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# BOOSTING AWAY SINGULARITIES FROM CONFORMAL STRING BACKGROUNDS

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**Abstract**

Generalizing our previous work, we show how $O(d,d)$ transformations can be used to "boost away" in new dimensions the physical singularities that occur generically in cosmological and/or black-hole conformal string backgrounds. As an example, we show how a recent model by Nappi and Witten can be made singularity-free via $O(3,3)$ boosts involving a fifth dimension.

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## 1. Introduction

A short while ago we have shown [1] how $O(d,d)$ transformations [2,3] acting on anisotropic but homogeneous (i.e. space-independent) string cosmologies in $D=d+1$ space-time dimensions, can turn trivial (i.e. flat) string backgrounds into non-trivial (i.e. curved) ones. We also noticed [1] that the $O(d,d)$ "boosted" backgrounds came out free of curvature singularities for non-exceptional values of the boost parameter $\gamma$.

In this note, we generalize the latter observation by showing that, even if the starting point is a generic, singular cosmology in $D=1+1$ dimensions or a $D=1+1$ black hole, the singularity gets "boosted away" by $O(2,2)$ transformations involving a third, originally flat dimension.

We shall then combine cosmological and black-hole solutions by considering the inhomogeneous, $D=4$ cosmological model recently discussed by Nappi and Witten [4], and by showing that the singularities of the model can be boosted away by $O(3,3)$ transformations involving a fifth dimension.

## 2. Boosting away singularities in $D=2$ backgrounds

The existence of torsion-free ($B=0$) $D=2$ string cosmologies or black holes is by now well known [5,6]. In a class of such models (and in a convenient reference frame) the corresponding metric $G_{\mu\nu}$ and dilaton $\phi$ backgrounds are given by:

$$ds^2 = dx^\mu dx^\nu G_{\mu\nu} = -dt^2 + \tanh^{\pm 2}(\sqrt{\Lambda}\ t/2)\ dx^2\ ,\ \Phi = -\ln\sinh(\sqrt{\Lambda}\ t) \qquad (1a)$$

$$ds^2 = -\tan^{\pm 2}(\sqrt{\Lambda}\ x/2)\ dt^2 + dx^2\ ,\ \Phi = -\ln\sin(\sqrt{\Lambda}\ x) \qquad (1b)$$

They are exact solutions of the field equations obtained from the low energy string effective action

$$S = \int d^D x \sqrt{|G|} e^{-\phi}(-\Lambda + R + G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho})$$

Here $H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + cyclic$, $\Phi = \phi - 1/2\ln\det G$ is the "shifted dilaton", which is inert under $O(d,d)$, and $\Lambda$ is the tree-level cosmological constant:

$$\Lambda = \frac{(c - c_{crit})}{3\alpha'}\ , \qquad (2)$$

where $c_{crit}=26$ (or 10) and $c$ is the total central charge of the matter fields, $c = 2 + c_{int}$. Of course, when $\Lambda < 0$, the replacements $\tan(\sqrt{\Lambda}\ t/2) \leftrightarrow \tanh(\sqrt{-\Lambda}\ t/2)$

etc. have to be applied to eqs. (1). The $\pm$ ambiguity occurring in (1) corresponds to cosmologies (or black holes) related to each other by scale-factor-duality [7] (or by its equivalent for black holes), a discrete subgroup of $O(d,d)$. Finally, eq. (1$b$) can be replaced by a Euclidean black hole simply by replacing $-dt^2$ by $d\tau^2$.

The scalar curvatures corresponding to the geometries (1), (2) are readily computed to be:

$$R = \Lambda(\tanh^{\pm 2}(\sqrt{\Lambda}\ t/2) - 1) \tag{3$a$}$$

$$R = -\Lambda(\tan^{\pm 2}(\sqrt{\Lambda}\ x/2) + 1) \tag{3$b$}$$

and thus exhibit singularities at particular values of $t$ or $x$.

Our strategy for removing the singularity (while remaining with a conformal background) consists in adding to the spacetime manifold a second flat spatial direction * $z$. Since, in both cases, the resulting backgrounds do not depend on two (out of the three) coordinates, there will be an $O(2,2)$ group [2,3] acting on the space of such conformal theories.

As already discussed in [1], the space of (gauge-inequivalent) solutions will be given by the coset $O(2,2)/GL(2) \times B_s$, where $B_s$ is the group of constant shifts of $B$. This coset is described by a single parameter $\gamma$, the boost parameter in the plane spanned by the two above-mentioned coordinates. Under $O(2,2)$:

$$M \to \Omega^T M \Omega, \qquad\qquad \Phi \to \Phi, \tag{4}$$

where, as usual [8, 2], $M$ is the $2d$ by $2d$ matrix

$$M = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix}, \tag{5}$$

(hereafter $G$ and $B$ stand for the $d$ by $d$ part of the metric and of the torsion) and, as in [1], the boost matrix $\Omega$ is taken as

$$\Omega(\gamma) = \frac{1}{2}\begin{pmatrix} 1+c & s & c-1 & -s \\ -s & 1-c & -s & 1+c \\ c-1 & s & 1+c & -s \\ s & 1+c & s & 1-c \end{pmatrix}, \tag{6}$$

with $c \equiv \cosh\gamma,\ \ s \equiv \sinh\gamma,\ \ 0 < \gamma < \infty$.

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* Either this dimension was already present or we have to readjust the value of $c_{int}$ in order not to change the value of $\Lambda$.

A straightforward calculation shows that, under the transformation (4), the backgrounds ($1\alpha$), acquire non-trivial components in the $x$–$z$ plane (including a non-vanishing $B$), given by:

$$G_\pm(\gamma) = \begin{pmatrix} \frac{(c-1)+(c+1)a^{\pm 2}}{(c+1)+(c-1)a^{\pm 2}} & \frac{-s(1+a^{\pm 2})}{(c+1)+(c-1)a^{\pm 2}} \\ \frac{-s(1+a^{\pm 2})}{(c+1)+(c-1)a^{\pm 2}} & 1 \end{pmatrix}$$

$$B_\pm(\gamma) = \begin{pmatrix} 0 & \frac{-s(1+a^{\pm 2})}{(c+1)+(c-1)a^{\pm 2}} \\ \frac{s(1+a^{\pm 2})}{(c+1)+(c-1)a^{\pm 2}} & 0 \end{pmatrix},$$

$$\phi_\pm(\gamma) = -\ln[1 + a^{\pm 2} \ \tanh^2(\gamma/2)] \ + \ \phi_\pm(0) \ + \ const \ , \tag{7}$$

where, for the metric ($1a$),

$$a = a(t) = \tanh(\sqrt{\Lambda}\ t/2)$$

$$\phi_+(0) = -2\ln\cosh(\sqrt{\Lambda}\ t/2) \quad , \quad \phi_-(0) = -2\ln\sinh(\sqrt{\Lambda}\ t/2). \tag{8a}$$

For the black-hole metric ($1b$) the same result holds, this time in the $t$–$z$ (or $\tau$–$z$ in the Euclidean case) plane and with:

$$a = a(x) = \tan(\sqrt{\Lambda}\ x/2)$$

$$\phi_+(0) = -2\ln\cos(\sqrt{\Lambda}\ x/2) \quad , \quad \phi_-(0) = -2\ln\sin(\sqrt{\Lambda}\ x/2). \tag{8b}$$

It is straightforward to compute the various curvature tensors for the boosted geometries, choosing e.g. the plus signs. For the curvature scalars we find, respectively:

$$R = \frac{\Lambda}{2}\frac{3+4c-7c^2-8c\cosh^2(\sqrt{\Lambda}\ t/2)}{[(c+1)\cosh^2(\sqrt{\Lambda}\ t/2)+(c-1)\sinh^2(\sqrt{\Lambda}\ t/2)]^2} \tag{9a}$$

$$R = \frac{\Lambda}{2}\frac{3c^2+4c-7-8c\cos^2(\sqrt{\Lambda}\ x/2)}{[(c+1)\cos^2(\sqrt{\Lambda}\ x/2)+(c-1)\sin^2(\sqrt{\Lambda}\ x/2)]^2} \tag{9b}$$

Similar expressions hold for the minus-sign choice in eq. (7). We note that, almost magically, all physical singularities have disappeared from the dilaton field (7) and from the scalar curvatures of the boosted metrics for generic values of the boost parameter $\gamma$. We have checked on the computer that the same is true for the various components of the Riemann and Ricci tensors, as well as for the other curvature invariants.

We may ask about the generality of this result, in particular whether or not it extends to higher-dimensional backgrounds. Some explicit examples indicate that, in more general cases, singularities are not removed by simply boosting in an extra dimension. An example is the one of an isotropic cosmology in $D = d+1$ with $\Lambda = 0$ and [5]

$$a(t) = (t/t_0)^{-\frac{1}{\sqrt{d}}} \quad , \quad \Phi = -\ln(t/t_0)$$

($t_0$ is an integration constant) which is singular at $t = 0$. By introducing an additional flat spatial direction, and by performing the same transformation as before, one obtains

$$\phi(\gamma) = -(1+\sqrt{d} - \frac{2}{\sqrt{d}}) \ln(t/t_0) - \ln[(t/t_0)^{\frac{2}{\sqrt{d}}} + \tanh^2(\gamma/2)]$$

which is still singular at the origin for $d > 1$.

On the contrary, our strategy can be applied, almost without modifications, to the four-dimensional model of Nappi and Witten (NW) [4]. This is not surprising since, as we shall explain below, the NW background itself can be obtained by an $O(2,2)$ boost of the direct product of a pair of two-dimensional models.

## 3. $O(2,2)$ derivation and $O(3,3)$ regularization of the Nappi-Witten model

Consider a string theory with vanishing $\Lambda$ [i.e. with $c = c_{crit}$, see eq. (2)] and containing, besides other degrees of freedom, a four-dimensional subspace of Minkowskian signature. A particular conformal background for such a theory consists of the direct product of a cosmological metric and of a Euclidean black hole, each one living in a two-dimensional subspace. The line element of such a model is thus a particular combination of metrics of the type (1$a$) and (1$b$):

$$ds^2 = dx^\mu dx^\nu G_{\mu\nu} = -dt^2 + dx^2 + \tan^{-2}(\sqrt{V}\ t/2)\ dy^2 + \tan^2(\sqrt{V}\ x/2)\ dz^2,$$

$$\Phi = -\ln\sin(\sqrt{V}\ t) - \ln\sin(\sqrt{V}\ x). \qquad (10)$$

where $V$ is a positive constant.

Taken by itself, the black-hole metric contained in (10) would require a positive cosmological constant $\Lambda = V$, while the cosmological-type background would need an opposite value for $\Lambda$. The complete background (10) is thus conformal for $\Lambda = 0$. Obviously, many other possibilities, with or without an overall cosmological constant, can be considered. For the sake of definiteness and in order to make contact with NW, we shall take in the following $V = 4$, in usual string units $2\alpha' = 1$.

Since the metric in (10) is independent of $y, z$, we can apply to it any $O(2,2)$ transformation to get new solutions. After moding out by gauge transformations we are left, as before, with a one-parameter family of gauge-inequivalent backgrounds given by:

$$M(\delta) = \Omega^T(\delta) M \Omega(\delta), \tag{11}$$

where:

$$\Omega(\delta) = \frac{1}{\sqrt{2}} \begin{pmatrix} \delta & 0 & 0 & \delta \\ 0 & 1 & -1 & 0 \\ 0 & \delta^{-1} & \delta^{-1} & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \tag{12}$$

is an $O(2,2)$ matrix. It is easy to check that $M(\delta)$ reproduces precisely * the family of inhomogeneous NW cosmologies after the following identification of their parameter $\alpha$:

$$\delta^2 = \frac{1 - \sin\alpha}{1 + \sin\alpha} \tag{13}$$

We see here a good example of the power of $O(d,d)$ in generating highly non-trivial conformal backgrounds. Nonetheless, if no extra dimension is called in, the boosts fail to remove the singularities occurring in the original background. As noticed by NW, their cosmological solutions do exhibit curvature singularities, for instance at $t = 0 = x$. Note, however, that the original background (10) was even more singular than the one of NW, since it had curvature singularities even at finite $x$ for $t \to 0$ (and similar ones at special values of $x$ and generic $t$).

We shall now show how to altogether eliminate the singularities of the NW background [or of that of eq. (10)] by introducing a fourth, originally flat spatial direction, parametrized by the coordinate $w$. The theory now acquires a larger symmetry isomorphic to $O(3,3)$.

Given the results of Section 2, it is natural to try to boost away the singularities of the NW model by performing, successively, a boost in the $z$–$w$ plane and one in the $y$–$w$ plane. The first will certainly remove the singularities along the $x$ axis, simply by repeating the steps of Section 2. The nice surprise is that the second boost eliminates also the singularities along the time direction without introducing back the ones already removed.

We shall now give some details of the actual calculations. Starting from the matrix $M$ corresponding to the metric (10) and $B = 0$, the boosted backgrounds

* This observation was first made by A. Giveon, see Note Added in ref. [4].

are given in terms of a boosted $M$ by:

$$\tilde{M}(\gamma_1,\gamma_2) = \Omega^T(\gamma_2)\Omega^T(\gamma_1)M\Omega(\gamma_1)\Omega(\gamma_2), \tag{14}$$

where $\Omega(\gamma_1)$ is the $z$–$w$ boost [here, in analogy with eq. (6), we use the notation $c_1 \equiv \cosh\gamma_1$ etc.]:

$$\Omega(\gamma_1) = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1+c_1 & s_1 & 0 & c_1-1 & -s_1 \\ 0 & -s_1 & 1-c_1 & 0 & -s_1 & 1+c_1 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & c_1-1 & s_1 & 0 & 1+c_1 & -s_1 \\ 0 & s_1 & 1+c_1 & 0 & s_1 & 1-c_1 \end{pmatrix}, \tag{15}$$

and $\Omega(\gamma_2)$ is a similar $y$–$w$ boost:

$$\Omega(\gamma_2) = \frac{1}{2}\begin{pmatrix} 1+c_2 & 0 & s_2 & c_2-1 & 0 & -s_2 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ -s_2 & 0 & 1-c_2 & -s_2 & 0 & 1+c_2 \\ c_2-1 & 0 & s_2 & 1+c_2 & 0 & -s_2 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ s_2 & 0 & 1+c_2 & s_2 & 0 & 1-c_2 \end{pmatrix}. \tag{16}$$

Computations can be done easily with some computer help and lead to the following doubly-boosted backgrounds (for simplicity $\gamma_1 = \gamma_2 = \gamma$, which is already sufficient for our purpose)

$$\tilde{G}_{11}(\gamma) = \frac{c-1+(c+1)a^2}{c+1+(c-1)a^2} \quad , \quad \tilde{G}_{12}(\gamma) = \frac{s^2(a^2-1)(b^2+1)}{[c+1+(c-1)a^2][c+1+(c-1)b^2]}$$

$$\tilde{G}_{13}(\gamma) = -\frac{s(1+a^2)}{c+1+(c-1)a^2} \quad , \quad \tilde{G}_{22}(\gamma) = \frac{c-1+(c+1)b^2}{c+1+(c-1)b^2}$$

$$\tilde{G}_{23}(\gamma) = -\frac{s(1+b^2)[c(a^2-1)-a^2-1]}{[c+1+(c-1)a^2][c+1+(c-1)b^2]} \quad , \quad \tilde{G}_{33}(\gamma) = 1$$

$$\tilde{B}(\gamma) = \begin{pmatrix} 0 & -\tilde{G}_{12}(\gamma) & \tilde{G}_{13}(\gamma) \\ \tilde{G}_{12}(\gamma) & 0 & \tilde{G}_{23}(\gamma) \\ -\tilde{G}_{13}(\gamma) & -\tilde{G}_{23}(\gamma) & 0 \end{pmatrix}$$

$$\tilde{\phi}(\gamma) = \Phi + \ln\frac{4ab}{[c+1+(c-1)a^2][c+1+(c-1)b^2]} \tag{17}$$

(the indices $1, 2, 3$ run over the set of coordinates $(y, z, w)$). Here $a^2 = \tan^{-2}(t)$, $b^2 = \tan^2(x)$, and $\Phi$ is the shifted dilaton of eq.(10), but the result (17) holds generally for any background whose metric may be written in the form $G_{\mu\nu} = diag\ (-1, 1, a^2, b^2)$.

At this point various components of the curvature tensor and their contractions can be computed with the help of a program (we have used MACSYMA) and some have been double checked either analytically or through consistency with the equations of motion (vanishing $\beta$-function conditions). In all cases we have found no singularity occurring in the boosted metrics.

As an example we quote here the curvature scalar and the coupling constant, which are given respectively by:

$$R = \frac{16\cos^2 t \sin^2 t - 20s^2}{[(c+1)\sin^2 t + (c-1)\cos^2 t]^2} + \frac{20s^2 - 16\cos^2 x \sin^2 x}{[(c+1)\cos^2 x + (c-1)\sin^2 x]^2}$$

$$e^{\phi} = [(c+1)\sin^2 t + (c-1)\cos^2 t]^{-1}[(c+1)\cos^2 x + (c-1)\sin^2 x]^{-1}. \qquad (18)$$

Strictly speaking, in order to obtain a NW-like model, we should still perform a third boost in the $y$–$z$ plane, with the same $\Omega(\delta)$ as in eq. (12). This would complicate the result (17), but it is quite obvious that it would not alter the conclusion that all singularities are indeed removed.

This result is not in contradiction, of course, with the classical singularity theorems [9]. By computing the components of the Ricci tensor for the doubly-boosted metric we have indeed, from eqs. (17),

$$\tilde{R}_{00} = \frac{2s^2 + 4(2c\ \cos^2 t - c - 1)}{[(c+1)\sin^2 t + (c-1)\cos^2 t]^2}. \qquad (19)$$

If $c = 1$, the expression $R_{00} = -2/\sin^2 t$ is recovered. It is valid for the original torsionless background of eq. (10). In that case $R_{00}$ is always negative, so that the strong energy condition is everywhere satisfied. If, instead, $c > 1$, one may see from eq. (19) that the sign of $R_{00}$ changes around the points that correspond to singularities of the original metric (such as, for instance, $\cos t = 1$). The strong-energy condition is thus violated, and this explains why the singularity theorems can be evaded. Moreover, the presence of torsion in the boosted background (17) seems to stress the crucial role played by this field in avoiding singularities, in agreement with our previous conjecture [1].

In conclusion, we have seen: i) how $O(d, d)$ transformations acting within the space-time dimensions of a given singular background can generate new non-trivial conformal backgrounds, without major effects on the singularities themselves, and ii) how $O(d+1, d+1)$ transformations involving an extra dimension (with originally trivial geometry) can instead "boost away" the original singularities by smearing out the curvature over a range of the original variables, and by involving in a non-trivial way the new dimension.

The full meaning and generality of our results remain to be understood. It would be nice, for instance, to know if they depend heavily on having started with a direct product of $D = 2$ backgrounds. We have seen that the most naive generalization of our procedure to genuine $D > 2$ backgrounds does not work, but we cannot exclude that a more general technique will be able to eliminate the singularities. In any event, we believe the phenomenon we have described to be a positive step in the quest for singularity-free and more realistic cosmological string scenarios.

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