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AMPLIFICATION OF VACUUM FLUCTUATIONS IN STRING COSMOLOGY BACKGROUNDS*

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Abstract

Inflationary string cosmology backgrounds can amplify perturbations in a more efficient way than conventional inflationary backgrounds, because the perturbation amplitude may grow - instead of being constant - outside the horizon. If not gauged away, the growing mode can limit the range of validity of a linearized description of perturbations. Even in the restricted linear range, however, this enhanced amplification may lead to phenomenological consequences unexpected in the context of the standard inflationary scenario. In particular, the production of a relic graviton background strong enough to be detected in future by LIGO, and/or the generation of a stochastic electromagnetic background strong enough to seed the cosmic magnetic fields and to be responsible for the observed large scale anisotropy.

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ABSTRACT

Inflationary string cosmology backgrounds can amplify perturbations in a more efficient way than conventional inflationary backgrounds, because the perturbation amplitude may grow - instead of being constant - outside the horizon. If not gauged away, the growing mode can limit the range of validity of a linearized description of perturbations. Even in the restricted linear range, however, this enhanced amplification may lead to phenomenological consequences unexpected in the context of the standard inflationary scenario. In particular, the production of a relic graviton background strong enough to be detected in future by LIGO, and/or the generation of a stochastic electromagnetic background strong enough to seed the cosmic magnetic fields and to be responsible for the observed large scale anisotropy.

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1. Introduction.

It is well known that the time-evolution of a cosmological background can amplify quantum fluctuations and generate primordial perturbation spectra [1]. Such a parametric amplification can be described, in a second-quantization language, as the production of pairs from the vacuum under the action of an external "pump" field. But it may also visualized, from a kinematic point of view, as a "stretching" process in which the comoving amplitude stays frozen (instead of decreasing adiabatically), while perturbations propagate under some effective potential barrier.

This is typically what happens in the standard inflationary scenario. In the inflationary backgrounds obtained from the string effective action, however, the perturbation amplitude may even grow [2] (instead of being frozen) during the stretching period. This leads to a more efficient amplification of perturbations, but the amplitude could grow too much, in such backgrounds, so as to prevent us from applying the standard linearized formalism.

In view of this aspect, the aim of this paper is twofold. On one hand I want to show that in some case this anomalous growth can be gauged away, so that perturbations can consistently linearized in an appropriate frame. I will discuss, in particular, the growing mode of scalar metric perturbations in a dilaton-driven background, which appears in the standard longitudinal gauge and which seems to complicate the computation of the spectrum [3]. On the other hand I want to show that, even if the growth is physical, and we have to restrict ourself to a reduced portion of parameter space in order to apply a linearized approach, such enhanced amplification is nevertheless rich of interesting phenomenological consequences.

I will discuss in particular three points: *i)* the production of a relic gravity wave background with a spectrum strongly enhanced in the high frequency sector, and its possible observation by large interferometric detectors [4] (such as LIGO and VIRGO); *ii)* the amplification of electromagnetic perturbations due to their direct coupling to the dilaton background, and the generation of primordial "seeds" for the galactic and extragalactic magnetic field [5]; *iii)* the generation of the large scale CMB anisotropy directly from the vacuum fluctuations of the electromagnetic field [6].

Throughout this paper the evolution of perturbations will be discussed in a type of background to which I shall refer to, for short, as "string cosmology background". At low energy such background represents a solution [7,8,9] of the tree-level, zeroth order in α', gravi-dilaton action

$$-\int d^{d+1}x\sqrt{|g|}e^{-\phi}\left(R+\partial_\mu\phi\partial^\mu\phi\right) \tag{1.1}$$

(possibly complemented by string matter sources). It describes the accelerated evolution from the string perturbative vacuum, with flat metric and vanishing dilaton coupling ($\phi=-\infty$), towards a phase driven by the kinetic energy of the dilaton field ($H^2\sim\dot{\phi}^2$), with negligible contribution from the dilaton potential. In this initial phase the curvature scale H^2 and the dilaton coupling e^ϕ are both growing, at a rate uniquely determined by the action (1.1).

The background can be consistently described in terms of the action (1.1), however, only up to the time $t=t_s$ when the curvature reaches the string scale, namely when $H\simeq H_s=(\alpha')^{-1/2}\equiv\lambda_s^{-1}$. At that time all orders in α' (i.e. all higher-derivative corrections) become important, and the background enters a truly "stringy" phase, whose kinematic details cannot be predicted on the ground of the previous simple action. The presence of this high-curvature phase cannot be avoided, as it is required [10] to stop the growth of the curvature, to freeze out the dilaton, and to arrange a smooth transition (at $t=t_1$) to the standard radiation-dominated evolution (where $\phi=$const).

In previous works (see for instance [11]) we assumed that the time scales t_s and t_1 (marking respectively the beginning of the string and of the radiation era) were of the same order, and we computed the perturbation spectrum in the sudden approximation, by matching directly the radiation era to the dilaton-driven phase. Here I will consider a more general situation in which the duration of the string era (t_1/t_s) is left completely arbitrary, and I will discuss its effects on the perturbation spectrum.

The evolution from the flat and cold initial state to the highly curved (and strongly coupled) final regime was previously called "pre-big-bang" [8], in order to stress the complementarity of that phase with respect to the standard, post-big-bang, decelerated scenario. During such a pre-big-bang epoch the accelerated evolution of the background can be invariantly characterized, from a kinematic point of view, as a phase of shrinking event horizons [2,8,9]. If, in particular, we parameterize the pre-big-bang scale factor (in cosmic time) as

$$a(t) \sim (-t)^{\beta}, \qquad -\infty < t < 0 \tag{1.2}$$

the existence condition for shrinking event horizons

$$\int_t^0 \frac{dt'}{a(t')} < \infty \tag{1.3}$$

imposes $\beta < 1$. So there are two classes of backgrounds in which the event horizon is shrinking.

For $\beta < 0$ we have a metric describing a phase of accelerated expansion and growing curvature,

$$\dot{a} > 0, \quad \ddot{a} > 0, \quad \dot{H} > 0 \tag{1.4}$$

of the type of pole-inflation [12], also called super-inflation ($H = \dot{a}/a$, and a dot denotes differentiation with respect to the cosmic time t). For $0 < \beta < 1$ we have instead a metric describing accelerated contraction and growing curvature scale,

$$\dot{a} < 0, \quad \ddot{a} < 0, \quad \dot{H} < 0 \tag{1.5}$$

The first type of metric provides a representation of the pre-big-bang scenario in the String (or Brans-Dicke) frame, in which test strings move along geodesic surfaces. The second in the Einstein frame, in which the gravi-dilaton action is diagonalized in the standard canonical form.

In both types of backgrounds the computation of the metric perturbation spectrum may become problematic, but the best frame to illustrate the difficulties is probably the Einstein frame, where the metric is contracting. It should be recalled, in this context, that the tensor perturbation spectrum for contracting backgrounds was first given by Starobinski [13], but the possible occurrence of problems, due to a growing solution of the perturbation equations, was pointed out only much later [14], in the context of dynamical dimensional reduction. The problem, however, was left unsolved.

2. The "growing mode" problem.

Consider the evolution of tensor metric perturbations, $\delta g_{\mu\nu} = a^2 h_{\mu\nu}$, in a $(3+1)$-dimensional conformally flat background, parameterized in conformal time ($\eta = \int dt/a$) by the scale factor

$$a(\eta) \sim (-\eta)^{\alpha}, \qquad -\infty < \eta < 0 \tag{2.1}$$

The Fourier components u_k of the correct variable obeying canonical commutation relations ($u_{\mu\nu} = a M_p h_{\mu\nu}$, M_p is the Planck mass) satisfy, for each of the two physical (transverse traceless) polarizations, the well known perturbation equation [1]

$$u_k'' + (k^2 - \frac{a''}{a})u_k = 0 \tag{2.2}$$

(a prime denotes differentiation with respect to η). In a string cosmology background the horizon is shrinking, so that all comoving length scales k^{-1} are "pushed out"of the horizon. For a mode k whose wavelength is larger than the horizon size (i.e. $|k\eta| << 1$), we have then the asymptotic solution

$$h_k = \frac{u_k}{aM_p} = A_k + B_k|\eta|^{1-2\alpha}, \qquad \eta \to 0_- \tag{2.3}$$

where A_k and B_k are integration constants.

The asymptotic behavior of the perturbation is thus determined by α. If $\alpha < 1/2$ the perturbation tends to stay constant outside the horizon, and the typical amplitude $|\delta_h|$ at the scale k^{-1}, for modes normalized to an initial vacuum fluctuation spectrum,

$$\lim_{\eta \to -\infty} u_k \sim \frac{1}{\sqrt{k}} e^{-ik\eta} \tag{2.4}$$

can be given as usual [15] in terms of the Hubble factor at horizon crossing ($k\eta \sim 1$)

$$|\delta_h| = k^{3/2}|h_k| \simeq \left(\frac{H}{M_p}\right)_{HC} \tag{2.5}$$

In this case the amplitude is always smaller than one provided the curvature is smaller than Planckian (this case includes, in particular, $\alpha < 0$, namely all backgrounds describing accelerated inflationary expansion, according to eq. (2.1)).

If, on the contrary, $\alpha > 1/2$, the second term is the dominant one in the solution (2.4), the perturbation amplitude tends to grow outside the horizon,

$$|\delta_h| = k^{3/2}|h_k| \simeq \left(\frac{H}{M_p}\right)_{HC} |k\eta|^{1-2\alpha}, \qquad \eta \to 0_- \tag{2.6}$$

and may become larger than one, thus breaking the validity of the perturbative approach. Otherwise stated: the energy density (in critical units) stored in the mode k, $\Omega(k) = d(\rho/\rho_c)/d\ln k$, may become larger than one in contrast with the hypothesis of negligible back-reaction of perturbations on the initial metric.

One might think that this problem - due to the dominance of the second term in eq. (2.3) - appears in the Einstein frame because of the contraction, but disappears in the String frame where the metric is expanding. Unfortunately this is not true because, in the String frame, the different metric background is compensated by a different perturbation equation, in such a way that the perturbation spectrum remains exactly the same [2,9].

This important property of perturbations can be easily illustrated by taking, as a simple example, an isotropic solution of the $(d+1)$-dimensional gravi-dilaton equations [2], obtained from the action (1.1) complemented by a perfect gas of long, stretched strings as sources (with equation of state $p = -\rho/d$).

In the Einstein frame the solution describes a contracting background for $\eta \to 0_-$,

$$a = (-\eta)^{2(d+1)/(d-1)(3+d)}, \quad \phi = -\frac{4d}{3+d}\sqrt{\frac{2}{d-1}}\ln(-\eta) \tag{2.7}$$

and the tensor perturbation equation

$$h_k'' + (d-1)\frac{a'}{a}h_k' + k^2 h_k = 0 \tag{2.8}$$

has an asymptotic solution (for $|k\eta| << 1$) which grows, according to eq. (2.3), as

$$\lim_{\eta \to 0_-} h_k \sim |\eta|^{(1-d)/(d+3)} \tag{2.9}$$

In the String frame the metric is expanding,

$$\tilde{a} = (-\eta)^{-2/(3+d)}, \qquad \tilde{\phi} = -\frac{4d}{3+d}\ln(-\eta) \tag{2.10}$$

but the perturbation is also coupled to the time-variation of the dilaton background [16],

$$h_k'' + \left[(d-1)\frac{\tilde{a}'}{\tilde{a}} - \tilde{\phi}'\right] h_k' + k^2 h_k = 0 \tag{2.11}$$

As a consequence, the explicit form of the perturbation equation is exactly the same as before,

$$h_k'' + \frac{2(d+1)}{d+3}\frac{h_k'}{\eta} + k^2 h_k = 0 \tag{2.12}$$

so that the solution is still growing, asymptotically, with the same power as in eq. (2.9).

It may be noted that in the String frame the growth of perturbations outside the horizon is due to the joint contribution of the metric and of the dilaton background to the "pump" field responsible for the parametric amplification process [17]. Such an effect is thus to be expected in generic scalar-tensor backgrounds, as noted also in [18]. The particular example chosen above is not much relevant, however, for a realistic scenario in which the phase of pre-big-bang inflation is long enough to solve the standard cosmological problems. In that case, in fact, all scales which are inside our present horizon crossed the horizon (for the first time) during the dilaton-driven phase or during the final string phase, in any case when the contribution of matter sources was negligible [9,11,19].

We shall thus consider, as a more significant (from a phenomenological point of view) background, the vacuum, dilaton-driven solution of the action (1.1), which in the Einstein frame (and in $d=3$) can be explicitly written as

$$a = (-\eta)^{1/2}, \qquad \phi = -\sqrt{3}\ln(-\eta), \qquad -\infty < \eta < 0 \tag{2.13}$$

In such a background one finds that the growth of tensor perturbations is simply logarithmic [3],

$$|\delta_h(\eta)| \simeq \left|\frac{H}{M_p}\right|_{HC} \ln|k\eta| \simeq \frac{H_s}{M_p}|k\eta_s|^{3/2}\ln|k\eta|, \qquad |k\eta_s| < 1,\ |\eta| > |\eta_s| \tag{2.14}$$

so that it can be easily kept under control, provided the curvature scale $H_s \sim (a_s\eta_s)^{-1}$ at the end of the dilaton phase is bounded.

The problem, however, is with scalar perturbations, described in the longitudinal gauge by the variable ψ such that [15]

$$(g_{\mu\nu} + \delta g_{\mu\nu})dx^\mu dx^\nu = a^2(1+2\psi)d\eta^2 - a^2(1-2\psi)(dx_i)^2 \tag{2.15}$$

The canonical variable v associated to ψ is defined (for each mode k) by [15]

$$\psi_k = -\frac{\phi'}{4k^2 M_p}\left(\frac{v_k}{a}\right)' \tag{2.16}$$

and satisfies a perturbation equation

$$v_k'' + (k^2 - \frac{a''}{a})v_k = 0 \tag{2.17}$$

which is identical to eq. (2.2) for the tensor canonical variable, with asymptotic solution

$$\frac{v_k}{a} \simeq \frac{1}{\sqrt{k}}\frac{\ln|k\eta|}{a_{HC}}, \qquad |k\eta| << 1 \tag{2.18}$$

Because of the different relation between canonical variable and metric perturbation, however, it turns out that the amplitude of longitudinal perturbations, normalized to an initial vacuum fluctuation spectrum,

$$\lim_{\eta\to-\infty} v_k \sim \frac{1}{\sqrt{k}} e^{-ik\eta} \tag{2.19}$$

grows, asymptotically, like η^{-2}. We have in fact, from (2.16),

$$|\delta_\psi(\eta)| = k^{3/2}|\psi_k| \simeq \left|\frac{H}{M_p}\right| |k\eta|^{-1/2} \simeq \left|\frac{H}{M_p}\right|_{HC} |k\eta|^{-2} \simeq$$

$$\simeq \left(\frac{H_s}{M_p}\right)\frac{|k\eta_s|^{3/2}}{|k\eta|^2} \sim \frac{1}{\eta^2}, \qquad \eta \to 0_- \tag{2.20}$$

This growth, as we have seen, cannot be eliminated by passing to the String frame. Neither can be eliminated in a background with a higher number of dimensions. In fact, in $d > 3$, the isotropic solution (2.13) is generalized as [9]

$$a = (-\eta)^{1/(d-1)}, \qquad \phi = -\sqrt{2d(d-1)}\ln a, \qquad -\infty < \eta < 0 \tag{2.21}$$

and the scalar perturbation equation in the longitudinal gauge [9]

$$\psi_k'' + 3(d-1)\frac{a'}{a}\psi_k' + k^2\psi_k = 0 \tag{2.22}$$

has the generalized asymptotic solution

$$\psi_k = A_k + B_k \eta a^{-3(d-1)} \tag{2.23}$$

By inserting the new metric (2.21) one thus finds the same growing time-behavior, $\psi_k \sim \eta^{-2}$, exactly as before. The same growth of ψ_k is also found in anisotropic, higher-dimensional, dilaton-dominated backgrounds [3].

Because of the growing mode there is always (at any given time η) a low frequency band for which $|\delta_\psi(\eta)| > 1$. In $d = 3$, in particular, such band is defined [from eq.(2.20)] as $k < \eta^{-1}(H/M_p)^2$. For such modes the linear approximation breaks down in the longitudinal gauge, and a full non-linear treatment would seem to be required in order to compute the spectrum. In spite of this conclusion, a linear description of scalar perturbations may remain possible provided we choose a different gauge, more appropriate to linearization than the longitudinal one.

A first signal that a perturbative expansion around a homogeneous background can be consistently truncated at the linear level, comes from an application of the "fluid flow" approach [20,21] to the perturbations of a scalar-tensor background. In this approach, the evolution of density and curvature inhomogeneities is described in terms of two covariant scalar variables, Δ and C, which are gauge invariant to all orders [22]. They are defined in terms of the momentum density of the scalar field, $\nabla\phi$, of the spatial curvature, ${}^{(3)}R$, and of their derivatives. By expanding around our homogeneous dilaton-driven background (2.13) one finds [3] for such variables the asymptotic solution ($|k\eta| << 1$), in the linear approximation,

$$\Delta_k = const, \qquad\qquad c_k = const + A_k \ln|k\eta| \tag{2.24}$$

showing that they tend to stay constant outside the horizon, with at most a logarithmic variation (like in the tensor case), which is not dangerous.

As a consequence, the amplitude of density and curvature fluctuations can be consistently computed in the linear approximation (for all modes) in terms of Δ and C, and their spectral distribution (normalized to an initial vacuum spectrum) turns out to be exactly the same as the tensor distribution (2.14), which is bounded.

What is important, moreover, is the fact that such a spectral distribution could also be obtained directly from the asymptotic solution of the scalar perturbation equations in the longitudinal gauge [3],

$$\psi_k = c_1 \ln|k\eta| + \frac{c_2}{\eta^2} \tag{2.25}$$

simply by neglecting the growing mode contribution (i.e. setting $c_2 = 0$). This may suggests that such growing mode has no direct physical meaning, and that it should be possible to get rid of it through an appropriate coordinate choice.

A good candidate to do the job is what we have called [3] off-diagonal gauge,

$$(g_{\mu\nu} + \delta g_{\mu\nu})dx^\mu dx^\nu = a^2\left[(1+2\varphi)d\eta^2 - (dx_i)^2 - 2\partial_i B dx^i d\eta\right] \tag{2.26}$$

which represents a complete choice of coordinates, with no residual degrees of freedom, just like the longitudinal gauge. In this gauge there are two variables for scalar perturbations, φ and B, and their asymptotic solution in the linear approximation is [3]

$$\varphi_k = c_1 \ln|k\eta| \sim \psi_k, \qquad B_k = \frac{c_2}{\eta} \sim \eta\psi_k, \qquad (\partial B)_k \sim |k\eta|\psi_k \tag{2.27}$$

(c_1 and c_2 are integration constants). The growing mode is thus completely gauged away for homogeneous perturbations (for which $\partial_i B = 0$). It is still present for non-homogeneous perturbations in the off-diagonal part of the metric, but it is "gauged down" by the factor $k\eta$ which is very small, asymptotically.

Fortunately this is enough for the validity of the linear approximation, as the amplitude of the off-diagonal perturbation, in this gauge,

$$|\delta_B| \simeq |k\eta||\delta_\psi| \simeq \left(\frac{H_s}{M_p}\right) |k\eta_s|^{1/2} \left|\frac{\eta_s}{\eta}\right| \tag{2.28}$$

stays smaller than one for all modes $k < |\eta_s|^{-1}$, and for the whole duration of the dilaton-driven phase, $|\eta| > |\eta_s|$. We have explicitly checked that quadratic corrections are smaller than the linear terms in the perturbation equations, but a full second order computation requires a further coordinate transformation [3]. The higher order problem is very interesting in itself, but a complete discussion of the problem is outside the scope of this paper. Having established that the vacuum fluctuations of the metric background, amplified by a phase of dilaton-driven evolution, can be consistently described (even in the scalar case) as small corrections of the homogeneous background solution, let me discuss instead some phenomenological consequence of such amplification. Scalar perturbations and dilaton production were discussed in [9,11,23]. Here I will concentrate, first of all, on graviton production.

3. "Thermal" graviton spectrum from dilaton-driven inflation.

Consider the amplification of tensor metric perturbation in a generic string cosmology background, of the type of that described in Sect. 1. Their present spectral energy distribution, $\Omega(\omega, t_0)$, can be computed in terms of the Bogoliubov coefficient determining their amplification (see Set. 5 below), or simply by following the evolution of the typical amplitude $|\delta_h|$ from the time of horizon crossing down to the present time t_0. For modes crossing the horizon in the inflationary dilaton-driven phase (i.e. for $t < t_s$), and reentering the horizon in the decelerated radiation era ($t > t_1$), one easily finds, from eq. (2.14)

$$\Omega(\omega, t) \equiv \frac{\omega}{\rho_c}\frac{d\rho}{d\omega} \simeq |\delta_h|^2 \simeq A\Omega_\gamma \left(\frac{H_s}{M_p}\right)^2 \left(\frac{\omega}{\omega_s}\right)^3 \ln^2\left(\frac{\omega}{\omega_s}\right), \quad \omega < \omega_s \tag{3.1}$$

Here $\omega = k/a$ is the red-shifted proper frequency for the mode k at time t, $\rho_c = M_p^2 H^2$ is the critical energy density, $\Omega_\gamma = (H_1/H)^2(a_1/a)^4 = \rho_\gamma/\rho_c$ is the radiation energy density in critical units, and $\omega_s = H_s a_s/a$ is the maximal amplified frequency during the dilaton-driven phase. Finally, A is a possible amplification factor due to the subsequent string phase ($t_s < t < t_1$), in case that the perturbation amplitude grows outside the horizon (instead of being constant) during such phase. This additional amplification does not modify however the slope of the spectrum, as we are considering modes that crossed the horizon before the beginning of the string phase.

An important property of the spectrum (3.1) is the universality of the slope ω^3 with respect to the total number d of spatial dimensions, and their possible anisotropy. Actually, the spectrum is also duality-invariant [4], in the sense that it is the same for all backgrounds, including those with torsion, obtained via $O(d,d)$ transformations [24] from the vacuum dilaton-driven background.

The spectrum (3.1) has also the same slope (modulo logarithmic corrections) as the low frequency part of a thermal black body spectrum, which can be written (in critical units) as

$$\Omega_T(\omega,t)=\frac{\omega^4}{\rho_c}\frac{1}{e^{\omega/T}-1}\simeq B\Omega_\gamma\left(\frac{H_s}{M_p}\right)^2\left(\frac{\omega}{\omega_s}\right)^3\frac{T}{\omega_s},\qquad \omega<T \tag{3.2}$$

Here $B=(H_s/H_1)^2(a_s/a_1)^4$ is a constant factor which depends on the time-gap between the beginning of the string phase and the beginning of the string era. We can thus parameterize the graviton spectrum (3.1) in terms of an effective temperature

$$T_s=(A/B)\omega_s \tag{3.3}$$

which depends on the initial curvature scale H_s, and on the subsequent kinematic of the high-curvature string phase.

For a negligible duration of the string phase, $t_s\sim t_1$, we have in particular $H_s\sim H_1\sim M_p$, and the spectrum (3.1) is peaked around a maximal amplified frequency $\omega_s\sim H_1a_1/a\sim 10^{11}$Hz, while it is exponentially decreasing at higher frequencies (where the parametric amplification is not effective). Moreover, $T_s\sim\omega_s\sim 1^oK$, so that this spectrum, produced by a geometry transition, is remarkably similar to that of the observed cosmic black body radiation [25,26] (see also Sect. 6.2 below).

The problem, however, is that we don't know the duration and the kinematics of the high curvature string phase. As a consequence, we know the slope (ω^3) of this "dilatonic" branch of the spectrum, but we don't know the position, in the (Ω,ω) plane, of the peak frequency ω_s. This uncertainty is, however, interesting, because the effects of the string phase could shift the spectrum (3.1) to a low enough frequency band, so as to overlap with the possible future sensitivity of large interferometric detectors such as LIGO [27] and VIRGO [28]. I will discuss this possibility in terms of a two-parameter model of background evolution, presented in the following Section.

4. Two-parameter model of background evolution.

Consider the scenario described in Sect. 1 (see also [9,11]), in which the initial (flat and cold) vacuum state, possibly perturbed by the injection of an arbitrarily small (but finite) density of bulk string matter, starts an accelerated evolution towards a phase of growing curvature and dilaton coupling, where the matter contribution becomes eventually negligible with respect to the gravi-dilaton kinetic energy. Such a phase is initially described by the low energy dilaton-dominated solution,

$$a=|\eta|^{1/2},\qquad \phi=-\sqrt{3}\ln|\eta|,\qquad -\infty<\eta<\eta_s \tag{4.1}$$

up to the time η_s, when the curvature reaches the string scale $H_s=\lambda_s^{-1}$, at a value of the string coupling $g_s=\exp(\phi_s/2)$. Provided the value of ϕ_s is sufficiently negative (i.e. provided the coupling g_s is sufficiently small to be still in the perturbative regime), such a value is also completely arbitrary, since there is no perturbative potential to break invariance under shifts of ϕ.

For $\eta > \eta_s$ the background enters a high curvature string phase of arbitrary (but unknown) duration, in which all higher-derivative (higher-order in $\alpha' = \lambda_s^2$) contributions to the effective action become important. During such phase the dilaton keeps growing towards the strong coupling regime, up to the time $\eta = \eta_1$ (at a curvature scale H_1), when a non-trivial dilaton potential freezes the coupling to its present constant value $g_1 = \exp(\phi_1/2)$. We shall assume, throughout this paper, that the time scale η_1 marks the end of the string era as well as the (nearly simultaneous) beginning of the standard, radiation-dominated evolution, where $a \sim \eta$ and $\phi =$ const (see however the last comment at the end of Sect. 6.2 for a possible alternative).

During the string phase the curvature is expected to stay controlled by the string scale, so that

$$|H| \simeq gM_p = \frac{e^{\phi/2}}{\lambda_p} = \frac{1}{\lambda_s}, \qquad \eta_s < \eta < \eta_1 \tag{4.2}$$

where λ_p is the Planck length. As a consequence, the curvature is increasing in the Einstein frame (where λ_p is constant), while it keeps constant in the string frame, where λ_s is constant and the Planck length grows like g from zero (at the initial vacuum) to its present value $\lambda_p \simeq 10^{-19}(GeV)^{-1}$. In both cases the final scale $H_1 \simeq g_1 M_p$ is fixed, and has to be of Planckian order to match the present value of the ratio λ_p/λ_s. One can estimate [29]

$$g_1 \simeq \frac{H_1}{M_p} \simeq \frac{\lambda_p}{\lambda_s} \simeq 0.3 - 0.03 \tag{4.3}$$

In analogy with the dilaton-driven solution (4.1), let us now parameterize, in the Einstein frame, the background kinematic during the string phase with a monotonic metric and dilaton evolution,

$$a = |\eta|^\alpha, \qquad \phi = -2\beta \ln|\eta|, \qquad \eta_s < \eta < \eta_1 \tag{4.4}$$

representing a sort of "average" time-behavior. Note that the two parameters α and β cannot be independent since, according to eq. (4.2),

$$\left|\frac{H_s}{H_1}\right| \simeq \frac{g_s}{g_1} \simeq \left|\frac{\eta_1}{\eta_s}\right|^{1+\alpha} \simeq \left|\frac{\eta_1}{\eta_s}\right|^{\beta} \tag{4.5}$$

from which

$$1+\alpha \simeq \beta \simeq -\frac{\log(g_s/g_1)}{\log|\eta_s/\eta_1|} \tag{4.6}$$

(note also that the condition $1+\alpha = \beta$ cannot be satisfied by the vacuum solutions of the lowest order string effective action [9], in agreement with the fact that all orders in α' are full operative in the high curvature string phase [10]).

The background evolution, for this class of models, is thus completely determined in terms of two parameters only, the duration (in conformal time) of the string phase, $|\eta_s/\eta_1|$, and the shift of the dilaton coupling (or of the curvature scale in Planck units) during the string phase, $g_s/g_1 = (H_s/M_p)/(H_1/M_p)$. I will use, for convenience, the decimal logarithm of these parameters,

$$\begin{aligned} x &= \log_{10} |\eta_s/\eta_1| = \log_{10} z_s \\ y &= \log_{10}(g_s/g_1) = \log_{10} \frac{(H_s/M_p)}{(H_1/M_p)} \end{aligned} \tag{4.7}$$

Here $z_s = |\eta_s/\eta_1| \simeq a_1/a_s$ defines the total red-shift associated to the string phase in the String frame, where the curvature is constant and the metric undergoes a phase of de Sitter-like expansion. It should be noted, finally, that the parameters (4.7) are completely frame-independent, as conformal time and dilaton field are exactly the same in the String and Einstein frame.

5. Parameterized graviton spectrum.

Consider the background discussed in the previous Section, characterized by the dilaton-driven evolution (4.1) for $\eta < \eta_s$, by the string evolution (4.4) for $\eta_s < \eta < \eta_1$, and by the standard radiation-dominated evolution for $\eta > \eta_1$. In these three regions, eq. (2.2) for the canonical variable u_k has the general exact solution

$$\begin{aligned} u_k &= |\eta|^{1/2} H_0^{(2)}(|k\eta|), && \eta < \eta_s \\ u_k &= |\eta|^{1/2} \left[A_+(k) H_\nu^{(2)}(|k\eta|) + A_-(k) H_\nu^{(1)}(|k\eta|)\right], && \eta_s < \eta < \eta_1 \\ u_k &= \frac{1}{\sqrt{k}} \left[c_+(k) e^{-ik\eta} + c_-(k) e^{ik\eta}\right], && \eta > \eta_1 \end{aligned} \tag{5.1}$$

where $\nu = |\alpha - 1/2|$, and $H^{(1,2)}$ are the first and second kind Hankel functions. We have normalized the solution to an initial vacuum fluctuation spectrum, containing only positive frequency modes at $\eta = -\infty$

$$\lim_{\eta \to -\infty} u_k = \frac{e^{-ik\eta}}{\sqrt{k}} \tag{5.2}$$

The asymptotic solution for $\eta \to +\infty$ is however a linear superposition of positive and negative frequency modes, determined by the so-called Bogoliubov coefficients $c_\pm(k)$ which parameterize, in a second quantization approach, the unitary transformation connecting $|in\rangle$ and $|out\rangle$ states. So, even starting from an initial vacuum state, it is possible to find a non-vanishing expectation number of produced particles (in this case gravitons) in the final state, given (for each mode k) by $\langle n_k \rangle = |c_-(k)|^2$.

We shall compute $c_\pm$ by matching the solutions (5.1) and their first derivatives at η_s and η_1. We observe, first of all, that a consistent growth of the curvature and of the coupling during the string phase (in the Einstein frame) can only be realized by choosing $|\eta_1| < |\eta_s|$, i.e. $\beta = 1 + \alpha > 0$ [see eq.(4.5)]. This corresponds to an inflationary string phase, characterized in the Einstein frame by accelerated expansion ($\dot{a} > 0$, $\ddot{a} > 0$, $\dot{H} > 0$) for $-1 < \alpha < 0$, and accelerated contraction ($\dot{a} < 0$, $\ddot{a} < 0$, $\dot{H} < 0$) for $\alpha > 0$,. It follows, in particular, that modes which "hit" the effective potential barrier $V(\eta) = a''/a$ of eq.(2.2) (otherwise stated: which cross the horizon) during the dilaton-driven phase, i. e. modes with $|k\eta_s| < 1$, stay under the barrier also during the string phase, since $|k\eta_1| < |k\eta_s| < 1$. In such case the maximal amplified proper frequency mode

$$\omega_1 = \frac{k_1}{a} \simeq \frac{1}{a\eta_1} \simeq \frac{H_1 a_1}{a} \simeq \left(\frac{H_1}{M_p}\right)^{1/2} 10^{11} Hz = \sqrt{g_1} 10^{11} Hz \qquad (5.3)$$

is related to the highest mode crossing the horizon in the dilaton phase, $\omega_s = H_s a_s/a$, by

$$\omega_s = \omega_1 \left|\frac{\eta_1}{\eta_s}\right| < \omega_1 \qquad (5.4)$$

For an approximate estimate of c_- we may thus consider two cases.

If $\omega_s < \omega < \omega_1$, i.e. if we consider modes crossing the horizon in the string phase, we can estimate $c_-(\omega)$ by using the large argument limit of the Hankel functions when matching the solutions at $\eta = \eta_s$, using however the small argument limit when matching at $\eta = \eta_1$. In this case the parametric amplification is induced by the second background transition only, as $A_+ \simeq 1$ and $A_- \simeq 0$, and we get

$$|c_-(\omega)| \simeq \left(\frac{\omega}{\omega_1}\right)^{-\nu-1/2}, \qquad \omega_s < \omega < \omega_1 \qquad (5.5)$$

(modulo numerical coefficients of order of unity). If, on the contrary, $\omega < \omega_s$, i.e. we consider modes crossing the horizon in the dilaton phase, we can use the small argument limit of the Hankel functions at both the matching epochs η_s and η_1. This gives $A_\pm = b_\pm |k\eta_s|^{-\nu} \ln |k\eta_s|$ ($b_\pm$ are numbers of order one), and

$$|c_-(\omega)| \simeq \left|\frac{\eta_s}{\eta_1}\right|^{\nu} \left(\frac{\omega}{\omega_1}\right)^{1/2} \ln\left(\frac{\omega_s}{\omega}\right), \qquad \omega < \omega_s \qquad (5.6)$$

We can now compute, in terms of $\langle n \rangle = |c_-|^2$, the spectral energy distribution $\Omega(\omega, t)$ (in critical units) of the produced gravitons, defined in such a way that the total graviton energy density ρ_g is obtained as $\rho_g = \rho_c \int \Omega(\omega) d\omega/\omega$. We have then

$$\begin{aligned} \Omega(\omega, t) &\simeq \frac{\omega^4}{M_p^2 H^2} |c_-(\omega)|^2 \simeq \\ &\simeq \left(\frac{H_1}{M_p}\right)^2 \left(\frac{H_1}{H}\right)^2 \left(\frac{a_1}{a}\right)^4 \left(\frac{\omega}{\omega_1}\right)^{3-2\nu}, \qquad \omega_s < \omega < \omega_1 \qquad (5.7) \\ &\simeq \left(\frac{H_1}{M_p}\right)^2 \left(\frac{H_1}{H}\right)^2 \left(\frac{a_1}{a}\right)^4 \left(\frac{\omega}{\omega_1}\right)^3 \left|\frac{\eta_s}{\eta_1}\right|^{2\nu} \ln^2\left(\frac{\omega_s}{\omega}\right), \quad \omega < \omega_s \end{aligned}$$

According to eqs. (4.6) and (4.7), moreover, $2\nu = |2\alpha - 1| = |3 + 2y/x|$ and $|\eta_s/\eta_1| = 10^x$. The tensor perturbation spectrum (5.7) is thus completely fixed in terms of our two free parameters, x, y, of the (known) fraction of critical energy density stored in radiation at time t, $\Omega_\gamma(t) = (H_1/H)^2(a_1/a)^4$, and of the (in principle known) present value of the ratio $g_1 = \lambda_p/\lambda_s$, as

$$\Omega(\omega,t) = g_1^2\Omega_\gamma(t)\left(\frac{\omega}{\omega_1}\right)^{3-|\frac{2y}{x}+3|}, \qquad 10^{-x} < \frac{\omega}{\omega_1} < 1 \tag{5.8}$$

$$\Omega(\omega,t) = g_1^2\Omega_\gamma(t)\left(\frac{\omega}{\omega_1}\right)^{3} 10^{|2y+3x|}, \qquad \frac{\omega}{\omega_1} < 10^{-x} \tag{5.9}$$

The first branch, with unknown slope, is due to modes crossing the horizon in the string phase, the second to modes crossing the horizon in the dilaton phase. Note that I have omitted, for simplicity, the logarithmic term in eq. (5.9), because it is not much relevant for the order of magnitude estimate that I want to discuss here. Note also that the same spectrum can also be obtained with a different approach, working in the String frame (see [4]).

Let us impose, on such spectrum, the condition of falling within the possible future sensitivity range of large interferometric detectors, namely [30]

$$\Omega(\omega_I) \gtrsim 10^{-10}, \qquad \omega_I = 10^2 Hz \tag{5.10}$$

which implies

$$|y+\frac{3}{2}x| > \frac{3}{2}x - \frac{(3+\log_{10}g_1)x}{9+\log_{10}g_1}, \qquad x > 9+\frac{1}{2}\log_{10}g_1$$

$$|2y+3x| > 21 - \frac{1}{2}\log_{10}g_1, \qquad x < 9+\frac{1}{2}\log_{10}g_1 \tag{5.11}$$

These conditions define an allowed region in our parameter space (x,y), which has to be further restricted however by the upper bound obtained from pulsar-timing measurements [31], namely

$$\Omega(\omega_P) \lesssim 10^{-6}, \qquad \omega_P = 10^{-8} Hz \tag{5.12}$$

which implies

$$|y+\frac{3}{2}x| < \frac{3}{2}x - \frac{(1+\log_{10}g_1)x}{19+\frac{1}{2}\log_{10}g_1}, \qquad x > 19+\frac{1}{2}\log_{10}g_1$$

$$|2y+3x| < 55 - \frac{1}{2}\log_{10}g_1, \qquad x < 19+\frac{1}{2}\log_{10}g_1 \tag{5.13}$$

We have to take into account, in addition, the asymptotic behavior of tensor perturbations outside the horizon. During the dilaton phase the growth is only logarithmic, but during the string phase the growth is faster (power-like) for backgrounds with $\alpha > 1/2$. Since the above spectrum has been obtained in the linear approximation, expanding around a homogeneous background, we must impose for consistency that the perturbation amplitude stays always smaller than one, so that perturbations have a negligible back-reaction on the metric. This implies $\Omega < 1$ at all ω and t. This bound, together with the slightly more stringent bound $\Omega < 0.1$ required by standard nucleosynthesis [32], can be automatically satisfied

- in view of the g_1^2 factor in eqs. (5.8), (5.9) - by requiring a growing perturbation spectrum, namely

$$y < 0, \qquad\qquad y > -3x \tag{5.14}$$

The conditions (5.11), (5.13) and (5.14) determine the allowed region of our parameter space compatible with the production of cosmic gravitons in the interferometric sensitivity range (5.10) (denoted by LIGO, for short). Such a region is plotted in **Fig.1**, by taking $g_1 = 1$ as a reference value. It is bounded below by the condition of nearly homogeneous background (5.14), and above by the same condition plus the pulsar bound (5.13). The upper part of the allowed region corresponds to a class of backgrounds in which the tensor perturbation amplitude stays constant, outside the horizon, during the string phase ($\alpha < 1/2$). The lower part corresponds instead to backgrounds in the the amplitude grows, outside the horizon, during the string phase ($\alpha > 1/2$).

We note, finally, that the area within the full bold lines refers to modes crossing the horizon in the dilaton phase; the area within the thin lines refers to modes crossing the horizon in the string phase, where the reliability of our predictions is weaker, as we used field-theoretic methods in a string-theoretic regime. Even neglecting all spectra referring to the string phase, however, we obtain a final allowed region which is non-vanishing, though certainly not too large.

The main message of this figure and of the spectrum (5.7) (irrespective of the particular value of the spectral index) is that graviton production, in string cosmology, is in general strongly enhanced in the high frequency sector (kHz-GHz). Such a frequency band, in our context, could be in fact all but the "desert" of relic gravitational radiation that one may expect on the ground of the standard inflationary scenario. In particular, a sensitivity of $\Omega \sim 10^{-4} - 10^{-5}$ in the KHz region (which does not seem out of reach in coincidence experiments between bars and interferometers [33]) could be already enough to detect a signal, so that a null result (in that band, at that level of sensitivity) would already provide a significant constraint on the parameters of the string cosmology background. This should encourage the study and the development of gravitational detectors (such as, for instance, microwave cavities [34]) with large sensitivity in the high frequency sector.

In the following Section I will compare the allowed region of **Fig. 1**, relative to graviton production (and their possible detection), to the allowed region relative to the amplification of electromagnetic perturbations (and to the production of primordial magnetic fields).

6. Parameterized electromagnetic spectrum

In string cosmology, the electromagnetic field $F_{\mu\nu}$ is directly coupled to the dilaton background. To lowest order, such coupling is represented by the string effective action as

$$\int d^{d+1}x\sqrt{|g|}e^{-\phi}F_{\mu\nu}F^{\mu\nu} \tag{6.1}$$

The electromagnetic field is also coupled to the metric background $g_{\mu\nu}$, of course, but in $d=3$ the metric coupling is conformally invariant, so that no parametric amplification of electromagnetic fluctuations is possible in a conformally flat background, like that of a typical inflationary model. One can try to break conformal invariance at the classical or quantum level - there are indeed various attempts in this sense [35,36] - but it turns out that it is very difficult, in general, to obtain a significant electromagnetic amplification from the metric coupling in a natural way, and in a realistic inflationary scenario.

In our context, on the contrary, the vacuum fluctuations of the electromagnetic field can be directly amplified by the time evolution of the dilaton background [5, 37]. Consider in fact the correct canonical variable ψ^μ representing electromagnetic perturbations [according to eq. (6.1)] in a $d=3$, conformally flat background, i.e. $\psi^\mu = A^\mu e^{-\phi/2}$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The Fourier modes ψ_k^μ satisfy, for each polarization component, the equation

$$\psi_k'' + \left[k^2 - V(\eta)\right]\psi_k = 0, \qquad V(\eta) = e^{\phi/2}(e^{-\phi/2})'' \tag{6.2}$$

obtained from the action (6.1) with the gauge condition $\partial_\nu(e^{-\phi}\partial^\mu A^\nu) = 0$. Such equation is very similar to the tensor perturbation equation (2.2), with the only difference that the Einstein scale factor is replaced by the inverse of the string coupling, $g^{-1} = e^{-\phi/2}$.

Consider now the string cosmology background of Sect. 4, in which the dilaton-driven phase (4.1) and the string phase (4.4) are followed by the radiation-driven expansion. For such background, the effective potential (6.2) is given explicitly by

$$
\begin{aligned}
V &= \frac{1}{4\eta^2}(3-\sqrt{12}), && \eta < \eta_s \\
V &= \frac{\beta(\beta-1)}{\eta^2}, && \eta_s < \eta < \eta_1 \\
V &= 0, && \eta > \eta_1
\end{aligned}
\tag{6.3}
$$

The exact solution of eq. (6.2), normalized to an initial vacuum fluctuation spectrum ($\psi_k \to e^{-ik\eta}/\sqrt{k}$ for $\eta \to -\infty$), is thus

$$
\begin{aligned}
\psi_k &= |\eta|^{1/2} H_\sigma^{(2)}(|k\eta|), && \eta < \eta_s \\
\psi_k &= |\eta|^{1/2} \left[B_+(k) H_\mu^{(2)}(|k\eta|) + B_-(k) H_\mu^{(1)}(|k\eta|)\right], && \eta_s < \eta < \eta_1 \\
\psi_k &= \frac{1}{\sqrt{k}} \left[c_+(k) e^{-ik\eta} + c_-(k) e^{ik\eta}\right], && \eta > \eta_1
\end{aligned}
\tag{6.4}
$$

where $\sigma = (\sqrt{3}-1)/2$, and $\mu = |\beta - 1/2|$.

For this model of background evolution the effective potential $V(\eta)$ grows in the dilaton phase, keeps growing in the string phase where it reaches a maximum $\sim \eta_1^{-2}$ around the transition scale η_1, and then goes rapidly to zero in the subsequent radiation phase, where $\phi = \phi_1 =$const. The maximum amplified frequency is of the same order as before, $\omega_1 = H_1 a_1/a = |\eta_s/\eta_1|\omega_s > \omega_s$, where $\omega_s = H_s a_s/a$ is the last mode hitting the barrier (or crossing the horizon) in the dilaton phase. For modes with $\omega > \omega_s$ the amplification is thus due to the second background transition only: we can evaluate $|c_-|$ by using the large argument limit of the Hankel functions when matching the solutions at η_s (which gives $B_+ \simeq 1$, $B_- \simeq 0$), using however the small argument limit when matching at η_1, which gives

$$
|c_-(\omega)| \simeq \left(\frac{\omega}{\omega_1}\right)^{-\mu-1/2}, \quad \omega_s < \omega < \omega_1 \tag{6.5}
$$

Modes with $\omega < \omega_s$, which exit the horizon in the dilaton phase, stay outside the horizon also in the string phase, so that we can use the small argument limit at both the matching epochs: this gives $B_\pm = b_\pm |k\eta_s|^{-\sigma-\mu}$ ($b_\pm$ are numbers of order of unity) and

$$
|c_-(\omega)| \simeq \left(\frac{\omega}{\omega_s}\right)^{-\sigma} \left(\frac{\omega}{\omega_1}\right)^{-1/2} \left|\frac{\eta_s}{\eta_1}\right|^{\mu}, \quad \omega < \omega_s \tag{6.6}
$$

We are interested, in particular, in the ratio

$$
r(\omega) = \frac{\omega}{\rho_\gamma}\frac{d\rho}{d\omega} \simeq \frac{\omega^4}{\rho_\gamma}|c_-(\omega)|^2 \tag{6.7}
$$

measuring the fraction of electromagnetic energy density stored in the mode ω, relative to the total radiation energy ρ_γ. By using the parameterization of Sect.

4 we have $2\mu = |2\beta - 1| = |1 + 2y/x|$ and $|\eta_s/\eta_1| = \omega_1/\omega_s = 10^x$, so that the electromagnetic perturbation spectrum is again determined by two parameters only, the duration of the string phase $|\eta_s/\eta_1|$, and the initial value of the string coupling, $g_s = g_1 10^y$. We find

$$r(\omega) = g_1^2 \left(\frac{\omega}{\omega_1}\right)^{3-|\frac{2y}{x}+1|}, \qquad 10^{-x} < \frac{\omega}{\omega_1} < 1 \tag{6.8}$$

for modes crossing the horizon in the string phase, and

$$r(\omega) = g_1^2 \left(\frac{\omega}{\omega_1}\right)^{4-\sqrt{3}} 10^{x(1-\sqrt{3})+|2y+x|}, \qquad \frac{\omega}{\omega_1} < 10^{-x} \tag{6.9}$$

for modes crossing the horizon in the dilaton phase. The same spectrum has been obtained, with a different approach, also in the String frame [5,6]. Note that, in this paper, the definition of g_1 has been rescaled with respect to [6], by absorbing into g_1 the 4π factor.

6.1. Seed magnetic fields

The above spectrum of amplified electromagnetic vacuum fluctuations has been obtained in the linear approximation, expanding around a homogeneous background. We have thus to impose on the spectrum the consistency condition of negligible back-reaction, $r(\omega) < 1$ at all ω. For $g_1 < 1$ this condition requires a growing perturbation spectrum, and imposes a rather stringent bound on parameter space,

$$y < x, \qquad y > -2x \tag{6.10}$$

(note that a growing spectrum also automatically satisfies the nucleosynthesis bound $r < 0.1$, in view of the g_1^2 factor which normalizes the strength of the spectrum, and of eq. (4.3)).

It becomes now an interesting question to ask whether, in spite of the above condition, the amplified vacuum fluctuations can be large enough to seed the dynamo mechanism which is widely believed to be responsible for the observed galactic (and extragalactic) magnetic fields [38]. Such a mechanism would require a primordial magnetic field coherent over the intergalactic Mpc scale, and with a minimal strength such that [35]

$$r(\omega_G) \gtrsim 10^{-34}, \qquad \omega_G = 10^{-14} Hz \tag{6.11}$$

This means, in terms of our parameters,

$$|y + \frac{x}{2}| > \frac{3}{2}x - \frac{(17 + \log_{10} g_1)x}{25 + \frac{1}{2}\log_{10} g_1}, \quad x > 25 + \frac{1}{2}\log_{10} g_1$$

$$x(1-\sqrt{3}) + |2y + x| > 23 - 0.87 \log_{10} g_1, \quad x < 25 + \frac{1}{2}\log_{10} g_1 \tag{6.12}$$

Surprisingly enough the answer to the previous question is positive, and this marks an important point in favor of the string cosmology scenario considered here, as it is in general quite difficult - if not impossible - to satisfy the condition (6.11) in other, more conventional, inflationary scenarios.

The allowed region of parameter space, compatible with the production of seed fields [eq. (6.12)] in a nearly homogeneous background [eq. (6.10)], is shown in **Fig. 2** (again for the reference value $g_1 = 1$). In the region within the full bold lines the seed fields are due to modes crossing the horizon in the dilaton phase, in the region within the thin lines to modes crossing the horizon in the string phase. In both cases the background satisfies $y < -x/2$, i.e. $\beta > 1/2$, so that the whole allowed region refers to perturbations which are always growing outside the horizon, even in the string phase.

We may see from **Fig. 2** that the production of seed fields require a very small value of the dilaton coupling at the beginning of the string phase,

$$g_s = e^{\phi_s/2} \lesssim 10^{-20} \tag{6.13}$$

This initial condition is particularly interesting, as it could have an important impact on the problem of freezing out the classical oscillations of the dilaton background (work is in progress). It also requires a long enough duration of the string phase,

$$z_s = |\eta_s/\eta_1| \gtrsim 10^{10} \tag{6.14}$$

which is not unreasonable, however, when z_s is translated in cosmic time and string units, $z_s = \exp(\Delta t/\lambda_s)$, namely $\Delta t \gtrsim 23\lambda_s$.

Also plotted in **Fig. 2**, for comparison, are the allowed regions for the production of gravitons falling within the interferometric sensitivity range, taken from the previous picture. Since there is no overlapping, a signal detected (for instance) by LIGO would seem to exclude the possibility of producing seed fields, and vice-versa. Such a conclusion should not be taken too seriously, however, because the allowed regions of **Fig. 2** actually define a "minimal" allowed area, obtained within the restricted range of parameters compatible with a linearized description of perturbations. If we drop the linear approximation, then the allowed area extends to the "south-western" part of the plane (x, y), and an overlap between electromagnetic and gravitational regions becomes possible. In that case, however, the perturbative approach around a homogeneous background could not be valid any longer, and we would not be able to provide a correct computation of the spectrum.

6.2. The CMB radiation and its anisotropy

In cosmological models based on the low energy string effective action, the spectrum of scalar and tensor metric perturbations grows in general too fast with frequency [3,9,16] to be able to explain the large scale anisotropy detected by COBE [39,40]. If we insist, however, in looking for an explanation of the anisotropy in terms of the quantum fluctuations of some primordial field (amplified by the background evolution), a possible - even if unconventional - explanation in a string cosmology context is provided by the vacuum fluctuations of the electromagnetic field [6].

Consider in fact electromagnetic perturbations, reentering the horizon ($|k\eta| \sim 1 \sim \omega/H$) after amplification. At the time of reentry H^{-1} they provide a field coherent over the horizon scale, which can seed the cosmic magnetic fields, as discussed in the previous Section. If reentry occurs before the decoupling era, however, soon after reentry perturbations are expected to thermalize and homogenize rapidly, because of their interactions with matter sources in thermal equilibrium. Modes crossing the horizon after decoupling, on the contrary, generate a stochastic perturbation background whose spectrum remains frozen until the present time t_0, and that may contribute to the observed inhomogeneities of the CMB radiation. In particular, for a complete electromagnetic origin of the observed anisotropy, $\Delta T/T$, at the present horizon scale, $\omega_0 \sim 10^{-18}$Hz, the perturbation amplitude should satisfy the condition

$$r(\omega_0)\Omega_\gamma(t_0) \sim (\Delta T/T)_0^2 \tag{6.15}$$

namely

$$r(\omega_0) \simeq 10^{-6}, \qquad \omega_0 = 10^{-18} Hz \tag{6.16}$$

According to our electromagnetic perturbation spectrum [eqs. (6.8), (6.9)] this condition can be satisfied consistently with the homogeneity bound (6.10), and without fine-tuning of parameters, provided the string phase is so long that all scales inside our present horizon crossed the horizon (for the first time) during the string phase, i. e. for $\omega_0 > \omega_s$ (or $z_s > 10^{29}$). If we accept this electromagnetic explanation of the anisotropy, we have then two important consequences.

The first follows from the fact that the peak value of the spectrum (6.8) is fixed, so that the spectral index n, defined by

$$r(\omega) = g_1^2 \left(\frac{\omega}{\omega_1}\right)^{n-1} \tag{6.17}$$

can be completely determined as a function of the amplitude at a given scale. For the horizon scale, in particular, we have from eqs. (6.15) and (5.3)

$$n \simeq \frac{25 + \frac{5}{2}\log_{10} g_1 - 2\log_{10}(\Delta T/T)_{\omega_0}}{29 + \frac{1}{2}\log_{10} g_1} \tag{6.18}$$

I have taken explicitly into account here the dependence of the spectrum on the the present value of the string coupling g_1 (which is illustrated in **Fig. 3**), to stress that such dependence is very weak, and that our estimate for n from $\Delta T/T$ is quite stable, in spite of the rather large theoretical uncertainty about g_1^2 (nearly two order of magnitude, recall eq. (4.3)).

In order to match the observed anisotropy, $\Delta T/T \sim 10^{-5}$, we obtain from eq. (6.18) (see also **Fig. 3**, where the relation (6.18) is plotted for three different values of g_1)

$$n \simeq 1.11 - 1.17 \tag{6.19}$$

This slightly growing (also called "blue" spectrum) is flat enough to be well compatible with the present analyses of the COBE data [39,40].

The second consequence follows from the fact that fixing a value of n in eq. (6.17) amounts to fix a relation between the parameters x and y of our background, according to eq. (6.8). If we accept, in particular, a value of n in the range of eq. (6.19), then we are in a region of parameter space which is also compatible with the production of seed fields, according to eq. (6.11). This means that we are allowed to formulate cosmological models in which cosmic magnetic fields and CMB anisotropy have the same common origin, in such a way to explain (for instance) why the energy density ρ_B of the observed cosmic magnetic fields is of the same order as that of the CMB radiation: in fact

$$\rho_B \sim \rho_\gamma \int^{\omega_1} r(\omega) d(\ln \omega) \sim \rho_{CMB} \tag{6.20}$$

A coincidence which is otherwise mysterious, to the best of my knowledge.

It is important to stress that the values of the parameters leading to eq.(6.19) are also automatically consistent with the bound following from the presence of strong magnetic fields at nucleosynthesis time [41], which imposes $r(\omega_N) \lesssim 0.05$ at the scale corresponding to the end of nucleosynthesis, $\omega_N \simeq 10^{-12}$Hz. By comparing photon and graviton production [eqs. (6.8) and (5.8)] we find, moreover, that for a background in which n lies in the range (6.19) the graviton spectrum grows fast enough with frequency ($\Omega \sim \omega^m$, $m = n + 1 = 2.11 - 2.17$) to be well compatible with the pulsar bound (5.12). Note that, with such a value of m, the metric perturbation contribution to the COBE anisotropy is completely negligible.

It should be mentioned the our electromagnetic perturbation spectrum, (6.8), (6.9), though obtained from the tree-level in g, lowest order in α', string effective action (6.1), is certainly stable with respect to loop corrections when applied in a range of parameters in which $g = \exp(\phi/2) << 1$, i. e. the dilaton is deeply inside the perturbative regime. As to α' corrections, they could modify (in principle) the string branch of the spectrum. However, since we are expanding around the vacuum background ($F_{\mu\nu} = 0$), no higher curvature term which can be written in the form of powers of the Maxwell Lagrangian, $(\alpha' F_{\mu\nu} F^{\mu\nu})^p, p \geq 2$, will affect the perturbation equations, as long as we are limited to the linear approximation.

We note, finally, that the class of backgrounds able to provide an electromagnetic explanation of the CMB anisotropy, can also account for the production of the CMB radiation itself, directly from the amplification of the vacuum fluctuations of the electromagnetic (and other gauge) fields.

Without introducing "ad hoc" some radiation source, suppose in fact that the gravi-dilaton background accelerates up to some maximum (nearly Planckian) scale H_1, corresponding to the peak of the effective potential $V(\eta)$ in the perturbation equations, and then decelerates, with corresponding decreasing of the potential barrier. This is enough for the production of a mixture of ultra-relativistic particles, with a spectrum which is thermal [25] (at a temperature $T_1 \sim H_1 a_1/a$) at high frequency ($\omega > T_1$), and possibly distorted by parametric amplification effects at low frequency.

The low frequency part of the spectrum remains frozen for those particles (like gravitons and dilatons) which interact only gravitationally, and then decouple soon after the transition; it is expected instead to approach rapidly a thermal distribution for all the other produced particles which go on interacting among themselves (and with the background sources) for a long enough time after the transition. For such particles the total energy, in critical units, is

$$\Omega_T(t) \sim \left(\frac{H_1}{M_p}\right)^2 \left(\frac{H_1}{H}\right)^2 \left(\frac{a_1}{a}\right)^4 \tag{6.21}$$

Even if, initially, $\Omega_T < 1$ (as $H_1 < M_p$), such a produced thermal radiation may thus become dominant ($\Omega_T = 1$), and identified with the presently observed radiation background, provided the scale factor, at the beginning of the decelerated epoch, grows in time more slowly than $H^{-1/2}$. This is indeed the case, for instance, of the time-reversed dilaton-dominated solution (4.1), which expands like $a(t) \sim t^{1/3}$ for $t \to +\infty$.

It should be stressed, however, that the identification of the radiation obtained from vacuum fluctuations with the observed radiation background implies a unique normalization of all perturbation spectra, $\Omega(\omega)$, at the maximum scale $\omega_1 \sim T_1$: at present time it imposes $\Omega(10^{11} Hz) \sim \Omega_T(t_0) \sim 10^{-4}$. On the other hand we know, from the existing phenomenological bounds mentioned in this paper, that at low frequencies $\Omega << 10^{-4}$. As a consequence, such a common origin of the CMB radiation and of its anisotropies can be consistently implemented only in a background which amplifies fluctuations with fast enough growing spectra (which is indeed the case of the string cosmology scenario discussed here).

7. Conclusion.

In inflationary string cosmology backgrounds perturbations can be amplified more efficiently than in conventional inflationary backgrounds, as the perturbation amplitude my even grow, instead of being constant, outside the horizon. In some case, like scalar metric perturbations in a dilaton-driven background, the effects of the growing mode can be gauged away. But in other cases the growth is physical, and can prevent a linearized description of perturbations.

In any case, such enhanced amplification is interesting and worth of further study, as it may lead to phenomenological consequences which are unexpected in the context of the standard inflationary scenario. For instance, the production of a relic graviton background strong enough to be detected by the large interferometric detectors, or the production of primordial magnetic fields strong enough to seed the galactic dynamo. Finally, the possible existence of a relic stochastic electromagnetic background, due to the amplification of the vacuum fluctuations of the electromagnetic field, strong enough to be entirely responsible for the observed large scale CMB anisotropy.

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