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Perturbations in a non-singular bouncing Universe

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Abstract

We complement the low-energy gravi-dilaton effective action of string theory with a non-local, general-covariant dilaton potential, and obtain homogeneous solutions describing a non-singular (bouncing-curvature) cosmology. We then compute, both analytically and numerically, the spectrum of amplified scalar and tensor perturbations, and draw some general lessons on how to extract observable consequences from pre-big bang and ekpyrotic scenarios.

In recent years string/M-theory has inspired new cosmological scenarios in which a long period of accelerated (growing-curvature) evolution, emerging from an almost trivial initial geometry, turns into a standard (decreasing-curvature) FRW-type cosmology, after going smoothly through a big bang-like event. There are by now several variations on this pre-big bang theme. Besides the original pre-big bang (PBB) scenario [1, 2, 3], based on the duality symmetries of string cosmology, new models incorporating brane and M-theory ideas have been proposed under the generic name of ekpyrotic (EKP) scenarios [4]. While different proposals differ in the way the scale factor behaves during the growing-curvature phase, they all share the feature of describing a bounce in $|H|$ (the absolute value of the Hubble parameter) or, in more geometrical terms, in the space-time curvature. The latter always starts vanishingly small, grows to a maximum at the would-be big bang, and then decreases again in the FRW phase. A common theoretical challenge to all these models is that of being able to describe the transition between the two regimes.

At a more phenomenological level, instead, the challenge is to compute, in a reliable way, the final spectrum of amplified quantum fluctuations to be compared with present data on CMB radiation and large-scale structure. In the PBB case it was admitted early on [5, 6] that adiabatic- curvature perturbations had too large a spectral index to be of any relevance at cosmologically interesting scales (while being possibly important for gravitational waves searches [7, 8]). Isocurvature perturbations (related to the Kalb–Ramond two-form) can instead be produced with an interestingly flat spectrum [9], but have to be converted into adiabatic-curvature perturbations through the so-called curvaton mechanism [10] before they can provide a viable scenario for large scale-anisotropies [11]. Proponents of the ekpyrotic scenario, while agreeing with the PBB result of a steep spectrum of tensor perturbations, have also repeatedly claimed [4, 12] to obtain "naturally" an almost scale-invariant spectrum of adiabatic-curvature perturbations, very much as in ordinary models of slow-roll inflation. These claims have generated a lot of heated discussion (see for instance [13]), with many arguments given in favour or against the phenomenological viability of EKP scenarios in the absence of a curvaton's help. The reasons for the disagreement can be ultimately traced back to the fact that the curvature bounce (hereafter simply referred to as the bounce) is put in by hand, rather than being derived from an underlying action. This leaves different authors to make different assumptions on how to smoothly connect perturbations across the bounce itself, and this often results in completely different physical predictions.

Our aim is to present a class of models where a regular bounce is derived from the field equations of a general-covariant, albeit non-local, action, and where perturbations can be studied from beginning to end. We are thus able to test, at least within the model, the assumptions made by different groups about how different perturbations should or should not behave across the bounce. In this letter we only consider a simple gravi-dilaton model and present the main results, working in the string frame and skipping their detailed derivation. In a longer, forthcoming paper [14] we shall give full details of the calculations in both the string and Einstein frames (with identical physical conclusions), and generalize the class of models to include a fluid component. We first present the action, the background-field equations, and a class of smooth, bouncing solutions. We then discuss, successively, the amplification of tensor and scalar perturbations from an initial spectrum of vacuum quantum fluctuations, and compare analytical and numerical results. We finally summarize our results and draw some conclusions.

We recall that the homogeneous cosmological solutions derived from the tree-level, low-energy string effective action exhibit in general a curvature singularity, disconnecting the pre-big bang branch from the post-big bang one [15]. Such a singularity is expected to be removed by higher-loops *and* higher-curvature α' corrections (see for instance [3]), but it is known [16] that non-singular solutions can be explicitly obtained already at low curvatures [2] in the presence of an appropriate *non-local* effective potential. An example of such a possibility can be illustrated by using a $(d+1)$-dimensional, general covariant gravi-dilaton effective action, which, in the string frame, reads:

$$S=-\frac{1}{2\lambda_{\rm s}^{d-1}}\int d^{d+1}x\sqrt{|g|}\; e^{-\varphi}\left[R+(\nabla\varphi)^2+V(\overline{\varphi})\right] \tag{1}$$

(metric conventions: $+---\ldots$). Here $\lambda_{\rm s}=\sqrt{\alpha'}=M_{\rm s}^{-1}$ is the string length scale, and the potential $V(\overline{\varphi}(x))$, a local function of $\overline{\varphi}$, is instead a non-local function (yet a scalar under general coordinate transformations) of the dilaton owing to the definitions:

$$V=V(e^{-\overline{\varphi}}),\qquad e^{-\overline{\varphi}(x)}=\int\frac{d^{d+1}y}{\lambda_{\rm s}^{d}}\sqrt{|g(y)|}\; e^{-\varphi(y)}\sqrt{\partial_\mu\varphi(y)\partial^\mu\varphi(y)}\;\delta\left(\varphi(x)-\varphi(y)\right). \tag{2}$$

Note that $e^{\overline{\varphi}}$ plays the role of a "reduced" coupling constant from $d+1$ to $0+1$ space-time dimensions. We may thus expect $V(e^{-\overline{\varphi}})$ to go like some inverse power of its argument as this becomes large (i.e. in perturbation theory). We shall discuss in [14] how such a non-local potential may be induced by loop corrections in (higher-dimensional) manifolds with compact spatial sections; here, we take this simply as a toy model that avoids the singularity, while staying all the time at low energy/curvature. For a background manifold isometric with respect to d spatial translations, it is known that action (1) leads to field equations that are covariant under scale-factor duality [1] and, in the presence of the two-form background $B_{\mu\nu}$, also under global $O(d,d)$ transformations [16].

Variation of the above action with respect to $g_{\mu\nu}$ and φ, though somewhat unusual, is straightforward and leads to the following field equations (in units of $2\lambda_{\rm s}^{d-1}=1$):

$$G_{\mu\nu}+\nabla_\mu\nabla_\nu\varphi+\frac{1}{2}g_{\mu\nu}\left[(\nabla\varphi)^2-2\nabla^2\varphi-V\right]-\frac{1}{2}e^{-\varphi}\sqrt{(\partial\varphi)^2}\;\gamma_{\mu\nu}I_1=0, \tag{3}$$

$$R+2\nabla^2\varphi-(\nabla\varphi)^2+V-\frac{\partial V}{\partial\bar\varphi}+e^{-\varphi}\frac{\hat\nabla^2\varphi}{\sqrt{(\partial\varphi)^2}}I_1-e^{-\varphi}V'I_2=0, \tag{4}$$

where (with a prime denoting differentiation with respect to the argument)

$$\begin{aligned} I_1&=\frac{1}{\lambda_s^d}\int d^{d+1}y\sqrt{|g(y)|}\;V'(e^{-\overline{\varphi}(y)})\;\delta\left(\varphi(x)-\varphi(y)\right), \\ I_2&=\frac{1}{\lambda_s^d}\int d^{d+1}y\sqrt{|g(y)|}\;\sqrt{\partial_\mu\varphi(y)\partial^\mu\varphi(y)}\;\delta'\left(\varphi(x)-\varphi(y)\right)\;, \end{aligned} \tag{5}$$

and we have also introduced the induced metric and Laplacian:

$$\gamma_{\mu\nu}=g_{\mu\nu}-\frac{\partial_\mu\varphi\partial_\nu\varphi}{(\partial\varphi)^2}, \qquad \hat\nabla^2\varphi=\gamma_{\mu\nu}\nabla^\mu\nabla^\nu\varphi. \tag{6}$$

For a homogeneous, isotropic and spatially flat background, we can set $g_{00}=1$, $g_{ij}=-a^2(t)\delta_{ij}$, $\varphi=\varphi(t)$, and obtain $e^{-\overline{\varphi}}=e^{-\varphi}a^d$, where we have absorbed into φ the dimensionless constant $-\ln(\int d^dy/\lambda_s^d)$, associated with the (finite) comoving spatial volume. In such a case, the time and space components of Eq.(3), and the dilaton equation (4), lead to a set of equations already studied in [16, 2, 5]:

$$\dot{\overline{\varphi}}^2-dH^2-V=0, \qquad \dot H-H\dot{\overline{\varphi}}=0, \tag{7}$$

$$2\ddot{\overline{\varphi}}-\dot{\overline{\varphi}}^2-dH^2+V-\frac{\partial V}{\partial\overline{\varphi}}=0, \tag{8}$$

where the third (dilaton) equation follows from the first two, *provided* $\dot{\overline{\varphi}}\neq 0$.

As noticed in [16], the above set of equations admits, quite generically, non-singular solutions. To recall this point and give an explicit analytical example we note that the two equations (7) can be reduced to quadratures, i.e.

$$H=me^{\overline{\varphi}}, \qquad t=\int^\lambda d\lambda'\left[d+\lambda'^2V(m\lambda')\right]^{-1/2}\;, \tag{9}$$

where m is an integration constant, and $\lambda=m^{-1}e^{-\overline{\varphi}}$. If the function appearing above between square brackets has a simple zero, we obtain a regular bouncing solution [16]. Furthermore, if $V\lambda^2\to 0$ at large $|\lambda|$ (corresponding to a potential generated beyond two loops), the asymptotic solutions approach the usual vacuum solutions at large $|t|$.

Consider, in particular, the class of potentials

$$V(m\lambda)=\lambda^{-2}\left(\left[\alpha-(m\lambda)^{-2n}\right]^{2-1/n}-d\right), \tag{10}$$

parametrized by the dimensionless coefficients α and by the "loop-counting" parameter n. For $\alpha>0$ and $n>0$, Eq. (10) leads to the general exact solution

$$H=me^{\overline{\varphi}}=m\left[\frac{\alpha}{1+(\alpha mt)^{2n}}\right]^{1/2n}. \tag{11}$$

As $|t| \to \infty$, the Hubble parameter behaves like $H|t| \sim \alpha^{(1-2n)/2n}$, so that the minimal pre-big bang solutions dominated by the dilaton kinetic energy are only recovered for $\alpha^{(2-1/n)} = d$ (see for instance [3]). If this condition is not satisfied, the background may still be non-singular, but the dilaton potential cannot be neglected, even asymptotically.

For the perturbation analysis of this paper it will be sufficient to use as a toy model the simple regular background associated to the four-loop potential $V(\overline{\varphi}) = -V_0 e^{4\overline{\varphi}}$. In this case the general solution, in $d = 3$ spatial dimensions, can be written as:

$$a(\tau) = \left[\tau + \sqrt{\tau^2 + 1}\right]^{1/\sqrt{3}}, \qquad \overline{\varphi} = -\frac{1}{2}\ln\left(1 + \tau^2\right) + \varphi_0, \qquad \tau = t/t_0, \tag{12}$$

where φ_0 is an integration constant and $t_0^{-1} = e^{\varphi_0}\sqrt{V_0}$. The solution is thus characterized by two relevant parameters, t_0^{-1} and e^{φ_0}, corresponding, up to some numerical factors, to the Hubble parameter and string coupling at the bounce, respectively (without loss of generality we have set $a(0) = 1$ with $t = 0$, the time at which the bounce occurs). We stress that the possibility of regular backgrounds is not limited to the class of potentials illustrated in Eq. (10), and that regular bouncing solutions can also be obtained by adding to the action (1) fluid matter sources, as already pointed out in [2] and illustrated in [14].

The evolution equation of tensor (transverse and traceless) metric perturbations can be obtained by perturbing to first order the (i, j) component of Eq. (3). For each polarization we obtain, in Fourier space,

$$\ddot{h}_k - \dot{\overline{\varphi}}\dot{h}_k + \omega^2 h_k = 0\,, \qquad \omega \equiv k/a. \tag{13}$$

We shall concentrate our attention on the case of regular solutions approaching asymptotically the well- known minimal gravi-dilaton model with negligible potential. The gravitational wave spectra arising in this case are characterized, in general, by a steep spectrum ($n_T > 1$), up to logarithmic corrections. In fact the asymptotic solution of Eq. (13) for large wavelengths can be written as

$$h_k = A_k + B_k \int_{t_{\mathrm{ex}}}^{t} \frac{e^{\varphi}}{a^3} dt, \tag{14}$$

where $t_{\mathrm{ex}} \sim \omega^{-1}$ denotes the time at which the perturbation exits the horizon. Since in the case under consideration $e^{\varphi} a^{-3} \sim |t|^{-1}$ for $t \to -\infty$, we must expect a $\ln \omega t$ growth of h_k, leading, ultimately, to $\ln k$ corrections in the power spectrum.

For an accurate camparison of analytical and numerical results, we shall restrict our attention to the specific regular bouncing solution (12). In this case, the evolution equation for the canonical normal mode $\mu_k = a e^{-\varphi/2} h_k$ has asymptotic solutions with normalization to an initial vacuum fluctuation spectrum given in terms of Hankel functions of index zero.

The application of the standard matching procedure leads to a Bogoliubov coefficient (see below for its exact definition) given by

$$|\beta_k|^2 = \epsilon_1 + \epsilon_2 \ln^2(k_1/k), \tag{15}$$

where ϵ_1 and ϵ_2 are numerical factors of order 1, and k_1 is a characteristic momentum scale. The associated power spectrum is

$$|\delta h_k|^2 = k^3 |h_k|^2 \sim \left(\frac{H(0)}{M_{\rm P}(0)}\right)^2 \left(\frac{k}{k_1}\right)^2 \ln^2\left(\frac{k}{k_1}\right), \tag{16}$$

where the normalization $H(0)/M_P(0) \sim (\lambda_s/t_0)\ e^{\varphi(0)/2}$ is controlled by the ratio between the curvature scale and the Planck mass at the bounce $t = 0$.

For the numerical calculation, it is convenient to work with the rescaled variable $\hat{h} = e^{-\varphi/2} a^{3/2} h = \sqrt{a}\mu$. In order to compute the amplification, we impose for $t \to -\infty$ the quantum mechanical initial conditions

$$\mu_k = \frac{1}{\sqrt{2k}} e^{-ik\eta} \qquad \to \qquad \hat{h}_k = \frac{1}{\sqrt{2\omega}} e^{-i\int \omega dt}, \tag{17}$$

where we recall that we are using units in which $M_{\rm s}^2 = 2$. The action of the gravi-dilaton background will produce a mixing in the positive and negative frequency modes so that, for $t \to +\infty$, the solution can be parametrized as

$$\hat{h}_k = \frac{1}{\sqrt{2\omega}} \left(\alpha_k e^{-i\int \omega dt} + \beta_k e^{i\int \omega dt}\right). \tag{18}$$

In a second quantized approach $|\beta_k|^2$ is nothing but the number of produced gravitons of given momentum k. The energy density of the produced gravitons will then be, up to numerical factors, $\int d\ln k\ k^4 |\beta_k|^2$. Thus, using the theoretical expectations given above, this corresponds to an energy spectrum going as k^4 (up to logarithmic corrections). The numerical integration in terms shows that both $\hat{h}$ and h pass regularly through the bounce; this is illustrated in Fig. 1, where we report the tensor power spectrum, $|\delta h_k|$, in the region around $t = 0$, clearly in qualitative agreement with Eq. (16).The absolute normalization of the tensor power spectrum has been fixed by imposing $H(0)/M_P(0) = 10^{-2}$.

The mixing coefficients of Eq. (18) can be obtained in terms of the asymptotic ($\tau \to +\infty$) values of the real and imaginary parts of $\hat{h}$, via the expressions:

$$|\alpha_k|^2 + |\beta_k|^2 = \left(\omega |\hat{h}_k|^2 + \frac{1}{\omega} \left|\dot{\hat{h}}_k - \frac{H}{2}\hat{h}_k\right|^2\right), \tag{19}$$

$$|\alpha_k|^2 - |\beta_k|^2 = i\Big(\dot{\hat{h}}_k \hat{h}_k^* - \dot{\hat{h}}_k^* \hat{h}_k\Big). \tag{20}$$

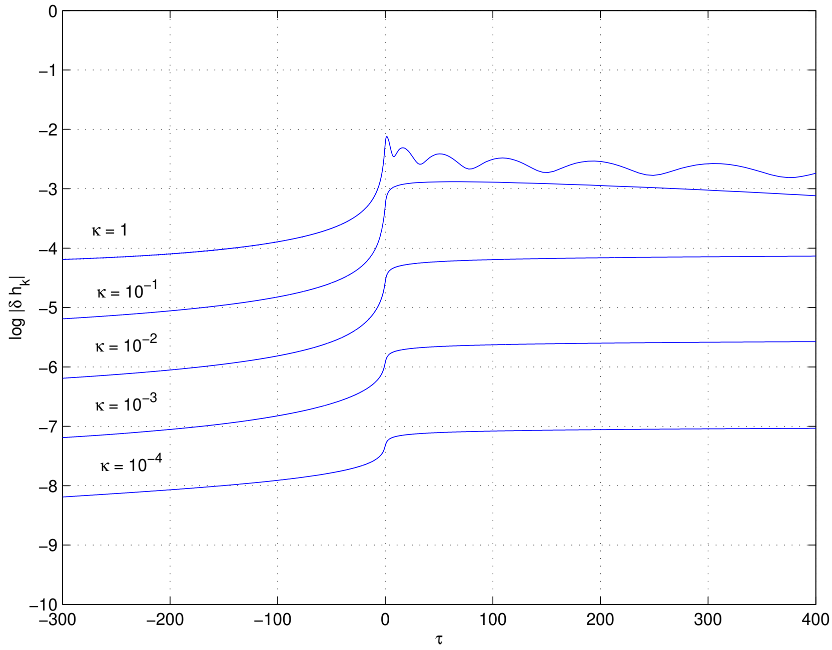

Figure 1: Behaviour of the tensor power spectrum $|\delta h_k|$, for different values of $\kappa = kt_0$, in the region around the bounce. In the present and in the following figures the log denotes the logarithm in basis ten.

We expect Eq. (20) to be identically equal to 1 thanks to the conservation of the Wronskian. On the other hand, Eq. (19) should approach a (k-dependent) constant only at late enough times. These two expectations are perfectly fulfilled, as illustrated in Fig. 2, which thus represents a highly non-trivial consistency check of our numerical procedure.

Clearly, smaller k re-enter the horizon (and thus reach their asymptotic value) later. The asymptotic value gives, for each k, the sought after Bogoliubov coefficient. The results are plotted in Fig. 3 (crosses) and fitted to the theoretical expectation (15). The fit is very good and the parameters $\epsilon_1, \epsilon_2, k_1$ are determined according to:

$$|\beta_k|^2 = 0.46 \ln^2(k/k_1) - 4.84, \qquad\qquad k_1 = 6.68/t_0. \tag{21}$$

We have not yet attempted to make a theoretical estimate of ϵ_2.

Unlike tensor perturbations, scalar fluctuations are, in general, neither gauge- nor frame-independent. Gauge-invariant perturbations can be defined, as usual, in each frame. Among these, the spatial curvature perturbation on uniform dilaton hypersurfaces $\mathcal{R}$, and the Bardeen potentials Ψ and Φ are particularly important for CMB phenomenology. Denoting by $\delta\varphi = \chi$ the dilaton perturbation, and introducing scalar metric perturbations in

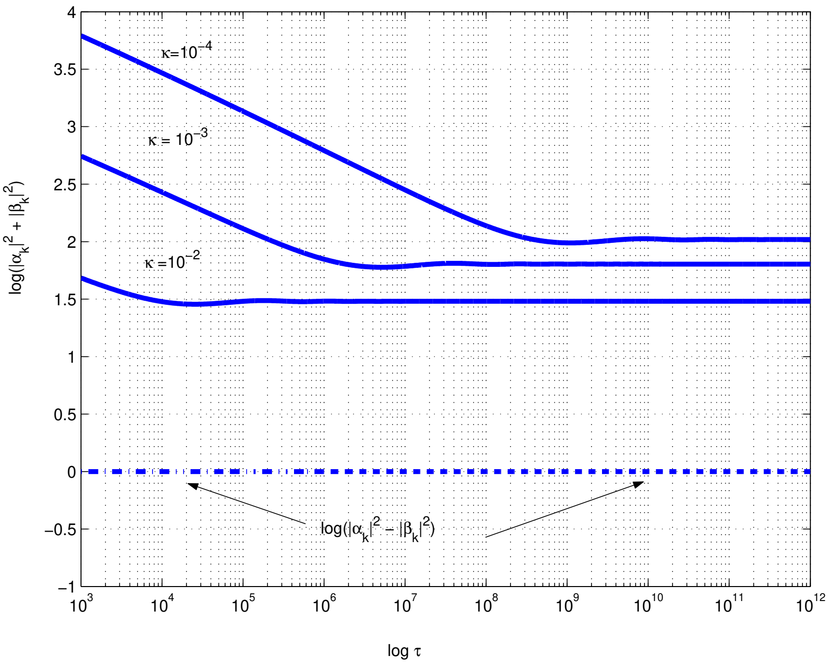

Figure 2: Asymptotic behaviours of the mixing coefficients given in Eqs. (19) and (20) for different values of $\kappa = kt_0$.

conformal time via the standard expressions

$$\delta g_{00} = 2a^2\phi, \qquad \delta g_{ij} = 2a^2(\psi\delta_{ij} - \partial_i\partial_j E), \qquad \delta g_{i0} = a^2\partial_i B\ , \tag{22}$$

we may express $\mathcal{R}$, Ψ and Φ as:

$$\mathcal{R} = -\psi - \frac{\mathcal{H}}{\varphi'}\chi, \quad \Psi = \psi + \mathcal{H}(E' - B), \quad \Phi = \phi - \mathcal{H}(E' - B) - (E' - B)', \tag{23}$$

where primes denote derivatives with respect to conformal time and $\mathcal{H} = a'/a$. Introducing the Einstein frame, with scale factor $a_E = ae^{-\varphi/2}$, we have $\phi_E = \phi_s - \frac{1}{2}\chi$, $\psi_E = \psi_s + \frac{1}{2}\chi$ (while B and E coincide in the two frames), and we can easily prove that $\mathcal{R}$ and $\Psi + \Phi$ are frame-independent, while $(\Psi - \Phi)_s = (\Psi - \Phi)_E - \chi + \varphi' E'$.

We shall work in the convenient uniform dilaton gauge $\chi = 0$, and also set $B = 0$, so that $\mathcal{R} = -\psi$ and $\Psi_E = \psi + \mathcal{H}_E E'$, where $\mathcal{H}_E = \mathcal{H} - \varphi'/2$. This gauge is particularly useful since all the perturbations variables in (22) coincide in the two frames. Another advantage is that, by perturbing the background equation (3), we find in this gauge $\delta\gamma_\mu^\nu = 0$, $\delta e^{-\overline{\varphi}} = 0$, $\delta I_1 = 0$. The perturbations of the background equations thus simply lead to the following set of evolution equations:

$$3(\varphi' - 2\mathcal{H})\psi' + (6\mathcal{H}\varphi' - \varphi'^2 - 6\mathcal{H}^2)\phi + 2\nabla^2\psi - (\varphi' - 2\mathcal{H})\nabla^2 E' = 0, \tag{24}$$

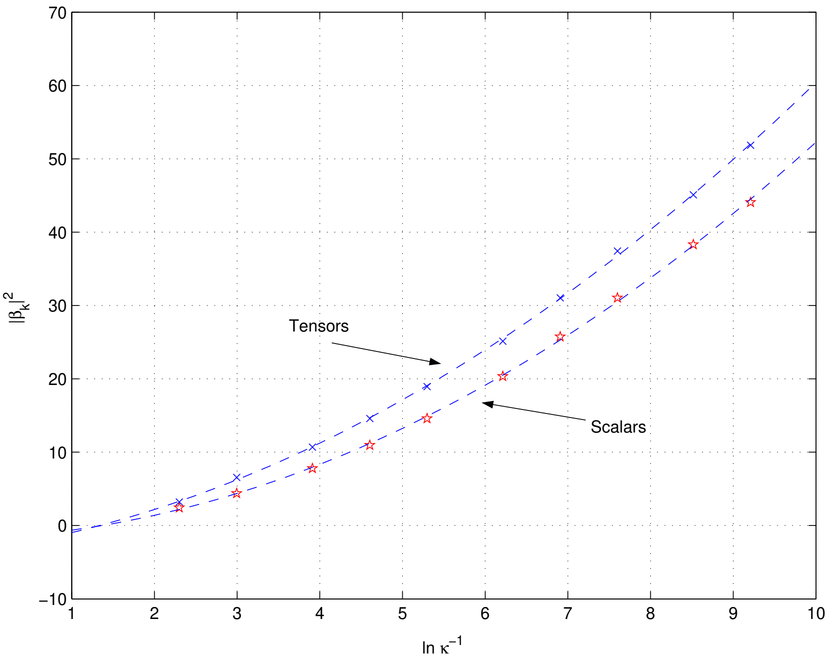

Figure 3: The numerically determined values of $|\beta_k|^2$ are fitted to theoretical expectations for tensor (crosses) and scalar (stars) perturbations.

$$(\varphi' - 2\mathcal{H})\phi = 2\psi', \tag{25}$$

$$E'' + (2\mathcal{H} - \varphi')E' + \psi - \phi = 0, \tag{26}$$

following, respectively, from the (00), $(0i)$ and $(i \neq j)$ components of the perturbed Eq. (3). Note that the last equation, upon using $\mathcal{H}_E = \mathcal{H} - \varphi'/2$, is nothing but $(\Psi - \Phi)_E = 0$, and that the additional equations obtained from the $(i = j)$ component and from the dilaton equation (4) are redundant.

We may combine the above system to get the interesting equation

$$\psi' = \frac{(\varphi' - 2\mathcal{H})}{\varphi'^2} \nabla^2 \left[(\varphi' - 2\mathcal{H})E' - 2\psi \right], \tag{27}$$

which turns out to be exactly the standard evolution equation for $\mathcal{R}$:

$$\mathcal{R}' = -\frac{4\mathcal{H}_e}{\varphi'^2} \nabla^2 \Psi_E. \tag{28}$$

Although this equation is *formally* the same as the standard equation of ordinary cosmological perturbation theory, the relation between φ'^2 to $(\mathcal{H}_E^2 - \mathcal{H}_E')$ is different in our case. Indeed, it can be easily checked, by transforming the background equations (7), (8) to the

Einstein frame, that:

$$4(\mathcal{H}_E' - \mathcal{H}_E^2) + \varphi'^2 + \frac{\partial V}{\partial \varphi} a_e^2 e^{\varphi} = 0. \tag{29}$$

This shows that the usual identity $\varphi'^2 \propto (\mathcal{H}' - \mathcal{H}^2)$ is not satisfied in our case. And indeed, in our class of bouncing solutions φ', unlike $\mathcal{H}_E' - \mathcal{H}_E^2$, *never* vanishes. As a consequence, the pre-factor on the right-hand side of Eq. (28) does not diverge during the bounce; we can thus infer, already at this stage, that $\mathcal{R}$ is likely to be conserved on super-horizon scales.

Let us now turn to the specific study of the background model already analyzed in the case of the tensor perturbations. On the basis of simple estimates based on the asymptotic behaviour of the solutions prior to the bounce and after, we expect the amplification coefficients for the scalar and the tensor modes of the geometry to be very similar, both qualitatively and quantitatively. Indeed, after some lengthy but trivial algebra, the following decoupled evolution equation for $\mathcal{R}$ can be obtained from Eqs.(24)–(26):

$$\mathcal{R}_k'' + 2\frac{z'}{z}\mathcal{R}_k' + k^2 C_s^2 \mathcal{R}_k = 0, \tag{30}$$

where

$$z = \frac{\varphi'}{\mathcal{H}_e} a e^{-\varphi/2}, \qquad C_s^2 = \left(1 + \frac{\partial V}{\partial \overline{\varphi}} \frac{a^2}{\varphi'^2}\right). \tag{31}$$

On the other hand, in order to enforce the correct quantum normalization, we note that $\mathcal{R}$ is related to the canonical normal mode of scalar perturbations, $v_k = z\mathcal{R}_k$. For large $|t|$, $V \to 0$, $z \propto a e^{-\varphi/2}$ and we are led exactly to the same equations as for tensor perturbations, the same asymptotic solutions, and thus the same spectrum for δR_k.

These expectations will now be checked numerically. We first notice that it is impossible to follow the evolution of perturbations directly through the canonical variable v (or the other often used variable u), since these become singular at the (Einstein-frame) bounce. This is not a problem, however, since v is only needed at very early or very late times, for normalization purposes. We can instead integrate directly a first-order system of differential equations, involving $\psi = -\mathcal{R}$ and E', which is completely regular throughout. Going over to (string-frame) cosmic time, and using the constraint (25) into Eqs. (24) and (26), we can eliminate ϕ. We then obtain:

$$\dot{\psi}_k = A_k(t)\psi_k + B_k(t)\mathcal{E}_k, \qquad \dot{\mathcal{E}}_k = C_k(t)\psi_k + D_k(t)\mathcal{E}_k, \tag{32}$$

where $\mathcal{E}_k = a^2 \dot{E}_k$, and

$$A_k(t) = \frac{2(\dot{\varphi} - 2H)}{\dot{\varphi}}\omega^2, \qquad B_k(t) = -\left(\frac{\dot{\varphi} - 2H}{\dot{\varphi}}\right)^2 \omega^2,$$
$$C_k(t) = \frac{4\omega^2}{\dot{\varphi}^2} - 1, \qquad D_k(t) = \dot{\varphi} - H - A_k(t). \tag{33}$$

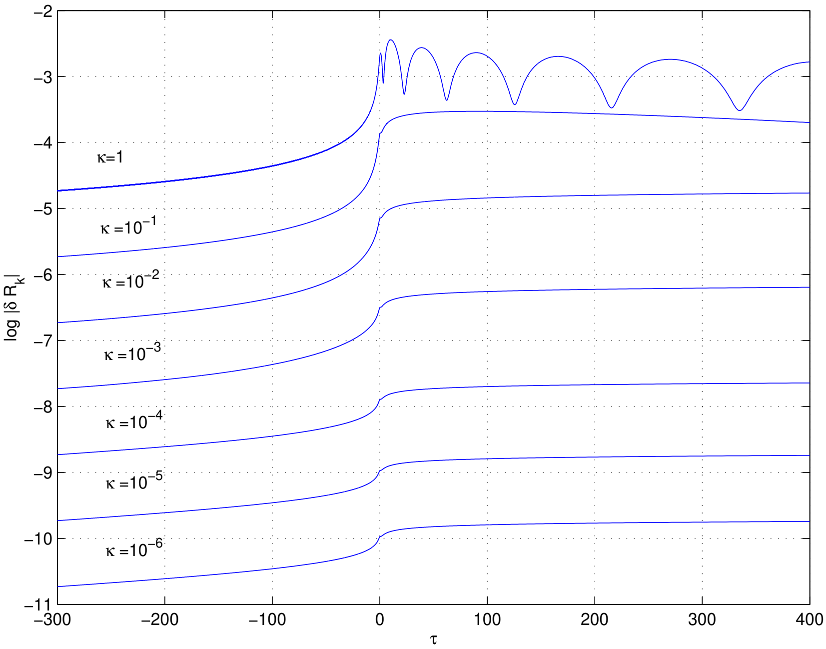

Figure 4: Behaviour of the power spectrum of $\mathcal{R}$, for different values of $\kappa = kt_0$.

This system can be solved by imposing quantum mechanical initial conditions for the fluctuations of the curvature. Using $v_k = z\mathcal{R}_k$ we expect, asymptotically

$$\mathcal{R}_k \sim \frac{1}{w}\frac{1}{\sqrt{2\omega}}\left(\alpha_k e^{-i\int \omega dt} + \beta_k e^{i\int \omega dt}\right), \tag{34}$$

where $w = \sqrt{a}z$.

Following the same lines the discussion as were developed in the case of the tensor modes of the geometry, we can obtain, in the asymptotic region after the bounce, the appropriate combinations that determine the mixing coefficients:

$$\begin{aligned} |\alpha_k|^2 - |\beta_k|^2 &= iw^2\left(\mathcal{R}_k^*\dot{\mathcal{R}}_k - \mathcal{R}_k\dot{\mathcal{R}}_k^*\right), \\ |\alpha_k|^2 + |\beta_k|^2 &= w^2\left(\omega|\mathcal{R}_k|^2 + \frac{1}{\omega}|Y\mathcal{R}_k + \dot{\mathcal{R}}_k|^2\right), \end{aligned} \tag{35}$$

where $Y = \dot{z}/z$. The numerically computed power spectrum of $\mathcal{R}$ is reported in Fig. 4. In full analogy with the procedure discussed before, we have also plotted the quantities appearing in Eqs. (35), checking that the first one gives identically 1, and that the second approaches a limiting value at late times. We can thus extract the corresponding value of $|\beta_k|^2$ and fit it to the theoretical expectation. The results of this analysis are reported in Fig. 3, together with the already discussed results for the tensors. The fit for the scalar

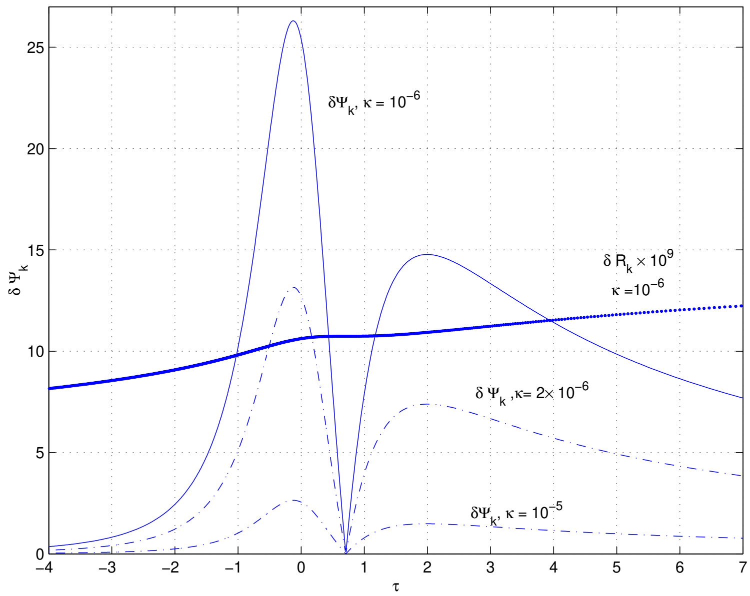

Figure 5: Time evolution of the power spectrum of the (Einstein-frame) Bardeen potential Ψ_E (thin and dashed curves). The bold curve represents the behaviour of an (appropriately rescaled) $\mathcal{R}$ mode.

case (starred points) gives:

$$|\beta_k|^2 = 0.46 \ln^2(k/k_2) - 2.22, \qquad\qquad k_2 = 2.2/t_0. \tag{36}$$

This formula is very similar to the one obtained in the case of tensors perturbations.

We now turn our attention to the behaviour of the (Einstein-frame) Bardeen potential. Its time evolution is shown in Fig. 5, where it is clearly visible that Ψ becomes much larger than one for small $\kappa\eta$, signalling a breakdown of perturbation theory in the longitudinal gauge (a point already emphasized by the first paper quoted in Ref. [13]). The two Bardeen potentials in the string frame, though unequal, exhibit similar pathological behaviours. We also see that, although all variables are regular across the bounce, the values of Ψ and Ψ', unlike those of $\mathcal{R}$, change drastically across the bounce. Thus, assuming continuity of Ψ and of its derivative in simplistic models for the bounce is very dangerous.

Let us finally turn to another popular variable, the spatial-curvature perturbation on constant-density hypersurfaces, ζ. It can be shown from its definition that it is not frame-independent, i.e. that

$$\zeta_s = -\psi_s - \frac{\mathcal{H}_s}{\rho_s'}\delta\rho_s \neq \zeta_E = -\psi_E - \frac{\mathcal{H}_E}{\rho_E'}\delta\rho_E. \tag{37}$$

In any case, in our gauge, ζ_E and ζ_s differ by $\mathcal{R}$ by terms proportional to ϕ. The latter, however, is very suppressed at large scales (cf. Eqs. (25) and (27)). We conclude that also ζ_s and ζ_E are well-behaved across the bounce. The fact that the power spectra of both ζ_E, ζ_s and $\mathcal{R}$ stay small across the bounce implies that perturbation theory remains valid at all times in our gauge, provided $H(0)/M_P(0) < 1$.

We now summarize our results and draw some conclusions:

- Having defined a non-singular bouncing cosmology, we are able to follow the behaviour of the various perturbation variables from beginning to end.

- Tensor perturbations behave as expected, with h becoming constant on superhorizon scales. Its power spectrum can be computed by studying the associated canonical variable μ that turns out to be also regular throughout the bounce. The numerically computed power spectrum is in perfect agreement with the analytic expectations (including log corrections).

- For scalar perturbations, the Bardeen potential Ψ, the curvature perturbation on uniform dilaton hypersurfaces $\mathcal{R}$, and the curvature perturbation on constant density hypersurfaces ζ, all go smoothly through the bounce.

- Both $\mathcal{R}$ and ζ stay constant on superhorizon scales, in agreement with general arguments [17], while Ψ does not.

- Provided that the ratio of the Hubble parameter to the effective Planck mass at the bounce ($t = 0$) is small, $H(0)/M_P(0) < 1$, $\mathcal{R}$ and ζ remain sufficiently small *at all scales* for perturbation theory to remain valid at all times in the comoving gauge.

- By contrast, Ψ becomes so large, at large scales, near the bounce that perturbation theory breaks down in the longitudinal gauge.

- The canonical variable v of scalar perturbations (as well as another often discussed variable, u) exhibits singularities at the bounce. This is not a problem since the only use of v is that of giving the initial normalization of the fluctuations and the final Bogolubov coefficient, and v is well behaved at sufficiently early or late times.

- We are thus able to compute numerically the scalar perturbation power spectrum. We find that it is very similar to that of tensor perturbations and in agreement with the analytic results that follow by assuming a smooth behaviour of $\mathcal{R}$ (or ζ) through the bounce (even when the background itself has discontinuous derivatives).

In conclusion, at least within the class of models considered in this paper, the procedure used to argue that, in bouncing Universes, the spectrum of adiabatic scalar perturbations can be much flatter than the one of tensor perturbations, appears to be unjustified.

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