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GRACEFUL EXIT IN QUANTUM STRING COSMOLOGY

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Abstract

We write an $O(d,d)$-covariant Wheeler-De Witt equation in the (d^2+1)-dimensional minisuperspace of low-energy cosmological string backgrounds. We discuss explicit examples of transitions between two duality-related cosmological phases, and we find a finite quantum transition probability even when the two phases are classically separated by a curvature singularity. This quantum approach is completely free from operator ordering ambiguities as a consequence of the duality symmetries of the string effective action.

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1 Introduction

The cosmological solutions of the low-energy string effective action, and their duality symmetry properties, have recently motivated the study of an inflationary scenario [1]-[3] in which the universe evolves from the string perturbative vacuum to the high curvature, strong coupling regime. One of the main problems, in this context, is the "graceful" exit from such an accelerated (also called pre-big-bang [1]) phase to the decelerated, decreasing curvature phase typical of the standard (post-big-bang) cosmological evolution. Both phases are present in the exact solutions of the tree-level, lowest order in α', string effective action [2, 3]. However, for vanishing torsion and dilaton potential, these phases correspond to different (duality-related) branches of the solution, defined over disconnected ranges of the time parameter and separated by a singularity of the curvature and of the coupling.

The graceful exit problem would be solved, at the classical level, by an exact cosmological solution connecting smoothly the two branches, and thus describing a continuous evolution from accelerated to decelerated expansion. Unfortunately, confirming a previous conjecture [4], it has been rigorously proved [5] that such a change of branch cannot be simply catalyzed by any (realistic) dilaton potential, if we limit ourselves to lowest order in the α' expansion of the string effective action. Such a no-go theorem has been recently extended, for spatially homogeneous and isotropic manifolds, to the case of non-vanishing torsion background [6] and non-vanishing spatial curvature [7].

A possible way of incorporating branch-changing in the pre-big-bang scenario is thus to resort to the conformal field theory approach [8], where all higher orders in α' are taken into account. On the other hand, in such a "stringy" regime dominated by higher-derivative terms in the effective action, the curvature is expected to approach the Planck scale and thus the quantum gravity regime. This suggests a quantum approach to the graceful exit problem (quantum cosmology methods in a string theory context were previously introduced also in [9, 10]).

By applying the Wheeler-De Witt (WDW) equation to the gravi-dilaton system, we show in this paper that the transition from a pre-big-bang to a post-big-bang classical solution corresponds to a reflection of the wave function in minisuperspace. A classical configuration describing branch-changing gives a reflection coefficient $R = 1$. The reflection probability is in general non-vanishing, however, even if the given classical background forbids branch-changing. This is the main result of this paper, which may allow a systematic classification of the initial conditions compatible with the present state of our Universe, even ignoring kinematical details during the quantum transition era.

The paper is organized as follows. In Section 2 we derive from the low energy string effective action the WDW equation for homogeneous, spatially flat cosmological backgrounds, including a non-trivial antisymmetric (torsion) tensor. We show, in particular, that the operator ordering problem is trivially solved because of the $O(d,d)$ covariance of the kinetic part of the Hamiltonian, which implies a globally flat minisuperspace metric. In Section 3 we study the free wave equation, and we identify the left and right moving modes in superspace with the two branches of the classical vacuum solution. The reader not interested in technical complications due to the presence of a non-trivial torsion background can move directly to Section 4, where we give two self-contained examples of "quantum" branch changing, in the two-dimensional minisuperspace parameterized by the dilaton field and by the (isotropic) metric scale factor. While the first example has a classic analogue, the second corresponds to a background configuration in which branch changing is classically forbidden. A brief summary, and our concluding remarks, are finally presented in Section 5.

2 O(d,d)-covariant Wheeler-De Witt equation

At low energy, the tree-level, $(d+1)$-dimensional (super)string effective action can be written as [11]

$$S = -\frac{1}{2\lambda_s^{d-1}} \int d^{d+1}x \sqrt{|g|} e^{-\phi} \left(R + \partial_\mu \phi \partial^\mu \phi - \frac{1}{12} H_{\mu\nu\alpha} H^{\mu\nu\alpha} + V \right). \tag{2.1}$$

Here ϕ is the dilaton field, $H_{\mu\nu\alpha}$ is the field strength of the two-index antisymmetric torsion tensor $B_{\mu\nu} = -B_{\nu\mu}$, and $\lambda_s \equiv (\alpha')^{1/2}$ is the fundamental string length parameter governing the high derivative expansion of the action. Note that we have included a possible dilaton potential V, and we have chosen to work in the String (or Brans-Dicke frame), whose metric coincides with the sigma-model metric to which strings are directly coupled. The more conventional choice of the Einstein frame leads to an equivalent description of the same cosmological scenario [3, 12], but it is less convenient for exploiting the duality symmetries of the underlying theory.

We shall consider, in this paper, homogeneous backgrounds with d Abelian isometries, for which a synchronous frame exists where $g_{00} = 1$, $g_{0i} = 0 = B_{0i}$, and the fields are independent of all space-like coordinates x^i $(i,j = 1,..,d)$. We also assume spatial sections of finite volume, $(\int d^dx\sqrt{|g|})_{t=\text{const}} < \infty$. For such backgrounds the action can be rewritten as [13]

$$S = -\frac{\lambda_s}{2}\int dte^{-\overline{\phi}}\left[(\dot{\overline{\phi}})^2 + \frac{1}{8}\text{Tr}\ \dot{M}(M^{-1})^{\dot{}} + V\right] \tag{2.2}$$

where a dot denotes differentiation with respect to the cosmic time t, and we have chosen to express length and energies in string units through λ_s. Here $\overline{\phi}$ is the " shifted" dilaton field,

$$\overline{\phi} = \phi - \ln|\det\ g_{\mu\nu}|^{1/2} \tag{2.3}$$

(we have absorbed into ϕ the constant shift $-\ln(\lambda_s^{-d}\int d^dx)$ required to secure the scalar behaviour of $\overline{\phi}$ under coordinate reparametrization). Finally, M is the $2d \times 2d$ matrix

$$M = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix} \tag{2.4}$$

where G and B are, respectively, matrix representations of the spatial part of the metric (g_{ij}) and of the antisymmetric tensor (B_{ij}).

For constant V, the whole action (2.2) is invariant under global $O(d,d)$ transformations [13]

$$\overline{\phi} \to \overline{\phi}, \qquad\qquad M \to \Omega^T M \Omega \tag{2.5}$$

where

$$\Omega^T\eta\Omega = \eta, \qquad\qquad \eta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{2.6}$$

In addition, M satisfies

$$M\eta M = \eta. \tag{2.7}$$

This $O(d,d)$ symmetry is preserved in the presence of bulk string matter [14] satisfying the string equations of motion, and it reduces to the scale factor duality symmetry [15, 16] (for torsionless, diagonal metric backgrounds) in the particular case in which we restrict Ω to η in eq. (2.5).

By using as time parameter τ, with $dt = e^{-\overline{\phi}}d\tau$, the action (2.2) leads to the Lagrangian (a prime denotes differentiation with respect to τ)

$$L(\tau) = -\frac{\lambda_s}{2}\left[(\overline{\phi}')^2 + \frac{1}{8}\text{Tr}\ M'(M^{-1})' + e^{-2\overline{\phi}}V\right] \tag{2.8}$$

whose corresponding Hamiltonian is

$$H = -\frac{1}{2\lambda_s}\Pi_{\overline{\phi}}^2 + \frac{4}{\lambda_s}\mathrm{Tr}\ (M\ \Pi_M M\ \Pi_M) + \frac{\lambda_s}{2} V e^{-2\overline{\phi}} \tag{2.9}$$

where $\Pi_{\overline{\phi}}$ and Π_M are the (dimensionless) canonical momenta

$$\Pi_{\overline{\phi}} = \frac{\delta L}{\delta \overline{\phi}'} = -\lambda_s \overline{\phi}', \qquad\qquad \Pi_M = \frac{\delta L}{\delta M'} = \frac{\lambda_s}{8} M^{-1} M' M^{-1}. \tag{2.10}$$

The variation of the action (2.1) with respect to the "lapse" function, $\sqrt{g_{00}}$, provides the canonical constraint $H = 0$. The WDW equation [17], implementing in superspace such a constraint through the differential representation $\Pi_{\overline{\phi}} = \pm i\delta/\delta\overline{\phi}$, $\Pi_M = \pm i\delta/\delta M$, would seem to be affected (as usual) by problems of quantum ordering, as $[M, \Pi_M] \neq 0$. In our context, however, the problems actually disappear because our (d^2+1)-dimensional minisuperspace is globally flat, as a consequence of the $O(d,d)$ symmetry. Indeed, by using the $O(d,d)$ property (2.7), we can always rewrite the M-dependent part of the kinetic operator as

$$\frac{1}{16}\mathrm{Tr}\ M'(M^{-1})' = \frac{1}{16}\mathrm{Tr}\ (M'\eta)^2. \tag{2.11}$$

The corresponding Hamiltonian

$$H = -\frac{1}{2\lambda_s}\Pi_{\overline{\phi}}^2 - \frac{4}{\lambda_s}\mathrm{Tr}\ (\eta\ \Pi_M\ \eta\ \Pi_M) + \frac{\lambda_s}{2} V e^{-2\overline{\phi}} \tag{2.12}$$

has a flat metric in momentum space, and leads to a WDW equation

$$\left[\frac{\delta^2}{\delta\overline{\phi}^2} + 8\mathrm{Tr}\ \left(\eta\frac{\delta}{\delta M}\eta\frac{\delta}{\delta M}\right) + \lambda_s^2 V e^{-2\overline{\phi}}\ \right]\Psi(\overline{\phi}, M) = 0, \tag{2.13}$$

which is manifestly free from problems of quantum ordering.

If we introduce curvilinear coordinates in minisuperspace, adopting for instance the parametrization of eq. (2.9), the ordering imposed by the $O(d,d)$ symmetry is equivalent to the general covariance of the Laplacian operator, as can be easily checked for the simple isotropic case $B = 0$, $G_{ij} = -a^2\delta_{ij}$. In that case the kinetic part of the Hamiltonian (2.12) is represented as

$$\lambda_s H_{\mathrm{Kin}} = -\frac{1}{2}\Pi_{\overline{\phi}}^2 - 4\mathrm{Tr}\ (\eta\Pi_M)^2 \equiv \frac{1}{2}\partial_{\overline{\phi}}^2 - 2d(a\partial_a + a^2\partial_a^2). \tag{2.14}$$

The parametrization of eq. (2.9), on the other hand, corresponds to the metric

$$\gamma_{\mu\nu} = \mathrm{diag}\ \left(-2,\ \frac{a^4}{4d},\ \frac{1}{4da^4}\right), \qquad\qquad \mu,\nu = 1,2,3 \quad , \tag{2.15}$$

in the three-dimensional space spanned by the differential operators

$$\Pi_\mu = i\partial_\mu = i\left(\frac{\partial}{\partial\overline{\phi}}, \frac{\partial}{\partial a^{-2}}, \frac{\partial}{\partial a^2}\right) \tag{2.16}$$

and the covariant Laplacian gives

$$-\nabla_\mu\nabla^\mu \equiv -\frac{1}{\sqrt{|\gamma|}}\partial_\mu\left(\sqrt{|\gamma|}\gamma^{\mu\nu}\partial_\nu\right) = \frac{1}{2}\partial^2_{\overline{\phi}} - 2d(a\partial_a + a^2\partial^2_a) \equiv \lambda_s H_{\mathrm{Kin}}. \tag{2.17}$$

The ordering fixed by the scale factor duality symmetry of the classical Hamiltonian is thus the same as that imposed by the requirement of general reparametrization invariance in minisuperspace (note that in our case there is no possible contribution to the ordered Hamiltonian from the scalar curvature of superspace [18], as the metric is globally flat).

3 Branch changing as wave reflection

We shall apply, in this paper, the WDW equation (2.13) to study the probability of transition from a given initial background configuration of the pre-big-bang type, to a final configuration typical of the standard cosmological scenario. This amounts to solving eq. (2.13) for a given value of the dilaton potential $V(\phi)$, with appropriate boundary conditions. The systematic study of the transition probability for a "realistic" (supersymmetry breaking) non-perturbative dilaton potential is postponed to a future work. The main goal of this paper is to show that in a quantum cosmology context it is possible to tunnel from one branch to another of the low energy classical solutions, even if the two branches are not smoothly connected, but they are separated by curvature singularities and unphysical regions of finite size.

To this aim we shall work in the simplifying hypothesis of $O(d,d)$ symmetry of the whole (in general non-local) action, assuming $V = V(\overline{\phi})$, because in that case the WDW equation (2.13) can be separated by setting

$$\Psi(\overline{\phi}, M) = \chi_A(M)\psi_A(\overline{\phi}) \tag{3.1}$$

where

$$(M\Pi_M)\chi_A \equiv iM\frac{\delta}{\delta M}\chi_A = -\left(\frac{1}{8}A\eta\right)\chi_A \tag{3.2}$$

and

$$\left[\frac{\delta^2}{\delta\overline{\phi}^2}+\frac{1}{8}\mathrm{Tr}\ (A\eta)^2+\lambda_s^2 V(\overline{\phi})e^{-2\overline{\phi}}\right]\psi_A(\overline{\phi})=0. \tag{3.3}$$

We have used here the momentum conservation $[M\Pi_M, H] = 0$, in agreement with the classical equations of motion obtained from the Lagrangian (2.8), which imply [13, 14]

$$M\Pi_M=-\frac{\lambda_s}{8}M\eta M'\eta=-\frac{1}{8}A\eta \tag{3.4}$$

where A is a constant $2d\times 2d$ matrix satisfying

$$M\eta A+A\eta M=0. \tag{3.5}$$

If we consider, in particular, the "free" wave equation ($V=0$), eq. (3.3) is easily solved by a linear superposition of left and right moving waves,

$$\psi_A^{\pm}(\overline{\phi})=\exp\left\{\pm\frac{i}{2}\overline{\phi}\left[\frac{1}{2}\mathrm{Tr}\ (A\eta)^2\right]^{1/2}\right\}. \tag{3.6}$$

For simplicity, we shall restrict our subsequent discussion to a diagonal, Bianchi I type vacuum background with $G_{ij}=-a_i^2(t)\delta_{ij}$ and $B=0$, corresponding to [13]

$$A=\begin{pmatrix}0 & -A_d\\ A_d & 0\end{pmatrix},\qquad (A_d)_{ij}=c_i\delta_{ij}\quad , \tag{3.7}$$

where c_i are arbitrary constants. The metric-dependent part of the wave function becomes in this case

$$\chi_A(a_j)=N\exp\left\{-\frac{i}{2}\sum_j c_j\ln a_j\right\}, \tag{3.8}$$

as one can check after realizing that the operator $M\Pi_M$ contains in this case only d independent variables. By defining

$$\alpha_j=c_j\left(\sum_j c_j^2\right)^{-1/2}\equiv c_j\left[\frac{1}{2}\mathrm{Tr}\ (A\eta)^2\right]^{-1/2},\qquad \sum_j\alpha_j^2=1 \tag{3.9}$$

the solutions of the WDW equation can finally be written in the form

$$\Psi_A^{(\pm)}(\overline{\phi},M)=\chi_A(M)\psi_A^{\pm}(\overline{\phi})=N_{\pm}\exp\left\{-\frac{i}{2}\left[\mathrm{Tr}\ (A_d)^2\right]^{1/2}\left(\sum_j\alpha_j\ln a_j\mp\overline{\phi}\right)\right\}, \tag{3.10}$$

where $N_{\pm}$ is an overall normalization coefficient.

For fixed α_j, namely for a given eigenstate of $M\Pi_M$, the left and right moving modes $\Psi^{(\pm)}$ correspond to different branches of the exact solution of the vacuum string cosmology equations [2, 3, 13, 15, 16, 19]

$$a_j = a_{j0}|t/\lambda_s|^{\pm\alpha_j}, \qquad \sum_j \alpha_j^2 = 1, \qquad \overline{\phi} - \phi_0 = -\ln|t/\lambda_s| = \mp \sum_j \alpha_j \ln a_j, \tag{3.11}$$

where a_{i0} and ϕ_0 are integration constants. What is important, for our purpose, is that if we apply the momentum operator $\Pi_{\overline{\phi}} = i\partial/\partial\overline{\phi}$ to the right moving wave $\Psi_A^{(+)}$ (the opposite sign with respect to the standard convention is due to the definition of $\Pi_{\overline{\phi}}$, eq. (2.10)), we reproduce the canonical momentum

$$\Pi_{\overline{\phi}} = -\lambda_s \dot{\overline{\phi}} e^{-\overline{\phi}} = -e^{-\phi_0} < 0 \tag{3.12}$$

of a classical configuration corresponding to an accelerated, expanding background, with growing curvature and dilaton coupling:

$$a_i \sim (-t)^{-\alpha_i}, \qquad t < 0, \qquad \alpha_i > 0, \qquad \overline{\phi} - \phi_0 = \sum_j \alpha_j \ln a_j, \qquad \dot{\overline{\phi}} > 0. \tag{3.13}$$

By applying $\Pi_{\overline{\phi}}$ to the left mover $\Psi_A^{(-)}$, labelled by the same eigenvalue A, we find instead a configuration with the opposite canonical momentum,

$$\Pi_{\overline{\phi}} = -\lambda_s \dot{\overline{\phi}} e^{-\overline{\phi}} = e^{\phi_0} > 0 \tag{3.14}$$

corresponding again to an expanding branch of the same classical solution, but decelerated and with decreasing curvature:

$$a_i \sim t^{\alpha_i}, \qquad t > 0, \qquad \alpha_i > 0, \qquad \overline{\phi} - \phi_0 = -\sum_j \alpha_j \ln a_j, \qquad \dot{\overline{\phi}} < 0. \tag{3.15}$$

A branch transition of the type required to solve the graceful exit problem, involving a scale factor duality transformation and time reversal [1], namely $a(t) \to a^{-1}(-t)$, is thus equivalent in this context to the spatial reflection of the WDW wave function in minisuperspace, $\Psi_A^{(+)} \to \Psi_A^{(-)}$.

4 Two simple examples

As no reflection is possible for free waves, let us introduce an appropriate dilaton potential, considering for simplicity a $d = 3$ isotropic background, and setting

$$B = 0, \qquad\qquad a(t) = \exp\left[\beta(t)/\sqrt{3}\right]. \tag{4.1}$$

The lowest order gravi-dilaton effective action,

$$S = -\frac{1}{2\lambda_s^2}\int d^4x\sqrt{-g}e^{-\phi}\left(R + \partial_\mu\phi\partial^\mu\phi + V\right), \tag{4.2}$$

after integrating by parts, and using as before the convenient time parameterization $dt = d\tau e^{-\overline{\phi}}$, reduces to (in the gauge $g_{00} = 1$):

$$S = -\frac{\lambda_s}{2}\int d\tau\left(\,\overline{\phi}'^2 - \beta'^2 + Ve^{-2\overline{\phi}}\,\right) \tag{4.3}$$

where

$$\overline{\phi} = \phi - \ln\int(d^3x/\lambda_s^3) - \sqrt{3}\beta. \tag{4.4}$$

The corresponding Hamiltonian

$$H = \frac{1}{2\lambda_s}\left(\,\Pi_\beta^2 - \Pi_{\overline{\phi}}^2 + \lambda_s^2 Ve^{-2\overline{\phi}}\,\right), \qquad \Pi_\beta = \lambda_s\beta', \qquad \Pi_{\overline{\phi}} = -\lambda_s\overline{\phi}' \tag{4.5}$$

is a particular case of eq. (2.12), for the torsionless isotropic background considered here. The WDW equation then takes the general form of a two-dimensional Schrödinger-like equation in the plane $(\overline{\phi}, \beta)$:

$$\left[\,\partial_{\overline{\phi}}^2 - \partial_\beta^2 + \lambda_s^2 V(\overline{\phi}, \beta)e^{-2\overline{\phi}}\,\right]\Psi(\overline{\phi}, \beta) = 0. \tag{4.6}$$

As anticipated in the previous section, we shall assume in this paper $V = V(\overline{\phi})$, in order to separate variables. Let us discuss first the particular case

$$V(\overline{\phi}) = -V_0 e^{4\overline{\phi}}, \qquad V_0 = \text{const}, \qquad V_0 > 0 \tag{4.7}$$

as a toy example of classical gravi-dilaton configuration allowing branch changing. A negative non-local dilaton potential, $V(\overline{\phi}) < 0$, although hard to motivate in a realistic superstring theory context, is indeed the only case in which exact analytical solutions are known [1, 20] connecting smoothly the pre- to the post-big-bang regime.

With the above potential, the classical equations of motion following from the Hamiltonian (4.5) imply momentum conservation along the β axis,

$$\Pi_\beta = \lambda_s \beta' = \lambda_s \dot{\beta} e^{-\overline{\phi}} = k = \text{const} \tag{4.8}$$

and are solved exactly by

$$\overline{\phi} = -\frac{1}{2} \ln \left(\frac{\lambda_s^2 V_0}{k^2} + \frac{k^2 t^2}{\lambda_s^2} \right), \qquad a = a_0 \left[\frac{k^2 t}{\lambda_s^2 \sqrt{V_0}} + \left(1 + \frac{k^4 t^2}{\lambda_s^4 V_0} \right)^{1/2} \right]^{1/\sqrt{3}} \tag{4.9}$$

(a_0 is a dimensionless integration constant). This is a regular "self-dual" solution, $a(t)/a_0 = a_0/a(-t)$, characterized by a bell-like shape of the curvature scale and of the coupling $e^{\overline{\phi}}$. It describes a background that evolves from an initial state of accelerated expansion and increasing curvature,

$$\begin{aligned} t \to -\infty, \qquad & a \sim (-t)^{-1/\sqrt{3}}, \quad \overline{\phi} \sim -\ln(-t) = \sqrt{3} \ln a = \beta \\ & \dot{a} > 0, \quad \ddot{a} > 0, \quad \dot{H} > 0 \end{aligned} \tag{4.10}$$

to a final state of decelerated expansion, decreasing curvature,

$$\begin{aligned} t \to +\infty, \qquad & a \sim t^{1/\sqrt{3}}, \quad \overline{\phi} \sim -\ln(-t) = -\sqrt{3} \ln a = -\beta \\ & \dot{a} > 0, \quad \ddot{a} < 0, \quad \dot{H} < 0. \end{aligned} \tag{4.11}$$

For the background generated by the potential (4.7), the WDW equation (4.6) can easily be separated by putting $\Psi(\overline{\phi}, \beta) = e^{-ik\beta} \psi_k(\overline{\phi})$, where k belongs to the continuous eigenvalue spectrum of Π_β,

$$\Pi_\beta \Psi_k = i \partial_\beta \Psi_k = k \Psi_k, \qquad [\Pi_\beta, H] = 0 \tag{4.12}$$

and ψ_k satisfies

$$\left(\partial_{\overline{\phi}}^2 + k^2 - \lambda_s^2 V_0 e^{2\overline{\phi}} \right) \psi_k(\overline{\phi}) = 0 \tag{4.13}$$

(note the role of time-like coordinate assigned to β, monotonically ranging from $-\infty$ to $+\infty$). The general solution for ψ_k is then a linear combination of modified Bessel functions $K_\nu(z)$, $I_\nu(z)$ [21], of complex index $\nu = ik$ and argument $z = \lambda_s \sqrt{V_0} e^{\overline{\phi}}$. We impose the regularity condition [22] $|\Psi_k| < 0$, corresponding to a vanishing wave function in the "impenetrable" region of infinite effective potential, $\psi_k(\overline{\phi}) \to 0$ for $\overline{\phi} \to +\infty$. This condition uniquely selects (modulo a normalization factor) the WDW solution as

$$\Psi_k(\overline{\phi}, \beta) = N K_{ik}(\lambda_s \sqrt{V_0} e^{\overline{\phi}}\ ) \ e^{-ik\beta}. \tag{4.14}$$

For $\overline{\phi} \to -\infty$, i.e. in the low energy regime, this solution contains asymptotically left and right moving waves, as [21]

$$\begin{aligned}
\lim_{\overline{\phi}\to-\infty} \Psi_k(\overline{\phi},\beta) &= -\frac{N\pi}{2\sin(ik\pi)}\left[\left(\frac{\lambda_s\sqrt{V_0}}{2}\right)^{ik}\frac{e^{-ik(\beta-\overline{\phi})}}{\Gamma(1+ik)} - \left(\frac{\lambda_s\sqrt{V_0}}{2}\right)^{-ik}\frac{e^{-ik(\beta+\overline{\phi})}}{\Gamma(1-ik)}\right] = \\
&= \Psi_k^{(+)} + \Psi_k^{(-)}
\end{aligned} \tag{4.15}$$

As discussed in the previous section, the right movers represent the accelerated, negative time branch (4.10), with $\beta = \overline{\phi}$, the left movers the decelerated, positive time branch (4.11), with $\beta = -\overline{\phi}$. The reflection coefficient, $R_k = |\Psi_k^{(-)}|^2/|\Psi_k^{(+)}|^2$, measures the probability of transition between the two branches of the low energy classical solution. In the low energy limit, according to eq. (4.15), $R_k \to 1$ for all k, as expected because we have considered an example in which the two branches are smoothly connected already at the classical level.

Consider now an example with the opposite potential,

$$V(\overline{\phi}) = V_0 e^{4\overline{\phi}}, \qquad V_0 = \text{const}, \qquad V_0 > 0. \tag{4.16}$$

In this case branch changing is classically forbidden. Indeed, the momentum conservation (4.8) is still valid, but the classical solution becomes

$$\overline{\phi} = -\frac{1}{2}\ln\left(\frac{k^2t^2}{\lambda_s^2} - \frac{\lambda_s^2 V_0}{k^2}\right), \qquad a = a_0\left|\frac{k^2 t}{\lambda_s^2\sqrt{V_0}} + \left(\frac{k^4t^2}{\lambda_s^4 V_0} - 1\right)^{1/2}\right|^{1/\sqrt{3}}. \tag{4.17}$$

The low energy (large time limit) branches (4.10) and (4.11) still exist, but they are now separated by an unphysical region, of extension $|t| < \lambda_s^2\sqrt{V_0}/k^2$, where the dominant energy condition is violated, and the expansion rate (H) and the dilaton coupling ($e^{\overline{\phi}}$) become imaginary. A curvature singularity is present at both ends of such a region, where the branches (4.10) and (4.11) respectively end and start.

Nevertheless, the quantum probability of transition between the two branches is non-vanishing. In fact, the solution of the WDW equation can be factorized as before, with the difference that eq. (4.13) is replaced by

$$\left(\,\partial_{\overline{\phi}}^2 + k^2 + \lambda_s^2 V_0 e^{2\overline{\phi}}\,\right)\psi_k(\overline{\phi}) = 0. \tag{4.18}$$

The general solution for ψ_k can now be written as a linear combination of first and second kind Hankel functions [21], $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$. By assuming for the Universe an initial pre-big-bang configuration, we impose that in the high-curvature limit $z \to \infty$ there are only right moving waves ($\dot{\overline{\phi}} > 0$, $\Pi_{\overline{\phi}} < 0$) approaching the singularity. This condition exactly coincides with the boundary conditions allowing tunnelling through classically forbidden regions of superspace [22] (which select only outgoing waves at the superspace boundary, where classical trajectories can end but not begin), and fixes the wave function as

$$\Psi_k(\overline{\phi}, \beta) = N H^{(1)}_{ik}(\lambda_s \sqrt{V_0} e^{\overline{\phi}}\ )\ e^{-ik\beta}. \tag{4.19}$$

Asymptotically, in the low curvature, perturbative regime $\overline{\phi} \to -\infty$, we then have

$$\begin{aligned} \lim_{\overline{\phi}\to-\infty} \Psi_k(\overline{\phi}, \beta) &= iN \csc(ik\pi) \left[e^{k\pi} \left(\frac{\lambda_s\sqrt{V_0}}{2}\right)^{ik} \frac{e^{-ik(\beta-\overline{\phi})}}{\Gamma(1+ik)} - \left(\frac{\lambda_s\sqrt{V_0}}{2}\right)^{-ik} \frac{e^{-ik(\beta+\overline{\phi})}}{\Gamma(1-ik)} \right] = \\ &= \Psi_k^{(+)} + \Psi_k^{(-)} \end{aligned} \tag{4.20}$$

and the relative amplitude of left and right modes defines the probability

$$R_k = \frac{|\Psi_k^{(-)}|^2}{|\Psi_k^{(+)}|^2} = e^{-2\pi k} \tag{4.21}$$

for transitions from the classical trajectory with $\beta = \overline{\phi}$ to the duality-related one, $\beta = -\overline{\phi}$.

By recalling the definition of k (eq. (4.8)) and of $\overline{\phi}$,

$$k = \frac{\sqrt{3}}{\lambda_s^2} \int d^3x \sqrt{-g} e^{-\phi} H = \text{const} \tag{4.22}$$

we can eventually express the above transition probability as

$$R(g_s, a_s) = \exp\left\{ -\frac{\sqrt{12}\pi}{g_s^2} \frac{\Omega(a_s)}{\lambda_s^3} \right\}. \tag{4.23}$$

Here $\Omega(a_s)$ and $g_s = e^{\phi_s/2}$ are, respectively, the values of the proper spatial volume and of the coupling, at the time $t = t_s$ at which $H = \lambda_s^{-1}$. For values of the coupling $g_s \sim 1$ the probability (4.21) is of order 1 for the formation of "bubbles" of unit proper size (or smaller) in string units at $t = t_s$.

The above example is not "realistic", in the sense that it does not describe the formation of a radiation-dominated or matter-dominated Universe similar to the one we live in today (we postpone the discussion of a more realistic scenario to a forthcoming paper). Nevertheless, it is an example of how the Universe can emerge from the inflationary phase in the right branch corresponding to decelerated expansion, and it is quite interesting that the probability of such a process is peaked in the strong coupling regime, with a typical instanton-like behaviour $R \sim \exp(-g^{-2})$.

We note, finally, that eq. (4.21) is valid for $k > 0$, namely for the transition between two expanding branches with $\text{sign}[H] = \text{sign}[\dot{\beta}] = \text{sign}[k] > 0$, and that it implies $0 < R_k < 1$ ($R_k \to 0$ for $k \to \infty$, as expected when the effective potential in eq. (4.18) becomes negligible). If we consider transitions between contracting branches, $k < 0$, the probability becomes $e^{2\pi k}$ so that, in general, $R_k = e^{-2\pi|k|}$. The appearance of an always negative argument in the exponential is a general consequence of applying tunneling boundary conditions in superspace, as clearly stressed recently also in [23].

5 Conclusion

In string cosmology, a classical description of the background evolution based on the low energy string effective action is allowed both at early and late times (i.e. at large time scales in string units), but it is not allowed in the intermediate epoch, when the background is expected to exit from the inflationary regime. A classical model of transition from the initial string perturbative vacuum to the present standard cosmological regime conflicts both with phenomenological constraints and with formal no-go theorems.

In this paper we have shown that such a transition can be studied quantum mechanically, and can be formulated as a problem of reflection of the Wheeler-De Witt wave function in superspace. By using tunneling boundary conditions, we find that the transition can occur (with finite probability) even in the case of background configurations in which it would be classically forbidden.

This quantum approach to the graceful exit problem is free from ambiguities of operator ordering, because of the underlying $O(d,d)$ symmetry of the kinetic part of the Hamiltonian. It thus seems to provide an appropriate tool for a systematic classification of the initial conditions, irrespective of the (unknown) kinematic details of the high curvature, strong coupling, transition regime.

Various problems, in this approach, are still to be solved, such as the physical interpretation of the wave function and the univocal choice of an appropriate time parameter (these problems affect in general the WDW approach to quantum cosmology, not only our scenario). We believe, however, that the result presented in this paper may improve our understanding of the birth and of the evolution of the Universe in terms of the basic principles of string theory.

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