# Kinematic interpretation of string instability in a background gravitational field

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Received 13 December 1990

The unstable regime in which the string oscillating modes develop imaginary frequencies is shown to be characterized, kinematically, by a positive relative acceleration among the different points of the string. Instability occurs when this acceleration, induced by the background curvature, is large enough to make the extension of the corresponding causally connected region smaller than the string maximal size. This kinematic characterization is applied, in particular, to discuss string instability in a static and spherically symmetric gravitational background.

## 1. Introduction

Recent studies [1,2] of the string in cosmological backgrounds have pointed out the possible emergence of instabilities, whenever the time evolution of the background is of the inflationary type. Indeed, by expanding the exact solution of the string equations around the geodesic motion of the center of mass, and considering the proper amplitude of the first order fluctuations, one finds for their Fourier components $\chi_n^i$ the equation [2]

$$\frac{d^2\chi_n^i}{dt^2}+\left(\frac{n^2}{\alpha'^2m^2}-\frac{1}{R}\frac{d^2R}{dt^2}\right)\chi_n^i=0\,, \tag{1}$$

where $R$ is the background scale factor, $t$ is the cosmic time (proportional to the world-sheet time), $m$ and $(\alpha')^{-2}$ are the string mass and the usual string tension. If the acceleration $d^2R/dt^2$ is positive (i.e. inflationary) and large enough, the oscillators develop imaginary frequencies and their proper amplitudes start to grow like the scale factor, while the comoving amplitudes become "frozen". The geodesic approximation turns out to break down [2] and one is led, asymptotically, to a highly unstable regime which requires, for its systematic description, a different approximation scheme based on the proportionality of the world-sheet and conformal times [3].

The main purpose of this paper is to present a kinematic interpretation of this instability, showing that instability is not a peculiar feature of inflationary backgrounds only, but of any physical situation in which effective repulsive forces tend to induce a positive relative acceleration between different points of the string.

It will be shown, in fact, that instability occurs when two ends of a string become causally disconnected because of the Rindler horizon associated to their relative acceleration. This interpretation thus explains [4] why $\alpha' m$, which defines the "maximal" size of the string, is the physical length required to characterize the onset of instability.

By using the previous definition it becomes possible, for any given external field, to predict instability a priori (i.e. without explicit reference to the string equations), only by means of a kinematic analysis. In this paper such a possibility will be applied, in particular, to the case of radially falling strings, in the field of a static and spherically symmetric source.

## 2. Cosmological instability and its kinematic interpretation

Let us recall, first of all, how instability arises in cosmological backgrounds. In the gauge in which the world-sheet metric is conformally flat, the equations of motion of a string, coupled to an external gravitational field, can be written as

$$\ddot{X}^A - X''^A + \Gamma_{BC}{}^A(\dot{X}^B + X'^B)(\dot{X}^C - X'^C) = 0 . \qquad (2.1)$$

[Conventions: $A, B, ... = 0, 1, ..., D-1$ and $i, j, ... = 1, ..., D-1$; a dot and a prime denote, respectively, differentiation with respect to the world-sheet time and space variables, $\tau$ and $\sigma$, and $\Gamma$ is the Christoffel connection for the background metric $G_{AB}(X)$.] By expanding the exact solution around a geodesic, i.e. by putting [1]

$$X^A(\sigma, \tau) = q^A(\tau) + \eta^A(\sigma, \tau) , \qquad (2.2)$$

where

$$\ddot{q}^A + \Gamma_{BC}{}^A(q)\dot{q}^B\dot{q}^C = 0 , \qquad (2.3)$$

one obtains for the first order fluctuations $\eta^A$ the linearized equations [1]

$$\ddot{\eta}^A - \eta''^A + 2\Gamma_{BC}{}^A(q)\dot{q}^B\dot{\eta}^C + \eta^D\, \partial_D\Gamma_{BC}{}^A(q)\dot{q}^B\dot{q}^C = 0 . \qquad (2.4)$$

Consider in particular a (spatially flat) Friedman–Robertson–Walker geometry, described (in the cosmic time gauge $X^0 = t$) by the metric

$$G_{AB} = \mathrm{diag}(1, -R^2(t)\delta_{ij}) \qquad (2.5)$$

and choose the geodesic field of an observer at rest in the comoving frame, i.e. $\dot{q}^A = \alpha' m \delta_0^A$. In this case, eq. (2.4) for the fluctuations along any spatial direction, $\eta^i$, can be written as

$$\ddot{\eta}^i - \eta''^i + 2\frac{\dot{R}}{R}\dot{\eta}^i = 0 . \qquad (2.6)$$

We can now perform a Fourier expansion,

$$\eta^A(\sigma, \tau) = \sum_n \eta_n^A(\tau)\, \mathrm{e}^{in\sigma} \qquad (2.7)$$

and define a "proper" amplitude $\chi_n^i = R\eta_n^i$. By using the proportionality of $\tau$ and $q^0$ we are thus led to eq. (1.1), which shows that instability occurs, for a given background $R(t)$, provided [2]

$$\frac{(\alpha' m)^2}{R}\frac{\mathrm{d}^2 R}{\mathrm{d}t^2} > 1 . \qquad (2.8)$$

This instability condition has a simple kinematic interpretation in terms of the relative acceleration between two ends of a string, induced by the external gravitational field. Consider in fact a string embedded in a curved background. In the absence of forces other than gravity, the string will be free falling. Different points of the string, however, will fall along different geodesics. Consider then the local free falling frame of one end (A) of the string: in this frame, the other end (B), at a proper distance $\lambda$ from A, will have an acceleration, relative to A, given by the equation of geodesic deviation (see chapter 11 of ref. [5]), i.e.

$$a^A = -R_{BCD}{}^A z^B u^C u^D , \qquad (2.9)$$

where $z^A$ is the spacelike separation vector, $z^A z_A = -\lambda^2$, $u^A$ is the geodesic velocity field, $u^A u_A = 1$, $z^A u_A = 0$, and $R_{ABCD}$ is the curvature tensor of the background manifold.

This acceleration defines, as usual, a local Rindler horizon (see chapter 6 of ref. [5]) at a proper distance $d = (-a^A a_A)^{-1/2}$ from the accelerated point (B). If the relative acceleration (2.9) is negative for all the points of the string, then the whole string will be always *inside* the causal horizon (see fig. 1a). But if the relative acceleration is positive it may be possible, for sufficiently large values of $|a|$, that A and B become causally disconnected because of the Rindler horizon interposed between them (see figs. 1b and 1c). This happens for $\lambda > d$.

For the background metric (2.5), and for two comoving geodesics ($u^A = \delta_0^A$) with proper spatial separation $\lambda$ oriented, for example, along the $X^i$ direction, $z^A = (\lambda/R)\delta_i^A$, the non-vanishing component of their relative acceleration (2.9) is

$$a^i = \frac{\lambda}{R^2}\frac{\mathrm{d}^2 R}{\mathrm{d}t^2} . \qquad (2.10)$$

This acceleration is positive only for inflationary backgrounds ($\mathrm{d}^2R/\mathrm{d}t^2 > 0$); for such backgrounds, the $\lambda > d$ condition becomes

$$\frac{\lambda^2}{R}\frac{\mathrm{d}^2 R}{\mathrm{d}t^2} > 1 , \qquad (2.11)$$

which, for $\lambda = \alpha' m$, corresponds exactly to the instability condition (2.8).

These kinematic arguments suggest that instability may be interpreted, classically, as a consequence of the fact that the various parts of the string become causally disconnected, because of the kinematic horizons induced by the background field. Since the extension of the causally allowed region decreases as the relative acceleration grows, the critical acceleration $a_c$ (i.e. the *minimal* background curvature) corre-

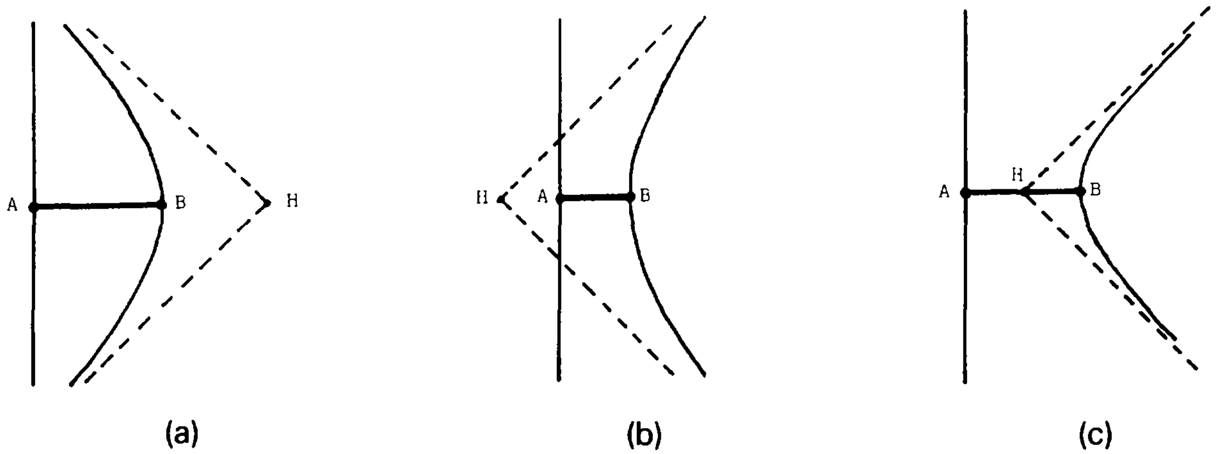


Fig. 1. The hyperbolic path of one end of the string (B) in the local free falling frame of the other end (A), and the associated Rindler horizon (represented by dashed lines). If the relative acceleration (induced by the background curvature) is negative, then the two points A and B are always inside the causal horizon (a). Instability requires a positive relative acceleration (b), and occurs when the horizon distance $d=$ HB is smaller than the length of the string $\lambda=$ AB (c).

sponding to the instability onset is thus fixed by the string *maximal* size (i.e. by the string tension) and is given by

$$a_c=(\alpha' m)^{-1} . \tag{2.12}$$

### 3. Another example

According to the previous discussion, we should expect the possible emergence of instability only when the background field induces an effective local repulsion, which tends to move apart from each other the various points of the string. This is confirmed by the following example.

Consider a static and spherically symmetric background, parametrized by polar coordinates $X^A=(T, R, \Theta, \Phi)$, with metric

$$G_{AB}=\mathrm{diag}(\mathrm{e}^{\nu}, -\mathrm{e}^{-\nu}, -R^2, -R^2\sin^2\Theta) , \tag{3.1}$$

where $\nu$ is a function of the radial coordinate only (henceforth we shall restrict the discussion, for simplicity, to $D=4$). Expand around a radial geodesic $q^A=(t, r, 0, 0)$, such that

$$\dot q^A=\alpha' m(k\,\mathrm{e}^{-\nu}, \sqrt{k^2-\mathrm{e}^{\nu}}, 0, 0) \tag{3.2}$$

[$k$ is a dimensionless integration constant, $\nu=\nu(r)$] and consider, in particular, eq. (2.4) for the angular fluctuations $\eta^{(2)}$:

$$\ddot\eta^{(2)}-\eta''^{(2)}+2\frac{\dot r}{r}\dot\eta^{(2)}=0 . \tag{3.3}$$

By Fourier expanding, and by putting $\eta_n^{(2)}=(\alpha' m/r)\chi_n^{(2)}$, we obtain

$$\ddot\chi_n^{(2)}+\left(n^2-\frac{\ddot r}{r}\right)\chi_n^{(2)}=0 . \tag{3.4}$$

Angular instability may thus occur, provided $\ddot r/r>1$, that is, by using the geodesic equation (2.3), provided

$$-\frac{\alpha'^2m^2}{2r}\frac{\mathrm{d}}{\mathrm{d}r}\mathrm{e}^{\nu}>1 . \tag{3.5}$$

Note that for a Schwarzschild field ($\mathrm{e}^{\nu}=1-2GM/r$) angular instability is clearly impossible ($\ddot r/r=-GM/r^3$ is always negative). It may become possible, however, in the field of a charged black hole: indeed, in a Reissner–Nordström metric,

$$\mathrm{e}^{\nu}=1-\frac{2GM}{r}+\frac{Q^2}{r^2} , \tag{3.6}$$

eq. (3.5) is satisfied, for a radially falling string, as soon as the string penetrates the spherical region defined by

$$r^4+(rGM-Q^2)(\alpha' m)^2<0 \tag{3.7}$$

[in agreement with the well known repulsive character, at short distances, of the metric (3.6)].

Let us check now that the instability condition (3.5) is exactly equivalent to the requirement that the relative acceleration between two free falling points, with radial geodesic velocity

$$u^A=(k\,e^{-\nu},\sqrt{k^2-e^\nu},0,0) \quad (3.8)$$

and spatial separation $|z|=\lambda=\alpha' m$ along the $X^{(2)}$ direction, i.e.

$$z^A=(0,0,\lambda/r,0) \quad (3.9)$$

be large enough to make them causally disconnected.

According to eq. (2.9) the relative acceleration between two such points is, in fact,

$$a^A=\left(0,0,-\frac{\lambda}{2r^2}\frac{d}{dr}e^\nu,0\right). \quad (3.10)$$

If $a^{(2)}$ is positive, their proper distance $\lambda=(-G_{AB}\times z^A z^B)^{1/2}$ may be larger than the distance of the Rindler horizon associated to the acceleration (3.10). This happens for $\lambda(-G_{AB}a^A a^B)^{1/2}>1$, namely

$$-\frac{\lambda^2}{2r}\frac{d}{dr}e^\nu>1\,, \quad (3.11)$$

in agreement with the instability condition (3.5) (for $\lambda=\alpha' m$), independently obtained from the previous analysis of the string equations.

## 4. Radial instability in the Schwarzschild field

In a Schwarzschild field, the angular acceleration $a^{(2)}$ between two radial geodesics is always negative and, as a consequence, instability for the angular fluctuations $\eta^{(2)}$ is forbidden. The situation is different, however, in the case of two free falling points with a non-zero separation along the *radial* direction (because of the gravitational attraction we expect in fact, for extended bodies, a tendency to be stretched radially).

Indeed, consider again the radial geodesic field (3.8), and two free falling points with proper separation $\lambda$ oriented along the radial direction. By imposing $z^A u_A=0$ and $(-z^A z_A)^{1/2}=\lambda$ we get, in the background (3.1),

$$z^A=\lambda(e^{-\nu}\sqrt{k^2-e^\nu},k,0,0)\,. \quad (4.1)$$

For these two points the radial component of their relative acceleration, according to (2.9), is then

$$a^{(1)}=-k\frac{\lambda}{2}\frac{d^2}{dr^2}e^\nu\,, \quad (4.2)$$

which, for a Schwarzschild field, is always positive. According to our previous kinematic interpretation we can thus predict, for a string falling in a Schwarzschild black hole, the possible occurrence of *radial* instability (even without explicit analysis of the $\eta^{(1)}$ fluctuation equation).

It is interesting, in particular, to compute the critical radius $r_c$ below which instability will develop (i.e. the ends of the string will become causally disconnected). The full acceleration vector, for the radial separation (4.1), is given by

$$a^A=-\frac{\lambda}{2}\frac{d^2e^\nu}{dr^2}(e^{-\nu}\sqrt{k^2-e^\nu},k,0,0) \quad (4.3)$$

and defines a Rindler horizon at a distance

$$d=(-a^A a_A)^{-1/2}=\left|\frac{\lambda}{2}\frac{d^2e^\nu}{dr^2}\right|^{-1}. \quad (4.4)$$

In the Schwarzschild case the instability condition $\lambda>d$ thus becomes $r<r_c$, where (for $\lambda=\alpha' m$)

$$r_c=(2GM\alpha'^2m^2)^{1/3}\,. \quad (4.5)$$

For sources of macroscopic mass $M$, this critical radius is much smaller than the Schwarzschild radius $r_s=2GM$; it is, however, possible, in principle, to have $r_c\sim r_s$ for black holes of very small mass. The occurrence of a string unstable regime, therefore, could be important both for the last stages of black hole evaporation, and for the conjectured string–black-hole transition [6].

## 5. Concluding remarks

In this paper I have pointed out a possible kinematic characterization of string instability. It has been shown that instability appears whenever the relative acceleration between two ends of a string, induced by the background curvature, is positive and large enough to make them causally disconnected. The critical value of the acceleration required for the instability onset is thus fixed by the string tension, according to eq. (2.12).

This kinematic approach provides a useful tool for determining if, and in which limit, instability may develop, without solving explicitly the string equations. For example, by using the fact that eq. (2.10) remains unchanged if the cosmological background has a non-vanishing (constant) spatial curvature, we may predict instability, according to (2.11), even for strings in closed (or negatively curved) inflationary models. In refs. [1–3], on the contrary, instability was deduced only for the spatially flat case.

For strings minimally coupled to a gravitational background, like in this paper, the induced acceleration among the points of the string is given by the equation of geodesic deviation (2.9). However, the kinematic interpretation of instability holds in general even if the ends of the string are accelerated by external interactions other than gravity. Indeed, for the case of a string with charge $\pm q$ on its ends, embedded in a background electric field $E$, it has been shown [7] that instability occurs for $E > 1/q\alpha'$. In this case the acceleration is $a = qE/m$, so that the instability condition again can be interpreted kinematically as $a > a_c$, with the same $a_c$ of eq. (2.12).

Finally, it should be stressed that this kinematic approach to instability is expected to apply, in general, not only to strings but also to extended bodies of arbitrary dimensions. Indeed, there is a corresponding instability for $p$-branes when they are embedded, for example, in inflationary geometries [8]. In such a case the instability condition is different, because of effects due to the world-volume curvature, which tends to increase the relative acceleration of the ends of the brane. But in the limit in which this curvature is neglected, one exactly recovers the same instability condition as in the string case.

## Acknowledgement

I wish to thank N. Sánchez and G. Veneziano for many useful discussions on string instability.

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