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HOMOGENEOUS CONFORMAL STRING BACKGROUNDS

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ABSTRACT

We present exact solutions characterised by Bianchi-type I, II, III, V, VI_0, VI_h four-dimensional metric, space-independent dilaton, and vanishing torsion background, for the low energy string effective action with zero central charge deficit. We show that, in such a context, curvature singularities cannot be avoided, except for the trivial case of flat spacetime and constant dilaton. We also provide a further example of the failure of the standard prescription for connecting conformal string backgrounds through duality transformations associated to non-semisimple, non-Abelian isometry groups.

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1 Introduction

It is well known that the request for conformal invariance of the sigma-model action for closed (super)string theory implies, at the tree level in the string loop expansion parameter e^{ϕ}, and to lowest order in the inverse of the string tension α', the background field equations of motion [1]

$$R_\mu{}^\nu + \nabla_\mu \nabla^\nu \phi - \frac{1}{4} H_{\mu\alpha\beta} H^{\nu\alpha\beta} = 0, \tag{1}$$

$$R - \nabla_\mu \phi \nabla^\mu \phi + 2\nabla_\mu \nabla^\mu \phi + V - \frac{1}{12} H_{\mu\nu\alpha} H^{\mu\nu\alpha} = 0, \tag{2}$$

$$\partial_\mu (e^{-\phi} \sqrt{|g|} H^{\mu\alpha\beta}) = 0. \tag{3}$$

Here V is a constant, ϕ is the dilaton field, $H_{\mu\nu\alpha} = 6\partial_{[\mu} B_{\nu\alpha]}$ is the field strength of the antisymmetric (torsion) tensor $B_{\mu\nu} = -B_{\nu\mu}$, and the covariant derivatives are performed with respect to the background metric $g_{\mu\nu}$. These equations can be derived from the low energy (D-dimensional) string effective action

$$S = -\int d^D X \sqrt{|g|} e^{-\phi} (R + \partial_\mu \phi \partial^\mu \phi + V - \frac{1}{12} H_{\mu\nu\alpha} H^{\mu\nu\alpha}). \tag{4}$$

In this paper we present a general procedure to integrate the equations (1)-(3) for the case of spatially homogeneous metric, space-independent dilaton, vanishing torsion ($H_{\mu\nu\alpha} = 0$) and critical dimension ($V = 0$). This last requirement does not exclude, of course, the phenomenologically interesting case of $d = D - 1 = 3$, provided one adds the right number of "spectator" dimensions in order to compensate the central charge deficit. For $d = 3$, in particular, our procedure can be applied to obtain exact solutions for anisotropic but homogeneous backgrounds, whose metric can be classified of type I, II, III, V, VI$_0$, VI$_h$ according to the Bianchi classification (see for instance [2, 3, 4]). These solutions may prove useful to study the correct implementation of the duality symmetry between conformal string backgrounds in the case of non-Abelian isometries [5] (see for instance the discussion in [6] of the particular Bianchi V model reported in [7]).

We recall that, in the hypothesis of spatial homogeneity, the d-dimensional spatial submanifold is invariant under the action of a d-parameter isometry group (generated by the d Killing vectors $\xi_i{}^\alpha$, $i = 1, 2, \ldots, d$), and the metric can be factorized (in a synchronous frame [3]) as

$$g_{00} = 1, \quad g_{0\alpha} = 0, \quad g_{\alpha\beta}(t, \vec{X}) = e_\alpha{}^i(\vec{X})\gamma_{ij}(t)e_\beta{}^j(\vec{X}), \tag{5}$$

($\alpha, \beta = 1, \dots, d$ are world indices in the spatial submanifold). All dependence on the spatial coordinates X^α is thus contained in the "spatial" *vielbein* $e_\alpha{}^i$, whose corresponding Ricci rotation coefficients

$$C_{ij}{}^k = e^\alpha{}_i e^\beta{}_j(\partial_\alpha e_\beta{}^k - \partial_\beta e_\alpha{}^k) \tag{6}$$

are constant and determined by the algebraic structure of the isometry group as [2, 3, 4]

$$[\xi_i, \xi_j] = C_{ij}{}^k \xi_k, \qquad \xi_i = \xi_i{}^\alpha \partial_\alpha. \tag{7}$$

Under the additional hypothesis that the dilaton field be space-independent, the background equations (1)-(3) can be reduced to ordinary time-differential equations for the variables $\gamma_{ij}(t), \phi(t)$. The spatial dependence of $R_\mu{}^\nu$ and $\nabla_\mu \nabla^\nu \phi$ is determined, indeed, by the choice of the isometry group, and can be factored out and eliminated through a projection on the spatial *vielbein* $e_\alpha{}^i$ [3, 4]:

$$R_\alpha{}^\beta \to R_i{}^j = e^\alpha{}_i R_\alpha{}^\beta e_\beta{}^j, \tag{8}$$
$$R_\alpha{}^0 \to R_i{}^0 = e^\alpha{}_i R_\alpha{}^0, \tag{9}$$
$$\nabla_\alpha \nabla^\beta \phi \to \nabla_i \nabla^j \phi = e^\alpha{}_i e_\beta{}^j \nabla_\alpha \nabla^\beta \phi, \tag{10}$$
$$\nabla_\alpha \nabla^0 \phi \to \nabla_i \nabla^0 \phi = e^\alpha{}_i \nabla_\alpha \nabla^0 \phi. \tag{11}$$

In particular, if we restrict our analysis to an anisotropic but diagonal matrix form for the invariant metric γ_{ij},

$$\gamma_{ij}(t) = -a_i^2(t)\delta_{ij}, \tag{12}$$

the projection gives

$$R_0{}^0 = R_0{}^0(\gamma_{mn}), \tag{13}$$
$$R_i{}^j = \left[R_i{}^i(\gamma_{mn}) + V_i(\gamma_{mn}, C_{mn}{}^r)\right] \delta_i{}^j, \tag{14}$$
$$R_i{}^0 = \frac{1}{2}(\delta_i{}^k C_{lj}{}^l - C_{ji}{}^k)\dot{\gamma}_{kl}\gamma^{lj}, \tag{15}$$
$$\nabla_0 \nabla^0 \phi = \ddot{\phi}, \tag{16}$$
$$\nabla_i \nabla^j \phi = \dot{\phi} H_i \delta_i{}^j, \tag{17}$$
$$\nabla_i \nabla^0 \phi = 0, \tag{18}$$

(no sum over i; a dot denotes differentiation with respect to the cosmic time t). Here $H_i = \dot{a}_i/a_i$, and $R_0{}^0(\gamma)$, $R_i{}^j(\gamma)$ are the time and space components of the Ricci tensor for the metric (12). The "effective potentials" $V_i(\gamma, C)$ (vanishing for $C_{ij}{}^k = 0$) represent the explicit contribution of the non-Abelian part of the isometry group and are related to the Riemann curvature of the spatial submanifold.

From the $(0,0)$ and (i,i) part of the background eq. (1) we thus obtain, respectively,

$$\sum_i (\dot{H}_i + H_i^2) - \ddot{\phi} = 0, \tag{19}$$

$$\dot{H}_i + H_i \sum_k H_k - H_i\dot{\phi} - V_i = 0, \tag{20}$$

while the mixed components $(i,0)$ give the constraint

$$\sum_{k=1}^{d} C_{ki}{}^k (H_i - H_k) = 0, \tag{21}$$

(no sum over i). The dilaton equation (2) moreover implies

$$2\ddot{\phi} - \dot{\phi}^2 + 2\dot{\phi}\sum_k H_k + \sum_k V_k - (\sum_k H_k)^2 - \sum_k H_k^2 - 2\sum_k \dot{H}_k = 0 \tag{22}$$

In the following section it will be shown that the above equations (19)-(22) can be integrated exactly, provided the potential functions $V_i(a_j)$ satisfy particular restrictions.

2 General integration method for a class of homogeneous backgrounds

In order to integrate the equations (19)-(22) we shall try to extend to the more general homogeneous case a procedure already successfully applied to space-independent metric backgrounds even in the presence of string sources [8, 9], non-vanishing torsion [10] and a particular class of dilaton potentials [8].

We introduce, first of all, the rescaled dilaton $\bar{\phi}$,

$$\bar{\phi} = \phi - \frac{1}{2}\ln|\det(\gamma_{ij})| = \phi - \sum_j \ln a_j, \tag{23}$$

which is exactly the duality-invariant variable defined in the context of the particular "scale factor" duality symmetry, for space-independent cosmological metrics [11, 12]. In terms of this variable the equations (19), (20), (22) become respectively:

$$\ddot{\bar{\phi}} - \sum_i H_i^2 = 0, \tag{24}$$

$$\dot{H}_i - H_i\dot{\bar{\phi}} - V_i = 0, \tag{25}$$

$$\dot{\bar{\phi}}^2 - 2\ddot{\bar{\phi}} + \sum_i H_i^2 - \sum_i V_i = 0. \tag{26}$$

The combination of eqs. (24) and (26) gives

$$\dot{\bar{\phi}}^2 - \sum_i H_i^2 - \sum_i V_i = 0. \tag{27}$$

By differentiating the equation above and using (24), (25) to eliminate $\ddot{\bar{\phi}}$, $\dot{H}_i$, we get

$$\sum_i (\dot{V}_i + 2H_iV_i) = 0, \tag{28}$$

which can be interpreted as a sort of covariant conservation equation for the effective "source density" $\sum_i V_i$, following from the Bianchi identities of the effective scalar-tensor theory.

We choose now eqs. (25)-(27) as independent equations, and we show that they can be integrated exactly for all V_i satisfying the condition

$$V_i = k_i \sum_{j=1}^{d} V_j, \tag{29}$$

where k_i can be arbitrary real numbers.

By combining eqs. (26), (27) we get in fact

$$(e^{-\bar{\phi}})^{\cdot\cdot} = e^{-\bar{\phi}} \sum_j V_j, \tag{30}$$

while eq. (25), using (29), can be rewritten as

$$(e^{-\bar{\phi}} H_i)^{\cdot} = e^{-\bar{\phi}} k_i \sum_j V_j. \tag{31}$$

If we substitute t for a new dimensionless time-like variable x, defined by

$$\frac{1}{L}\frac{dx}{dt} = e^{-\bar{\phi}} \sum_j V_j, \tag{32}$$

(L is an appropriate dimensional constant), eqs. (30) and (31) can be integrated a first time to give

$$(e^{-\bar{\phi}})' e^{-\bar{\phi}} \sum_j V_j = \frac{(x+x_0)}{L^2}, \tag{33}$$

$$\frac{a_i'}{a_i} e^{-\bar{\phi}} \sum_j V_j = \frac{e^{\bar{\phi}}}{L^2}\Gamma_i, \tag{34}$$

where

$$\Gamma_i = k_i x + x_i \tag{35}$$

(x_i, x_0 are integration constants, and a prime denotes differentiation with respect to x). Moreover, using eqs. (29) and (34), the identity (28) can be written as

$$\sum_j V_j' = -\frac{e^{2\bar{\phi}}}{L^2} \sum_j (\Gamma_j^2)'. \tag{36}$$

By adding eqs. (33), (36), and integrating, we thus obtain the important constraint

$$L^2 e^{-2\bar{\phi}} \sum_j V_j = \beta + (x+x_0)^2 - \sum_j \Gamma_j^2, \tag{37}$$

which allows the separation of variables in eqs. (33), (34) and which, as we shall see, ultimately defines the range of validity of our solution with respect to the x coordinate (β is an integration constant).

The constant β appearing in eq. (37) is not arbitrary. Indeed, out of the three independent equations (25)-(27) we have used, up to now, only eq. (25) and a linear combination of eqs. (26) and (27). We still have the freedom to impose that eq. (27) be also separately satisfied by the result of our first integration, eqs. (33), (34). By computing $\dot{\bar{\phi}}$ and H_i from eqs. (32)-(34), and inserting their values into eq. (27), we find that this last equation is identically satisfied, and compatible with eq. (37), if and only if $\beta = 0$. Using eq. (37) (with $\beta = 0$) the system of coupled differential equations (33),

(34) can be consistently reduced to quadratures, and we are eventually led to

$$\bar{\phi}' = -\frac{x + x_0}{D(x)}, \tag{38}$$

$$\frac{a_i'}{a_i} = \frac{\Gamma_i}{D(x)}, \tag{39}$$

where the quadratic form $D(x)$ must satisfy the condition

$$D(x) \equiv (x + x_0)^2 - \sum_i \Gamma_i^2 = L^2 e^{-2\bar{\phi}} \sum_j V_j. \tag{40}$$

Our background equations can thus be integrated exactly for all homogeneous metrics satisfying eq. (29), and the solution is valid for the range of x compatible with the constraint (40). Moreover, the allowed values of the constant "charges" k_i, and of the integration constants x_i, are further restricted by the mixed components of the background equations, $R_i{}^0 = 0$. The insertion of eq. (39) into eq. (21) gives in fact the additional constraints on the solution

$$\sum_k C_{ki}{}^k (k_i - k_k) = 0, \quad \sum_k C_{ki}{}^k (x_i - x_k) = 0 \tag{41}$$

(no sum over i).

We finally note that our integration procedure obviously applies also to the trivial case $V_i = 0$ (Abelian isometry group of spatial translations). In this case, however, there is no need to introduce a new time variable and from eqs. (30), (31) we obtain directly

$$e^{\bar{\phi}} = \frac{L}{c_0 t + t_0}, \qquad H_i = \frac{c_i}{c_0 t + t_0} \tag{42}$$

where c_i, c_0, t_0, L are integration constants, related by the condition

$$c_0^2 = \sum_i c_i^2, \tag{43}$$

which is required in order to satisfy separately also eq. (27). One thus recovers the well-known "Kasner-like" anisotropic background [11, 13], first derived in the context of the Brans-Dicke solutions in vacuum [14].

In the following section we shall apply the integration procedure just outlined to the case of homogeneous cosmological backgrounds in $d = 3$ spatial dimensions.

3 Bianchi-type solutions and curvature singularities

Homogeneous manifolds with $d = 3$ spatial dimensions can be classified in nine different Bianchi types [2, 3, 4], according to the structure of their isometry groups. By considering the explicit form of the potential functions $V_i(a_j)$ for the various metric types (see for instance [15]), one finds that the conditions of applicability of our integration procedure are met for Bianchi types I, II and VI_h (in the notations of Ref. [2]). This last case includes Bianchi types III, V and VI_0, corresponding to $h = 0$, 1 and -1 respectively.

Bianchi I type is characterised by an Abelian isometry group, $V_i = 0$, and in this case the integration of eqs. (42) leads to the previously quoted solution [11, 13, 14]. For a Bianchi II metric there is only one non-vanishing structure constant,

$$C_{31}{}^{2} = 1 = -C_{13}{}^{2} \tag{44}$$

and eq. (29) is satisfied with

$$k_i = (-1, 1, 1), \qquad L^2 \sum_j V_j = \frac{a_1^2}{2a_2^2 a_3^2} \geq 0 \tag{45}$$

For Bianchi VI_h the structure constants are

$$C_{21}{}^{2} = 1 \quad , \qquad C_{31}{}^{3} = h \tag{46}$$

and eq. (29) is satisfied with

$$k_i = \frac{1}{2(1+h+h^2)}(1+h^2, 1+h, h+h^2), \quad L^2 \sum_j V_j = \frac{2}{a_1^2}(1+h+h^2) \geq 0 \tag{47}$$

One can easily verify that the constants k_i of the above Bianchi models also automatically satisfy the constraint (41).

In the case of Bianchi II and Bianchi VI_h metric, the general form of the background solution is thus provided by the explicit integration of eqs. (38), (39). By calling $x_\pm$ the two real zeros of $D(x)$ (the case of complex roots, and of real but coincident roots $x_+ = x_-$, will be discussed below) we obtain

$$\frac{a_i}{a_{i0}} = |(x - x_+)(x - x_-)|^{\frac{k_i}{2\alpha}} \left| \frac{x - x_+}{x - x_-} \right|^{\frac{\alpha_i}{2}}, \tag{48}$$

$$e^{\bar{\phi}} = e^{\bar{\phi}_0} |(x - x_+)(x - x_-)|^{-\frac{1}{2\alpha}} \left| \frac{x - x_+}{x - x_-} \right|^{-\frac{1}{2}\sum_i k_i \alpha_i}, \tag{49}$$

where a_{i0}, $\bar{\phi}_0$ are integration constants, and

$$\alpha = 1 - \sum_i k_i^2, \tag{50}$$

$$\alpha_i = \frac{\alpha x_i + k_i(\sum_j k_j x_j - x_0)}{\alpha\sqrt{(\sum_j k_j x_j - x_0)^2 + \alpha(\sum_j x_j^2 - x_0^2)}}, \tag{51}$$

$$\sum \alpha_i k_i = \frac{\sum_i k_i x_i - x_0 \sum_i k_i^2}{\alpha\sqrt{(\sum_j k_j x_j - x_0)^2 + \alpha(\sum_j x_j^2 - x_0^2)}}, \tag{52}$$

$$x_\pm = \frac{1}{\alpha}\left(\sum_j k_j x_j - x_0 \pm \sqrt{(\sum_j k_j x_j - x_0)^2 + \alpha(\sum_j x_j^2 - x_0^2)}\right). \tag{53}$$

The coefficients k_i are given by eqs. (45) and (47) for Bianchi types II and VI$_h$ respectively, and the integration constants x_i must satisfy the constraint (41), which for the Bianchi VI$_h$ type reads explicitly

$$(1+h)x_1 = x_2 + h x_3. \tag{54}$$

Further restrictions on the solutions follow from eq. (40), which in the Bianchi II case imposes a relation among the integration constants a_{i0}, $\bar{\phi}_0$, and which in the Bianchi VI$_h$ case also defines the allowed range of h, for any given choice of the integration constants. It is interesting to note that for $h = 1$ (Bianchi V), a possible choice is the particular case $x_1 = x_2 = x_3$, which leads to an isotropic homogeneous solution with $a_1 = a_2 = a_3$. Such solution represents a Friedman-Robertson-Walker conformal string background with constant (negative) spatial curvature, while the isotropic version of the Bianchi I solution (42) represents the corresponding background with vanishing spatial curvature.

The temporal range of validity of the solution (48), (49) is also determined by eq. (40), which implies

$$\mathrm{sign}(D) = \mathrm{sign}(\sum_j V_j) \geq 0. \tag{55}$$

For a Bianchi II metric we have $\alpha < 0$ (see eq. (45)), and the solution is thus defined in the limited range

$$x_- < x < x_+. \tag{56}$$

For a Bianchi VI_h metric we must treat separately the particular case $h = -1$ (Bianchi VI_0), for which $\alpha = 0$ and the quadratic form $D(x)$ degenerates in a line which crosses the x axis at

$$x = x_c = \frac{\sum_j x_j^2 - x_0^2}{2(x_0 - x_1)}. \tag{57}$$

The solution is defined, in this case, on the half-line $x > x_c$. For all other values of h we have $\alpha > 0$ and the solution is characterised by two branches, defined on the two half-lines

$$x < x_-, \qquad x > x_+. \tag{58}$$

In correspondence of the two roots of $D(x)$ both H_i and $\exp(\bar{\phi})$ diverge, and the background solutions run into a singularity of both the curvature and the effective string coupling constant. A similar singularity occurs for the Bianchi I solution (42), which is characterised by two branches, defined on the two half-lines

$$t < -t_0/c_0, \qquad t > -t_0/c_0 \tag{59}$$

and separated by a curvature singularity at $t = -t_0/c_0$. Such singularities cannot be avoided in the context of the low energy string effective action considered here, except for the trivial case of flat spacetime and constant dilaton solution.

Indeed, necessary conditions to prevent divergences of the curvature and dilaton background turn out to be 1) the absence of real zeros of $D(x)$ or 2) the coincidence of the two real zeros of $D(x)$ among themselves and with the zeros of the two numerators at the right-hand-side of eqs. (38), (39), namely $x + x_0 = 0 = k_i x + x_i$, where $D(x) = 0$.

If the quadratic form $D(x)$

$$D(x) = (x + x_0)^2 - \sum_i (k_i x + x_i)^2 = \alpha(x - x_+)(x - x_-) \tag{60}$$

has no real zeros, however, it must be always negative. Therefore, the first requirement cannot be satisfied neither by Bianchi II nor by Bianchi VI_h solutions, as it would be in contradiction with the condition (55). In the Bianchi I case the first requirement could be satisfied by the choice $c_0 = 0$, but this implies that all the constants c_i are vanishing, namely that the solution is trivial (see eqs. (42), (43)).

The second requirement can be met by choosing the integration constants x_i in such a way that the two real roots of $D(x)$ coincide with x_0, namely for

$$x_i = k_i x_0, \qquad x_+ = x_- = -x_0, \qquad D(x) = \alpha(x+x_0)^2 \tag{61}$$

In this case, however, the Bianchi II and Bianchi VI_0 solutions are consistently defined only on a point (where $\sum_j V_j = 0$), according to eq. (55). For a Bianchi VI_h metric ($h \neq -1$), on the contrary, the range of validity is non-trivial, and the solution is defined by the equations

$$\bar{\phi}' = -\frac{1}{\alpha(x+x_0)}, \qquad \frac{a_i'}{a_i} = \frac{k_i}{\alpha(x+x_0)}, \qquad \alpha = 1 - \sum_i k_i^2, \tag{62}$$

with the coefficients k_i of eq. (47). Their integration gives

$$\bar{\phi} = \bar{\phi}_0 + \ln|x+x_0|^{-\frac{1}{\alpha}}, \qquad a_i = a_{i0}|x+x_0|^{\frac{k_i}{\alpha}}, \tag{63}$$

where $\bar{\phi}_0$ and a_{i0} are integration constants.

This solution, however, is only valid for the set of values of $\bar{\phi}_0$, a_{i0} and h satisfying the constraint (40). As a consequence, its dynamical content is trivial, as one can easily check by noting first of all that the dilaton background is constant (according to the definition (23)),

$$\phi = \bar{\phi} + \sum_j \ln a_j = \bar{\phi}_0 + \sum_i \ln a_{i0} = const, \tag{64}$$

since $\sum_i k_i = 1$. Moreover, choose for instance the integration constants in such a way that the scale factors, when expressed in cosmic time according to eq. (32), are given by

$$a_i(t) = |t|^{\beta_i}, \qquad \beta_i = (1, \frac{1+h}{1+h^2}, \frac{h(1+h)}{1+h^2}) \tag{65}$$

and the full Bianchi VI_h metric ($h \neq -1$) takes the form

$$g_{\mu\nu}(\vec{X}, t) = \mathrm{diag}(1, -t^2, -t^{2\beta_2}e^{-2X}, -t^{2\beta_3}e^{-2hX}). \tag{66}$$

The constraint (40) implies then a condition on h which is only satisfied, for real values of the parameter, by $h = 0$ and 1 (see the Appendix). In both cases, the solution (66) is identically Ricci flat and Riemann flat (see the Appendix), showing that also the metric background is trivial.

4 Conclusion

In this paper we have presented a procedure for obtaining homogeneous background solutions for the low energy string effective action. Such solutions are characterized by a spatial, generally non-Abelian transitive isometry group, and may be useful for investigating possible extensions of the $O(d,d)$ covariance (see [16] and references therein) associated to backgrounds with Abelian translational symmetry. Moreover, in $d=3$ spatial dimensions such solutions correspond to homogeneous Bianchi type models, which may be of some phenomenological interest for applications to a very early cosmological regime with non-vanishing anisotropy and time-varying dilaton field. The explicit form of the metric and dilaton field, for the particular case of Bianchi I, II, III, V and VI_0 models, is given explicitly in **Table I**.

The solutions reported in the table refer to the case in which the zeros of $D(x)$ are real and both different from the zeros of Γ_i and of $x+x_0$ (otherwise the dilaton is constant, and the metric globally flat up to reparametrizations). The solutions (except those of the Bianchi II and Bianchi VI_0 type) in general exhibit two branches, characterised respectively by a final and an initial curvature singularity (a similar behaviour is also typical of Bianchi I backgrounds with nontrivial torsion, $H_{\mu\nu\alpha}\neq 0$, as recently discussed in [17]). The singularities cannot be avoided in this context, but they could be eventually cured by higher order corrections in α' and in the string loop expansion parameter, which become important when approaching the high curvature, strong coupling regime surrounding the singularity.

We finally note that the trivial solution (64)-(66) suggests particularly simple examples of conformal backgrounds suitable for performing duality transformations with respect to a non-Abelian isometry group. The case of a Bianchi V metric ($h=1$) was already discussed in [7]. The Bianchi III case ($h=0$),

$$g_{\mu\nu}=\mathrm{diag}(1,-t^2,-t^2e^{-2X},-1), \qquad \phi=const, \tag{67}$$

also corresponds to a non-semisimple, non-Abelian group of isometries, with ${C_{21}}^2=1$ as the only non-vanishing structure constant. By following the standard prescriptions [5, 7], the non-Abelian duality transformations applied to eq.(67) lead to a dual metric which is still diagonal,

$$\tilde{g}_{\mu\nu}=\mathrm{diag}(1,-\frac{t^2}{\Delta},-\frac{t^2}{\Delta},-1), \qquad \Delta=t^4+Y^2, \tag{68}$$

but also to a non-vanishing torsion and a non-trivial dilaton field,

$$\tilde{B}_{12} = \frac{Y}{\Delta} = -\tilde{B}_{21}, \qquad \tilde{\phi} = -\ln\Delta + const. \tag{69}$$

Since

$$e^{-\tilde{\phi}}\sqrt{|\tilde{g}|} = t^2, \quad \tilde{H}^{201} = -\frac{4Y}{t}, \quad \tilde{H}^{301} = 0 = \tilde{H}^{321}, \tag{70}$$

it follows that the dual background is not conformal, as one can easily check by noting for instance that the component $\alpha = 0$, $\beta = 1$ of eq.(3) is not satisfied,

$$\partial_2\left(e^{-\tilde{\phi}}\sqrt{|\tilde{g}|}\tilde{H}^{201}\right) = -4t \neq 0. \tag{71}$$

By following the same procedure as in [7] one can show, in particular, that no possible choice of the transformed dilaton can restore conformal invariance for the dual background $\{\tilde{g}, \tilde{B}\}$ defined in (68), (69). This confirms a recent analysis [18] showing that, in the case of non-semisimple groups, an additional anomaly cancellation condition is to be imposed for the consistency of non-abelian duality.

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6 Appendix

In order to compute the allowed values of h for the particular solution (64), (65),

$$a_i(t) = t^{\beta_i}, \quad \beta_i = (1, \frac{1+h}{1+h^2}, \frac{h(1+h)}{1+h^2}), \quad h \neq -1, \tag{72}$$

$$\phi = c = const, \qquad \bar{\phi} = c - \sum_i \beta_i \ln t, \tag{73}$$

we rewrite it in terms of the x coordinate. By recalling that, for a Bianchi type VI_h,

$$V_i = \frac{1}{t^2}(1+h^2, 1+h, h+h^2), \tag{74}$$

we obtain from eq. (32)

$$x + x_0 = \frac{2(1+h+h^2)}{\sum_k \beta_k - 1} t^{\sum_k \beta_k - 1} e^{-c} \tag{75}$$

(we have put $L = 1$ for simplicity). It follows that

$$a_i = a_{i0}(x + x_0)^{\frac{\beta_i}{\sum_k \beta_k - 1}}, \qquad e^{-\bar{\phi}} = e^{-\bar{\phi}_0}(x + x_0)^{\frac{\sum_i \beta_i}{\sum_k \beta_k - 1}}, \tag{76}$$

where

$$a_{i0} = \left[\frac{\sum_k \beta_k - 1}{e^{-c} 2(1+h+h^2)}\right]^{\frac{\beta_i}{\sum_k \beta_k - 1}}, \quad e^{-\bar{\phi}_0} = e^{-c}\left[\frac{\sum_k \beta_k - 1}{e^{-c} 2(1+h+h^2)}\right]^{\frac{\sum_i \beta_i}{\sum_k \beta_k - 1}}. \tag{77}$$

By inserting these values into the constraint (40) we thus obtain the condition

$$\alpha = \frac{2}{a_{10}^2}(1+h+h^2) e^{-2\bar{\phi}_0} \tag{78}$$

which reads explicitly

$$\frac{(1+h)^4(1+h+h^2)}{(1+h^2)^2} = 2(1+h+h^2) - (1+h+2h^2+h^3+h^4) \tag{79}$$

and which, for h real, is only satisfied by $h = 0, 1$ ($h = -1$ is also allowed, but this value is to be excluded for the particular solution we are considering, see Sect. 3). For these two values of h the full Bianchi metric (66) is identically Ricci flat,

$$R_1{}^1 = \frac{(h-1)h(2+h+h^2)}{(1+h^2)t^2} \equiv 0, \qquad R_0{}^0 \equiv 0, \tag{80}$$

$$R_2{}^2 = \frac{1+h}{1+h^2} R_1{}^1, \qquad R_3{}^3 = \frac{h(1+h)}{1+h^2} R_1{}^1 \tag{81}$$

as required by a solution of the background field equations with constant dilaton. However, the spacetime manifold is also globally flat, since for the metric (66) all the components of the Riemann tensor are proportional to $h(h-1)$, and thus identically vanishing for $h = 0, 1$. In particular,

$$R_{1212} = \frac{h(h-1)}{1+h^2} t^{\frac{2(1+h)}{1+h^2}} e^{-2X}, \tag{82}$$

$$R_{1220} = -R_{1212}t^{-1}, \tag{83}$$

$$R_{1313} = (1 + h + h^2)R_{1212}t^{\frac{2(h^2-1)}{1+h^2}}e^{2X(1-h)}, \tag{84}$$

$$R_{1330} = (1 + h + h^2)^{-1}R_{1313}t^{-1}, \tag{85}$$

$$R_{2020} = \frac{1 + h}{1 + h^2}R_{1220}t^{-1}, \tag{86}$$

$$R_{3030} = \frac{1 + h}{1 + h^2}R_{1330}t^{-1}, \tag{87}$$

$$R_{2323} = \frac{h(2 + h + h^2)}{1 + h}R_{3030}t^{\frac{-2(1+h)}{1+h^2}}e^{-2X}. \tag{88}$$

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7 Table caption

Explicit form of the metric and dilaton field in $d = 3$ for Bianchi types I, II, III, V and VI_0 ($\alpha_i, x_\pm$ and x_c are defined respectively by eqs.(51), (53) and (57)). The solutions reported here refer to the non-trivial case in which the zeros of $D(x)$ are real and different from $-x_0$ and from the zeros of Γ_i. The range of validity of such solutions is discussed in Section 3.

Table 1

Bianchi I

$$ds^2 = dt^2 - a_1^2 dX^2 - a_2^2 dY^2 - a_3^2 dZ^2$$

$$a_i = a_{i0}(t_0 + c_0 t)^{c_i/c_0}, \qquad \sum_i c_i^2 = c_0^2, \qquad i = 1, 2, 3$$

$$e^{\bar{\phi}} = \frac{e^{\phi}}{a_1 a_2 a_3} = L(c_0 t + t_0)^{-1}$$

Bianchi II

$$ds^2 = dt^2 - a_1^2 dX^2 - a_2^2 (dY - X dZ)^2 - a_3^2 dZ^2$$

$$a_i = a_{i0} |(x - x_+)(x - x_-)|^{\frac{-k_i}{4}} \left| \frac{x - x_+}{x - x_-} \right|^{\frac{\alpha_i}{2}}, \qquad k_i = (-1, 1, 1)$$

$$e^{\bar{\phi}} = e^{\bar{\phi}_0} |(x - x_+)(x - x_-)|^{\frac{1}{4}} \left| \frac{x - x_+}{x - x_-} \right|^{\frac{1}{2}(\alpha_1 - \alpha_2 - \alpha_3)}$$

Bianchi III

$$ds^2 = dt^2 - a_1^2 dX^2 - a_2^2 e^{-2X} dY^2 - a_3^2 dZ^2$$

$$a_i = a_{i0} |(x - x_+)(x - x_-)|^{k_i} \left| \frac{x - x_+}{x - x_-} \right|^{\frac{\alpha_i}{2}}, \qquad k_i = (\frac{1}{2}, \frac{1}{2}, 0)$$

$$e^{\bar{\phi}} = e^{\bar{\phi}_0} |(x - x_+)(x - x_-)|^{-1} \left| \frac{x - x_+}{x - x_-} \right|^{-\frac{1}{4}(\alpha_1 + \alpha_2)}$$

Bianchi V

$$ds^2 = dt^2 - a_1^2 dX^2 - a_2^2 e^{-2X} dY^2 - a_3^2 e^{-2X} dZ^2$$

$$a_i = a_{i0} |(x - x_+)(x - x_-)|^{\frac{3k_i}{4}} \left| \frac{x - x_+}{x - x_-} \right|^{\frac{\alpha_i}{2}}, \qquad k_i = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$$

$$e^{\bar{\phi}} = e^{\bar{\phi}_0} |(x - x_+)(x - x_-)|^{-\frac{3}{4}} \left| \frac{x - x_+}{x - x_-} \right|^{-\frac{1}{6}(\alpha_1 + \alpha_2 + \alpha_3)}$$

Bianchi VI$_0$

$$ds^2 = dt^2 - a_1^2 dX^2 - a_2^2 e^{-2X} dY^2 - a_3^2 e^{2X} dZ^2$$

$$a_1 = a_{10}(x - x_c)^{\frac{x_c + x_1}{2(x_0 - x_1)}} e^{\frac{x}{2(x_0 - x_1)}}$$

$$a_2 = a_{20}(x - x_c)^{\frac{x_2}{2(x_0 - x_1)}}$$

$$a_3 = a_{30}(x - x_c)^{\frac{x_3}{2(x_0 - x_1)}}, \qquad\qquad x_2 = x_3$$

$$e^{\bar{\phi}} = e^{\bar{\phi}_0}(x - x_c)^{-\frac{x_c + x_0}{2(x_0 - x_1)}} e^{-\frac{x}{2(x_0 - x_1)}}$$