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ENTROPY PRODUCTION IN THE COSMOLOGICAL AMPLIFICATION OF THE VACUUM FLUCTUATIONS

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Abstract

We estimate the entropy associated to a background of squeezed cosmic gravitons, and we argue that the process of cosmological pair production from the vacuum may explain the large amount of entropy of our present universe.

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Entropy production in the cosmological amplification of the vacuum fluctuations

A successful inflationary model is expected to solve the various problems of the standard cosmological scenario and then, in particular, must provide some explanation to the unnaturally large value of the entropy to-day stored in the cosmic microwave background (CMB). The solution to the entropy problem relies, conventionally, on the so-called "reheating" process [1], occurring at the end of the inflationary era, and realized in various model-dependent ways such as bubble collisions and/or inflaton decay. In view of the difficulties encountered by some model in providing an efficient enough reheating phase [1,2], however, it seems interesting to point out an additional source of entropy in the inflationary scenario: the parametric amplification of the vacuum fluctuations, or (equivalently stated, in a second quantization formalism) the pair production from the vacuum, due to changes in the background metric field.

As discussed in previous papers [3-5], the growth of the average number of field quanta, due to pair production, is naturally associated to the growth of the entropy of that field. On the other hand the pairs spontaneously emitted from the vacuum, in an evolving cosmological background, necessarily appear in a "squeezed" quantum state [6,7], namely in a state in which the quantum fluctuations of a given field operator Q are suppressed with respect to the vacuum, while the fluctuations of the canonically conjugate operator P (called "superfluctuant") are correspondingly enhanced (see, for instance, [8,9]).

In order to compute the entropy associated to cosmological pair production, we shall thus propose in this paper a "coarse graining" approach to non-equilibrium entropy, valid for squeezed states, in which the loss of information associated to the reduced density matrix is represented by the increased dispersion of the superfluctuant operator P. Such a quantum-mechanical approach is different from a recent field-theoretic computation of the entropy of the cosmological perturbations [10,11]. The two methods provide results which coincide in the large squeezing limit; our expression for the entropy, however, is always positive definite (unlike the classical approach of [10]), and seems then to be valid even in the small squeezing regime.

The definition of entropy we shall provide holds in general for any mechanism of particle production from the vacuum (more generally, from any coherent initial state), not necessarily of gravitational origin, which can be represented as a mixing of positive and negative frequency modes through an appropriate Bogoliubov transformation. We shall apply this definition, in particular, to estimate the entropy change associated to the formation of a background of cosmic gravitons. By using a very general model of cosmological evolution, we shall find an entropy which may be as large as the CMB electromagnetic entropy, provided the transition from inflation to the radiation-dominated phase occurs at a curvature scale of the Planck order.

Let us recall, first of all, that the quantum description [6,7] of the amplification of scalar or tensor fluctuations is based on the separation of the field into background solution and first order perturbations, and on the expansion of the solution to the perturbed wave equation into $|in>$ and $|out>$ modes. The complex coefficients of this expansion are interpreted, in a second-quantization formalism, as annihilation and creation operators for a particle $(b, b^\dagger)$ and the corresponding anti-particle $(\tilde{b}, \tilde{b}^\dagger)$. The relation between $|in>$ and $|out>$ mode solutions can thus be expressed, for each mode k, as a Bogoliubov transformation between the $|in>$ operators $(b, b^\dagger, \tilde{b}, \tilde{b}^\dagger)$ and the $|out>$ ones $(a, a^\dagger, \tilde{a}, \tilde{a}^\dagger)$:

$$\begin{aligned} a_k &= c_+(k) b_k + c_-^*(k) \tilde{b}^\dagger_{-k} \\ \tilde{a}^\dagger_{-k} &= c_-(k) b_k + c_+^*(k) \tilde{b}^\dagger_{-k} \end{aligned} \tag{1}$$

where $|c_+|^2 - |c_-|^2 = 1$. By parametrizing the Bogoliubov coefficients $c_\pm$ in terms of the two real numbers $r \geq 0$ and θ,

$$c_+(k) = \cosh r(k) \quad , \quad c_-^*(k) = e^{2i\theta_k} \sinh r(k) \tag{2}$$

the relations (1) can be re-written as unitary transformations generated by the (momentum-conserving) two-mode squeezing operator Σ_k,

$$\Sigma_k = \exp(z_k^* b_k \tilde{b}_{-k} - z_k b_k^\dagger \tilde{b}^\dagger_{-k}) \quad , \quad z_k = r(k) e^{2i\theta_k} \tag{3}$$

(r is the so-called squeezing parameter) as

$$a_k = \Sigma_k b_k \Sigma_k^\dagger \tag{4}$$

(and related expressions for $\tilde{a}^\dagger, a^\dagger$ and $\tilde{a}$). Starting with the $|in>$ vacuum state $|0>$, such that $b_k|0>= 0 = \tilde{b}_{-k}|0>$, the pair production process leads then, for each mode, to the "squeezed vacuum" state [6,7] $|z_k>= \Sigma_k|0>$, which satisfies $a_k|z_k>= 0 = \tilde{a}_{-k}|z_k>$, and which has an averaged particle number

$$\overline{n}_k =< z_k|b_k^\dagger b_k|z_k>=< z_k|\tilde{b}^\dagger_{-k}\tilde{b}_{-k}|z_k>=< 0|a_k^\dagger a_k|0>= |c_-(k)|^2 = \sinh^2 r(k) \tag{5}$$

For the purpose of this paper it is convenient to note that the two-mode transformation (4) can be factorized as the product of two one-mode transformations, as usually done in squeezed-state theory [9], by defining a new pair of annihilation operators $(a_1, \tilde{a}_1)$ through the complex rotation

$$a = \frac{1}{\sqrt{2}}(a_1 - i\tilde{a}_1) \quad , \quad \tilde{a} = \frac{1}{\sqrt{2}}(a_1 + i\tilde{a}_1) \tag{6}$$

and a similar expression for $b, \tilde{b}$,

$$b = \frac{1}{\sqrt{2}}(b_1 - i\tilde{b}_1) \quad , \quad \tilde{b} = \frac{1}{\sqrt{2}}(b_1 + i\tilde{b}_1) \tag{7}$$

(obviously, by definition, $[a_1, a_1^\dagger] = [\tilde{a}_1, \tilde{a}_1^\dagger] = [b_1, b_1^\dagger] = [\tilde{b}_1, \tilde{b}_1^\dagger] = 1$, $[a_1, \tilde{a}_1] = [b_1, \tilde{b}_1] = 0$). The transformation (4) can thus be rewritten, for each mode, as

$$a = b_1 \cosh r + b_1^\dagger e^{2i\theta} \sinh r - i(\tilde{b}_1 \cosh r + \tilde{b}_1^\dagger e^{2i\theta} \sinh r) = \Sigma_1 b_1 \Sigma_1^\dagger - i\tilde{\Sigma}_1 \tilde{b}_1 \tilde{\Sigma}_1^\dagger \tag{8}$$

where Σ_1 and $\tilde{\Sigma}_1$ are the one-mode squeezing operators for b_1 and $\tilde{b}_1$,

$$\Sigma_1 = \exp(\frac{z^*}{2} b_1^2 - \frac{z}{2} b_1^{\dagger 2}) \quad , \quad \tilde{\Sigma}_1 = \exp(\frac{z^*}{2} \tilde{b}_1^2 - \frac{z}{2} \tilde{b}_1^{\dagger 2}) \tag{9}$$

Given any two-mode squeezed state $|z>$, with squeezing parameter $r > 0$, it is always possible to introduce two operators x and $\tilde{x}$ whose variances $(\Delta x)_z$, $(\Delta \tilde{x})_z$ are amplified with respect to the vacuum value $(\Delta x)_0$, $(\Delta \tilde{x})_0$, namely [8,9] $(\Delta x)_z/(\Delta x)_0 = (\Delta \tilde{x})_z/(\Delta \tilde{x})_0 = \exp(r)$, where $(\Delta x^2)_z = < z|x^2|z >$-$(< z|x|z >)^2$ (and the same for $\tilde{x}$). In terms of these operators, b_1 and $\tilde{b}_1$ have the differential representation

$$b_1 = \frac{e^{i(\theta+\pi/2)}}{\sqrt{2}}(x + \partial_x) \quad , \quad \tilde{b}_1 = \frac{e^{i(\theta+\pi/2)}}{\sqrt{2}}(\tilde{x} + \partial_{\tilde{x}}) \tag{10}$$

where the phase has been chosen according to eq.(2), in such a way to identify the x and $\tilde{x}$ operators with the superfluctuant ones.

Therefore, in the $(x, \tilde{x})$-space representation the wave function ψ_{z_k} for the two-mode squeezed vacuum, determined by the condition $a\psi_z = 0$, can be factorized according to eq.(8) as

$$\psi_{z_k}(x, \tilde{x}) = < x\tilde{x}|z_k > = \psi^1_{z_k}(x)\psi^1_{z_k}(\tilde{x}) \tag{11}$$

where $\psi^1_z(x)$ is fixed by the differential equation

$$\cosh r(x + \partial_x)\psi^1 = \sinh r(x - \partial_x)\psi^1 \tag{12}$$

With the normalization condition $< z_k|z_k >= 1$ one obtains

$$\psi_{z_k}(x,\tilde{x}) = (\frac{\sigma_k}{\pi})^{1/2} exp[-\frac{\sigma_k}{2}(x^2+\tilde{x}^2)] \tag{13}$$

where the real number $\sigma_k = exp[-2r(k)]$ measures the departure of ψ_{z_k} from the vacuum wave function ($\sigma = 1$).

The dynamical evolution leading to the formation of squeezed states is thus accompanied, in general, by a loss of information corresponding to a larger dispersion of x and $\tilde{x}$ around their mean values, $\Delta x = \Delta \tilde{x} = 1/\sqrt{2\sigma}$. Such an increase in uncertainty is measured, in the $(x, \tilde{x})$ representation, by the "flattening" of the gaussian (13) with respect to the vacuum wave function, with a probability distribution $P_k(x,\tilde{x}) = |< x\tilde{x}|z_k >|^2$ which defines the reduced density operator

$$\rho_k = \int dx d\tilde{x} |x\tilde{x}>< x\tilde{x}|z_k>< z_k|x\tilde{x}>< x\tilde{x}| \quad , \quad Tr\rho_k = 1 \tag{14}$$

The entropy production associated to the transition $|0> \to |z_k>$, according to this reduction scheme, can now be computed by applying the usual information-theoretic definition of entropy ($S = -Tr\rho \ln \rho$), and by subtracting the constant contribution of the vacuum state,

$$S_0 = -\frac{1}{\pi}\int dx d\tilde{x} e^{-(x^2+\tilde{x}^2)} \ln(\frac{e^{-(x^2+\tilde{x}^2)}}{\pi}) = 1 + \ln \pi \tag{15}$$

The result can be simply expressed in terms of the averaged particle number $\overline{n}_k$ as

$$S(k) = -Tr\rho_k \ln \rho_k - S_0 = -\int dx d\tilde{x} P_k(x,\tilde{x}) \ln P_k(x,\tilde{x}) - S_0 =$$
$$= 2r(k) = 2\sinh^{-1}|c_-(k)| = 2\ln(\sqrt{\overline{n}_k} + \sqrt{1+\overline{n}_k}) \tag{16}$$

This expression for the entropy density per mode holds even if the initial vacuum is replaced in general by a coherent state $|\alpha_k>$, such that $b_k|\alpha_k> = \alpha_k|\alpha_k>$; moreover, it is always positive definite, and can be applied for any value of the squeezing parameter r. In the large $\overline{n}_k$ (or, equivalently, large squeezing) limit, eq.(16) gives

$$S(k) \simeq \ln |c_-(k)|^2 = \ln \overline{n}_k \tag{17}$$

which coincides exactly (for $\overline{n}_k >> 1$) with the entropy associated to an oscillator in a state of thermal equilibrium, and with the result of [10,11] (such a relation between the entropy and the mixing Bogoliubov coefficient was first suggested in [3]). The same result can be obtained if the reduction procedure, from the pure squeezed state down to a statistical mixture, is performed (mode by mode) in the Fock space spanned by the eigenstates $|n_k\tilde{n}_{-k}>$ of the number operators $N_k = b_k^\dagger b_k$, $\tilde{N}_k = \tilde{b}_{-k}^\dagger \tilde{b}_{-k}$.

Indeed, as stressed in [6], the number N is a superfluctuant operator in the state $|z_k>$, while the variance of the conjugate phase operator is squeezed. For large values of r, the loss of information with respect to N is measured, as discussed also in [5,11], by the density operator

$$\rho_k = \sum_n P_k(n)|n_k\tilde{n}_{-k}><n_k\tilde{n}_{-k}| \tag{18}$$

where $P_k(n) = |<n\tilde{n}|z_k>|^2$. It is well known, on the other hand, that any process of pair production from the vacuum described by the Bogoliubov transformation (1) is characterized by a probability distribution [12]

$$P_k(n) = \frac{|c_-(k)|^{2n}}{(1+|c_-(k)|^2)^{n+1}} \tag{19}$$

One thus obtain, for $|c_-| >> 1$, that $S(k) = -Tr\rho_k \ln \rho_k \simeq \ln |c_-(k)|^2$, in agreement with eq.(17).

The expression (16) provides the entropy density per mode for all cases of particle production (not necessarily of gravitational origin) parametrized by the Bogoliubov transformation (1). The total entropy S in a proper spatial volume V is thus obtained by summing the squeezing contributions over all modes of proper frequency ω (following the coarse graining approach of [4,5], we have neglected any correlation among modes)

$$S = \frac{V}{\pi^2}\int_{\omega_0}^{\omega_1} r(\omega)\omega^2 d\omega \tag{20}$$

where ω_0 and ω_1 delimit the frequency interval in which the squeezing mechanism is effective.

We shall apply this result in order to estimate the entropy growth associated to the cosmological productions of gravitons in a homogeneous and isotropic background, whose scale factor $a(t)$ describes the evolution from a primordial inflationary phase to the standard radiation and matter-dominated expansion. In such case proper and comoving frequency (k) are related by $\omega = k/a(t)$, and the graviton spectrum is determined by the Bogoliubov coefficient as follows [13-15] (we follow the notations of [13])

$$\begin{aligned} |c_-(\omega)| &\simeq (\frac{\omega}{\omega_1})^{-|\delta|} \quad &, \quad \omega_2 < \omega < \omega_1 \\ |c_-(\omega)| &\simeq (\frac{\omega}{\omega_1})^{-|\delta|}(\frac{\omega}{\omega_2})^{-1} \quad &, \quad \omega_0 < \omega < \omega_2 \end{aligned} \tag{21}$$

Here δ is an order of unity parameter depending on the kinematical behaviour of the inflationary phase ($|\delta| = 2$, for instance, in the case of de Sitter inflation); ω_1 is the maximum cutoff frequency depending on the curvature scale H_1 at the beginning ($t = t_1$) of the radiation-dominated evolution; ω_2 is the frequency corresponding to the radiation-matter transition ($t = t_2$); finally, $\omega_0 \simeq H_0 \sim 10^{-18} sec^{-1}$ is the minimum frequency fixed by the to-day value of the Hubble radius ($\omega_1 >> \omega_2 >> \omega_0$).

In this example $|c_-| >> 1$, so that $r(\omega) \simeq \ln |c_-(\omega)|$. By integrating the squeezing parameter according to eq.(20) we find that the total entropy per comoving volume is constant, just like the black- body entropy. By keeping only the dominant terms, we obtain for the graviton entropy S_g in a comoving volume ℓ^3

$$S_g = \frac{|\delta|}{(3\pi)^2}(a\ell\omega_1)^3 \tag{22}$$

Neglecting numerical factors of order unity, the total entropy $(a\ell\omega_1)^3$ can be easily estimated at any given observation time t_0 during the matter-dominated era, by rescaling down $\omega_1(t_0)$ to ω_0 as [13-15] $\omega_1(t_0)/\omega_0 = k_1/k_0 \simeq (H_1 a_1)/(H_0 a_0)$, where $H_1 = H(t_1)$, $a_1 = a(t_1)$ and so on. By recalling that $a \sim H^{-1/2}$ in the radiation phase, we obtain

$$\frac{\omega_1}{\omega_0} = (\frac{H_1}{H_2})^{1/2}_{rad}(\frac{H_2 a_2}{H_0 a_0})_{mat} = (\frac{H_1}{M_P})^{1/2}(\frac{M_P a_P}{H_0 a_0}) \tag{23}$$

where the transition curvature scale H_1 has been conveniently expressed in units of Planck mass M_P. By introducing the black-body temperature T_γ, which evolves adiabatically ($a(t)T_\gamma(t) = const$) from $T_\gamma(t_0)$ to $T_\gamma(t_P) = M_P$, we find from eq.(23)

$$\omega_1(t_0) = (\frac{H_1}{M_P})^{1/2} T_\gamma(t_0). \tag{24}$$

S_g can thus be rewritten

$$S_g \simeq |\delta| S_\gamma (\frac{H_1}{M_P})^{3/2} \tag{25}$$

where $S_\gamma \simeq [a(t)\ell T_\gamma(t)]^3 = const$ is the usual black-body entropy of the CMB radiation (in terms of the to-day parameters, $(a_0 \ell T_{\gamma 0})^3 \sim (T_{\gamma 0}/H_0)^3 \sim (10^{29})^3$).

Eq.(25), which represent the main result of this paper, holds in general for particle production with a spectrum $\overline{n}(\omega) \sim \omega^{-|\delta|}$, quite irrespective of the kind of fluctuations which are amplified (even for photons, in a suitable non-conformally flat background), and is only weakly dependent on the background kinematics (through the parameter $|\delta|$, which is typically of order unity [13-15]). Particle production from the vacuum is thus a process able to explain the observed cosmological level of entropy, *provided the curvature scale at the inflation-radiation transition is of the order of the Planck one, $H_1 \simeq M_P$.*

In the standard inflationary scenario, based on an accelerated phase of de Sitter-like exponential expansion, there are well known phenomenological bounds [16] which imply $H_1 \lesssim 10^{-4} M_P$, and which rule out the mechanism discussed here as a possible explanation of the observed entropy. However, as recently stressed in [13,17,18], such constraints are evaded if the de Sitter and/or the radiation-dominated phase are preceeded by a phase of accelerated evolution and growing curvature, like in the recently proposed "pre-big-bang" models [18,19] of a duality-symmetric string cosmology. We thus conclude that the result obtained in this paper provides an additional reason of interest for such models, where the maximum curvature scale is only constrained by [13,17,18] $H_1 \lesssim 10^2 M_P$.

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Note added

After this paper was submitted we received a preprint by R. Brandenberger, T.Prokopec and V.Mukhanov, *"The entropy of the gravitational field"* (Brown-HET-849, August 1992), whose content overlap to some extent with ours, and in which a result similar to our equation (25) is obtained through a different procedure.

References

1. E.W.Kolb and M.S.Turner, The early universe (Addison-Wesley, Redwood City, CA 1990), Chapt.8;
   K.A.Olive, Phys.Rep.190(1990)307
2. E.W.Kolb, in Proc. of the 79th Nobel Symposium, Physica Scripta T36(1991)199;
   M.S.Turner, in Proc of the First Erice School "D.Chalonge" on astrofundamental physics (September 1991), ed. by N.Sanchez (World Scientific, Singapore)
3. B.L.Hu and D.Pavon, Phys.Lett.B180(1986)329
4. B.L.Hu and H.E.Kandrup, Phys.Rev.D35(1987)1776;
   H.E.Kandrup, Phys.Lett.B185(1987)382;
   H.E.Kandrup, Phys.Lett.B202(1988)207;
   H.E.Kandrup, J.Math.Phys.28(1987)1398
5. H.E.Kandrup, Phys.Rev.D37(1988)3505
6. L.P.Grishchuk and Y.V.Sidorov, Phys.Rev.D42(1990)3413;
   L.P.Grishchuk, "Quantum mechanics of the primordial cosmological perturbations", in Proc of the 6th Marcel Grossmann Meeting (Kyoto, June 1991);
   L.P.Grishchuk, "Squeezed states in the theory of primordial gravitational waves", in Proc. of the Workshop on squeezed states and uncertainty relations (Maryland Univ.), ed. by D.Han, Y.S.Kim and W.W.Zachary (Nasa Conf. Pub. No.3135, 1992) p.329
7. L.P.Grishchuk and Y.V.Sidorov, Class. Quantum Grav.6(1991) L161;
   L.P.Grishchuk, in Proc of the VIth Brazilian School on cosmology and gravitation (Rio de Janeiro, July 1989)
8. J.Grochmalicki and M.Lewenstein, Phys.Rep.208(1991)189
9. B.L.Schumaker, Phys.Rep.135(1986)317
10. R.Brandenbeger, V.Mukhanov and T.Prokopec, "Entropy of a classical stochastic field and cosmological perturbations", BROWN-HET-859 (June 1992)
11. T.Prokopec, "Entropy of the squeezed vacuum", BROWN-HET-861 (June 1992)
12. M.Gasperini, Prog.Theor.Phys.84(1990)899
13. M.Gasperini and M.Giovannini, Phys.Lett.B282(1992)36;
    "Dilaton contributions to the cosmic gravitational wave background",

DFTT-58/92 (to appear in Phys.Rev.D)
14. B.Allen, Phys.Rev.D37(1988)2078
15. V.Sanhi, Phys.Rev.D42(1990)453
16. V.A.Rubakov, M.V.Sazhin and A.V.Veryanskin, Phys.Lett.B115 (1982)189; R.Fabbri and M.D.Pollock, Phys.Lett.B125(1983)445; L.F.Abbot and M.B.Wise, Nucl.Phys.B244(1984)541
17. M.Gasperini and M.Giovannini, Class. Quantum Grav.9(1992)L137
18. M.Gasperini and G.Veneziano, "Pre-big-bang in string cosmology", CERN TH.6572/92 (July 1992)
19. M.Gasperini, N.Sanchez and G.Veneziano, Nucl.Phys.B364(1991) 365; G.Veneziano, Phys.Lett.B265(1991)287; M.Gasperini, J.Maharana and G.Veneziano, Phys.Lett.B272(1991)277; M.Gasperini and G.Veneziano, Phys.Lett.B277(1992)256; G.Veneziano, in Proc of the 6th Int. Workshop on Theor.Phys. "String quantum gravity" (Erice, June 1992), ed. by N.Sanchez (World Scientific, Singapore)