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A GEOMETRIC REGULARIZATION PROCEDURE FOR THE CURVATURE OF COSMOLOGICAL BACKGROUNDS *

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ABSTRACT

A formal procedure is presented, which can regularize a given cosmological metric by avoiding the occurrence of curvatures and densities larger than the Planck ones. The method is applied, in particular, to power-law and super inflationary backgrounds and it is shown that, in both cases, the regularized metric describes a smooth evolution from (or towards) an asymptotic de Sitter phase of constant maximal curvature.

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ABSTRACT

A formal procedure is presented, which can regularize a given cosmological metric by avoiding the occurrence of curvatures and densities larger than the Planck ones. The method is applied, in particular, to power-law and super inflationary backgrounds and it is shown that, in both cases, the regularized metric describes a smooth evolution from (or towards) an asymptotic de Sitter phase of constant maximal curvature.

1.Introduction

According to the standard model, the universe is gravitationally dominated by a perfect fluid source with dust-like or radiation-like equation of state, and the cosmological evolution should start from an initial phase characterized by an infinite (or arbitrarily high) value of the density and of the background curvature.

It is true that, in order to explain the present homogeneity of the cosmic microwave radiation, and to avoid fine-tuning problems with the initial conditions, the validity of the standard model cannot be indefinitely extended backward in time. It seems very likely, in particular, that the present decelerated expansion should be replaced, at early enough times, by an inflationary (i.e. accelerated) time evolution. Even in this case, however, the occurrence of a singular phase with an infinitely large curvature cannot be excluded (unless inflation is exactly of the de Sitter type, and correspond then to a maximally symmetric manifold of constant, finite curvature).

In this talk I want to present a possible geometric procedure for regularizing the curvature of the early universe, based on the embedding of the given singular geometry into a larger, curved manifold whose coordinates are positions and velocities. I shall call it "phase-spacetime" [1]. According to such procedure, the regularized geometry that one obtains always coincides, asymptotically, with a de Sitter phase of maximal constant curvature, whose radius represents the only free parameter of the whole formalism. By fixing, in particular, this radius equal to the Planck length one can thus avoid the occurrence of energy densities larger than the Planck one.

2. Maximal curvature and maximal acceleration

My starting point is the observation that the problem of limiting the curvature of a given background is equivalent to the problem of limiting the relative accelerations between two points of an elementary extended object minimally coupled to that background.

Suppose indeed that there are no interactions other than gravity, so that the given object is free falling. Different points of the object will fall, however, along different geodesics. If z^μ is the (spacelike) separation vector between two points, their relative acceleration, according to the equation of geodesic deviation [2], is then

$$a^\mu = -R_{\nu\alpha\beta}{}^{\mu} z^\nu u^\alpha u^\beta \tag{1}$$

(u^μ is the velocity vector, $z_\mu u^\mu = 0$). For a given, finite proper distance $|z| > 0$, the proper strength of this relative acceleration depends only on the components of the background curvature tensor, $R_{\mu\nu\alpha\beta}$, and then it diverges if and only if the curvature diverges.

In order to impose a bound on the curvature let me recall, therefore, that there is a simple way to impose over spacetime a causal structure in which proper accelerations are limited [1]. Suppose in fact that the spacetime V_4, with coordinates ξ^μ and metric $g_{\mu\nu}(\xi)$, is a four-dimensional hypersurface locally embedded in a eight-dimensional manifold M_8, whose coordinates X^A are the relativistic positions and velocity four-vectors:

$$X^A = (x^\mu, \lambda u^\mu) \qquad , \qquad u^\mu = dx^\mu/ds \tag{2}$$

where $ds = (dx^\mu dx_\mu)^{1/2}$, and λ is a constant length required for dimensional homogeneity (conventions: $\mu\nu = 1,..4$; $A, B = 1, ...8$; $\hbar = c = 1$).

In this context the usual relativistic invariant ds is to be replaced by a more general invariant interval, which I shall call $d\tilde{s}$, representing the infinitesimal distance in M_8. By taking for M_8 the metric $g_{AB} = g_{\mu\nu} \bigotimes g_{\mu\nu}$, the generalized invariant becomes

$$d\tilde{s}^2 = g_{AB} dX^A dX^B = g_{\mu\nu}(dx^\mu dx^\nu + \lambda^2 du^\mu du^\nu) \qquad (3)$$

Along any path, defined by a given $x^\mu(s)$ and $u^\mu(s)$, we have $d\tilde{s}^2 = ds^2(1-\lambda^2|a|^2)$, where $|a|^2 = |du^\mu/ds|^2$. The imposition of a causal structure over M_8, $d\tilde{s}^2 > 0$, thus implies that proper accelerations must be bounded, $|a| < \lambda^{-1}$.

Let me note that the introduction of the generalized invariant (3) was first proposed by Das [3] and, independently, by Caianiello [4], who was the first to point out the existence of a "maximal proper acceleration", λ^{-1}, as a consequence of this geometric structure. Henceforth I shall assume that the minimal length λ determining the maximal allowed acceleration coincides with the Planck length, $\lambda = L_p$, in agreement with previous suggestions [5] (for other possible choices see, however, ref.[1]).

I should mention that there are many important motivations for generalizing in this way the spacetime geometry [1], such as the possibility of obtaining a geometric interpretation of the position- momentum commutation rules of quantum mechanics. Some of these motivations, investigated by Marmo, de Filippo, Vilasi, Feoli, Guz, Brandt and others [1], have been already discussed, in their previous talks, by Caianiello and Scarpetta. For the purpose of my talk I will only note that the embeddding of V_4 into M_8, defined by the eight parametric equations

$$x^\mu = x^\mu(\xi) \qquad , \qquad u^\mu = u^\mu(\xi) \qquad (4)$$

induces locally over spacetime, according to eq.(3), a generalized metric $\tilde{g}$ such that $d\tilde{s}^2 = \tilde{g}_{\mu\nu} d\xi^\mu d\xi^\nu$, where

$$\tilde{g}_{\mu\nu} = g_{\alpha\beta}\left(\frac{\partial x^\alpha}{\partial \xi^\mu}\frac{\partial x^\beta}{\partial \xi^\nu} + L_p^2 \frac{\partial u^\alpha}{\partial \xi^\mu}\frac{\partial u^\beta}{\partial \xi^\nu}\right) \qquad (5)$$

This metric depends both on the coordinates ξ^μ chosen to parametrize V_4, and on the velocity field defined on it. Different fields define different embeddings [1]. In particular, I shall call "geodesic" the embedding if u^μ is a geodesic velocity field [6]. The main purpose of this paper is to show that the geodesic embedding of a cosmological metric (g) into M_8 may provide, according to eq.(5), a modified metric $\tilde{g}$ which is regularized, in the sense that its curvature cannot exceed the limit fixed by the Planck length.

3.Geodesic embedding of an inflationary metric into "phase-spacetime"

I shall consider, for simplicity, an explicit example in which the given cosmological background V_4 is conformally flat. In the conformal parametrization $\xi^\mu = (\eta, \vec{x})$ its metric is given by

$$g_{\mu\nu}(\xi) = diag\ R^2(\eta)(1, -1, -1, -1) \tag{6}$$

where R is the usual Friedman-Robertson-Walker scale factor, and η is the conformal time, related to the cosmic time coordinate t by $R = dt/d\eta$. By choosing the co-moving geodesic field

$$u^\mu(\xi) = (R^{-1}, 0, 0, 0) \tag{7}$$

the embedding procedure defined previously leads to a generalized metric which, written in terms of the original scale factor $R(\eta)$, takes the form

$$\tilde{g}_{\mu\nu}(\xi) = diag\ R^2(1 + L_p^2 \frac{R'^2}{R^4}, -1, -1, -1) \tag{8}$$

(a prime denotes differentiation with respect to η, a dot with respect to t).

This new metric has, at least, two interesting properties. First of all, its curvature is regularized. This can be explicitly verified by considering the following scalar quantity, $S = -(g_{\mu\nu} R^{\mu\nu})/12$, proportional to the trace of the Ricci tensor. For the old metric one has

$$S = \frac{R''}{2R^3} \tag{9}$$

while, for the new metric, this scalar can be expressed as

$$\tilde{S} = \frac{S + L_p^2 H^4}{(1 + L_p^2 H^2)^2} \tag{10}$$

where $H = R'/R^2 \equiv \dot{R}/R$ denotes the usual Hubble parameter.

Supposing that the time evolution of the early universe corresponds to an accelerated expansion, I shall consider, in particular, the case in which the original metric is of the inflationary type (the regularization of a standard, decelerated expansion is discussed in Ref. [6]). The asymptotic behaviour of the scale factor near the singularity can thus parametrized, in the cosmic time gauge, by $R(t) = (t/t_0)^\alpha$, where t_0 is a reference value fixing a convenient unit of time. This scale factor describes a power-law inflationary phase ($\ddot{R} > 0, \dot{H} < 0$) for $\alpha > 1, t > 0, t_0 > 0$, and a phase of super-inflation ($\ddot{R} > 0, \dot{H} > 0$) for $\alpha < 0, t < 0$ and $t_0 < 0$.

In both cases the scalar curvature $S(t)$ of eq.(9) diverges like t^{-2} as $t \to 0$. For the new metric, on the contrary, $\tilde{S}(t)$ approaches the limiting value L_p^{-2} when $t \to 0$, as can be easily verified by using the explicit expression

$$\tilde{S}(t) = \frac{1}{(1 + L_p^2 \frac{\alpha^2}{t^2})^2}(\frac{2\alpha^2 - \alpha}{2t^2} + L_p^2 \frac{\alpha^4}{t^4}) \tag{11}$$

Quite independently of the value of α, the Planck curvature is then the maximum allowed curvature for the regularized metric (8). The behaviour of the scalar curvature $S(t)$ and $\tilde{S}(t)$ (in units of L_p^{-2}), as a function of the cosmic time, are shown in **Fig.1** for two cases of power-law inflation, corresponding to $\alpha = 2$ and $\alpha = 3$. Dashed curves correspond to the curvature S of the original geometry before the embedding, solid curves refer to the curvature $\tilde{S}$ of the regularized geometry, after the embedding. The same is shown in **Fig.2**, for two cases of super-inflation corresponding to $\alpha = -1$ and $\alpha = -2$.

The other important property of the metric (8) that I want to stress is the fact that the regularized geometry always tends to coincide, asymptotically, with a de Sitter manifold of minimal radius L_p. In other words, the phase of maximal curvature corresponds necessarily to a maximally symmetric phase of constant curvature, quite independently of the particular form of the metric before the embedding.

This property can be easily verified in the comoving frame of the regularized geometry, by exploiting the relation between the cosmic time and the conformal time η of the "old" metric, that is

$$\frac{dt}{d\eta} = (R^2 + L_p^2 \frac{R'^2}{R^2})^{1/2} \tag{11}$$

(following from the definition (8)). Consider, in fact, the regime in which the embedding corrections become dominant, namely the regime in which the size of the event horizon $\sim H^{-1}$, is much smaller than L_p, $HL_p >> 1$. In this limit one gets from eq.(11) the relation $L_p(\dot{R}/R) = 1$, which defines just the scale factor for a de Sitter manifold of radius $H^{-1} = L_p = const$, corresponding to a cosmological constant $\Lambda = 3/L_p^2$.

This important property of the regularized geometry is clearly displayed in the following two pictures, where I have plotted the behaviour of the scale factors, as a function of time, before and after the embedding. The scale factor for the regularized metric, $\tilde{R}$, is obtained by integrating eq.(11), after inserting the explicit expression of the original metric, $R = (t/t_0)^\alpha$. One thus obtains that $\tilde{R}$, in the cosmic time gauge, is defined implicitly through the relation

$$t = \sqrt{\alpha^2 L_p^2 + t_0^2 \tilde{R}^{2/\alpha}} + \frac{\alpha}{2} L_p ln(\frac{\sqrt{\alpha^2 L_p^2 + t_0^2 \tilde{R}^{2/\alpha}} - \alpha L_p}{\sqrt{\alpha^2 L_p^2 + t_0^2 \tilde{R}^{2/\alpha}} + \alpha L_p}) \tag{12}$$

Note that at large values of $\tilde{R}^{2/\alpha}$ one recovers the original metric, i.e. $\tilde{R} \to R$. At small enough values of $\tilde{R}^{2/\alpha}$, on the contrary, eq.(12) provides the expected de Sitter behaviour, i.e. $t = const + ln\tilde{R}$.

Fig.3 shows the modification of the scale factors in two cases of power-law inflation, $\alpha = 2$ and $\alpha = 3$. In both cases the regularized geometries (solid curves) clearly converge, for $\tilde{R} \to 0$, to a de Sitter phase of exponential expansion (represented in the plot by a dotted line). The same is shown in **Fig.4**, for two super- inflationary backgrounds corresponding to $\alpha = -1$ and $\alpha = -2$. The only difference, in the case of super-inflation, is that the de Sitter phase is approached in the large $\tilde{R}$ limit.

At small curvature (where the embedding corrections are negligible) eq. (12) provides, as already noted, the original scale factor. The regularized metric (8) describes then a smooth evolution of the given geometry from, or towards, a de Sitter manifold of minimal size L_p, depending on if the initial geometry is, respectively, of the power-law ($\dot{H} < 0$) or of the super ($\dot{H} > 0$) inflationary type.

In any case, it is important to stress that the maximal curvature of the asymptotic de Sitter phase cannot be exceeded even if one starts with a geometry which, before the embedding, is already of the de Sitter type, and is characterized by a cosmological constant larger than $3/L_p^2$.

Suppose to start, indeed, with the metric

$$ds^2 = \frac{3}{\Lambda \eta^2}(d\eta^2 - d\vec{x}^2) \tag{13}$$

describing (in the conformal gauge) a de Sitter manifold with arbitrarily large value of Λ. After the embedding the regularized metric (8), written again in a conformally flat form by means of a new conformal coordinate $\tilde{\eta}$, is given by

$$d\tilde{s}^2 = \frac{3 + L_p^2 \Lambda}{\Lambda \tilde{\eta}^2}(d\tilde{\eta}^2 - d\vec{x}^2) \tag{14}$$

The geometry is still of the de Sitter type, but with a new cosmological constant

$$\tilde{\Lambda} = \frac{3\Lambda}{3 + L_p^2 \Lambda} \tag{15}$$

which is bounded, $\tilde{\Lambda} \leq 3/L_p^2$, even if $\Lambda \to \infty$. After the embedding, therefore, the curvature of the regularized de Sitter geometry is always smaller than the Planck limit, and $\Lambda_M = 3/L_p^2$ is the maximum allowed cosmological constant.

4. Summary and conclusion

The embedding of spacetime into the eight-dimensional phase-spacetime provides a formal procedure to regularize the cosmological models of the early universe. This procedure leads, automatically, to a modified metric (eq.(8)) characterized by a maximum allowed curvature, $S \leq 1/L_p^2$ (see Fig.1,2), and a maximum allowed cosmological constant, $\Lambda \leq 3/L_p^2$. (For cosmological applications I have fixed at the Planck scale the length representing the only free parameter of this procedure , see Section 2).

The singular phase of infinite density (and infinite curvature) is smoothed out and replaced by a phase of exponential inflation, i.e. by a de Sitter manifold of minimal size $H^{-1} = L_p$. This de Sitter manifold is approached asymptotically in the large R limit (i.e. at the end of the evolution, see Fig.4) if the regularized background is of the super-inflationary type ($\dot{H} > 0$), and in the small R limit (i.e. at the beginning of the evolution, see Fig.3) if the regularized background is power-law inflating ($\dot{H} < 0$).

In both cases the regularizing procedure (i.e. the embedding into phase-spacetime) has a **deflationary** effect, in the sense of Ref.[7], since the regularized metric always describes a smooth evolution from a **higher** to a **lower** level of accelerated expansion.

Deflationary metrics, on the other hand, seem to be typical of string- driven cosmology [7], and then of phases dominated by the contribution of finite size objects. This may suggest that the embedding I have presented, more than a formal procedure, could be regarded also as an effective model for the perturbations induced by extended objects (of size $\sim L_p$) on the background geometry, and arising when the curvature scale is comparable to (or smaller than) their proper size. I hope to discuss this possibility in future works.

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