CERN-TH/95-40
DFTT-16/95

Evolution of a String Network in Backgrounds with Rolling Horizons

M. Gasperini, M. Giovannini
Dipartimento di Fisica Teorica, Università di Torino,
Via P.Giuria 1, 10125 Turin, Italy,
K. A. Meissner
Institute of Theoretical Physics, ul. Hoza 69, 00-681 Warsaw, Poland
and
G. Veneziano
TH. Division, CERN, CH-1211 Geneva 23, Switzerland

Abstract

We discuss the temporal variation of the equation of state of a classical string network, evolving in a background in which the Hubble radius H^{-1} shrinks to a minimum and then re-expands to infinity. We also present a method to look for self-consistent non-vacuum string backgrounds, corresponding to the simultaneous solution of the gravi-dilaton background equations and of the string equations of motion.

To appear in the **"String gravity and physics at the Planck energy scale "**
(World Scientific, Singapore, 1995)

CERN-TH/95-40
February 1995

EVOLUTION OF A STRING NETWORK IN BACKGROUNDS WITH ROLLING HORIZONS

M. Gasperini, M. Giovannini
Dipartimento di Fisica Teorica, Università di Torino,
Via P.Giuria 1, 10125 Turin, Italy,
K. A. Meissner
Institute of Theoretical Physics, ul. Hoza 69, 00-681 Warsaw, Poland
and
G. Veneziano
TH. Division, CERN, CH-1211 Geneva 23, Switzerland

ABSTRACT

We discuss the temporal variation of the equation of state of a classical string network, evolving in a background in which the Hubble radius H^{-1} shrinks to a minimum and then re-expands to infinity. We also present a method to look for self-consistent non-vacuum string backgrounds, corresponding to the simultaneous solution of the gravi-dilaton background equations and of the string equations of motion.

1. Introduction.

Extensive studies of string propagation in cosmological backgrounds [1-4], have shown that the qualitative behaviour of classical strings depends dramatically on the presence of event horizons in the given background geometry. In this case it is possible to define two physically distinct regimes for the classical solutions, according to whether the proper size L of a string, $L \sim \alpha' E$, is smaller or larger than the size of the horizon L_h. It turns out that, for $L << L_h$, the solutions are oscillatory with constant proper size, while the regime $L >> L_h$ is characterized by non-oscillatory solutions, in which the string proper size evolves in time like the scale factor, $a(t)$, of the background geometry (see also [5]).

This latter regime, first discussed in the context of a de Sitter-type manifold [1,2,5], is of crucial importance for all those metric backgrounds in which the size of the event horizon shrinks to zero. Classically, in such geometries, strings of any size asymptotically become larger than the horizon. The classical description, however, is expected to break down at some point. Thus, while the string proper size is expected to be always larger than $\lambda_s = \sqrt{\alpha' \hbar}$ as a result of the uncertainty principle [6], classical geometries with horizon size smaller than λ_s are also expected to suffer large quantum corrections. Since we want to keep our discussion here entirely classical we shall use λ_s as a short distance cut-off both for the string size distribution and for the horizon of the background geometry.

The shrinking of the event horizon is typical of inflationary scenarios (also called "pre-big-bang" scenarios) which follow from the low energy string effective action [7-8], the accelerated evolution of the metric being driven by the kinetic energy of the dilaton field. A question which arises naturally in that context is whether the back reaction of string matter, evolving in the presence of shrinking horizons, could lead to a string-plus-dilaton-driven inflationary scenario which solves simultaneously, and self-consistently, both the background field equations and the string equations of motion (very recent work on similar issues can be found in [9]).

In order to avoid confusion, we recall that the shrinking of the event horizon can occur in two types of backgrounds [8], characterized, respectively, by a superinflationary accelerated expansion,

$$\dot{a} > 0, \quad \ddot{a} > 0, \quad \dot{H} > 0 \tag{1.1}$$

or by an accelerated contraction

$$\dot{a} < 0, \quad \ddot{a} < 0, \quad \dot{H} < 0 \tag{1.2}$$

of the scale factor $a(t)$. Here $H = \dot{a}/a$, and a dot denotes differentiation with respect to cosmic time t.

Consider, in fact, a spatially flat homogeneous and isotropic metric,

$$g_{\mu\nu} = diag(1, -a^2\delta_{ij}), \qquad i,j = 1, ..., d \geq 3 \tag{1.3}$$

parameterized by

$$a(t) \sim (-t)^\alpha \tag{1.4}$$

for $t \to 0_-$. The proper size of the event horizon

$$L_h(t) = a(t) \int_t^0 dt' a^{-1}(t') \tag{1.5}$$

is finite, and shrinks linearly in cosmic time, $L_h(t) \simeq (-t)$, for all $\alpha < 1$. The negative part of this range ($\alpha < 0$) corresponds to superinflation, eq.(1.1), while the positive part ($0 < \alpha < 1$) corresponds to accelerated contraction, eq.(1.2).

Both superinflation and accelerated contraction provide a representation of the pre-big-bang scenario [7,8], the former in the so-called string frame, (in which weakly coupled test strings move along geodesic surfaces of the background metric), the latter in the Einstein frame (in which the dilaton is minimally coupled to the metric and the corresponding effective action is diagonalized in the standard canonical form).

Obviously, the abovementioned phenomena, being of purely geometrical nature, refer just to the string-frame metric (the dilaton controls instead the strength of the mutual interaction among strings, assumed to be negligible in the following). For all these reasons we will limit our attention to a background which describes (in the string frame) an initial superinflationary era followed by the time-reversed version of an accelerated contraction: the latter phase describes nothing more than the standard (non-inflationary) decelerated expansion ($\dot{a} > 0$, $\ddot{a} < 0$) of the present epoch. It was shown in [10] that a transition between the two epochs cannot occur in the low curvature regime: an intermediate genuinly stringy epoch is needed as well.

Consider now the string equations of motion which, in the gauge in which the world-sheet metric is conformally flat, can be written explicitly as [1-3]

$$\frac{d^2x^\mu}{d\tau^2} - \frac{d^2x^\mu}{d\sigma^2} + \Gamma^\mu_{\alpha\beta}(\frac{dx^\alpha}{d\tau} + \frac{dx^\alpha}{d\sigma})(\frac{dx^\beta}{d\tau} - \frac{dx^\beta}{d\sigma}) = 0$$

$$g_{\mu\nu}(\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}) = 0, \quad g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\sigma} = 0 \tag{1.6}$$

where $x^\mu(\sigma,\tau)$ are the string coordinates, τ and σ the usual world-sheet time and space variables, and $\Gamma^\mu_{\alpha\beta}$ is the Christoffel connection for the background metric (1.3). When the string proper size is negligible with respect to the horizon, the exact solution of these equations can be expressed as an expansion around the point-particle motion of the string center of mass [1]. If the horizon is shrinking, however, such an expansion breaks down, and the explicit solution of eqs.(1.6) shows that the string comoving size becomes frozen, asymptotically, while the proper size evolves in time like the scale factor, $L(t) \sim a(t)$ for $L(t) >> L_h(t) \sim (-t)$.

This occurs both for superinflation and for accelerated contraction [3]. The crucial difference between the two types of background, however, is that, in an expanding geometry, the asymptotic solutions of eqs.(1.6) are characterized by

$$\left|\frac{\partial x^0}{\partial\tau}\right| >> \left|\frac{\partial x^0}{\partial\sigma}\right|, \qquad \left|\frac{\partial x^i}{\partial\tau}\right| << \left|\frac{\partial x^i}{\partial\sigma}\right| \tag{1.7}$$

while, in the contracting case, they satisfy

$$\left|\frac{\partial x^0}{\partial \tau}\right| >> \left|\frac{\partial x^0}{\partial \sigma}\right|, \qquad \left|\frac{\partial x^i}{\partial \tau}\right| >> \left|\frac{\partial x^i}{\partial \sigma}\right| \tag{1.8}$$

As a consequence, the string energy-momentum tensor

$$T^{\mu\nu}(x) = \frac{1}{\pi\alpha' \sqrt{|g|}} \int d\sigma d\tau \left(\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} - \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}\right)\delta^{d+1}(x - x(\sigma,\tau)) \tag{1.9}$$

($(\alpha')^{-1}$ is the string tension) satisfies respectively the asymptotic conditions

$$T_0{}^0 \simeq \sum_i T_i{}^i, \quad \alpha < 0, \qquad T_0{}^0 \simeq -\sum_i T_i{}^i, \quad 0 < \alpha < 1, \tag{1.10}$$

In the perfect fluid approximation, $T_0{}^0 = \rho$, $T_i{}^j = -p\delta_i^j$, one thus obtain for a diluted gas of classical strings the effective equations of state [3]

$$\frac{p}{\rho} \simeq -\frac{1}{d}, \quad \alpha < 0, \qquad \frac{p}{\rho} \simeq \frac{1}{d}, \quad 0 < \alpha < 1, \tag{1.11}$$

The relationship between the two regimes can be easily understood in terms of the so-called scale-factor duality property of string motions in cosmological backgrounds [7].

We note, for comparison, that in the opposite regime $L(t) << L_h(t)$ the string solutions are instead characterized by the condition

$$\left|\frac{\partial x^0}{\partial \tau}\right| >> \left|\frac{\partial x^0}{\partial \sigma}\right|, \qquad \left|\frac{\partial x^i}{\partial \tau}\right| \simeq \left|\frac{\partial x^i}{\partial \sigma}\right| \tag{1.12}$$

which implies

$$T_0{}^0 >> \sum_i T_i{}^i, \qquad \frac{p}{\rho} \simeq 0 \tag{1.13}$$

As it is clear from the way they are obtained, the effective equations of state (1.11) and (1.13), characterized by a constant ratio $\gamma = p/\rho$, are only valid asymptotically when strings are, respectively, far outside or well inside the horizon. The purpose of this paper is to discuss instead the time-variation of the ratio p/ρ (for a gas of classical strings) versus the time-variation of the event horizon, considering in particular a background in which the Hubble radius H^{-1} shrinks to a minimum and then re-expands to infinity. To this aim we shall propose in Section 2 an effective evolution equation for the density of strings of arbitrary proper size, and we shall present in Section 3 some examples of explicit solution of such equation.

A possible approach to the problem of obtaining simultaneous and self-consistent solutions to the system of background field equations and string equations of motion will be discussed in Section 4. Our main results will be finally summarized in Section 5.

2. Evolution equation for the string distribution.

Let us consider a cosmological metric background of the type (1.3) which, starting from an asymptotically flat initial state at $t = -\infty$, evolves towards a high curvature regime [7-8]. The curvature reaches at $t = 0$ a maximum scale of order $\lambda_s^{-1} \lesssim \ell_p^{-1}$ (where ℓ_p is the Planck length), and then decreases approaching zero as $t \to +\infty$. In such a background, the Hubble length $D(t) \equiv H^{-1}$ (which we shall assume here to be always positive) shrinks to a minimum $D(0) \simeq \lambda_s$, and then re-expands to infinity.

Consider, in this background, a string of initial proper size $L_{-\infty}$. According to the results reviewed in the previous Section [1-5], the string size stays constant inside the horizon, and evolves like $a(t)$ outside the horizon. The time-dependence of the string proper size $L(t)$ can thus be fixed by the equation

$$L(t) = L_{-\infty}\theta(-t)\theta(D(t) - L(t)) + L_{-\infty}\theta(L(t) - D(t))\frac{a(t)}{a(t_{out})}+$$

$$+L_{+\infty}\theta(D(t) - L(t))\theta(t) \tag{2.1}$$

where θ is the Heaviside step function,

$$L_{+\infty} = L_{-\infty}\frac{a(t_{in})}{a(t_{out})} \tag{2.2}$$

and t_{out} and t_{in} are, respectively , the times at which the string leaves and re-enters the horizon, determined by the conditions $D(t_{out}) = L_{-\infty}$, $D(t_{in}) = L_{+\infty}$.

Consider next, in the same background, a network of N such non-interacting strings, occupying initially a certain comoving volume. Let us denote by $L^i(t)$ ($i = 1, 2, ..., N$) the individual proper sizes at time t. The number distribution $n(L, t)$ of strings of proper size L, at time t, such that $N = \int n dL$, is obviously given by

$$n(L, t) = \sum_{i=1}^{N} \delta(L - L^i(t)) \tag{2.3}$$

We now differentiate this distribution with respect to L and t and take into account that, from eq.(2.1),

$$\dot{L}^{(i)} = H(t)\theta(L^{(i)}(t) - D(t))L^{(i)} \tag{2.4}$$

We thus find that the number distribution of a diluted gas of classical strings, in the considered background, must satisfy the approximate evolution equation

$$\frac{\partial n(L,t)}{\partial t} + H(t)\frac{\partial}{\partial L}[n(L,t)L\theta(L-D)] = 0 \tag{2.5}$$

Equivalently, by using a instead of t as evolution parameter,

$$a\frac{\partial n(L,a)}{\partial a} + \frac{\partial}{\partial L}[n(L,a)L\theta(L-D)] = 0 \tag{2.6}$$

The general solution of this equation can be conveniently expressed in terms of the initial string distribution, $n_{-\infty}(L)$, as

$$n(L,a) = n_{-\infty}(L)\theta(D-L) + \frac{1}{a}h(\frac{L}{a})\theta(L-D) \tag{2.7}$$

where

$$h(\xi) = a(\xi)n_{-\infty}(D(\xi))\left[1 + \frac{\partial \ln a(\xi)}{\partial \ln H(\xi)}\right]^{-1} \tag{2.8}$$

($\xi = \frac{D}{a}$). Once the background $a(t)$ and the initial string distribution $n_{-\infty}$ are given one can thus compute all physical observables, and in particular the energy density of strings which are smaller and larger than the horizon (which we shall call for simplicity stable and unstable, respectively).

By recalling that the string energy is proportional to the proper size L, the energy of stable and unstable strings in our ensemble can be estimated, respectively, as

$$\begin{aligned} E_s &= \int d^3x\sqrt{|g|}\rho_s = \frac{1}{\pi\alpha'}\int Ln(L,a)\theta(D-L)dL \\ E_u &= \int d^3x\sqrt{|g|}\rho_u = \frac{1}{\pi\alpha'}\int Ln(L,a)\theta(L-D)dL \end{aligned} \tag{2.9}$$

(the integral extends of course over the whole range of L, from the minimal allowed proper size λ_s up to infinity). The corresponding pressure, on the other hand, is given asymptotically by the effective equations of state (1.13) and (1.11), respectively. The time evolution of the ration $\gamma = p/\rho$ for the full string distribution, therefore, can be approximated as

$$\gamma(t) = \pm\frac{E_u}{dE} = \pm\frac{1}{d}\frac{\int_{\lambda_s}^{\infty} Ln\theta(L-D)dL}{\int_{\lambda_s}^{\infty} LndL} \tag{2.10}$$

(here $E = E_s + E_u$, and the sign of γ depends on the type of background, according to eq.(1.11)).

The time-dependence of γ determined in this way relies on the evolution equation (2.6), obtained in the sudden approximation in which the transition from the stable to unstable regime is very roughly parameterized by the step function $\theta(L-D)$. Such a transition, however, could be be approximated in a continuous way by introducing an appropriate function $\theta(L,a)$ which goes to 0 for $LH << 1$ and to 1 for $LH >> 1$, and which interpolates smoothly between strings smaller and larger than the horizon. The evolution equation for n, written in terms of such a smooth function,

$$a\frac{\partial n(L,a)}{\partial a} + \frac{\partial}{\partial L}[n(L,a)L\theta(L,a)] = 0 \tag{2.11}$$

has then the general solution

$$n(L,a) = -\sigma(z)\frac{\partial z}{\partial L} \tag{2.12}$$

where σ is an arbitrary function of z, and $z(L,a)$ is a particular solution of the differential equation

$$a\frac{\partial z}{\partial a} + \theta(L,a)L\frac{\partial z}{\partial L} = 0 \tag{2.13}$$

In the following Section we shall provide examples of solution of the evolution equation for the string number distribution, considering both the sudden approximation, eq.(2.6), and the continuous one, eq.(2.11).

3. Examples of exact solutions.

We shall consider first an accelerated background corresponding to a phase of pre-big-bang evolution, [7-8], parameterized by the scale factor (1.4) with $\alpha < 1$, and $-\infty \le t \le -t_1$, where $t_1 \simeq \lambda_s$. We shall assume, moreover, a power-law behavior for the initial string distribution,

$$n_{-\infty}(L) = \Lambda^{\beta-1}L^{-\beta} \tag{3.1}$$

Here $\beta > 2$ for the convergence of the energy integral $\int^{\infty} LndL$, and Λ is an appropriate length parameter, related to the total string number N by

$$N = \int_{\lambda_s}^{\infty} n_{-\infty}(L)dL = \frac{1}{\beta-1}\left(\frac{\Lambda}{\lambda_s}\right)^{\beta-1} \tag{3.2}$$

In this background $(\partial \ln a/\partial \ln H) = -\alpha$, and from the general solution (2.7) we find that the energy of stable and unstable strings is given, respectively, by

$$\pi\alpha' E_s = \int_{\lambda_s}^{D} L n dL = \frac{\Lambda^{\beta-1}}{\beta-2}\left(\lambda_s^{2-\beta} - D^{2-\beta}\right)$$

$$\pi\alpha' E_u = \int_{D}^{\infty} L n dL = \frac{\Lambda^{\beta-1}}{\beta+\alpha-2} D^{2-\beta} \tag{3.3}$$

(we have supposed $\beta + \alpha > 2$). Note that $\dot{E} = HE_u$, in agreement with the covariant conservation of the stress tensor ($\dot{\rho} + dH(\rho + p) = 0$), and with the fact that only unstable strings contribute to the total pressure, with $p = -\rho/d$. As a consequence of eqs.(3.2), (3.3), the ratio $\gamma(t)$ evolves in time (according to eq.(2.10)) as

$$\gamma(t) = \pm\frac{1}{d}\frac{\beta-2}{(\beta+\alpha-2)|H\lambda_s|^{2-\beta} - \alpha}, \quad \beta > 2, \quad \beta+\alpha > 2 \tag{3.4}$$

Asymptotically,

$$\gamma = \pm\frac{1}{d}\frac{\beta-2}{\beta+\alpha-2}|H\lambda_s|^{\beta-2}, \qquad |H\lambda_s| \to 0 \tag{3.4a}$$

$$\gamma = \pm\frac{1}{d}, \qquad |H\lambda_s| \to 1 \tag{3.4b}$$

Here $sign\{\gamma\} = sign\{\alpha\}$, so that in case of superinflationary expansion γ approaches zero from negative values as $t \to -\infty$, while in case of accelerated contraction γ approaches zero from positive values. If we consider a background which evolves from a phase of pre-big-bang accelerated expansion,

$$a(t) \sim (-t)^{\alpha}, \qquad \alpha < 0, \quad t < 0 \tag{3.5}$$

to a phase of standard decelerated expansion,

$$\tilde{a}(t) \sim t^{\delta}, \qquad 0 < \delta < 1, \quad t > 0 \tag{3.6}$$

we must expect, therefore, a change of trend in the equation of state, corresponding to the evolution of γ which starts from 0 at $t = -\infty$, reaches a negative minimum of order unity at $t = 0$ (when $H_1 \simeq \lambda_s^{-1}$), and then starts to increase towards positive values, approaching zero as $t \to +\infty$.

In order to describe the time evolution of γ in the decelerated regime following the phase of pre-big-bang inflationary expansion, we write first the general solution for n in the background (3.6) and then demand that it joins smoothly to the final distribution originating from the preceding inflationary background (3.5). One easily gets, at $t > 0$

$$n(L, \tilde{a}) = n_{+\infty}(L)\theta(D - L) + \frac{1}{\tilde{a}} h(\frac{L}{\tilde{a}})\theta(L - D) \tag{3.7}$$

where $n_{-\infty}$ has been replaced by the final asymptotic string distribution for $t \to +\infty$,

$$n_{+\infty}(\xi) = (1 - \delta)\frac{h(\xi)}{\tilde{a}(\xi)} \tag{3.8}$$

(according to eq.(2.8)), and

$$h(\xi) = A\xi^{(\alpha-\beta)/(1-\alpha)} \tag{3.9}$$

Here the ξ-dependence of h is obtained from eq.(2.8) written in the background (3.5), and A is a constant coefficient which we shall fix by normalizing the total number of strings to that of the initial distribution, $N = \int n_{-\infty} dL$.

For the decelerated background (3.6) $\xi \sim D^{1-\delta}$. From eq.(3.8) we get, therefore,

$$n_{+\infty}(L) = c_{+\infty} L^{[\delta(\beta-1)+\alpha-\beta]/(1-\alpha)} \tag{3.10}$$

where the number $c_{+\infty}$ depends on A, α, β and δ. On the other hand, if we start at $t = -\infty$ with the initial distribution (3.1), we have

$$n_{+\infty}(L)dL = \frac{c_{+\infty}}{\Lambda^{\beta-1}} \left(\frac{1-\alpha}{1-\delta}\right) n_{-\infty}(L^{\frac{1-\delta}{1-\alpha}})d(L^{\frac{1-\delta}{1-\alpha}}) \tag{3.11}$$

so that we can always choose A in such a way that the total number of strings is conserved in the transition from a to $\tilde{a}$. This in agreement with a classical description of string propagation in curved backgrounds, where any quantum string creation or string decay process is completely neglected (see [9] for a discussion which includes the effects of quantum decay through string splitting).

We note, finally, that for the general solution (3.7) the energy of stable and unstable strings is given, respectively, by

$$\pi\alpha' E_s = \frac{c_{+\infty}}{k-2}\left(\lambda_s^{2-k} - D^{2-k}\right)$$

$$\pi\alpha' E_u = \frac{c_{+\infty}}{k+\delta-2} D^{2-k}, \qquad k = -\frac{\delta(\beta-1)+\alpha-\beta}{1-\alpha} \tag{3.12}$$

($k > 2$ for a finite total energy). The corresponding ratio $\gamma(t)$ increases towards zero, for $t \to +\infty$, as

$$\gamma(t) = -\frac{1}{d}\frac{k-2}{k+\delta-2}(H\lambda_s)^{k-2} \tag{3.13}$$

The transition from a phase of shrinking horizons to a phase of expanding horizons is associated, therefore, to a switching of $\dot{\gamma}$ from negative to positive values, and to a subsequent decreasing of $|\gamma|$ as more and more strings re-enter the horizon.

This behavior of γ, obtained in the context of the sudden approximation, can be qualitatively confirmed by using a different approach based on the smoothed evolution equation (2.11). Consider in fact the continuous scale factor

$$a = \left(\frac{t+\sqrt{t^2+t_1^2}}{t_1}\right)^{1/2}, \quad D \equiv H^{-1} = t_1(a^2+a^{-2}), \quad -\infty \leq t \leq +\infty \tag{3.14}$$

which is self-dual [7,8], i.e. satisfies $a(t) = a^{-1}(-t)$. It connects smoothly the standard radiation-dominated expansion, $a \sim t^{1/2}$ for $t \to +\infty$, to the duality-related inflationary regime, $a \sim (-t)^{-1/2}$ for $t \to -\infty$. By choosing

$$\theta(L,a) = \frac{L}{L+H^{-1}} \tag{3.15}$$

as interpolating function between the stable ($LH >> 1$) and unstable ($LH << 1$) regime, one can easily check that

$$z(L,a) = \frac{2t_1}{L} + \arctan\left(a^2 + \frac{L}{2t_1}\right) - \frac{\pi}{2} \tag{3.16}$$

is a particular solution of eq.(2.13). We choose, moreover,

$$\sigma(z) = \frac{N\pi z}{\left(z+\frac{\pi}{2}\right)^3} \tag{3.17}$$

as simple example of distribution which leads to a finite total number of strings and to a finite total energy for $L \to 0$ and $L \to \infty$.

From the solution (2.12) of the smoothed evolution equation we obtain, in particular,

$$N = \int_0^\infty dL\ n(L,a) = \int_0^\infty \sigma(z)dz$$

$$\pi\alpha' E = \int_0^\infty dL\ Ln(L,a) \to c_1 + c_2 a^2, \qquad a \to 0$$

$$\pi\alpha' E = \int_0^\infty dL\ Ln(L,a) \to \frac{4Nt_1}{\pi} - \frac{16Nt_1}{\pi^2 a^2}\ln\left(\frac{\pi a^2}{2}\right), \qquad a \to \infty$$

where c_1 and c_2 are positive constants. On the other hand, in d isotropic dimensions, the pressure to energy density ratio for a perfect gas contained inside a proper volume V can be expressed as

$$\gamma = \frac{p}{\rho} = -\frac{V}{E}\frac{\partial E}{\partial V} = -\frac{a}{dE}\frac{\partial E}{\partial a} \tag{3.19}$$

The behavior of γ over the whole range of t can then be found by integrating numerically the given string energy distribution. For what concerns the asymptotic behavior of γ, however, we can easily obtain from eqs.(3.18) that

$$t \to -\infty, \quad a \to 0 \quad \Longrightarrow \quad \gamma \sim -a^2 \sim -\frac{1}{(-t)} \sim -H_\infty$$

$$t \to +\infty, \quad a \to \infty \quad \Longrightarrow \quad \gamma \sim -\frac{1}{a^2}\ln a^2 \sim -H_\infty \ln H_\infty \tag{3.20}$$

where $H_\infty \equiv H(t = \pm\infty) = 1/(2|t|)$ (see eq.(3.14). Therefore, γ starts decreasing from zero at $t \to -\infty$, and then eventually increases to approach zero from negative values as $t \to +\infty$, in full qualitative agreement with our previous conclusions. It should be noted, moreover, that γ is always negative, in agreement with eq.(3.19) and with the fact that the total string energy is growing,

$$\dot{E} = -H\int_0^\infty LdL\frac{\partial}{\partial L}(nL\theta) = H\int_0^\infty dLnL\theta \equiv HE_u > 0 \tag{3.21}$$

over the whole time range $-\infty \leq t \leq +\infty$.

The initial distributions considered in this section consist, essentially, of short tests strings. The mean length of the strings is $O(\lambda_s)$ in the case of the distribution (3.2), and $O(t_1)$ in the case of the distribution (3.17). The same results of the previous examples can be obtained, however, also starting with a distribution of strings with arbitrary mean length L_0. In this case, if L_0 is sufficiently large compared to λ_s, then the ratio $\frac{p}{\rho}$ approaches the minimum value $\frac{-1}{d}$ as soon as D becomes much smaller than L_0.

Consider for instance an accelerated expanding background described by the scale factor (3.5), and assume the following initial distribution :

$$n_{-\infty}(L) = \frac{N}{L_0}e^{-\frac{L}{L_0}} \tag{3.22}$$

In this case the mean string length is:

$$< L >= \frac{\int_{\lambda_s}^\infty Ln_{-\infty}(L)dL}{\int_{\lambda_s}^\infty n_{-\infty}(L)dL} = L_0 + \lambda_s \tag{3.23}$$

The energy of stable and unstable strings is then, from the general solution (2.7),

$$\frac{1}{N}\pi\alpha' E_s = \int_{\lambda_s}^{D} dL L n(L,a) = (\lambda_s + L_0)e^{-\frac{\lambda_s}{L_0}} - (D+L_0)e^{\frac{-D}{L_0}} \tag{3.24}$$

$$\frac{1}{N}\pi\alpha' E_u = \int_{D}^{+\infty} dL L n(L,a) = (\frac{D}{L_0})^{\alpha} L_0 \Gamma\left(2-\alpha, \frac{1}{L_0}(-\alpha)^{\frac{-\alpha}{\alpha-1}}(\frac{D}{a})^{\frac{1}{1-\alpha}}\right) \tag{3.25}$$

(Γ is the incomplete Euler Gamma function).

In order to give a particularly simple example we will choose now an inflationary background with $\alpha = -1$, so that E_u reduces to:

$$\frac{1}{N}\pi\alpha' E_u = L_0(\frac{L_0}{D})[(\frac{D}{L_0})^2 + 2(\frac{D}{L_0}) + 2]e^{\frac{-D}{L_0}} \tag{3.26}$$

and the ratio $\gamma(t)$ becomes, according to equation (2.10),

$$\gamma(t) = -\frac{1}{d}\frac{1}{(1 + \frac{(\frac{\lambda_s}{L_0}+1)e^{-\frac{\lambda_s}{L_0}} - (\frac{D}{L_0}+1)e^{-\frac{D}{L_0}}}{(\frac{D}{L_0}+2\frac{L_0}{D}+2)e^{\frac{-D}{L_0}}})} \tag{3.27}$$

From this equation we can easily check that if $\lambda_s \simeq L_0$ then $\gamma \to 0$ for $D >> \lambda_s$ and $\gamma \to -\frac{1}{d}$ for $D \to \lambda_s$, in agreement with our previous examples. If, on the contrary, we assume an initial distribution with $L_0 >> \lambda_s$ then we have $\gamma \simeq -\frac{1}{d}$ as soon as $D << L_0$. This means that, for a gas of sufficiently long strings, the minimal negative pressure of the unstable regime can be reached even before attaining the quantum string limit.

4. Self-consistent string sources for the background equations.

The field equations obtained from the low energy string effective action for a spatially flat, homogeneous and isotropic metric and dilaton background, with matter sources but vanishing dilaton potential, can be written (in d spatial dimensions) as [7,8,11]

$$\begin{gathered} \dot{\overline{\phi}}^2 - 2\ddot{\overline{\phi}} + dH^2 = 0 \\ \dot{\overline{\phi}}^2 - dH^2 = \overline{\rho}e^{\overline{\phi}} \\ 2(\dot{H} - H\dot{\overline{\phi}}) = \overline{p}e^{\overline{\phi}} \end{gathered} \tag{4.1}$$

Here $\overline{\phi}$ is the duality-invariant shifted dilaton field,

$$\overline{\phi} = \phi - \ln\sqrt{|g|} = \phi - d\ln a \tag{4.2}$$

and the rescaled energy density and pressure

$$\overline{\rho} = \rho\sqrt{|g|}, \qquad\qquad \overline{p} = p\sqrt{|g|} \tag{4.3}$$

satisfy the conservation equation

$$\dot{\overline{\rho}} + dH\overline{p} = 0 \ . \tag{4.4}$$

As already discussed in [8,11], these equations can be integrated a first time to give

$$\frac{1}{a}\frac{da}{dx} = \frac{2\Gamma}{(x+x_0)^2 - d\Gamma^2} \tag{4.5}$$

$$\frac{d\overline{\phi}}{dx} = -\frac{2(x+x_0)}{(x+x_0)^2 - d\Gamma^2} \tag{4.6}$$

$$\overline{\rho} = \frac{e^{\overline{\phi}}}{4\ell^2}[(x+x_0)^2 - d\Gamma^2] \tag{4.7}$$

where ℓ, x_0 are integration constants, x a new (dimensionless) time parameter related to cosmic time by

$$\overline{\rho} = \frac{1}{\ell}\frac{dx}{dt} \tag{4.8}$$

and

$$\Gamma(x) = \int^x \gamma(x')dx' \tag{4.9}$$

($\gamma = p/\rho = \overline{p}/\overline{\rho}$ as before). For any given time-dependent equation of state $\gamma(x)$ one can thus obtain, from eqs.(4.5)-(4.7), the general exact solution for $a(x), \phi(x)$ and $\rho(x)$, which can be eventually expressed in cosmic time through eq.(4.8).

If we are looking, however, for a simultaneous solution of the background equations and of the string equations of motion, the functional form of γ to be inserted into eq.(4.9) must be consistent with the time-evolution of the given initial string distribution. In particular, in the context of the fluid model discussed in the previous Sections, γ is to be obtained by solving the evolution equation (2.6) or (2.11), and it will be expressed in general as a function of the background scale factor, $\gamma = \gamma(a)$. It turns out, therefore, that a more convenient variable to look for solutions of the background equations (4.5)-(4.7), with a string gas as a consistent matter sources, is $y = \ln a$ instead of x.

By defining a new function $\omega(y)$ such that

$$(x+x_0) = -2\Gamma\frac{\omega'}{\omega} \tag{4.10}$$

(a prime denotes differentiation with respect to y), eq.(4.5) implies then

$$\omega'' + \frac{\Gamma'}{\Gamma}\omega' - \frac{d}{4}\omega = 0 \tag{4.11}$$

For any given $\Gamma(y)$ this equation uniquely determines $\omega(y)$, which inserted into eq.(4.10) provides $x = x(y)$. By inverting such relation we can eventually obtain $a = e^y = a(x)$. Moreover, the dilaton equation (4.6) can be rewritten

$$\overline{\phi}' = -\frac{2(x+x_0)}{(x+x_0)^2 - d\Gamma^2}\left(\frac{dx}{dy}\right) = -\frac{x+x_0}{\Gamma} = 2\frac{\omega'}{\omega} \tag{4.12}$$

so that $\overline{\phi}$ is also completely determined in terms of y as

$$\overline{\phi}(y) = \phi_0 + 2\ln\omega(y) \tag{4.13}$$

(ϕ_0 is an integration constant). The corresponding energy density is then, from eqs.(4.7), (4.10),

$$\overline{\rho}(y) = \left(\frac{\Gamma}{\ell}\right)^2 e^{\phi_0}\left({\omega'}^2 - \frac{d}{4}\omega^2\right) \tag{4.14}$$

We note, finally, that for any given $\Gamma(y)$ the corresponding time-evolution of the ratio $\gamma = p/\rho$ is determined by eqs.(4.9) and (4.5) as

$$\gamma(y) = \frac{d\Gamma(y)}{dx} = \frac{\Gamma'}{2\Gamma}\left(\frac{{\omega'}^2}{\omega^2} - \frac{d}{4}\right)^{-1} \tag{4.15}$$

A simple, but important application of this method to obtain self-consistent string-driven backgrounds, is the linear case $\Gamma(y) \sim y$. This case corresponds, according to eqs.(4.8), (4.9), to a phase in which $dx/dt = const$, and in which $\gamma \sim dy/dt = d(\ln a)/dt = H$, as suggested by the results of the previous Section (see eqs.(3.20) and (3.4a) with $\beta = 3$). In this case the solution of eq.(4.11) can be expressed in terms of modified Bessel functions, and the full system of equations can be easily integrated.

On the other hand, a background with $dx/dt \sim \overline{\rho} = const$ corresponds to an exact solution of the cosmological equations (4.1) obtained by imposing, as initial conditions at $t = -\infty$, the string perturbative vacuum (flat spacetime, $\phi = -\infty$), with a small but finite initial density of dust-like sources, $\rho > 0$, $p = 0$ [11]. Such a solution can be written explicitly, in the isotropic case, as

$$a(t) = a_0\left|\frac{t-2T}{t}\right|^{\pm 1/\sqrt{d}}, \qquad e^{\overline{\phi}} = \frac{16\ell^2 e^{-\phi_0}}{|t(t-2T)|},$$

$$\overline{\rho} = \frac{1}{\ell}\frac{dx}{dt} = \frac{e^{\phi_0}}{4\ell^2} = const, \qquad p = 0, \quad t < 0 \tag{4.16}$$

where a_0, ϕ_0 and T are positive integration constants (the plus and minus sign in the exponent corresponds, for $t \to 0_-$, to a metric describing accelerated expansion and contraction, respectively).

This background is certainly consistent with with a string-driven evolution in the limit $t \to -\infty$, and in particular for $|t| >> T$. Indeed, in this regime the metric becomes flat,

$$a = const, \qquad \phi \sim -2\ln(-t), , \qquad \rho = const \tag{4.17}$$

so that there are no horizons, all strings are stable and are thus represented by a pressureless stress tensor. When $|t| \sim T$, however, the curvature scale begins to increase, and all strings progressively enter the non-oscillating unstable regime. The time variation of the ratio p/ρ has then to be taken into account, in general, for an exact string-driven solution which may describe the background evolution away from the string perturbative vacuum, consistently with the string equations of motion. What is remarkable, however, is that the simple background (4.16) may remain a good zeroth-order approximation to the exact string-driven solution, if the duration of the accelerated regime (from T to λ_s) is long enough to satisfy the phenomenological constraints on inflation [11].

By adopting an iterative procedure, let us assume indeed the solution (4.16) to be a zeroth-order approximation, and let us compute the first-order corrections by inserting that solution into the string evolution equation (2.6), in order to obtain the corresponding value of $\gamma(t)$. We shall take for simplicity an initial string distribution with $\beta = 3$, but our final conclusion turns out to be independent of the particular choice of β. Following the procedure illustrated in Section 3 we find, for $|t| >> T$, that $\xi \sim t^2$, so that $\pi\alpha' E_u = \Lambda^2 D^{-1}$. As a consequence we have, from eq.(2.10),

$$\gamma(t) = -\frac{1}{d} H\lambda_s \tag{4.18}$$

Note that γ goes to zero like $-H$ as $t \to -\infty$, in agreement with our previous results.

We now insert this expression of $\gamma(t)$ into the right-hand side of the background equations (4.5)-(4.7), by recalling that for the pressureless background (4.16) one has, to zeroth-order,

$$\Gamma^{(0)} = \pm\frac{T}{4\ell\sqrt{d}} e^{\phi_0} \tag{4.19}$$

Therefore, to next order,

$$\Gamma = \Gamma^{(0)} + \int_{-\infty}^{x} \gamma(x')dx' = \Gamma^{(0)} - \frac{\lambda_s e^{\phi_0}}{d4\ell} \int_{-\infty}^{t} H(t')dt' =$$

$$= \Gamma^{(0)} \left(1 - \frac{\lambda_s}{dT} \ln \left| \frac{t-2T}{t} \right| \right) \tag{4.20}$$

According to our iterative approach, the integration of eqs. (4.5)-(4.7) with the new expression (4.20) for γ provides a first-order approximation to the background fields $a(t)$, $\phi(t)$. The first-order corrections to (4.16) due to a non-vanishing string effective pressure are certainly negligible in the regime $|t| >> T$. However, as clearly shown by eq. (4.20), the corrections may remain small over the whole time range if $T >> \lambda_s$.

In connection with this last point, it is important to note that the ratio T/λ_s measures the duration of the inflationary phase associated to the solution (4.16). For $|t| < T$ the metric background describes indeed a dilaton-dominated, accelerated evolution, which can be approximated by

$$a(t) \sim (-t)^{\mp 1/\sqrt{d}} \tag{4.21}$$

During such a phase the event horizon (1.5) shrinks linearly, and the ratio r of the proper size of a causally connected region to the proper size of the horizon grows in time like $r(t) \sim (-t)^{-1\mp 1/\sqrt{d}}$, for $t \to 0_-$. On the other hand, the horizon problem of the standard cosmological model is solved if the growth of $r(t)$, for $|t|$ ranging from T to λ_s, is large enough to compensate the decrease of $r(t)$ in the subsequent phase of expanding horizons, down to the present time t_0. This requires [11], by assuming that the end of inflation at $t = -\lambda_s$ is followed by the standard radiation-dominated and matter-dominated evolution,

$$\left(\frac{\lambda_s}{T}\right)^{-1\mp 1/\sqrt{d}} \gtrsim 10^{30} \left(\frac{\ell_p}{\lambda_s}\right)^{1/2} \tag{4.22}$$

We can thus conclude that if the integration constant T is chosen in such a way that the solution (4.16) may describe a phenomenologically interesting inflationary background, then $T >> \lambda_s$. In particular, for $\lambda_s \sim \ell_p$,

$$T \gtrsim 10^{30\sqrt{d}/(\sqrt{d}\pm 1)} \lambda_s \tag{4.23}$$

In that case, according to eq.(4.20), the background (4.16) also represents a good zeroth-order approximation to an exact simultaneous solution of the background equations and of the string equations of motion, having the string perturbative vacuum as initial condition.

5. Conclusion.

In this paper we have studied the number distribution of a classical string network, evolving in a background in which the causal correlation length H^{-1} shrinks to a minimum and then re-expands. Such kind of background emerges naturally from the low energy string effective action when looking for cosmological models based on the string perturbative vacuum as initial condition.

In such a background, the string proper size L is constant outside the horizon, while it varies like the scale factor outside the horizon. As a consequence, the number distribution $n(L)$ varies in time, leading to a variation of the effective equation of state associated to the string network. We have presented two possible approaches to compute the ratio $\gamma = p/\rho$ in terms of the initial distribution and of the background scale factor. In both cases we have checked that, for a test string distribution, γ goes to zero with a power-like behaviour as $|H|^{-1}$ increases; moreover, γ ranges over negative values when the metric describes accelerated expansion, and over positive values when the metric describes accelerated contraction. In the first case, the transition from shrinking to expanding horizons is associated to a transition from decreasing to increasing pressure. The pressure, however, stays always negative, unless quantum effects (such as string decay into radiation) are included, which would eventually bring the equation of state to that of relativistic matter, $\gamma = 1/d$.

We have discussed, finally, a possible method to look for simultaneous solutions of the background equations and of the string propagation equation. We have shown that the simple dust-like solution can approximate, self-consistently, the exact string driven background describing the evolution from the initial string perturbative vacuum to a final highly curved, strong coupling regime. The corrections to such a zeroth-order approximation keep small over the whole temporal range, provided the time extension T of the inflationary regime is much larger than one when measured in string units, $T/\lambda_s >> 1$.

References.

1. H. J. De Vega and N. Sánchez, Phys. Lett. B197 (1987) 320.
2. N. Sánchez and G. Veneziano, Nucl. Phys. B333 (1990) 253.
3. M. Gasperini, N. Sánchez and G. Veneziano, Int. J. Mod. Phys. A6 (1991) 3853; Nucl. Phys. B364 (1991) 365;
   Nguyen Suan Han and G. Veneziano, Mod. Phys. Lett A6 (1991) 1993.
4. M. Gasperini, Phys. Lett. B258 (1991) 70.
5. N. Turok and P. Bhattacharjiee, Phys. Rev. D29 (1984) 1557;
   N. Turok, Phys. Rev. Lett. 60 (1988) 543.
6. G. Veneziano, Europhys. Lett. 2 (1986) 133.
7. G. Veneziano, Phys. Lett. B265 (1991) 287.
8. M. Gasperini and G. Veneziano, Astropart. Phys. 1 (1993) 317; Mod. Phys. Lett. A8 (1993) 3701.
9. H. J. De Vega and N. Sánchez, Phys. Rev. D50 (1994) 7202.
10. R. Brustein and G. Veneziano, Phys. Lett. B329 (1994) 429.
11. M. Gasperini and G. Veneziano, Phys. Rev. D50 (1994) 2519.