JOURNAL OF MATHEMATICAL PHYSICS **51**, 022103 (2010)

# Quantum Zeno effect and dynamics

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(Received 13 November 2009; accepted 14 December 2009; published online 2 February 2010)

If frequent measurements ascertain whether a quantum system is still in a given subspace, it remains in that subspace and a quantum Zeno effect takes place. The limiting time evolution within the projected subspace is called quantum Zeno dynamics. This phenomenon is related to the limit of a product formula obtained by intertwining the time evolution group with an orthogonal projection. By introducing a novel product formula, we will give a characterization of the quantum Zeno effect for finite-rank projections in terms of a spectral decay property of the Hamiltonian in the range of the projections. Moreover, we will also characterize its limiting quantum Zeno dynamics and exhibit its (not necessarily bounded from below) generator as a generalized mean value Hamiltonian. © *2010 American Institute of Physics.* [doi:10.1063/1.3290971]

## I. INTRODUCTION

Frequent measurements can slow down the evolution of a quantum system and eventually hinder any transition to states different from the initial one. This phenomenon, first considered by Beskow and Nilsson[1] in their study of the decay of unstable systems, was named quantum Zeno effect (QZE) by Misra and Sudarshan,[12] who suggested a parallelism with the paradox of the arrow by the philosopher Zeno of Elea.

Since then, QZE has received constant attention by physicists and mathematicians, who explored different facets of the phenomenon. The whole field is very active. For an up-to-date review of the main mathematical and physical aspects, see Ref. 6 and references therein.

QZE has been observed experimentally in a variety of systems, on experiments involving photons, nuclear spins, ions, optical pumping, photons in a cavity, ultracold atoms, and Bose–Einstein condensates. In all the abovementioned implementations, the quantum system is forced to remain in its initial state through a measurement associated with a one-dimensional projection. No experiment has been performed so far in order to check the multidimensional QZE and the quantum Zeno dynamics (QZD), that is, the effective limiting dynamics inside the measured subspace. However, these ideas might lead to remarkable applications, e.g., in quantum computation and in the control of decoherence.

From the mathematical point of view, QZD is related to the limit of a product formula obtained by intertwining the dynamical time evolution group with the orthogonal projection associated with the measurements performed on the system. It can be viewed as a generalization of Trotter–Kato product formulas[2,10,16,17] to more singular objects in which one semigroup is replaced by a projection.

Since the seminal paper by Misra and Sudarshan,[12] the main object of interest has been the limit of the QZD product formula. Its structure has been thoroughly investigated and has been well

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characterized under quite general hypotheses. In particular, by assuming that the Hamiltonian is bounded from below and the limit is strongly continuous, one obtains a unitary group within the projected subspace.[3,12]

On the other hand, the much more difficult question of the existence of this limit, for infinite dimensional projections and unbounded Hamiltonian, is still open. Since this product formula and its properties are of great importance in the study of quantum dynamical semigroups and have remarkable consequences both in mathematical physics and operator theory, there have been many investigations from different perspectives and motivations. See, for example, Refs. 5, 7–9, 11, 14, and 15.

In 2005, Exner and Ichinose[4] proved the existence of a QZD when the Hamiltonian is positive and the domain of its square root has a dense intersection with the range $\mathcal{H}_P$ of the projection. However, this result was proved in the $L^2_{\mathrm{loc}}(\mathbb{R},\mathcal{H}_P)$ topology, instead of the more natural strong operator topology. As a corollary of the main result, they solved the problem in the norm operator topology when the projections are finite dimensional.

The first main result that we present in this paper is a complete characterization of the multidimensional QZE for Hamiltonians that are not necessarily bounded from below, through the introduction of a novel product formula directly related to the QZE. We show that the existence of the limit is related to a falloff property of the spectral measure of the Hamiltonian in the range of the projection.

Then, we also exhibit a characterization of the QZD in terms of the abovementioned energy fall-off property and of the existence of a mean value Hamiltonian in a generalized sense.

This paper is organized as follows. In Sec. II we discuss the relation between the QZE and its limiting dynamics, in particular, we recall the product formula related to the QZD. Then, we introduce a new product formula that is directly related to the QZE and present our first theorem on the characterization of the existence of its limit. Finally, in the second theorem, we will give a characterization of the related QZD. Moreover, we consider an example that explains the differences between the conditions that imply the QZD and the QZE. The proofs of the theorems are postponed to Sec. III.

## II. QZE VERSUS QZD: RESULTS

Consider a quantum system $Q$, whose states are described by density operators that are positive operators with unit trace, in a complex separable Hilbert space $\mathcal{H}$. The time evolution of the system is governed by a unitary group $U(t)=\exp(-itH)$, where $H$ is a time-independent self-adjoint Hamiltonian. Consider also an orthogonal projection $P$ that describes the measurement process that is performed on $Q$. This kind of measurement ascertains whether the system is in the subspace $\mathcal{H}_P := P\mathcal{H}$. Assume that the initial density operator $\rho_0$ of the system has support in $\mathcal{H}_P$, namely,

$$\rho_0 = P\rho_0 P, \quad \mathrm{tr}(\rho_0 P) = 1.$$

The state of the system at time $\tau$ is

$$\rho(\tau) = U(\tau)\rho_0 U(\tau)^*$$

and after a measurement, if the outcome is positive, it becomes

$$\frac{P\rho(\tau)P}{p(\tau)} = \frac{V(\tau)\rho_0 V(\tau)^*}{p(\tau)},$$

where $V(\tau)=PU(\tau)P$ and $p(\tau)=\mathrm{tr}(V(\tau)\rho_0 V(\tau)^*)$. Observe that, since $P$ is not assumed to commute with the Hamiltonian, when $[P,H]\neq 0$, the unitary evolution drives the system outside $\mathcal{H}_P$, and $p(\tau)$ is, in general, smaller than unity.

If we perform a series of $P$-observations on $Q$ at time $\tau_j = jt/N$, $j\in\{1,\ldots,N\}$, its state after $N$ positive measurements is, up to a normalization,

$$\rho_N(t) = V_N(t)\rho_0 V_N(t)^*,$$

where $V_N(t)=(PU(t/N)P)^N$ and the survival probability in $\mathcal{H}_P$ reads as

$$p_N(t) = \mathrm{tr}(V_N(t)\rho_0 V_N(t)^*). \tag{1}$$

Our interest is focused on the following question: under what conditions

$$p_N(t) \to 1 \quad \text{for } N \to +\infty? \tag{2}$$

Misra and Sudarshan[12] baptized this problem QZE: repeated $P$-observations in succession inhibit transitions outside the observed subspace $\mathcal{H}_P$. That is, rephrasing the Greek philosopher Zeno, the observed quantum arrow does not move.

Since the seminal paper, Ref. 12, the main object of interest has been the limit of the following product formula:

$$V_N(t) = (PU(t/N)P)^N \tag{3}$$

and, in particular, whether $U_Z(t)=\lim_N V_N(t)$ exists and is given by a unitary group in $\mathcal{H}_P$. The existence of a unitary limit is tantamount to the presence of a QZD. If this is the case, one immediately gets

$$\lim_{N\to\infty} p_N(t) = \mathrm{tr}(U_Z(t)\rho_0 U_Z(t)^*) = \mathrm{tr}(P\rho_0) = 1$$

by the cyclic property of the trace. Namely, QZD implies QZE.

The following theorem, due to Exner and Ichinose,[4] about the existence of the limit of the QZD product formula (3) when $P$ is a finite rank projection and $H$ is positive, provides a sufficient condition for the QZE. For a simple proof of this result, see Ref. 5.

**Theorem 1:** (Reference 4) *Let $\mathcal{H}$ be a complex Hilbert space and $H$ a positive self-adjoint operator with dense domain $D(H)\subset\mathcal{H}$. Let $P$ be an orthogonal finite-rank projection onto $\mathcal{H}_P=P\mathcal{H}$. If $\mathcal{H}_P\subset D(H^{1/2})$, where $D(H^{1/2})$ is the domain of the square root of $H$, then*

$$\lim_{N\to+\infty} V_N(t) = P\exp(-it(H^{1/2}P)^*(H^{1/2}P))$$

*uniformly for $t$ in finite intervals of $\mathbb{R}$.*

The hypothesis $\mathcal{H}_P\subset D(H^{1/2})$ on the pair Hamiltonian-projection can be regarded as a condition on the spectral measure of $H$ over the range of $P$ in the following way: for every $\psi\in\mathcal{H}$, one gets

$$\langle H\rangle_{P\psi} \coloneqq (H^{1/2}P\psi, H^{1/2}P\psi) = \int_{[0,+\infty)} \lambda\, \mathrm{d}(P\psi, P_\lambda^H P\psi) < +\infty, \tag{4}$$

where $\{P_\Omega^H\}$ is the projection-valued measure associated with $H$.

Therefore, the above result can be summarized as follows: whenever a positive Hamiltonian has a finite mean value $\langle H\rangle$ on vector states in the range of $P$, frequently $P$-observations force the state of the system to remain in the subspace $\mathcal{H}_P$ and the limiting dynamics in this space is given by the unitary group $U_Z(t)=P\exp(-it(H^{1/2}P)^*(H^{1/2}P))$. As a consequence, finite-energy states exhibit a QZE. One can ask if the sufficient condition (4) is also necessary for the QZE. We will show that the answer to this question is negative. Indeed, the QZE implies a condition weaker than (4) on the spectral measure of $H$.

The first result of this paper is a characterization of the multidimensional QZE. In order to achieve our goal, we look at the problem from a different perspective. Instead of considering the product formula (3), let us move back our attention to Eqs. (1) and (2). By invoking the cyclic property of the trace, we will study the limit of the following product formula:

$$Z_N(t) = V_N(t)^* V_N(t). \tag{5}$$

One gets that the QZE (2) takes place if and only if

$$Z_N(t) \to P \quad \text{for } N \to +\infty. \tag{6}$$

We will call (5) *QZE product formula*, as opposed to the QZD product formula (3).

The next theorem is on the equivalence between the QZE and a certain fall-off condition on the spectral measure associated to the Hamiltonian, which is weaker than (4). Let us denote, as usual, with $o(s)$ an operator-valued function defined in a neighborhood of 0 and such that $\|o(s)\|/s=0$, for $s\to 0$. Let us also use the notation $A^c=\mathbb{R}\setminus A$ for any subset $A\subset\mathbb{R}$.

**Theorem 2:** *Consider a self-adjoint operator $H$ and an orthogonal finite-rank projection $P$ in a complex Hilbert space $\mathcal{H}$. Let $\{P_\Omega^H\}$ be the projection-valued spectral measure of $H$ and $\{U(t)=e^{-itH}\}_{t\in\mathbb{R}}$ the one-parameter unitary group generated by $H$. Consider the product formula $Z_N(t)=V_N(t)^*V_N(t)$, where $V_N(t)=(PU(t/N)P)^N$ with $t\in\mathbb{R}$ and $N\in\mathbb{N}^*$. The following statements are equivalent:*

(1)

$$PP^H_{(-\Lambda,\Lambda)^c}P = o\left(\frac{1}{\Lambda}\right) \quad \text{for } \Lambda \to +\infty,$$

(2)

$$\frac{\mathrm{d}}{\mathrm{d}s} Z_1(s)\big|_{s=0} = 0,$$

(3)

$$\lim_{N\to+\infty} Z_N(t) = P$$

*uniformly for $t$ in finite intervals of $\mathbb{R}$.*

Two comments are now in order. First, by taking the matrix element of condition 1, one obtains that for any $\psi\in\mathcal{H}$

$$(\psi, PP^H_{(-\Lambda,\Lambda)^c}P\psi) = \int_{(-\Lambda,\Lambda)^c} \mathrm{d}(P\psi, P^H_\lambda P\psi) = o\left(\frac{1}{\Lambda}\right),$$

that is,

$$\Lambda \int_{(-\Lambda,\Lambda)^c} \mathrm{d}(P\psi, P^H_\lambda P\psi) \to 0 \quad \text{for } \Lambda \to +\infty. \tag{7}$$

Let us compare (7) with condition (4). Under the hypotheses of Theorem 1, namely, $H\geq 0$ and $\mathcal{H}_P\subset D(H^{1/2})$, one gets

$$\Lambda \int_{(-\Lambda,\Lambda)^c} \mathrm{d}(P\psi, P^H_\lambda P\psi) = \Lambda \int_{[\Lambda,+\infty)} \mathrm{d}(P\psi, P^H_\lambda P\psi) \leq \int_{[\Lambda,+\infty)} \lambda \mathrm{d}(P\psi, P^H_\lambda P\psi) \to 0$$

when $\Lambda\to+\infty$. Therefore, condition (7) is implied by (4), but it is weaker than the latter.

Second, when the measurement projection is one dimensional, we can write $P=\psi(\psi,\cdot)=|\psi\rangle\langle\psi|$ for some $\psi\in\mathcal{H}$ and $\|\psi\|=1$. Physically, this projection checks whether the system is in the pure state $\psi$. In this case, we get

$$V(s) = PU(s)P = \mathcal{A}(s)P, \tag{8}$$

where

$$s \in \mathbb{R} \mapsto \mathcal{A}(s) = (\psi, U(s)\psi) = (\psi, e^{-iHs}\psi)$$

is the *survival probability amplitude* in the state $\psi$. Its associated probability is

$$s \in \mathbb{R} \mapsto p(s) = |\mathcal{A}(s)|^2,$$

and represents the probability of finding at time $s$ in state $\psi$ a system that started in $\psi$ at time 0.

Note that the survival amplitude can be rewritten as

$$\mathcal{A}(s) = \int_{\mathbb{R}} e^{-is\lambda} \mathrm{d}\mu_\psi^H(\lambda),$$

where $\mu_\psi^H(\Omega)=(\psi, P_\Omega^H\psi)$, for every Borel set $\Omega \subset \mathbb{R}$, is the spectral measure of $H$ at $\psi$. Therefore, $\mathcal{A}(s)$ is nothing but the Fourier transform of the spectral measure $\mu_\psi^H$, i.e., a characteristic function, in probabilistic jargon.

Since $V_N(t)=[V(t/N)]^N$, the QZE product formula (5) reads as

$$Z_N(t) = V_N(t)^* V_N(t) = [p(t/N)]^N P.$$

Thus for one-dimensional projections the occurrence of the QZE (6) is equivalent to the limit of the survival probability

$$[p(t/N)]^N \to 1 \quad \text{for } N \to +\infty. \tag{9}$$

Physically, (9) asserts that the system stays frozen in the initial state.

The following proposition, stated for a generic Borel probability measure on $\mathbb{R}$, gives a characterization of the limit (9) in terms of the falloff property of the spectral measure $\mu_\psi^H$ for large energy values. The proof makes use of the equivalent condition of vanishing derivative of the survival probability at $s=0$. Interestingly enough, the first step in the proof of our main Theorem 2 is Proposition 1 (which is a special case of the former!).

*Proposition 1: Let $\mu$ be a Borel measure on $\mathbb{R}$ with $\mu(\mathbb{R})=1$. Define for every $s \in \mathbb{R}$*

$$\mathcal{A}(s) = \int_{\mathbb{R}} e^{-is\lambda} \mathrm{d}\mu(\lambda)$$

*and*

$$p(s) = |\mathcal{A}(s)|^2.$$

*Then, the following assertions are equivalent:*

(1)

$$\mu((-\Lambda,\Lambda)^c) = o\left(\frac{1}{\Lambda}\right) \quad \text{for } \Lambda \to +\infty,$$

(2)

$$p'(0) = 0,$$

(3)

$$\lim_{N\to+\infty} [p(t/N)]^N = 1$$

*uniformly for $t$ in finite intervals of $\mathbb{R}$.*

*Remark 1:* Let $\mu$ be a Borel measure on $\mathbb{R}$ with $\mu(\mathbb{R})=1$. Suppose that $\mu$ satisfies one of the conditions of Proposition 1. Observe that for all $s \in \mathbb{R}$

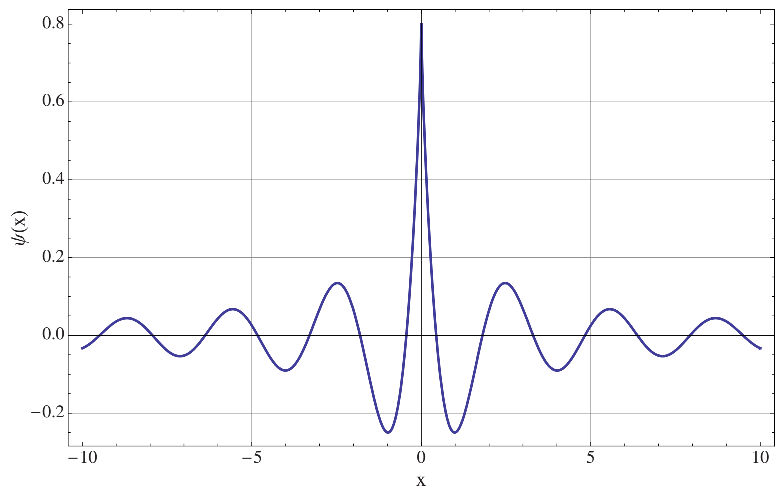


FIG. 1. (Color online) Wave packet $\psi(x)$ with the fall-off property 1 of Proposition 1, but with infinite mean energy.

$$p'(s) = \mathcal{A}'(s)\overline{\mathcal{A}(s)} + \mathcal{A}(s)\overline{\mathcal{A}'(s)},$$

Therefore,

$$p'(0) = 2\ \mathrm{Re}\ \mathcal{A}'(0) = \lim_{s\to 0} 2\left(\frac{\mathrm{Re}\ \mathcal{A}(s) - 1}{s}\right) = \lim_{s\to 0}\frac{2}{s}\int_{\mathbb{R}} (\cos(\lambda s) - 1)\mathrm{d}\mu(\lambda).$$

Then, the real part of $\mathcal{A}'(0)$ that can be rewritten as

$$\mathrm{Re}\ \mathcal{A}'(0) = -\lim_{s\to 0}\frac{2}{s}\int_{\mathbb{R}} \sin^2\left(\frac{\lambda s}{2}\right)\mathrm{d}\mu(\lambda),$$

must be equal to 0, while there are no constraints on the imaginary part of $\mathcal{A}'(0)$ given by

$$\mathrm{Im}\ \mathcal{A}'(0) = -\lim_{s\to 0}\int_{\mathbb{R}} \frac{\sin(\lambda s)}{s}\mathrm{d}\mu(\lambda).$$

We will show that it can also diverge. □

*Example 1:* Let $a>1$. We consider as $\mu$ the following probability measure:

$$\mu(E) = a\log a\int_{E\cap[a,+\infty)} \frac{1+\log\lambda}{\lambda^2\log^2\lambda}\mathrm{d}\lambda \tag{10}$$

for every Borel set $E\subset\mathbb{R}$.

Physically, one can implement the above example with a free particle in $n$ dimension subjected to a suitable one-dimensional projective measurement. Indeed, consider the free Hamiltonian of a particle (with mass $m=1/2$) $H=-\Delta$ with domain $H^2(\mathbb{R}^n)\subset L^2(\mathbb{R}^n)$. Consider a projection $P=\psi(\psi,\cdot)=|\psi\rangle\langle\psi|$ over the (radially symmetric) wave function $\psi\in L^2(\mathbb{R}^n)$, whose Fourier transform reads for $p\in\mathbb{R}^n$, $|p|>a^{1/2}>1$ as

$$\hat{\psi}(p) = \sqrt{\frac{2a\log a}{|S^{n-1}|}}\frac{\sqrt{1+\log|p|^2}}{|p|^{n/2+1}\log|p|^2},$$

and $\hat{\psi}(p)=0$ otherwise, where $|S^{n-1}|$ is the area of the unit sphere. The wave packet $\psi(x)$ for $n=1$ is plotted in Fig. 1. By using Fourier transform, it is not difficult to show that the spectral measure of the free Hamiltonian at state $\psi$ yields exactly the measure $\mu_{\psi}^{H}=\mu$ in (10), which satisfies the hypothesis of Theorem 1, because $\mu(\mathbb{R})=1$ and

$$\lim_{\Lambda\to+\infty}\Lambda\mu((-\Lambda,\Lambda)^{c})=\lim_{\Lambda\to+\infty}\Lambda a\log a\int_{\Lambda}^{+\infty}\frac{1+\log\lambda}{\lambda^{2}\log^{2}\lambda}\mathrm{d}\lambda=\lim_{\Lambda\to+\infty}\frac{a\log a}{\log\Lambda}=0.$$

However, observe that

$$\int_{\mathbb{R}}\lambda\mathrm{d}\mu(\lambda)=a\log a\int_{a}^{+\infty}\frac{1+\log\lambda}{\lambda\log^{2}\lambda}\mathrm{d}\lambda=a\log a\int_{\log a}^{+\infty}\frac{1+z}{z^{2}}\mathrm{d}z=+\infty.$$

Therefore, despite the fact that $\psi$ does not belong to $D(H^{1/2})=H^{1}(\mathbb{R}^{n})$ and thus has infinite energy, by a Zeno limit, one can freeze its dynamics in the initial state.

Now we show that in this case Im $\mathcal{A}'$ diverges. In fact, Im $\mathcal{A}$ has a cusp at the origin. Observe that

$$\lim_{s\to0^{+}}\int_{\mathbb{R}}\frac{\sin(\lambda s)}{s}\mathrm{d}\mu(\lambda)=\lim_{s\to0^{+}}\int_{|\lambda|\le1/s}\frac{\sin(\lambda s)}{s}\mathrm{d}\mu(\lambda)+\lim_{s\to0^{+}}\int_{|\lambda|>1/s}\frac{\sin(\lambda s)}{s}\mathrm{d}\mu(\lambda).\qquad(11)$$

The second limit on the right hand side of (11) vanishes because

$$0\le\int_{|\lambda|>1/s}\frac{\sin(\lambda s)}{s}\mathrm{d}\mu(\lambda)\le\frac{1}{s}\int_{|\lambda|>1/s}\mathrm{d}\mu(\lambda)\to0,$$

while the first limit equals $+\infty$ because

$$\sin 1\int_{|\lambda|\le1/s}\lambda\mathrm{d}\mu(\lambda)\le\int_{|\lambda|\le1/s}\frac{\sin(\lambda s)}{s}\mathrm{d}\mu(\lambda)$$

and

$$\lim_{s\to0^{+}}\int_{|\lambda|\le1/s}\lambda\mathrm{d}\mu(\lambda)=\int_{\mathbb{R}}\lambda\mathrm{d}\mu(\lambda)=+\infty.$$

Thus,

$$\lim_{s\to0^{+}}\mathrm{Im}\,\mathcal{A}'(s)=-\lim_{s\to0^{+}}\int_{\mathbb{R}}\frac{\sin(\lambda s)}{s}\mathrm{d}\mu(\lambda)=-\infty.$$

Similarly, one can prove that

$$\lim_{s\to0^{-}}\mathrm{Im}\,\mathcal{A}'(t)=+\infty.$$

□

Note that in Theorem 2 there is no mention to a lower bound of the Hamiltonian. Boundedness from below is something of red herring. It has played a crucial role in QZD; in fact, it has been always advocated in the literature, and indeed the limiting (Zeno) Hamiltonian which engenders the effective dynamics in Theorem 1 is nothing but the Friedrich's extension of the operator *PHP*. However, if one is concerned with the QZE per se, such hypothesis is quite unnatural. Therefore, one can wonder whether lower boundedness is really a physical requirement for QZD, or rather it is just a—very convenient—technical hypothesis. Our second main result answers this question. It gives a characterization of the QZD in which boundedness from below plays no role.

**Theorem 3:** *Consider a self-adjoint operator H and an orthogonal finite-rank projection P in a complex Hilbert space* $\mathcal{H}$. *Let* $\{P_{\Omega}^{H}\}$ *be the projection-valued spectral measure of H and* $\{U(t)=e^{-itH}\}_{t\in\mathbb{R}}$ *the one-parameter unitary group generated by H. Consider the product formula*

*$V_N(t)=(PU(t/N)P)^N$ with $t\in\mathbb{R}$ and $N\in\mathbb{N}^*$ and the family of self-adjoint operators $H^{(\Lambda)}=HP^H_{(-\Lambda,\Lambda)}$ with $\Lambda>0$. The following statements are equivalent:*

(1)

$$PP^H_{(-\Lambda,\Lambda)^c}P=o\left(\frac{1}{\Lambda}\right)\quad\text{for }\Lambda\to+\infty,$$

*and the limit*

$$H_Z=\lim_{\Lambda\to+\infty}PH^{(\Lambda)}P$$

*exists and is bounded,*

(2)

$$\frac{\mathrm{d}}{\mathrm{d}s}V(s)\big|_{s=0}=-iH_Z,$$

(3)

$$\lim_{N\to+\infty}V_N(t)=Pe^{-itH_Z}$$

*uniformly for t in finite intervals of $\mathbb{R}$.*

Therefore, the existence of QZD is equivalent to the energy falloff property, which assures the existence of QZE, *and* to the existence of a limit mean value Hamiltonian $H_Z=\lim_\Lambda PH^{(\Lambda)}P$.

*Remark 2:* In the one-dimensional case, when $P=|\psi\rangle\langle\psi|$, by using Eq. (8), the QZD product formula (3) reads as

$$V_N(t)=[\mathcal{A}(t/N)]^NP.$$

In such a case, QZD is trivial and its existence is equivalent to the existence of the numerical limit

$$\lim_{N\to+\infty}[\mathcal{A}(t/N)]^N=e^{-itE_Z}\tag{12}$$

with a finite phase $E_Z\in\mathbb{R}$. This has to be compared with QZE, where one looks at the modulus of Eq. (12), and thus the existence of a finite mean energy is not necessary, as shown also in Example 1.

Theorem 3 states that the phase $E_Z$ is finite if and only if the limit

$$H_Z=\lim_{\Lambda\to+\infty}(\psi,H^{(\Lambda)}\psi)P$$

exists and is bounded, and in such a case, one has

$$E_Z=\lim_{\Lambda\to+\infty}(\psi,H^{(\Lambda)}\psi).$$

□

From Theorem 3 one immediately gets that if the attention is restricted to positive Hamiltonians, the condition given in Theorem 1 on the domain of the square root is both necessary and sufficient. Indeed, we have the following.

*Corollary 1: Let H be a positive self-adjoint operator and P be an orthogonal finite-rank projection onto $\mathcal{H}_P=P\mathcal{H}$. Then,*

$$\mathcal{H}_P\subset D(H^{1/2})\Leftrightarrow\lim_{N\to+\infty}V_N(t)=P\exp(-it(H^{1/2}P)^*(H^{1/2}P)).$$

*Proof:* One implication is the content of Theorem 1. The other follows by Theorem 3 after noting that, when $H \geq 0$,

$$PH^{(\Lambda)}P = P\int_{[0,\Lambda)} \lambda \mathrm{d}P_\lambda^H P,$$

and thus the existence of a bounded limit $\lim_\Lambda PH^{(\Lambda)}P$ implies that $\|H^{1/2}P\| < \infty$. □

*Remark 3:* By looking at Corollary 1, one might think that the results for positive operators hold true in the general unbounded case by replacing the condition $\mathcal{H}_P \subset D(H^{1/2})$ with $\mathcal{H}_P \subset D(|H|^{1/2})$. Unfortunately, this is not true. The condition $\mathcal{H}_P \subset D(|H|^{1/2})$ is stronger than condition (1) in Theorem 3 and, in fact, is a sufficient condition for QZD, but is not necessary. Indeed, it is easy to construct a probability Borel measure $\mu_\psi$ associated with the Hamiltonian $H$ at some $\psi \in \mathcal{H}$ such that

$$\lim_{\Lambda \to +\infty} \int_{(-\Lambda,\Lambda)} \lambda \mathrm{d}\mu_\psi(\lambda) < +\infty,$$

while

$$\lim_{\Lambda \to +\infty} \int_{(-\Lambda,\Lambda)} |\lambda| \mathrm{d}\mu_\psi(\lambda) = +\infty.$$

Observe that in this case if one considers the projection $P = |\psi\rangle\langle\psi|$, one gets that $\mathcal{H}_P \not\subset D(|H|^{1/2})$, despite the fact that the pair projection-Hamiltonian $(P,H)$ satisfies statement 1 of Theorem 3. □

## III. PROOFS OF THE THEOREMS

Let us now turn to the proofs of our characterizations of the QZE and its dynamics, Theorems 2 and 3. First of all let us prove a preliminary lemma that will be useful in the following. We note, incidentally, that this Tauberian result is interesting in itself and is probably known in the probability community. However, we will give here a purely analytical proof.

*Lemma 1: Let $\mu$ be a Borel measure on $\mathbb{R}$ with $\mu(\mathbb{R}) = 1$. The following assertions are equivalent:*

(1)

$$\mu((-\Lambda,\Lambda)^c) = o\left(\frac{1}{\Lambda}\right) \quad \text{for } \Lambda \to +\infty,$$

(2)

$$\frac{1}{\Lambda^{k+1}} \int_{(-\Lambda,\Lambda)} \lambda^{k+1} \mathrm{d}\mu(\lambda) = o\left(\frac{1}{\Lambda}\right) \quad \text{for } \Lambda \to +\infty \quad \textit{for every } k \in \mathbb{N}^*.$$

*Proof:*

$(1) \Rightarrow (2)$. Let $k \in \mathbb{N}^*$, then for every $\Lambda > 0$, by using an integration by parts formula (see, e.g., Ref. 13), we have that

$$\begin{aligned}\frac{1}{\Lambda^k} &\int_{(-\Lambda,\Lambda)} \lambda^{k+1} \mathrm{d}\mu(\lambda) \\ &= \frac{1}{\Lambda^k}\Bigg[\Lambda^{k+1}\mu((-\infty,\Lambda]) - (-\Lambda)^{k+1}\mu((-\infty,-\Lambda]) \\ &\quad - (k+1)\int_{(-\Lambda,\Lambda)} \lambda^k \mu((-\infty,\lambda])\mathrm{d}\lambda\Bigg]\end{aligned}$$

$$= \Lambda\mu((-\infty,\Lambda]) + (-1)^k\Lambda\mu((-\infty,-\Lambda]) - \frac{k+1}{\Lambda^k}\int_{(0,\Lambda)} \lambda^k(\mu((-\infty,\lambda]) + (-1)^k\mu((-\infty,-\lambda]))\mathrm{d}\lambda.$$

We can write

$$\begin{aligned}\frac{1}{\Lambda^k}\int_{(-\Lambda,\Lambda)} &\lambda^{k+1}\mathrm{d}\mu(\lambda)\\ &= \Lambda\mu((-\Lambda,\Lambda]) - \frac{k+1}{\Lambda^k}\int_{(0,\Lambda)} \lambda^k\mu((-\lambda,\lambda])\mathrm{d}\lambda\\ &+ (1+(-1)^k)\left(\Lambda\mu((-\infty,-\Lambda]) - \frac{k+1}{\Lambda^k}\int_{(0,\Lambda)} \lambda^k\mu((-\infty,-\lambda])\mathrm{d}\lambda\right).\end{aligned} \tag{13}$$

The second line of (13) reads as

$$\begin{aligned}\Lambda[1-\mu((-\Lambda,\Lambda]^c)] &- \frac{k+1}{\Lambda^k}\int_{(0,\Lambda)} \lambda^k(1-\mu((-\lambda,\lambda]^c))\mathrm{d}\lambda\\ &= -\Lambda\mu((-\Lambda,\Lambda]^c) + \frac{k+1}{\Lambda^k}\int_{(0,\Lambda)} \lambda^k\mu((-\lambda,\lambda]^c)\mathrm{d}\lambda\\ &\le -\Lambda\mu((-\Lambda,\Lambda]^c) + \frac{k+1}{\Lambda}\int_{(0,\Lambda)} \lambda\mu((-\lambda,\lambda]^c)\mathrm{d}\lambda \to 0 \quad \text{for } \Lambda\to+\infty,\end{aligned}$$

while in the third line of (13), which is nonzero only for $k$ even, one gets

$$\Lambda\mu((-\infty,-\Lambda]) \le \Lambda\mu((-\Lambda,\Lambda)^c) \to 0$$

and

$$\frac{1}{\Lambda^k}\int_{[0,\Lambda]} \lambda^k\mu((-\infty,-\lambda])\mathrm{d}\lambda \le \frac{1}{\Lambda}\int_{[0,\Lambda]} \lambda\mu((-\lambda,\lambda)^c)\mathrm{d}\lambda \to 0.$$

Therefore, we have that

$$\lim_{\Lambda\to+\infty}\frac{1}{\Lambda^k}\int_{(-\Lambda,\Lambda)} \lambda^{k+1}\mathrm{d}\mu(\lambda) = 0.$$

(2)$\Rightarrow$(1). Let us choose $k=1$ and fix $\epsilon>0$. By hypothesis, we have that there exists a real $\Lambda_0>0$ such that for every $\Lambda>\Lambda_0$

$$\frac{1}{\Lambda}\int_{(-\Lambda,\Lambda)} \lambda^2\mathrm{d}\mu(\lambda) < \epsilon.$$

Thus, for every $\Lambda>\Lambda_0$,

$$\begin{aligned}\Lambda\int_{(-\Lambda,\Lambda)^c}\mathrm{d}\mu(\lambda) &= \Lambda\sum_{k=0}^{+\infty}\int_{2^k\Lambda\le|\lambda|<2^{k+1}\Lambda}\mathrm{d}\mu(\lambda)\\ &\le \Lambda\sum_{k=0}^{+\infty}\int_{2^k\Lambda\le|\lambda|<2^{k+1}\Lambda}\left(\frac{\lambda}{2^k\Lambda}\right)^2\mathrm{d}\mu(\lambda)\\ &\le \sum_{k=1}^{+\infty}\frac{1}{2^{k-1}}\frac{1}{2^{k+1}\Lambda}\int_{|\lambda|<2^{k+1}\Lambda}\lambda^2\mathrm{d}\mu(\lambda)\\ &\le 2\epsilon.\end{aligned}$$

□

*Remark 4:* Observe that in order to prove that (2)⇒(1), it is sufficient that

$$\frac{1}{\Lambda^3}\int_{(-\Lambda,\Lambda)}\lambda^2\mathrm{d}\mu(\lambda) = o\left(\frac{1}{\Lambda}\right) \quad \text{for } \Lambda\to+\infty.$$

□

Now we prove Proposition 1 on the characterization of the one-dimensional QZE. We will use it as the first step in the proof of the multidimensional case, Theorem 2.

### A. Proof of Proposition 1

Let us start with the proof of the first equivalence (1)⇔(2).

Observe that $\mathcal{A}$ is a (uniformly) continuous function and that $\mathcal{A}(0)=1$. Define for every $s \in \mathbb{R}$

$$z(s) = \mathcal{A}(s) - 1$$

so that $z$ is a continuous function with $z(0)=0$. Recall that

$$p'(0) = \lim_{s\to 0} 2\frac{\mathrm{Re}(z(s))}{s} \tag{14}$$

and

$$\frac{\mathrm{Re}(z(s))}{s} = \frac{1}{s}\int_{\mathbb{R}}(\cos(\lambda s)-1)\mathrm{d}\mu(\lambda) = -\frac{2}{s}\int_{\mathbb{R}}\sin^2\left(\frac{\lambda s}{2}\right)\mathrm{d}\mu(\lambda).$$

We can write

$$\frac{2}{|s|}\int_{\mathbb{R}}\sin^2\left(\frac{\lambda s}{2}\right)\mathrm{d}\mu(\lambda) = g(s) + h(s),$$

where

$$g(s) = \frac{2}{|s|}\int_{|\lambda|<2/|s|}\sin^2\left(\frac{\lambda s}{2}\right)\mathrm{d}\mu(\lambda)$$

and

$$h(s) = \frac{2}{|s|}\int_{|\lambda|\ge 2/|s|}\sin^2\left(\frac{\lambda s}{2}\right)\mathrm{d}\mu(\lambda).$$

Therefore, since $g,h\ge 0$, one has that

$$\lim_{s\to 0} 2\frac{\mathrm{Re}(z(s))}{s} = 0 \Leftrightarrow \lim_{s\to 0} g(s) = \lim_{s\to 0} h(s) = 0. \tag{15}$$

Observe that, since $x^2 \sin^2 1 \le \sin^2 x \le x^2$ for $|x| < 1$,

$$\sin^2 1 \frac{|s|}{2}\int_{|\lambda|<2/|s|} \lambda^2 \mathrm{d}\mu(\lambda) \le g(s) \le \frac{|s|}{2}\int_{|\lambda|<2/|s|} \lambda^2 \mathrm{d}\mu(\lambda).$$

Therefore,

$$\lim_{s\to 0} g(s) = 0 \Leftrightarrow \lim_{s\to 0}\frac{|s|}{2}\int_{|\lambda|<2/|s|} \lambda^2 \mathrm{d}\mu(\lambda) = 0. \tag{16}$$

(1)$\Rightarrow$(2). Using Lemma 1 and (16), one gets that $g(s)\to 0$ for $s\to 0$. Moreover,

$$0 \le h(s) \le \frac{2}{|s|}\mu((-2/|s|,2/|s|)^c) \to 0, \quad s \to 0.$$

(2)$\Rightarrow$(1). Observe that using (15), we have that

$$p'(0) = 0 \Rightarrow \lim_{s\to 0} g(s) = 0.$$

Thus, by using (16) and Remark 4, we prove the thesis.

Now we prove the second equivalence (2)$\Leftrightarrow$(3).

Observe that

$$[p(t/N)]^N - 1 = \sum_{k=0}^{N-1} [p(t/N)]^k (p(t/N) - 1) = S_N(t) N (p(t/N) - 1), \tag{17}$$

where

$$S_N(t) = \frac{1}{N}\sum_{k=0}^{N-1} [p(t/N)]^k.$$

(2)$\Rightarrow$(3). Since $0 \le S_N(t) \le 1$, we have from (17) that

$$|[p(t/N)]^N - 1| \le N|p(t/N) - 1| \to 0, \quad N \to +\infty$$

uniformly for $t$ in finite intervals of $\mathbb{R}$.

(3)$\Rightarrow$(2). We know that $[p(t/N)]^N \to 1$ for $N\to+\infty$ uniformly in $t$ in finite intervals of $\mathbb{R}$. Observe that, since $0 \le p(t/N) \le 1$,

$$1 \ge S_N(t) \ge [p(t/N)]^N \to 1$$

uniformly for $t$ in finite intervals of $\mathbb{R}$. Therefore, from (17),

$$\lim_{N\to+\infty} N(p(t/N) - 1) = 0,$$

and thus $p'(0)=0$. □

Now that we have gathered all necessary ingredients, let us conclude this section with the proofs of our main results, Theorems 2 and 3.

### B. Proof of Theorem 2

(1)$\Rightarrow$(2). From $Z_1(s) = V(s)^* V(s)$, one gets

$$Z_1'(s) = \left(\frac{\mathrm{d}}{\mathrm{d}s}V(s)^*\right)V(s) + V(s)^*\left(\frac{\mathrm{d}}{\mathrm{d}s}V(s)\right).$$

Therefore, for every $\phi \in \mathcal{H}$, we have

$$(\phi, Z_1'(0)\phi) = \frac{\mathrm{d}}{\mathrm{d}s}[(e^{isH}P\phi, P\phi) + (e^{-isH}P\phi, P\phi)]_{s=0}.$$

If $P\phi \neq 0$, let us define the probability Borel measure on $\mathbb{R}$

$$\mathrm{d}\mu(\lambda) = \frac{1}{\|P\phi\|^2}\mathrm{d}(P\phi, P_\lambda^H P\phi)$$

and the survival amplitude

$$\mathcal{A}(s) = \int_{\mathbb{R}} e^{-is\lambda}\mathrm{d}\mu(\lambda).$$

Therefore,

$$(\phi, Z_1'(0)\phi) = 2\|P\phi\|^2 \frac{\mathrm{d}}{\mathrm{d}s}\mathrm{Re}(\mathcal{A}(s))\Big|_{s=0}. \tag{18}$$

By condition (1), we get that $\mu$ satisfies condition (1) of Proposition 1. Therefore, the right side of (18) vanishes and, by the polarization identity, it follows that $Z_1'(0)=0$.

(2)$\Rightarrow$(1). Let $\psi \in \mathcal{H}$, $\|P\psi\|=1$, and consider the Borel probability measure

$$\mathrm{d}\mu(\lambda) = \mathrm{d}(P\psi, P_\lambda^H P\psi).$$

Define for every $s \in \mathbb{R}$

$$\mathcal{A}(s) = \int_{\mathbb{R}} e^{-is\lambda}\mathrm{d}\mu(\lambda) \quad \text{and} \quad p(s) = |\mathcal{A}(s)|^2.$$

Observe that

$$p'(0) = (P\psi, Z_1'(0)P\psi) = 0,$$

thus, using the equivalence proved in Proposition 1, we have

$$\lim_{\Lambda\to+\infty} \Lambda \int_{(-\Lambda,\Lambda)^c} \mathrm{d}\mu(\lambda) = \lim_{\Lambda\to+\infty} \Lambda(\psi, PP^H_{(-\Lambda,\Lambda)^c}P\psi) = 0.$$

Since $\mathcal{H}_P$ is a finite dimensional space we have proved Condition 1.

(2)$\Rightarrow$(3). Use the telescopic sum,

$$\begin{aligned} Z_N(t) - P &= V_N(t)^* V_N(t) - P \\ &= \sum_{k=0}^{N-1}\left[V\left(\frac{t}{N}\right)^*\right]^k \left[V\left(\frac{t}{N}\right)^* V\left(\frac{t}{N}\right) - P\right]\left[V\left(\frac{t}{N}\right)\right]^{N-1-k} \\ &= \sum_{k=0}^{N-1}\left[V\left(\frac{t}{N}\right)^*\right]^k \left[Z_1\left(\frac{t}{N}\right) - P\right]\left[V\left(\frac{t}{N}\right)\right]^{N-1-k}. \end{aligned} \tag{19}$$

Therefore, since $\|V(t/N)\| \leq 1$, one gets

$$\|Z_N(t)-P\| \leq N\|Z_1(t/N)-P\| \to 0$$

uniformly for $t$ in finite intervals of $\mathbb{R}$ by hypothesis.

(3)⇒(2). We want to prove that

$$\lim_{N\to+\infty} N(Z_1(t/N)-P) = \lim_{N\to+\infty} N(V(t/N)^* V(t/N)-P) = 0 \tag{20}$$

uniformly for $t$ in finite intervals of $\mathbb{R}$. Observe that (19) can be expanded also in this way

$$Z_N(t)-P = N\left[V\left(\frac{t}{N}\right)^* S_N(t) V\left(\frac{t}{N}\right) - S_N(t)\right], \tag{21}$$

where

$$S_N(t) = \frac{P}{N}\sum_{k=0}^{N-1}\left[V\left(\frac{t}{N}\right)^*\right]^k\left[V\left(\frac{t}{N}\right)\right]^k.$$

Let us first prove that the ergodic sum $S_N(t)$ tends to $P$,

$$\lim_{N\to+\infty} S_N(t) = P$$

uniformly for $t$ in finite intervals of $\mathbb{R}$. It is easy to see that for every $k \geq l \geq 0$

$$\left[V\left(\frac{t}{N}\right)^*\right]^k\left[V\left(\frac{t}{N}\right)\right]^k \leq \left[V\left(\frac{t}{N}\right)^*\right]^l\left[V\left(\frac{t}{N}\right)\right]^l,$$

whence, for every $k \in \{0,1,\ldots,N-1\}$,

$$0 \leq P - P\left[V\left(\frac{t}{N}\right)^*\right]^k\left[V\left(\frac{t}{N}\right)\right]^k \leq P - Z_N(t).$$

Therefore,

$$0 \leq P - S_N(t) = \frac{1}{N}\sum_{k=0}^{N-1}\left(P - P\left[V\left(\frac{t}{N}\right)^*\right]^k\left[V\left(\frac{t}{N}\right)\right]^k\right) \leq P - Z_N(t) \to 0$$

by hypothesis.

Assume that (20) is not valid. By taking the trace of (21) and by using its cyclic property, we get

$$\mathrm{tr}(Z_N(t)-P) = \mathrm{tr}\left[N\left(V\left(\frac{t}{N}\right)V\left(\frac{t}{N}\right)^* - P\right)S_N(t)\right].$$

Since the ergodic sum $S_N(t)$ is a positive operator whose limit is $P$, the right hand side does not tend to 0, while the left hand side vanishes by hypothesis, and we get a contradiction. □

## C. Proof of Theorem 3

Let us start with the proof of the first equivalence (1)⇔(2).

Let $\mathrm{Re}\, V(s)=(V(s)+V(s)^*)/2$ and $\mathrm{Im}\, V(s)=(V(s)-V(s)^*)/2i$ for all $s \in \mathbb{R}$. Observe that by Theorem 2 it follows that

$$PP^H_{(-\Lambda,\Lambda)^c}P = o\left(\frac{1}{\Lambda}\right) \Leftrightarrow \frac{\mathrm{d}}{\mathrm{d}s}V(s)^*V(s)\big|_{s=0} = 2\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{Re}\, V(s)\big|_{s=0} = 0. \tag{22}$$

Now we prove that

$$H_Z = \lim_{\Lambda\to+\infty} PH^{(\Lambda)}P \Leftrightarrow -\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{Im}\, V(s)|_{s=0} = H_Z. \tag{23}$$

Let us denote $\mathrm{d}Q_\lambda = PdP_\lambda^H P$. Observe that

$$-\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{Im}\, V(s)|_{s=0} = \lim_{s\to 0}\frac{1}{s}P\,\sin(sH)P = \lim_{s\to 0}\int_{\mathbb{R}}\frac{\sin(\lambda s)}{s}\mathrm{d}Q_\lambda .$$

In order to prove (23), we will prove that

$$\lim_{s\to 0} PH^{(\pi/s)}P - \frac{1}{s}P\,\sin(sH)P = 0. \tag{24}$$

We have that, if $s>0$

$$\begin{aligned}
&PH^{(\pi/s)}P - \frac{1}{s}P\,\sin(sH)P\\
&\quad= \int_{(-\pi/s,\pi/s)}\lambda \mathrm{d}Q_\lambda - \frac{1}{s}\int_{\mathbb{R}}\sin(\lambda s)\mathrm{d}Q_\lambda\\
&\quad= \int_{(-\pi/s,\pi/s)}\lambda\left(1-\frac{\sin(\lambda s)}{\lambda s}\right)\mathrm{d}Q_\lambda - \frac{1}{s}\int_{(-\pi/s,\pi/s)^c}\sin(\lambda s)\mathrm{d}Q_\lambda .
\end{aligned}$$

Therefore, since $1-\sin x/x \geq 0$, we get

$$\begin{aligned}
&-\frac{\pi}{s}\int_{(-\pi/s,\pi/s)}\left(1-\frac{\sin(\lambda s)}{\lambda s}\right)dQ_\lambda - \frac{1}{s}\int_{(-\pi/s,\pi/s)^c}dQ_\lambda\\
&\quad\leq PH^{(\pi/s)}P - \frac{1}{s}P\,\sin(sH)P\\
&\quad\leq \frac{\pi}{s}\int_{(-\pi/s,\pi/s)}\left(1-\frac{\sin(\lambda s)}{\lambda s}\right)dQ_\lambda + \frac{1}{s}\int_{(-\pi/s,\pi/s)^c}dQ_\lambda .
\end{aligned}$$

By noting that

$$0 \leq \frac{\pi}{s}\int_{(-\pi/s,\pi/s)}\left(1-\frac{\sin(\lambda s)}{\lambda s}\right)\mathrm{d}Q_\lambda \leq \frac{\pi s}{6}\int_{(-\pi/s,\pi/s)}\lambda^2 \mathrm{d}Q_\lambda$$

and by using (22) and Lemma 1, we obtain that (24)) holds when $s\to 0^+$. With the same argument, one can prove the thesis when $s\to 0^-$.

(2)⇒(3). Observe that

$$\begin{aligned}
\|V_N(t) - Pe^{-itH_Z}\| &= \|(V(t/N))^N - (Pe^{-itH_Z/N})^N\|\\
&= \left\|\sum_{k=0}^{N-1}(V(t/N))^{N-1-k}(V(t/N) - Pe^{-itH_Z/N})Pe^{-iktH_Z/N}\right\|\\
&\leq N\|V(t/N) - Pe^{-itH_Z/N}\| \to 0
\end{aligned}$$

uniformly for $t$ in finite intervals of $\mathbb{R}$.

(3)⇒(2). Let $z>0$. We will prove that

$$\lim_{N\to+\infty}(z - N(V(t/N) - P))^{-1}P = (z + itH_Z)^{-1}P$$

uniformly for $t$ in finite intervals of $\mathbb{R}$. This implies that

$$\lim_{N\to+\infty} N(V(t/N)-P) = -iH_Z$$

uniformly for $t$ in finite intervals of $\mathbb{R}$, and thus

$$\frac{\mathrm{d}}{\mathrm{d}s}V(s)\Big|_{s=0} = -iH_Z.$$

Indeed, observe that

$$(z-N(V(t/N)-P))^{-1}P = \frac{1}{N}\sum_{k=0}^{+\infty}\frac{V(t/N)^k}{(1+z/N)^{k+1}}P = \int_0^{+\infty}\frac{V(t/N)^{[sN]}}{(1+z/N)^{[sN]+1}}P\mathrm{d}s, \tag{25}$$

where $[\cdot]$ denotes the integer part function. By the dominated convergence theorem, the right hand side of (25) converges to

$$\int_0^{+\infty} e^{-sz}Pe^{-istH_Z}P\mathrm{d}s = (z+itH_Z)^{-1}P$$

uniformly for $t$ in finite intervals of $\mathbb{R}$. □

## ACKNOWLEDGMENTS

We would like to thank Sandro Graffi, Andrzej Kossakowski, Hiromichi Nakazato, Saverio Pascazio, and Shuichi Tasaki for stimulating discussions. This work was partly supported by the European Union through the Integrated Project EuroSQIP.

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