DFTT-47/92
August 1992

Gravity Waves From Primordial Dimensional Reduction *

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Abstract

We show that the dynamical contraction of n "internal" spacelike dimensions leads, in general, to the formation of a cosmic graviton background, similar to that produced by inflation. By using a simple phenomenological model we discuss how to obtain information and constraints on the total number of extra dimensions, on their compactification scale and on the kinematics of the dimensional reduction process.

To appear in **Class. Quantum Grav.**

* Essay written for the 1992 Awards for Essays on Gravitation (Gravity Research Foundation, Wellesley Hills, Ma), and selected for Honorable Mention

Gravity waves from primordial dimensional reduction.

It is well known that the spectral density of relic gravitons, produced by the cosmological evolution [1], provides important constraints on inflationary models [2] and, more generally, provides direct information on the behaviour of the Hubble parameter of the very early Universe [3,4]. Less known, or less studied (up to now) at least, is the connection between the present energy distribution of cosmic gravitons and the past occurrence of a dimensional reduction process. The aim of this paper is to compute the graviton production according to a simple phenomenological model of dynamical dimensional reduction, and to show that one can obtain from present observations significant constraints on the number of contracting dimensions, on their rate of shrinking, and on the final compactification scale.

The model is based on a $(4+n)$-dimensional manifold, with coordinates $x^A = (x^\mu, y^a)$, in which n "internal" dimensions shrink with an arbitrary power-law behaviour $b(x^0) = (x^0)^\beta$, while the three "physical" dimensions, with scale factor $a(x^0)$, are frozen (conventions:$\mu, \nu = 0,1,2,3; a,b = 1,2,...n$). In the conformal time gauge ($x^0 = \eta$), this evolution is parametrized by the metric

$$g_{AB} = (a^2(\eta)\eta_{\mu\nu}, -b^2(\eta)\overline{g}_{ab}(x^c)) \tag{1}$$

where $\eta_{\mu\nu} = (+---)$ is the 4-dimensional Minkowski metric, $\overline{g}_{ab}$ is the metric of a maximally symmetric n- dimensional manifold, and ($\beta > 0$)

$$\begin{array}{rcl} a(\eta) = 1 \quad , & b(\eta) = (-\eta/\eta_0)^\beta \quad , & \eta < -\eta_0 < 0 \\ a(\eta) = (\eta + 2\eta_0)/\eta_0 \quad , & b(\eta) = 1 \quad , & \eta > -\eta_0 \end{array} \tag{2}$$

(note that conformal and cosmic time coincide for $\eta \le -\eta_0$).

This metric describes the decoupling of internal and external dimensions down to a final curvature scale $H_0 \simeq \eta_0^{-1}$. As a consequence of this effective dimensional reduction the spacetime becomes (for $\eta > -\eta_0$) the direct product of a static internal space with constant curvature, and a four-dimensional (spatially flat) manifold whose geometry evolves according to the standard, radiation-dominated FRW cosmology. We have chosen this particular behaviour of $a(\eta)$ as an example of physical evolution in which the production of gravitational radiation is entirely due to the shrinking of the internal dimensions, without contributions from the expansion of three-dimensional space.

In the background (1) a first order gravitational perturbation h_{AB} must satisfy the linearized wave equation [1]

$$\partial_C(\sqrt{|g|}g^{CD}\partial_D)h_{AB} = 0 \tag{3}$$

The four-dimensional components, $h_{\mu\nu}(x^\alpha)$, are thus decoupled from the internal ones. By performing a Fourier expansion, $h^{\mu\nu} = \sum_k h_k^{\mu\nu}(\eta)e^{ikx}$, and defining

$$h_k(\eta) = \mu_k(\eta)a^{-1}(\eta)b(\eta)^{-n/2} \tag{4}$$

we are led to the equation, for each "external" mode μ_k,

$$\mu_k'' + [k^2 - V(\eta)]\mu_k = 0 \tag{5}$$

where

$$V(\eta) = \frac{a''}{a} + \frac{nb''}{2b} + \frac{n}{4}(n-2)\frac{b'^2}{b^2} + \frac{na'b'}{ab} \tag{6}$$

(a prime denotes differentiation with respect to η). With the choice (2) of the scale factors we have then an effective potential barrier

$$\begin{aligned} V(\eta) = \frac{n\beta}{4\eta^2}(n\beta - 2) \quad &, \quad \eta < -\eta_0 \\ V(\eta) = 0 \quad &, \quad \eta > -\eta_0 \end{aligned} \tag{7}$$

which leads to the amplification [1] of the perturbation amplitude (4), or (equivalently) to the mixing of positive and negative frequency modes and then to particle production. (Note that for $\beta = 0$ or $n = 0$ there is no barrier so that no graviton production takes place, as anticipated. The same happens when $n\beta = 2$).

The frequency mixing coefficient $c_-(k)$, determining the spectral number of produced particles [5], can be explicitly computed by matching at $\eta = -\eta_0$ the general "out" solution of eq.(5),

$$\mu_k = \frac{1}{\sqrt{k}}(Ae^{-ik(\eta+\eta_0)} + Be^{ik(\eta+\eta_0)}) \quad , \quad \eta > -\eta_0 \tag{8}$$

to the "in" solution of eq.(5) which represents positive frequency modes in the $\eta \to -\infty$ limit (and correspond to the Bunch-Davies "conformal" vacuum [5])

$$\mu_k = N\eta^{1/2} H_\nu^{(2)}(k|\eta|)$$
$$\nu = \frac{1}{2}|\beta n - 1| \quad , \quad \eta < -\eta_0 \tag{9}$$

(N is a constant normalization coefficient, and $H_\nu^{(2)}$ is the Hankel function). According to this "sudden" approximation we have (neglecting numerical factors of order unity)

$$|c_-(k)| = \frac{|B|}{|N|} \simeq (k\eta_0)^{-|\nu|-1/2} \quad , \quad k\eta_0 \lesssim 1 \tag{10}$$

For $k\eta_0 > 1$ this approximation is no longer valid, and the mixing coefficient is to be computed by replacing the potential step with a smooth transition from $V(\eta_0)$ to 0 (in order to avoid ultraviolet divergences). In this way one finds, however, that for $k\eta_0 > 1$ the mixing is exponentially suppressed [5], so that it may be ignored for the purpose of this paper.

The spectral energy density, $\rho(k) = k(d\rho_g/dk) \simeq k^4|c_-|^2$, which is the variable usually adopted to characterize the distribution of the produced gravitons [3,4], for our model of dimensional reduction is then given by

$$\begin{aligned} \rho(k) &= k^{2+\beta n}\eta_0^{\beta n-2} \quad , \quad 0 < \beta n < 1 \\ \rho(k) &= k^{4-\beta n}\eta_0^{-\beta n} \quad , \quad \beta n > 1 \end{aligned} \tag{11}$$

It is interesting to note that for $\beta n = 4$ we have a flat graviton spectrum, like that produced by an epoch of de Sitter inflation [6]. For $\beta n > 4$ the energy decreases with frequency like in the case of power- law inflation [7], while for $\beta n < 4$ it grows with frequency, like in the case of superinflation [8].

The transition time, η_0, can be conveniently expressed in terms of the final curvature scale, H_0, corresponding to the end of the compactification phase: $\eta_0^{-1} \simeq |b'/b|_{\eta=-\eta_0} \equiv H_0$. In the same way, a frequency k can be expressed in terms of the curvature scale, evaluated at the epoch $\eta_k \simeq -k^{-1}$ at which the given mode "hits" the effective potential barrier: $k = |V(\eta_k)|^{1/2} \simeq |b'/b|_{\eta=\eta_k} \equiv H_k$. Measured in units of critical energy density, $\rho_c \simeq H_0^2/G$, the spectrum (11) can thus be rewritten in the following form, useful for the comparison with experimental constraints ($\Omega(k) \equiv \rho(k)/\rho_c$)

$$\begin{aligned} \Omega(k) &= GH_k^2(\frac{H_k}{H_0})^{\beta n} \quad , \quad 0 < \beta n < 1 \\ \Omega(k) &= GH_k^2(\frac{H_k}{H_0})^{2-\beta n} \quad , \quad \beta n > 1 \end{aligned} \tag{12}$$

(again numerical factors of order unity have been neglected).

The energy density of a cosmic graviton background is presently constrained, in general, by three types of observational data [3]. First of all by the isotropy of the electromagnetic microwave background radiation [2,3], which provides the most significant constraint ($\Omega \lesssim 10^{-8}$) at the minimal frequency ($\sim 10^{-18}$ Hertz) corresponding to the inverse of the present Hubble radius. Pulsar timing data [3,9] imply, moreover, $\Omega \lesssim 10^{-6}$ for a small range of frequencies around 10^{-8} Hertz. Finally, there is the overall bound $\Omega \lesssim 1$ (to avoid that the produced gravitons over-close our present universe), which provides the most effective constraint at the frequency corresponding to the highest value of $\Omega(k)$. For growing spectra, this occurs at the final frequency k_0, which can be related to the minimal value k_m (fixed by the present Hubble radius) by an adiabatic rescaling down to the transition time η_0, i.e.

$$\frac{k_0}{k_m} = \frac{H_0 a_0}{H(t)a(t)} \simeq 10^{29} \tag{13}$$

($H(t)$ and $a(t)$ are the present values of the Hubble parameter $\dot{a}/a$ and of the cosmological scale factor). With the same procedure one obtains, for the frequency k_p corresponding to the pulsar bound, $k_p/k_0 \simeq 10^{-19}$.

Since the quoted phenomenological constraints are to be applied at different frequencies, their relative importance depends on the slope of the gravitational spectrum. For the power-law behaviour of eq.(11) it turns out, in particular, that the most significant constraints on Ω is provided by the closure density bound if $\beta n < 70/19$, by the pulsar bound if $70/19 < \beta n < 19/5$, and by the isotropy bound if $\beta n > 19/5$. By expressing these bounds as limits on the final value of the compactification scale H_0 (using eqs. (12,13)), we find ($M_p = G^{-1/2}$ is the Planck mass)

$$
\begin{aligned}
H_0/M_p &\leq 1 \qquad , \qquad 0 < \beta n < 70/19 \\
H_0/M_p &\leq 10^{35-\frac{19}{2}\beta n} \qquad , \qquad 70/19 < \beta n < 19/5 \\
H_0/M_p &\leq 10^{54-\frac{29}{2}\beta n} \qquad , \qquad 19/5 < \beta n
\end{aligned}
\tag{14}
$$

These results are illustrated in **Fig.1**. They seem to suggest, in particular, that the compactification process must be slow enough ($\beta n \lesssim 3.8$ at fixed n) if the size of the internal space has to shrink down close to the Planck scale.

These results depend, of course, on the particular model of dimensional reduction we have chosen (see eq.(2)). More generally, the shrinking of the internal dimensions may be accompanied by a simultaneous (possibly inflationary) expansion of the external space, which may itself contribute to the graviton production. The spectral density one obtains, in that case, leads to results which are qualitatively the same as those of eq.(14), but quantitatively different [10]. Moreover, a realistic model should also include the graviton production due to the transition from the radiation-dominated to the matter-dominated regime [3,4,7,11], which modifies the low-frequency tail of the spectrum in the range $10^{-16} - 10^{-18}$ Hertz.

Already from this simple model we can see, however, that the present phenomenological bounds on a background of cosmic gravitons provide interesting information and constraints on the possible number of contracted dimensions, on their compactification scale, and on the dynamical model describing their decoupling from macroscopic three-dimensional space.

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Figure caption

Fig.1

The maximum allowed value of the compactification scale H_0 (in units of Planck mass M_p) plotted versus βn, where n is the total number of internal dimensions, and $\beta > 0$ is the power describing their (isotropic) shrinking according to eq.(2). H_0 is constrained by the cosmic microwave background isotropy, by pulsar timing data, and by the critical density bound. The allowed region with $H_0 \leq M_p$ extends from $\beta n \simeq 3.68$ down to $\beta n = 0$.