\rangle \\ -i\langle\check{a}_{2l}\check{a}_{-1}\rangle & 0 \end{pmatrix} = \begin{pmatrix} 0 & g_l \\ -g_{-l} & 0 \end{pmatrix}, \tag{8}$$

where the real coefficients g_l are given, for an infinite chain, by

$$g_l = \frac{1}{2\pi}\int_0^{2\pi} d\phi e^{i\phi l} \frac{(\cos\phi-\lambda)+i\sin\phi}{\sqrt{(\cos\phi-\lambda)^2+\sin^2\phi}}. \tag{9}$$

Thus Γ_L^A is a real, skew-symmetic $2L\times 2L$ matrix, i.e., $(\Gamma_L^A)^T=-\Gamma_L^A$, since all the blocks Π_l have the property $(\Pi_l)^T=-\Pi_{-l}$.

III. ENTROPY OF TWO BLOCKS OF SPINS

The entropy of a single block of L contiguous spins in the critical regime can be obtained by very accurate numerical results [4,8] and analytical conformal field theory calculations [9,10]. Given the ground state $|\psi_0\rangle$, one finds the reduced density matrix

$$\varrho_L = \mathrm{Tr}_{\neg L}\{|\psi_0\rangle\langle\psi_0|\}, \tag{10}$$

where the trace is over all spins that do not belong to the block, and its entropy

$$S_L = -\mathrm{Tr}\{\varrho_L \log \varrho_L\}. \tag{11}$$

For definiteness, in this paper we will fix the base of logarithms to 2. The calculations of Refs. [4,8], yielding an expression of the reduced density matrix ϱ_L and the entanglement entropy S_L of L adjacent spins in the ground state $|\psi_0\rangle$ are reviewed in Appendix A. The key point is the following: the block entropy (11) is the sum of L terms

$$S_L = \sum_{l=1}^{L} H\left(\frac{1+\nu_l}{2}\right), \tag{12}$$

where

$$H(x) = -x\log x - (1-x)\log(1-x) \tag{13}$$

is the Shannon entropy of a bit, and $\pm i\nu_l$, with $1\leqslant l\leqslant L$, are the pairs of (purely imaginary) eigenvalues of the block correlation matrix Γ_L^A of Eq. (7).

In the continuous limit

$$S_L = \frac{1}{6}\log L + \mathcal{K}(L), \tag{14}$$

where $1/6$ derives from the central charge $c=1/2$ of a free massless fermionic field and

$$\mathcal{K}(L) = \mathcal{K} + O\left(\frac{1}{L}\right), \quad L \to \infty, \tag{15}$$

$\mathcal{K}$ being a constant. In this section we want to extend this approach and construct the density matrix $\varrho_{L,d}$ of a subsystem of two blocks of L adjacent spins situated at a distance d, studying their entanglement with the rest of the chain. See Fig. 1. To this aim, one starts by computing the matrix Γ^A_L of a single block of adjacent spins (7) and then traces out the central d spins as follows.

We define the $4L \times 4L$ correlation matrix $\Gamma^A_{L,d}$ of two blocks, each of L spins, situated at a distance d (d, like L, are expressed in units of the distance between adjacent spins, and are therefore dimensionless)

$$\Gamma^A_{L,d} = \begin{pmatrix} A^{(L)}_0 & A^{(L)}_{-L-d} \\ A^{(L)}_{L+d} & A^{(L)}_0 \end{pmatrix} = \begin{pmatrix} \Gamma^A_L & A^{(L)}_{-L-d} \\ A^{(L)}_{L+d} & \Gamma^A_L \end{pmatrix}, \tag{16}$$

where $A^{(L)}_x = A^{(L,L)}_x$ with

$$A^{(L,M)}_x = \begin{pmatrix} \Pi_x & \Pi_{x-1} & \cdots & \Pi_{x-M+1} \\ \Pi_{x+1} & \Pi_x & \cdots & \Pi_{x-M+2} \\ \vdots & & \ddots & \vdots \\ \Pi_{x+L-1} & \Pi_{x+L-2} & \cdots & \Pi_{x-M+L} \end{pmatrix}. \tag{17}$$

The matrix $A^{(L)}_x = (-A^{(L)}_{-x})^T$ is a Toeplitz matrix, and $\Gamma^A_{L,d}$ has the property

$$\Gamma^A_{L,0} = \Gamma^A_{2L}, \tag{18}$$

i.e., when the distance between the two blocks is zero, $\Gamma^A_{L,d}$ becomes equal to the matrix (7) of a single block of size $2L$. The matrix $\Gamma^A_{L,d}$ in Eq. (16) is obtained by tracing out the d rows and d columns that are labeled with $L \leqslant x < L+d$ in the $(2L+d) \times (2L+d)$ matrix Γ^A_{2L+d},

$$\Gamma^A_{2L+d} = \begin{pmatrix} A^{(L)}_0 & A^{(L,d)}_{-L} & A^{(L)}_{-L-d} \\ A^{(d,L)}_L & A^{(d)}_0 & A^{(d,L)}_{-d} \\ A^{(L)}_{L+d} & A^{(L,d)}_d & A^{(L)}_0 \end{pmatrix}. \tag{19}$$

For example, let us consider the case of two blocks of $L=2$ spins at a distance $d=3$. In this case the 7×7 matrix $\Gamma^A_{2L+d} = \Gamma^A_7$ reads

$$\Gamma^A_7 = \begin{pmatrix} \Pi_0 & \Pi_{-1} & \Pi_{-2} & \Pi_{-3} & \Pi_{-4} & \Pi_{-5} & \Pi_{-6} \\ \Pi_1 & \Pi_0 & \Pi_{-1} & \Pi_{-2} & \cdots & & \Pi_{-5} \\ \Pi_2 & \Pi_1 & \ddots & & & & \vdots \\ \Pi_3 & \Pi_2 & & \ddots & & & \vdots \\ \Pi_4 & \vdots & & & & & \vdots \\ \Pi_5 & & & & & \ddots & \vdots \\ \Pi_6 & \cdots & & \cdots & & \cdots & \Pi_0 \end{pmatrix} \tag{20}$$

and we have to cancel the columns whose first element is labeled by -2, -3, and -4 and the rows whose first element is labeled by 2, 3, and 4, obtaining

$$\Gamma^A_{2,3} = \begin{pmatrix} \Pi_0 & \Pi_{-1} & \Pi_{-5} & \Pi_{-6} \\ \Pi_1 & \Pi_0 & \Pi_{-4} & \Pi_{-5} \\ \Pi_5 & \Pi_4 & \Pi_0 & \Pi_{-1} \\ \Pi_6 & \Pi_5 & \Pi_1 & \Pi_0 \end{pmatrix}, \tag{21}$$

that is again a real skew-symmetrix matrix.

The entanglement of the two blocks of spins reads

$$S(L,d) = -\operatorname{Tr}(\varrho_{L,d} \log \varrho_{L,d}), \tag{22}$$

where $\varrho_{L,d}$ is the density matrix of two blocks of L adjacent spins at a distance d. Exactly as for a single block, the above entropy can be given an explicit expression in terms of the eigenvalues $\pm i\nu_l$, with $1 \leqslant l \leqslant 2L$, of $\Gamma^A_{L,d}$, analogous to Eq. (12),

$$S(L,d) = \sum_{l=1}^{2L} H\left(\frac{1+\nu_l}{2}\right), \tag{23}$$

with H given by Eq. (13). Before investigating the behavior of Eq. (22), it is instructive to look first at some simple examples.

IV. ANALYTICAL RESULTS

A. Entanglement of two single spins

We consider the Ising chain in a critical transverse magnetic field $\lambda_c = 1$. At $\lambda = \lambda_c$, the coefficients g_l of the reduced correlation matrix $\Gamma^A_{L,d}$, defined in Sec. III, and given by Eq. (9), can be computed analytically, yielding

$$g_l = \frac{1}{2\pi}\int_0^{2\pi} d\phi e^{i\phi l} i e^{i\phi/2} = -\frac{1}{\pi\left(l+\frac{1}{2}\right)}. \tag{24}$$

In order to compute the entanglement entropy $S(L,d)$ of two single spins at a distance d, we have to calculate the eigenvalues of the correlation matrix $\Gamma^A_{L,d}$, where $L=1$ and $d \geqslant 0$. Equation (16) reads

$$\begin{aligned} \Gamma^A_{1,d} &= \begin{pmatrix} \Pi_0 & \Pi_{-(1+d)} \\ \Pi_{1+d} & \Pi_0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & g_0 & 0 & g_{-l} \\ -g_0 & 0 & -g_l & 0 \\ 0 & g_l & 0 & g_0 \\ -g_{-l} & 0 & -g_0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \frac{-2}{\pi} & 0 & \frac{-2}{\pi(-2l+1)} \\ \frac{2}{\pi} & 0 & \frac{2}{\pi(2l+1)} & 0 \\ 0 & \frac{-2}{\pi(2l+1)} & 0 & \frac{-2}{\pi} \\ \frac{2}{\pi(-2l+1)} & 0 & \frac{2}{\pi} & 0 \end{pmatrix}, \end{aligned} \tag{25}$$

where $l=d+1$. The eigenvalues are the solutions to the characteristic equation

$$\det(\Gamma^A_{L,d}-\mu)=0. \tag{26}$$

For $L=1$, there are four eigenvalues $\pm i\nu_1$ and $\pm i\nu_2$, with

$$\nu_{1,2}=\frac{2}{\pi}\frac{\sqrt{(4l^2-1)^2+4l^2}\pm 1}{4l^2-1}. \tag{27}$$

The eigenvalues of the reduced density matrix $\varrho_{L,d}$ are $(1\pm\nu_k)/2$ and the von Neumann entropy reads

$$S(1,d)=H\left(\frac{1+\nu_1}{2}\right)+H\left(\frac{1+\nu_2}{2}\right), \tag{28}$$

where H is given by Eq. (13).

It is interesting to consider the cases $d=0$ and $d\gg 1$. In the former case we have $\nu_{1,2}=2(\sqrt{13}\pm 1)/(3\pi)$, hence

$$\begin{aligned}S(1,0)&=H\left(\frac{\sqrt{13}+2}{3\pi}\right)+H\left(\frac{\sqrt{13}}{3\pi}\right)\\&=-\frac{1}{2}\log\left(\frac{1}{16}+\frac{16-7\pi^2}{9\pi^4}\right)\\&\quad-\frac{\sqrt{13}}{3\pi}\log\left(1+\frac{8\sqrt{13}\pi}{16-4\sqrt{13}\pi+3\pi^2}\right)\\&\quad-\frac{1}{3\pi}\log\left(1+\frac{8\pi}{3\pi^2-4\pi-16}\right).\end{aligned} \tag{29}$$

In the latter case, since the eigenvalues $\nu_{1,2}=2/\pi+O(1/d^2)$ coincide in the limit $d\to+\infty$, we have

$$\begin{aligned}S(1,d)&=2H\left(\frac{\pi+2}{2\pi}\right)+O\left(\frac{1}{d^2}\right)\\&=\frac{2}{\pi}\log\left(\frac{\pi-2}{\pi+2}\right)+\log\left(\frac{4\pi^2}{\pi^2-4}\right)+O\left(\frac{1}{d^2}\right).\end{aligned} \tag{30}$$

More on this phenomenon later.

B. Entanglement of two blocks of $L=2$ spins

If $L=2$, we have to compute the eigenvalues of

$$\Gamma^A_{2,d}=\begin{pmatrix}\Pi_0 & \Pi_{-1} & \Pi_{-(2+d)} & \Pi_{-(3+d)}\\ \Pi_1 & \Pi_0 & \Pi_{-(1+d)} & \Pi_{-(2+d)}\\ \Pi_{(2+d)} & \Pi_{(1+d)} & \Pi_0 & \Pi_{-1}\\ \Pi_{(3+d)} & \Pi_{(2+d)} & \Pi_1 & \Pi_0\end{pmatrix}=\begin{pmatrix}\Pi_0 & \Pi_{-1} & \Pi_{-(1+l)} & \Pi_{-(2+l)}\\ \Pi_1 & \Pi_0 & \Pi_{-l} & \Pi_{-(1+l)}\\ \Pi_{(1+l)} & \Pi_l & \Pi_0 & \Pi_{-1}\\ \Pi_{(2+l)} & \Pi_{(1+l)} & \Pi_1 & \Pi_0\end{pmatrix}, \tag{31}$$

with $l=d+1$. The characteristic equation (26) is of eighth degree, but can be reduced to a quartic equation in $t=\mu^2$,

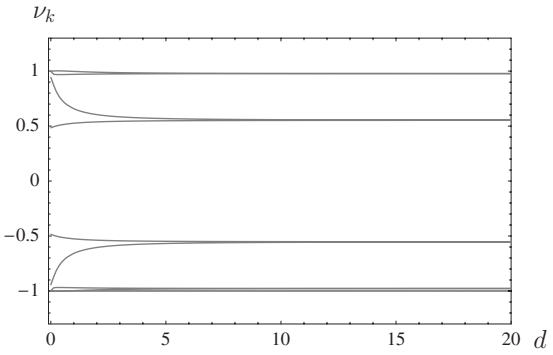

FIG. 2. Eigenvalues of the reduced density matrix of two blocks of $L=2$ spins versus their distance d at the critical point $\lambda_c=1$. Note that the ν_k's reach a saturation value at $d\simeq 5$. Here and in the following figures, d is expressed in units of the distance between adjacent spins, and is therefore dimensionless.

$$t^4+pt^3+qt^2+rt+s=0, \tag{32}$$

which has an exact solution. The coefficients p, q, r, and s are functions of the distance d and are explicitly written in Appendix B.

The matrix $\Gamma^A_{L,d}$ will have the eight eigenvalues $\pm\mu_k=\pm i\nu_k$, with $k=1,2,3,4$, two for each (negative) root of t. The eigenvalues of the reduced density matrix $\varrho_{2,d}$ are $(1\pm\nu_k)/2$ and the entropy reads

$$S(L,d)=\sum_{k=1}^{4}H\left(\frac{1+\nu_k}{2}\right), \tag{33}$$

with H given by Eq. (13).

In Fig. 2 we plot the eigenvalues ν_k versus the distance d. Note that the eigenvalues quickly saturate at a distance $d\simeq 5$ between the blocks. This means that the entanglement between the two $L=2$ blocks reaches its asymptotic value for $d\gtrsim 5$.

The asymptotic values of the eigenvalues are solutions to the equation obtained by taking the limit $d\to\infty$ of Eq. (32). One gets

$$\begin{aligned}\nu_1(\infty)=\nu_3(\infty)&=\frac{2}{\pi}\frac{\sqrt{13}+1}{3},\\ \nu_2(\infty)=\nu_4(\infty)&=\frac{2}{\pi}\frac{\sqrt{13}-1}{3},\end{aligned} \tag{34}$$

in agreement with Fig. 2. Note that they coincide with Eq. (27) evaluated at $l=1$, i.e., with the eigenvalues of the reduced density matrix of two spins at a distance $d=l-1=0$. Therefore for $d\to\infty$ the eigenvalues coalesce into pairs and the spectrum of two blocks (of two spins) coincides with the spectrum (with degeneracy 2) of a single block. This phenomenon, which implies the asymptotic additivity of block entropy, is independent of the blocks dimension L and will be discussed in full generality in Sec. V B 2.

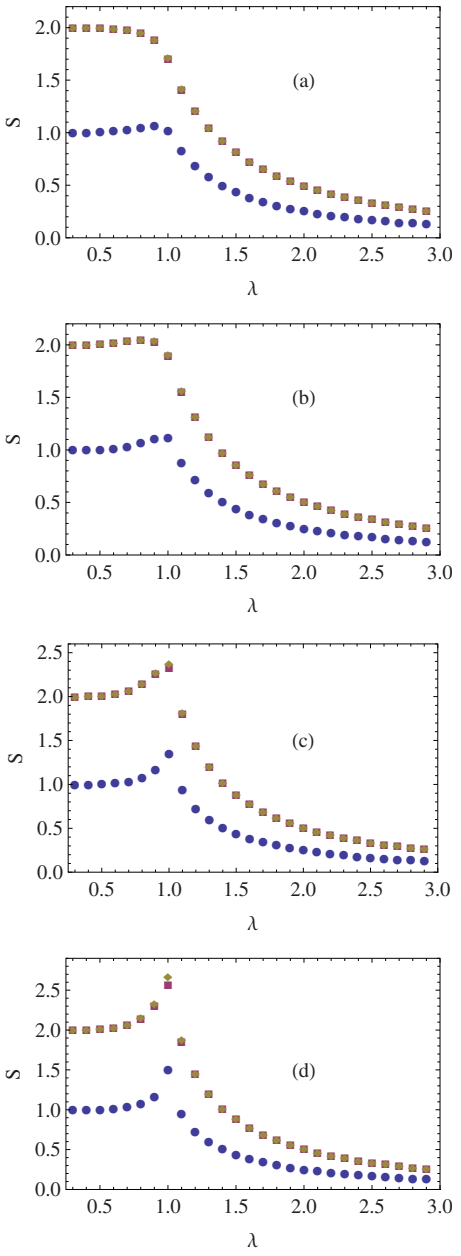

FIG. 3. (Color online) Entropy of the reduced density matrix for blocks of (a) $L=2$, (b) $L=3$, (c) $L=8$, and (d) $L=15$ spins versus the external magnetic field λ for several distances d between the blocks. Circles: $d=0$, squares: $d=10$, and diamonds: $d=50$. Squares and diamonds are indistinguishable in (a) and (b), and are barely distinguishable in (c) and (d) only around the critical value $\lambda=1$.

V. ENTROPY OF TWO BLOCKS OF SPINS

In this section we look at the entanglement entropy $S(L,d)$ of two blocks of spins when the magnetic field varies. At the critical point, $\lambda_c=1$, we will find an expression for $S(L,d)$ in terms of the entropy S_L of a single block, given in Eqs. (11)–(14), and investigate its limits $d\to 0$, $d\to\infty$. We will combine numerical estimates with analytical methods.

A. Entropy versus λ

We start by evaluating the entanglement entropy $S(L,d)$ versus the magnetic field λ. The entanglement between two single spins at a distance d and the remaining part of the chain was investigated in Refs. [23–25] as a function of the magnetic field λ. A generalization to a comb of m spins, spaced d sites apart can be found in [26]. We now generalize these results to the case of two arbitrary blocks of L spins at a distance d.

In general, the presence of a gap between the blocks yields a larger entropy for all values of the magnetic field λ. Let us start examining the situation at zero magnetic field. At $\lambda=0$, from Eq. (9) one gets that $g_l=\delta_{l,-1}$. Therefore, since $\Pi_l=0$ for $l\neq\pm 1$ and $\Pi_{\pm 1}=\sigma^{\mp}=(\sigma_1\mp i\sigma_2)/2$, Eqs. (16) and (17) greatly simplify. In particular, for any $L>1$, the off-diagonal blocks in Eq. (16) read, for $d=0$,

$$A_L^{(L)}=-(A_{-L}^{(L)})^T=\begin{pmatrix}0 & 0 & \dots & \Pi_{+1}\\ \vdots & & \ddots & \vdots\\ 0 & 0 & \dots & 0\end{pmatrix} \tag{35}$$

and yield a tridiagonal block matrix

$$\Gamma_{L,0}^A=\Gamma_{2L}^A=\begin{pmatrix}0 & \Pi_{-1} & 0 & \dots & 0\\ \Pi_{+1} & 0 & \Pi_{-1} & \dots & 0\\ \vdots & \vdots & \ddots & & \vdots\\ 0 & 0 & 0 & \dots & 0\end{pmatrix}, \tag{36}$$

while, for any $d>0$, $A_{L+d}^{(L)}=-(A_{-L-d}^{(L)})^T=0$ and one gets

$$\Gamma_{L,d}^A=\Gamma_L^A\oplus\Gamma_L^A. \tag{37}$$

Thus at $d=0$, the characteristic polynomial is

$$\det(\Gamma_{L,0}^A-\mu)=\det(\Gamma_{2L}^A-\mu)=\mu^2(\mu^2+1)^{2L-1}, \tag{38}$$

hence $\nu_1=0$ and $\nu_l=1$ for $2\leqslant l\leqslant 2L$, so that

$$S(L,0)=H\left(\frac{1}{2}\right)+\sum_{l=2}^{2L}H(1)=H\left(\frac{1}{2}\right)=\log 2=1. \tag{39}$$

On the other hand, for $d\geqslant 1$, one gets

$$\det(\Gamma_{L,d}^A-\mu)=\det(\Gamma_L^A-\mu)^2=\mu^4(\mu^2+1)^{2L-2}, \tag{40}$$

hence $\nu_1=\nu_2=0$ and $\nu_l=1$ for $3\leqslant l\leqslant 2L$ and

$$S(L,d)=2S(L,0)=2H\left(\frac{1}{2}\right)=2. \tag{41}$$

This is intuitively clear: at zero transverse field $S(L,d)$ follows exactly an area law [27] and in one dimension the presence of a gap doubles the area of the boundary, doubling the entropy. See Fig. 3. For nonzero values of the magnetic field, there are corrections to the area law, due to correlations between the two blocks. An entropy increase is still natural, but it turns out to be smaller than the factor of 2 that one would naively expect for a doubled boundary. We will show that the factor of 2 can be recovered in the limit of large gap d: this is the phenomenon of asymptotic additivity of entropy mentioned at the end of Sec. IV B. See Sec. V B.

We show the results of some numerical investigations in Fig. 3. Let us first look at the case of small $L(\simeq 2-3)$. For $d>0$ entropy is not maximum at the critical point $\lambda=1$, but rather for some $\lambda<1$. On the other hand, for bigger L, the

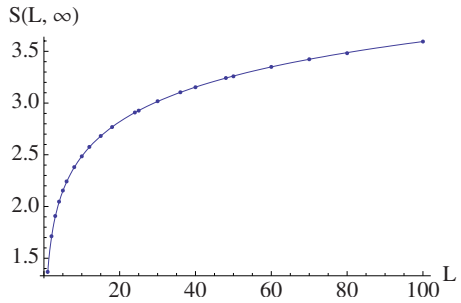

FIG. 4. (Color online) Saturation value of the critical block entropy ($\lambda=1$). The numerical values of $S(L,\infty)$ are fitted by $\frac{1}{3}\log L+2\mathcal{K}(L)$.

maximum entropy is always reached at the critical point $\lambda=1$ and its value grows with the size L of the blocks.

One also notices that, far from criticality, the entropy has a very weak dependence on the value of the gap $d\geqslant 1$, and thus an area law is a very good approximation. The largest deviations are at the critical point, when the correlation length diverges. At fixed L, the ordinate of the cusp at $\lambda=1$ is always an increasing function of d. We now turn to the study of the critical case and endeavor to find some interesting analytical expressions.

B. Critical chain

1. Blocks of contiguous spins ($d\to 0$)

In the limit $d\to 0$, the entropy of entanglement (22) must reproduce the single-block result (11) as a particular case:

$$S(L,0)=S_{2L}. \tag{42}$$

This is a simple consistency check and was numerically verified when the magnetic field is critical, $\lambda_c=1$. As a by-product, this enables us to obtain the value of the constant $\mathcal{K}$ via the logarithmic fit

$$S(L,0)=\frac{1}{6}\log 2L+\mathcal{K}+O\left(\frac{1}{L}\right). \tag{43}$$

We obtain

$$\mathcal{K}=0.690\,413, \tag{44}$$

with an error $\simeq 9\times 10^{-6}$, corroborating the results in [4]. An accurate fit enables us to give a precise estimate of the corrections in $1/L$, but more on this later.

2. Asymptotic additivity of entropy ($d\to\infty$)

The plot in Fig. 4 shows that in the limit $d\to\infty$ the two-block entropy is accurately fitted by

$$S(L,\infty)=\frac{1}{3}\log L+2\mathcal{K}(L)=2S_L. \tag{45}$$

Entropy becomes therefore additive at large distances d. The physical meaning of this result is that the entanglement entropy of two separated blocks of L spins becomes twice the entropy of a single block L when the distance between the blocks approaches ∞, i.e., it becomes much larger than the size of a block. Therefore, at the critical point, the quantum correlations between two finite blocks of spins saturate at a certain distance. We now explain this result, which turns out to be valid for every value of the magnetic field λ.

Note that from Eq. (9) one gets

$$\lim_{l\to\infty} g_l=0 \tag{46}$$

by the Riemann-Lesbegue lemma. Thus from Eq. (8)

$$\lim_{l\to\infty}\Pi_l=0, \tag{47}$$

and from Eq. (17)

$$\lim_{x\to\infty} A_x^{(L)}=0,\quad \forall\, L. \tag{48}$$

$A_x^{(L)}$ accounts for the residual correlation of two blocks of spin at a distance x.

Therefore the limit of the reduced correlation matrix of two blocks (16) reads

$$\begin{aligned}\Gamma_{L,\infty}^A&=\lim_{d\to+\infty}\Gamma_{L,d}^A\\&=\lim_{d\to+\infty}\begin{pmatrix}A_0^{(L)} & A_{-L-d}^{(L)}\\ A_{L+d}^{(L)} & A_0^{(L)}\end{pmatrix}\\&=\begin{pmatrix}A_0^{(L)} & 0\\ 0 & A_0^{(L)}\end{pmatrix}\\&=\Gamma_L^A\oplus\Gamma_L^A,\end{aligned} \tag{49}$$

for all L. The two blocks become independent and the limiting spectrum of $\Gamma_{L,d}^A$ is given by the spectrum of a single block Γ_L^A, with degeneracy 2. As a consequence, the entropy (23) becomes additive in the limit

$$S(L,\infty)=\sum_{l=1}^{2L}H\left(\frac{1+\nu_l(\Gamma_{L,\infty}^A)}{2}\right)=2\sum_{l=1}^{L}H\left(\frac{1+\nu_l(\Gamma_L^A)}{2}\right)=2S_L, \tag{50}$$

where $\pm i\nu_l(\Gamma_{L,d}^A)$ with $1\leqslant l\leqslant 2L$ denote the eigenvalues of $\Gamma_{L,d}^A$ and $\pm i\nu_m(\Gamma_L^A)$ with $1\leqslant m\leqslant L$ denote the eigenvalues of Γ_L^A. We stress again that these results are valid for all values of the magnetic field λ. See the introductory comments in Sec. V A. The area law discussed there is restored for sufficiently large d when the correlations between the two blocks are negligible and the two boundaries become "independent."

3. General behavior of S(L,d) at the critical point

We now turn to the problem of describing the entanglement of two blocks of L spin at an arbitrary distance d with the remaining part of the critical Ising chain. In order to find a function of L and d we make two assumptions: we require that the entropy be a function of *all* the scales of the problem; moreover, the dependence must be logarithmic.

Let therefore

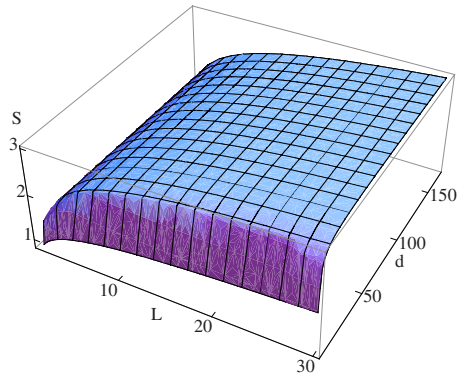

FIG. 5. (Color online) Critical entropy ($\lambda=1$) between two blocks of L spins at a distance d and the remaning part of the (infinite) chain.

$$S(L,d)=\frac{1}{6}[2\log(L-\alpha)-2\log(L+d)+\log(2L+d-\alpha)+\log(d+\alpha)+\beta], \tag{51}$$

with $0<\alpha<1$ and $\beta\in\mathbb{R}$. See Fig. 1. The quantity α fixes the position of the end of each block, $\alpha=0$ corresponding to the central point between two adjacent spins, while $\alpha=1$ to the position of the last (or first) spin. Clearly, α detects granularity in the chain and the corrections due to α will be important for small values of L and/or d. We obtain

$$S(L,0)=\frac{1}{6}\log(2L-\alpha)+\frac{1}{3}\log\left(1-\frac{\alpha}{L}\right)+\frac{1}{6}(\log\alpha+\beta)\sim\frac{1}{6}\log(2L)+\frac{1}{6}(\log\alpha+\beta),\quad L\to\infty, \tag{52}$$

hence

$$\log\alpha+\beta=6\mathcal{K}. \tag{53}$$

On the other hand,

$$S(L,+\infty)=\frac{1}{6}[2\log(L-\alpha)+\beta]\sim\frac{1}{3}\log(L)+\frac{1}{6}\beta,\quad L\to\infty, \tag{54}$$

hence

$$\beta=12\mathcal{K}. \tag{55}$$

In conclusion,

$$S(L,d)=\frac{1}{6}[2\log(L-\alpha)-2\log(L+d)+\log(2L+d-\alpha)+\log(d+\alpha)-2\log\alpha], \tag{56}$$

with

$$\alpha=2^{-6\mathcal{K}}=0.056\,622\,6. \tag{57}$$

Notice that there are *no* free parameters. Moreover, in the realm of validity of CFT, when $d,L\gg1$, one gets

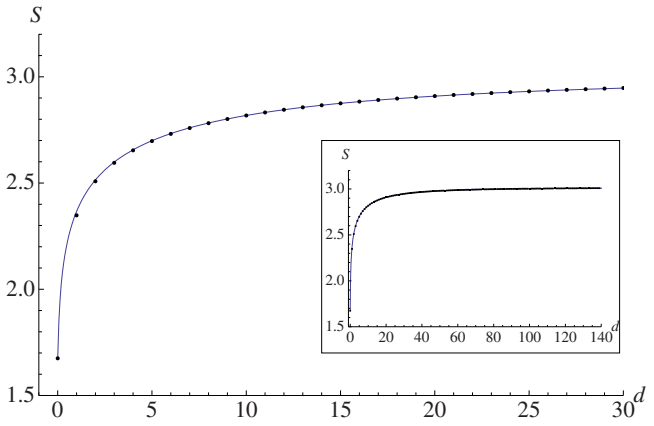

FIG. 6. (Color online) Critical entropy ($\lambda=1$) between two blocks of $L=30$ spins and the remaning part of the chain versus d. Inset: $L=30$ with $0<d<140$.

$$S(L,d)\sim\frac{1}{6}[2\log L-2\log(L+d)+\log(2L+d)+\log d]+\mathcal{K} \tag{58}$$

that agrees with the results of Calabrese and Cardy [10] when one adds a missing addendum in their formula (3.32).

From Eq. (56) we get

$$S(L,0)=\frac{1}{6}\log(2L)+\mathcal{K}(L), \tag{59}$$

where

$$\mathcal{K}(L)=-\frac{1}{6}\log\alpha+\frac{1}{6}\log\left(1-\frac{\alpha}{2L}\right)+\frac{1}{3}\log\left(1-\frac{\alpha}{L}\right)\sim\mathcal{K}-\frac{5}{12}\frac{\alpha}{L}-\frac{3}{16}\frac{\alpha^2}{L^2}-\frac{17}{144}\frac{\alpha^3}{L^3},\quad L\to\infty. \tag{60}$$

This formula is in excellent agreement with numerical results. It provides the explicit expression in Eq. (15) and was used in Eqs. (43) and (44) and Fig. 4. See also the discussion at the end of Sec. V B 1.

The global behavior of

$$\begin{aligned}S(L,d)&=\frac{1}{6}[2\log(L-\alpha)-2\log(L+d)+\log(2L+d-\alpha)+\log(d+\alpha)]\\&=\frac{1}{6}[2\log L-2\log(L+d)+\log(2L+d)+\log d]+\mathcal{K}(L),\end{aligned} \tag{61}$$

with α and $\mathcal{K}(L)$ given by Eqs. (57) and (60), respectively, is displayed in Fig. 5. The fit is accurate up to one part in 10^3 for small L (<10) and one part in 10^6 for $L>10$. Notice the logarithmic L dependence for $d=0$ and the saturation effect for $L/d\ll1$. A section of Fig. 5 is displayed in Fig. 6. In particular, the inset shows the asymptotic behavior of the entropy and its saturation.

4. Behavior of the critical entropy for small d

In Figs. 5 and 6 one notices for all values of L a sharp entropy increase at small values of d from $d=0$ to 1. This corresponds to the two ordinates of the cusps in Fig. 3. Let us endeavor to interpret this phenomenon on the basis of the formulas derived in this section. Equation (56) yields

$$\begin{aligned}\Delta S &= S(L,1) - S(L,0) \\ &= \frac{1}{6}\left(-2\log\frac{L+1}{L} + \log\frac{2L+1-\alpha}{2L-\alpha} + \log\frac{1+\alpha}{\alpha}\right) \\ &\overset{L\gg 1}{\sim} \frac{1}{6}\log\frac{1+\alpha}{\alpha} \simeq -\frac{1}{6}\log\alpha = \mathcal{K},\end{aligned} \tag{62}$$

where we used Eq. (57) in the last equalities. This agrees very well with Fig. 6 and explains why ΔS is largely independent of L in Fig. 5. More to this, the final result in Eq. (62) yields a suggestive interpretation of the fitting parameter α in Eq. (57) and of the constant $\mathcal{K}$: they turn out to be related to the entropy increase ΔS associated with the opening of a $d=1$ gap (one qubit) in an interval of $2L$ contiguous spins. It is therefore not surprising that $\mathcal{K}=-(1/6)\log\alpha$, being $O(\Delta S)$, is also necessarily of order 1.

VI. CONCLUSIONS

We provided an analytic and numerical treatment of the entanglement entropy of two disjoint blocks of spins as a function of their length and distance in the quantum Ising model with a transverse magnetic field. We gave an analytic expression of the entropy at the critical point, for two blocks of 1 and 2 spins at a generic distance. We showed that the presence of a gap always yields an entropy increase. At criticality, due to a logarithmic correction to the area law, this increase is less than a factor of 2 for all values of d, and becomes 2 for large d, when asymptotic additivity takes place. We also showed that, interestingly, the entropy of the two blocks can be written in terms of the entropy of a single block of spins, that, at the quantum phase transition, grows logarithmically with the size of the block. We have also given an accurate idea of the general features of the entropy of two blocks as a function of their size and distance.

The behavior of the entanglement in a critical spin chain agrees with well-known results in conformal field theory, where the geometric entropy (analogous to the spin block entropy, but defined in the continuum) can be computed for $(1+1)$-dimensional theories [15,16]. The translation of field theoretical methods and ideas in the language of quantum information will hopefully enable us to make use of additional results for an arbitrary number of disjoint intervals [10,22]. This would of great interest from the point of view of multipartite entanglement. The study of the statistical distribution of bipartite entanglement for different bipartitions [28] is a useful tool for the analysis of multipartite entanglement. A deeper comprehension of the dependence of entropy on distances and sizes of blocks of spins could yield information about the role of quantum phase transitions in the generation of multipartite entangled states [29,30].

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APPENDIX A: COMPUTATION OF S_L

We review here the computation of the entropy of the reduced density matrix $\varrho_L \equiv \mathrm{Tr}_{\neg L}|\psi_0\rangle\langle\psi_0|$ for L adjacent spins [4,8]. In the limit of infinite chain ($N\to\infty$), a given finite section of the chain is fully translational invariant and ϱ_L describes the state of any block of L contiguous spins. The density matrix ϱ_L can be reconstructed from the restricted $2L\times 2L$ correlation matrix Γ_L^A of Eq. (7). In particular, a direct way to compute the spectrum of ϱ_L and its entropy S_L from Γ_L^A is the following.

The matrix Γ_L^A can be put into a block-diagonal form by an orthogonal transformation and its eigenvalues are purely imaginary and come in pairs, $\pm i\nu_l$, and $|\nu_l|\leqslant 1$, with $1\leqslant l\leqslant L$. Let $V\in \mathrm{SO}(2L)$ be the special orthogonal matrix such that $\Gamma_L^C = V\Gamma_L^A V^T$ is block diagonal,

$$\Gamma_L^C = \bigoplus_{l=1}^{L} \nu_l \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{A1}$$

Then, V defines a new set of Majorana operators, $\check{c}_m^\dagger = \check{c}_m$,

$$\check{c}_m \equiv \sum_{n=1}^{2L} V_{m,n}\check{a}_n, \tag{A2}$$

that satisfy the same anticommutation relations as the $\check{a}_n$'s and have correlation matrix Γ_L^C. The structure of Γ_L^C implies that mode $\check{c}_{2l-1}$ is only correlated to mode $\check{c}_{2l}$. In the language of fermionic operators, one gets L spinless fermionic modes

$$\hat{c}_l \equiv \frac{\check{c}_{2l-1} + i\check{c}_{2l}}{2},$$

$$\{\hat{c}_l, \hat{c}_m\} = 0, \quad \{\hat{c}_l^\dagger, \hat{c}_m\} = \delta_{mn}, \tag{A3}$$

that, by construction, fulfill

$$\langle \hat{c}_l \hat{c}_m\rangle = 0, \quad \langle \hat{c}_l^\dagger \hat{c}_m\rangle = \delta_{lm}\frac{1+\nu_l}{2}. \tag{A4}$$

Thus the L (nonlocal) fermionic modes are *uncorrelated*, so that the reduced density matrix can be written as a product

$$\varrho_L = \rho_1 \otimes \cdots \otimes \rho_L. \tag{A5}$$

Now, the density matrix ρ_l, $1\leqslant l\leqslant L$, has eigenvalues $(1\pm\nu_l)/2$ and entanglement entropy

$$S(\rho_l) = H\left(\frac{1+\nu_l}{2}\right), \tag{A6}$$

where H is the Shannon entropy of a bit (13). Therefore the spectrum of ϱ_L results from the product of the spectra of the density matrices ρ_l, and the entropy of ϱ_L is the sum of the entropies of the L uncorrelated modes,

$$S_L = \sum_{l=1}^{L} H\left(\frac{1+\nu_l}{2}\right). \tag{A7}$$

This is Eq. (12) of the text. Summarizing, for arbitrary values of the magnetic field λ and in the thermodynamic limit, $N \to \infty$, the block entropy S_L of the ground state of the Ising model is given by the sum (A7), where $\pm i\nu_l$ are the pairs of imaginary eigenvalues of the block correlation matrix Γ_L^A of Eq. (7).

APPENDIX B: EIGENVALUES OF $\Gamma_{2,d}^A$

The coefficients in Eq. (32) are functions of the distance d between the blocks and read

$$p(d) = \frac{2^6(3+2d)^4(5+2d)^4}{\pi^2 A}\left[5303 + \frac{24\,314B}{3} + \frac{41\,528B^2}{9} + \frac{10\,144B^3}{9} + \frac{896B^4}{9}\right],$$

$$q(d) = \frac{2^{12}(3+2d)^2(5+2d)^2}{\pi^4 A}\left[203\,297 + 391\,466B + \frac{2\,841\,841B^2}{9} + \frac{3\,652\,160B^3}{27} + \frac{2\,617\,216B^4}{81} + \frac{329\,984B^5}{81} + \frac{17\,152B^6}{81}\right],$$

$$r(d) = \frac{2^{22}(2+d)^4}{\pi^6 A}\left[12\,271 + \frac{68\,116B}{3} + \frac{158\,795B^2}{9} + \frac{198\,074B^3}{27} + \frac{139\,000B^4}{81} + \frac{17\,312B^5}{81} + \frac{896B^6}{81}\right],$$

$$s(d) = \frac{2^{32}}{3^4\pi^8 A}(1+d)^4(2+d)^8(3+d)^4, \tag{B1}$$

where $A=(1+2d)^2(3+2d)^6(5+2d)^6(7+2d)^2$ and $B=d(4+d)$. We obtain μ^2 by solving Eq. (32). The eight eigenvalues of $\Gamma_{2,d}^A$ will be $\pm\mu_k = \pm i\nu_k$, with $0 \leqslant \nu_k \leqslant 1$, for $k=1,2,3,4$.

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