A Dirac gamma matrices

This section collects the essential identities involving Dirac gamma matrices that are needed for QED cross-section calculations. We work in $d = 4 - 2\epsilon$ dimensions for compatibility with dimensional regularization.

Box A.1 Convention notice: $d = 4 - 2\epsilon$ vs $d = 4 - \varepsilon$

Important. Two slightly different conventions for the dimensional regulator coexist in these notes. To avoid factor-of-two confusion in UV poles, here is the explicit dictionary.

Definitions:

- **This appendix** (Aitchison/Peskin–Schroeder style): $d = 4 - 2\epsilon_{\text{app}}$, so $\epsilon_{\text{app}} = (4-d)/2$.
- **Chapters 6–7 (renormalization) and Sec. 4 of Ch. 4 (Feynman rules)**: $d = 4 - \varepsilon_{\text{ch}}$, so $\varepsilon_{\text{ch}} = 4 - d$.

Explicit translation bridge:

$$\boxed{\varepsilon_{\text{ch}} = 2\,\epsilon_{\text{app}}\,, \qquad \frac{1}{\epsilon_{\text{app}}} = \frac{2}{\varepsilon_{\text{ch}}}} \tag{A.1}$$

Equivalently, a residue/log expansion in the appendix written as $\Gamma(\epsilon_{\text{app}}) = 1/\epsilon_{\text{app}} - \gamma_E + O(\epsilon_{\text{app}})$ becomes $\Gamma(\varepsilon_{\text{ch}}/2) = 2/\varepsilon_{\text{ch}} - \gamma_E + O(\varepsilon_{\text{ch}})$ in the renormalization chapters.

Practical rule. When importing a formula from the appendix into a renormalization calculation (or vice versa), substitute $\epsilon_{\text{app}} \to \varepsilon_{\text{ch}}/2$ (and accordingly $1/\epsilon_{\text{app}} \to 2/\varepsilon_{\text{ch}}$). The $\ln(\mu^2/m^2)$ accompanying the pole is *unaffected* by the rescaling (the factor of 2 in the pole residue is compensated by the half-shift in d when expanding μ^{d-4}).

Clifford algebra and basic properties

The gamma matrices γ^μ satisfy the **Clifford algebra**:

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\mathbb{1}}\,. \tag{A.2}$$

In d dimensions, $g^{\mu\nu} = \mathrm{diag}(+1, -1, -1, \ldots, -1)$ with $g^\mu{}_\mu = d$.

Hermiticity properties (in the Dirac representation):

$$(\gamma^0)^\dagger = \gamma^0\,, \qquad (\gamma^i)^\dagger = -\gamma^i \quad (i = 1, 2, 3)\,, \tag{A.3}$$

or equivalently:

$$\boxed{(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0}\,. \tag{A.4}$$

Slash notation: For any four-vector a^μ, we define

$$\not{a} \equiv a_\mu\gamma^\mu = a^0\gamma^0 - a^1\gamma^1 - a^2\gamma^2 - a^3\gamma^3\,. \tag{A.5}$$

From the Clifford algebra, it follows that:

$$\boxed{\not{a}\not{b} + \not{b}\not{a} = 2(a\cdot b)\mathbb{1}}\,, \tag{A.6}$$

and in particular:

$$\not{a}\not{a} = a^2\mathbb{1}\,. \tag{A.7}$$

Contraction identities

The following identities are essential for simplifying products of gamma matrices. We derive them systematically from the Clifford algebra (A.2).

Identity 1: $\gamma^\mu\gamma_\mu = d$
Proof. Contract the Clifford algebra with $g_{\mu\nu}$:

$$\begin{aligned}\gamma^\mu\gamma_\mu &= g_{\mu\nu}\gamma^\mu\gamma^\nu = \tfrac{1}{2}g_{\mu\nu}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)\\ &= \tfrac{1}{2}g_{\mu\nu}\cdot 2g^{\mu\nu} = g^\mu{}_\mu = d\,.\end{aligned} \qquad \square$$

Identity 2: $\gamma^\mu\gamma^\nu\gamma_\mu = (2-d)\gamma^\nu$
Proof. Use the Clifford algebra to move γ_μ past γ^ν:

$$\begin{aligned}\gamma^\mu\gamma^\nu\gamma_\mu &= \gamma^\mu(2g^\nu{}_\mu - \gamma_\mu\gamma^\nu)\\ &= 2\gamma^\nu - \gamma^\mu\gamma_\mu\gamma^\nu = 2\gamma^\nu - d\gamma^\nu = (2-d)\gamma^\nu\,.\end{aligned} \qquad \square$$

Identity 3: $\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4g^{\nu\rho} - (4-d)\gamma^\nu\gamma^\rho$
Proof. Apply Identity 2 twice:

$$\begin{aligned}\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu &= \gamma^\mu\gamma^\nu(2g^\rho{}_\mu - \gamma_\mu\gamma^\rho) = 2\gamma^\rho\gamma^\nu - \gamma^\mu\gamma^\nu\gamma_\mu\gamma^\rho \\ &= 2\gamma^\rho\gamma^\nu - (2-d)\gamma^\nu\gamma^\rho \\ &= 2(2g^{\nu\rho} - \gamma^\nu\gamma^\rho) - (2-d)\gamma^\nu\gamma^\rho \\ &= 4g^{\nu\rho} - 2\gamma^\nu\gamma^\rho - (2-d)\gamma^\nu\gamma^\rho = 4g^{\nu\rho} - (4-d)\gamma^\nu\gamma^\rho.\end{aligned}$$

□

Identity 4: $\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu + (4-d)\gamma^\nu\gamma^\rho\gamma^\sigma$
Proof. Apply the previous identities:

$$\begin{aligned}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu &= \gamma^\mu\gamma^\nu\gamma^\rho(2g^\sigma{}_\mu - \gamma_\mu\gamma^\sigma) = 2\gamma^\sigma\gamma^\nu\gamma^\rho - \gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu\gamma^\sigma \\ &= 2\gamma^\sigma\gamma^\nu\gamma^\rho - [4g^{\nu\rho} - (4-d)\gamma^\nu\gamma^\rho]\gamma^\sigma \\ &= 2\gamma^\sigma\gamma^\nu\gamma^\rho - 4g^{\nu\rho}\gamma^\sigma + (4-d)\gamma^\nu\gamma^\rho\gamma^\sigma.\end{aligned} \tag{A.8}$$

Now use $\gamma^\sigma\gamma^\nu\gamma^\rho = 2g^{\sigma\nu}\gamma^\rho - \gamma^\nu\gamma^\sigma\gamma^\rho = 2g^{\sigma\nu}\gamma^\rho - 2g^{\sigma\rho}\gamma^\nu + \gamma^\nu\gamma^\rho\gamma^\sigma$, but a simpler approach is to note that $\gamma^\sigma\gamma^\nu\gamma^\rho = -\gamma^\sigma\gamma^\rho\gamma^\nu + 2g^{\nu\rho}\gamma^\sigma$. Thus:

$$\begin{aligned}&= 2(-\gamma^\sigma\gamma^\rho\gamma^\nu + 2g^{\nu\rho}\gamma^\sigma) - 4g^{\nu\rho}\gamma^\sigma + (4-d)\gamma^\nu\gamma^\rho\gamma^\sigma \\ &= -2\gamma^\sigma\gamma^\rho\gamma^\nu + (4-d)\gamma^\nu\gamma^\rho\gamma^\sigma.\end{aligned}$$

□

In $d = 4$ **dimensions**, these simplify to:

$$\boxed{\begin{aligned}\gamma^\mu\gamma_\mu &= 4 \\ \gamma^\mu\gamma^\nu\gamma_\mu &= -2\gamma^\nu \\ \gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu &= 4g^{\nu\rho} \\ \gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu &= -2\gamma^\sigma\gamma^\rho\gamma^\nu.\end{aligned}} \tag{A.9}$$

Contractions with slashed vectors follow immediately by replacing $\gamma^\nu \to \not{a} = a_\nu\gamma^\nu$:

$$\gamma^\mu\not{a}\gamma_\mu = (2-d)\not{a} = -2\not{a} \quad (\text{in } d=4) \tag{A.10}$$

$$\gamma^\mu\not{a}\not{b}\gamma_\mu = 4(a\cdot b) - (4-d)\not{a}\not{b} = 4(a\cdot b) \quad (\text{in } d=4) \tag{A.11}$$

$$\begin{aligned}\gamma^\mu\not{a}\not{b}\not{c}\gamma_\mu &= -2\not{c}\not{b}\not{a} + (4-d)\not{a}\not{b}\not{c} \\ &= -2\not{c}\not{b}\not{a} \quad (\text{in } d=4).\end{aligned} \tag{A.12}$$

Trace identities

The trace of gamma matrices is fundamental for computing spin-summed amplitudes squared. The gamma matrices are 4×4 in four dimensions (or $2^{d/2}\times2^{d/2}$ in d dimensions, analytically continued).

Basic trace rules:

$$\boxed{\begin{aligned} \mathrm{Tr}(\mathbb{1}) &= 4 \quad (\text{or } 2^{d/2} \text{ in } d \text{ dim.}) \\ \mathrm{Tr}(\text{odd \# of } \gamma\text{'s}) &= 0 \end{aligned}} \tag{A.13}$$

Proof that the trace of an odd number of gamma matrices vanishes. In $d = 4$, we use the fact that γ^5 anticommutes with all γ^μ and $(\gamma^5)^2 = \mathbb{1}$. For any string Γ of n gamma matrices:

$$\mathrm{Tr}(\Gamma) = \mathrm{Tr}(\gamma^5\gamma^5\Gamma) = \mathrm{Tr}(\gamma^5\Gamma\gamma^5) = (-1)^n\mathrm{Tr}(\gamma^5\gamma^5\Gamma) = (-1)^n\mathrm{Tr}(\Gamma), \tag{A.14}$$

where we used the cyclic property of the trace and moved γ^5 past n gamma matrices, picking up $(-1)^n$. For odd n, this gives $\mathrm{Tr}(\Gamma) = -\mathrm{Tr}(\Gamma)$, hence $\mathrm{Tr}(\Gamma) = 0$. □

Trace of two gamma matrices:

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}\,. \tag{A.15}$$

Proof. From the Clifford algebra, $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\mathbb{1}$. Taking the trace and using $\mathrm{Tr}(\gamma^\mu\gamma^\nu) = \mathrm{Tr}(\gamma^\nu\gamma^\mu)$ (cyclic property):

$$2\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 2g^{\mu\nu}\mathrm{Tr}(\mathbb{1}) = 8g^{\mu\nu} \implies \mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}.$$ □

Trace of four gamma matrices:

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})\,. \tag{A.16}$$

Proof. Use the Clifford algebra to anticommute γ^μ through:

$$\begin{aligned} \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= \mathrm{Tr}[(2g^{\mu\nu} - \gamma^\nu\gamma^\mu)\gamma^\rho\gamma^\sigma] \\ &= 2g^{\mu\nu}\mathrm{Tr}(\gamma^\rho\gamma^\sigma) - \mathrm{Tr}(\gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma). \end{aligned} \tag{A.17}$$

Continue anticommuting γ^μ in the second term:

$$\begin{aligned} \mathrm{Tr}(\gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma) &= 2g^{\mu\rho}\,\mathrm{Tr}(\gamma^\nu\gamma^\sigma) \\ &\quad - \mathrm{Tr}(\gamma^\nu\gamma^\rho\gamma^\mu\gamma^\sigma) \\ &= 2g^{\mu\rho}\,\mathrm{Tr}(\gamma^\nu\gamma^\sigma) \\ &\quad - 2g^{\mu\sigma}\,\mathrm{Tr}(\gamma^\nu\gamma^\rho) \\ &\quad + \mathrm{Tr}(\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu). \end{aligned} \tag{A.18}$$

By the cyclic property of the trace,

$$\mathrm{Tr}(\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu) = \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma)\,.$$

Denoting $T \equiv \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma)$:

$$T = 2g^{\mu\nu}\cdot 4g^{\rho\sigma} - 2g^{\mu\rho}\cdot 4g^{\nu\sigma} + 2g^{\mu\sigma}\cdot 4g^{\nu\rho} - T. \tag{A.19}$$

Solving: $2T = 8(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$, hence the result. □

The result (A.16) can be written more symmetrically as:

$$\boxed{\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right)}. \tag{A.20}$$

Traces with slashed vectors follow by contraction with the momenta:

$$\mathrm{Tr}(\not{a}\not{b}) = a_\mu b_\nu \mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4a_\mu b_\nu g^{\mu\nu} = 4(a\cdot b) \tag{A.21}$$

$$\mathrm{Tr}(\not{a}\not{b}\not{c}\not{d}) = 4\big[(a\cdot b)(c\cdot d) - (a\cdot c)(b\cdot d) + (a\cdot d)(b\cdot c)\big]. \tag{A.22}$$

Remark. [Recursive trace formula] The trace of $2n$ gamma matrices can be computed recursively:

$$\mathrm{Tr}(\gamma^{\mu_1}\cdots\gamma^{\mu_{2n}}) = \sum_{k=2}^{2n}(-1)^k g^{\mu_1\mu_k}\mathrm{Tr}(\gamma^{\mu_2}\cdots\hat{\gamma}^{\mu_k}\cdots\gamma^{\mu_{2n}}) \tag{A.23}$$

where $\hat{\gamma}^{\mu_k}$ means that γ^{μ_k} is omitted. This follows by repeatedly anticommuting γ^{μ_1} through the string using the Clifford algebra, then using the cyclic property of the trace.

Trace of six gamma matrices can be computed using the recursive formula. The result is:

$$\begin{aligned}\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\alpha\gamma^\beta) = 4\big[&g^{\mu\nu}(g^{\rho\sigma}g^{\alpha\beta} - g^{\rho\alpha}g^{\sigma\beta} + g^{\rho\beta}g^{\sigma\alpha})\\ &- g^{\mu\rho}(g^{\nu\sigma}g^{\alpha\beta} - g^{\nu\alpha}g^{\sigma\beta} + g^{\nu\beta}g^{\sigma\alpha})\\ &+ g^{\mu\sigma}(g^{\nu\rho}g^{\alpha\beta} - g^{\nu\alpha}g^{\rho\beta} + g^{\nu\beta}g^{\rho\alpha})\\ &- g^{\mu\alpha}(g^{\nu\rho}g^{\sigma\beta} - g^{\nu\sigma}g^{\rho\beta} + g^{\nu\beta}g^{\rho\sigma})\\ &+ g^{\mu\beta}(g^{\nu\rho}g^{\sigma\alpha} - g^{\nu\sigma}g^{\rho\alpha} + g^{\nu\alpha}g^{\rho\sigma})\big]\end{aligned} \tag{A.24}$$

The γ^5 matrix

In four dimensions, the chiral matrix γ^5 is defined as:

$$\boxed{\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma}, \tag{A.25}$$

where $\epsilon^{0123} = +1$, consistently with Chapter 9.

Properties of γ^5:

$$\boxed{\begin{aligned} (\gamma^5)^2 &= \mathbb{1} \\ (\gamma^5)^\dagger &= \gamma^5 \\ \{\gamma^5, \gamma^\mu\} &= 0 \quad \text{(anticommutes with all } \gamma^\mu\text{)} \end{aligned}} \tag{A.26}$$

Proof of $(\gamma^5)^2 = \mathbb{1}$. Direct computation:

$$(\gamma^5)^2 = (i\gamma^0\gamma^1\gamma^2\gamma^3)(i\gamma^0\gamma^1\gamma^2\gamma^3) = -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3. \tag{A.27}$$

Move γ^0 to the left using $\gamma^3\gamma^0 = -\gamma^0\gamma^3$, etc. Each anticommutation gives a minus sign, and we need to move γ^0 past $\gamma^3, \gamma^2, \gamma^1$: three minus signs. Then $\gamma^0\gamma^0 = \mathbb{1}$:

$$= -(-1)^3\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 = \gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3. \tag{A.28}$$

Similarly, move γ^1 past γ^3, γ^2 (two minus signs) and use $\gamma^1\gamma^1 = -\mathbb{1}$:

$$= (-1)^2(-1)\gamma^2\gamma^3\gamma^2\gamma^3 = -\gamma^2\gamma^3\gamma^2\gamma^3. \tag{A.29}$$

Move γ^2 past γ^3 (one minus sign) and use $\gamma^2\gamma^2 = -\mathbb{1}$:

$$= -(-1)(-1)\gamma^3\gamma^3 = -\gamma^3\gamma^3 = -(-\mathbb{1}) = \mathbb{1}. \qquad \square$$

Proof of $(\gamma^5)^\dagger = \gamma^5$. Using $(\gamma^0)^\dagger = \gamma^0$ and $(\gamma^i)^\dagger = -\gamma^i$:

$$\begin{aligned} (\gamma^5)^\dagger &= (i\gamma^0\gamma^1\gamma^2\gamma^3)^\dagger = -i(\gamma^3)^\dagger(\gamma^2)^\dagger(\gamma^1)^\dagger(\gamma^0)^\dagger \\ &= -i(-\gamma^3)(-\gamma^2)(-\gamma^1)(\gamma^0) = i\gamma^3\gamma^2\gamma^1\gamma^0. \end{aligned} \tag{A.30}$$

Now anticommute to restore the original order. Moving γ^0 to the front requires passing it through $\gamma^1, \gamma^2, \gamma^3$ (three anticommutations, giving $(-1)^3$):

$$= i(-1)^3\gamma^0\gamma^3\gamma^2\gamma^1 = -i\gamma^0\gamma^3\gamma^2\gamma^1. \tag{A.31}$$

Moving γ^1 to the second position (past γ^2, γ^3, two anticommutations):

$$= -i(-1)^2\gamma^0\gamma^1\gamma^3\gamma^2 = -i\gamma^0\gamma^1\gamma^3\gamma^2. \tag{A.32}$$

Finally, swap γ^2 and γ^3:

$$= -i(-1)\gamma^0\gamma^1\gamma^2\gamma^3 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5. \qquad \square$$

Proof of $\{\gamma^5, \gamma^\mu\} = 0$. For $\mu = 0$:

$$\gamma^5\gamma^0 = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0 = i\gamma^0\gamma^1\gamma^2(-\gamma^0\gamma^3) = \cdots = -i\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^0\gamma^5. \tag{A.33}$$

Each spatial gamma anticommutes with γ^0, giving three minus signs total for the three anticommutations needed to bring γ^0 to the front. For $\mu = i$ (spatial), the factor γ^i already appears in γ^5. Moving γ^i through γ^5 produces an odd number of anticommutations (with the other spatial gammas and γ^0), giving an overall minus sign. Thus $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$ for all μ. □

Chirality projectors:

$$P_L = \frac{1-\gamma^5}{2}, \qquad P_R = \frac{1+\gamma^5}{2}, \tag{A.34}$$

satisfying $P_L + P_R = \mathbb{1}$, $P_L^2 = P_L$, $P_R^2 = P_R$, and $P_L P_R = 0$.
Proof of projector properties. The sum is obvious. For P_L^2:

$$P_L^2 = \frac{(1-\gamma^5)^2}{4} = \frac{1-2\gamma^5+(\gamma^5)^2}{4} = \frac{2-2\gamma^5}{4} = \frac{1-\gamma^5}{2} = P_L.$$ □

Trace identities involving γ^5:

$$\mathrm{Tr}(\gamma^5) = 0 \tag{A.35}$$

$$\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu) = 0 \tag{A.36}$$

$$\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = -4i\epsilon^{\mu\nu\rho\sigma}\,. \tag{A.37}$$

Proof of Tr$(\gamma^5) = 0$. Write $\gamma^5 = \gamma^5\gamma^\mu\gamma_\mu/4$ (using $\gamma^\mu\gamma_\mu = 4$). Then:

$$\mathrm{Tr}(\gamma^5) = \frac{1}{4}\mathrm{Tr}(\gamma^5\gamma^\mu\gamma_\mu) = \frac{1}{4}\mathrm{Tr}(\gamma_\mu\gamma^5\gamma^\mu) = -\frac{1}{4}\mathrm{Tr}(\gamma^5\gamma_\mu\gamma^\mu) = -\mathrm{Tr}(\gamma^5), \tag{A.38}$$

where we used the cyclic property and $\gamma_\mu\gamma^5 = -\gamma^5\gamma_\mu$. Hence $\mathrm{Tr}(\gamma^5) = 0$. □
Proof of Tr$(\gamma^5\gamma^\mu\gamma^\nu) = 0$. The trace has three gamma matrices plus $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, giving a total of 7 gamma matrices. Since the trace of an odd number of gamma matrices vanishes (proven above), this trace is zero. □
Proof of Tr$(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = -4i\epsilon^{\mu\nu\rho\sigma}$. The trace vanishes unless all four indices are distinct (otherwise, use the Clifford algebra to reduce the number of gamma matrices, giving a trace with γ^5 and fewer than 4 gammas). For distinct indices, the trace is completely antisymmetric (anticommuting two gammas past each other gives a minus sign). The only completely antisymmetric tensor is $\epsilon^{\mu\nu\rho\sigma}$, so:

$$\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = c\cdot\epsilon^{\mu\nu\rho\sigma}. \tag{A.39}$$

To find c, evaluate $\mathrm{Tr}(\gamma^5\gamma^0\gamma^1\gamma^2\gamma^3)$:

$$\mathrm{Tr}(\gamma^5\gamma^0\gamma^1\gamma^2\gamma^3) = \mathrm{Tr}\Big(i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3\Big) = -i\,\mathrm{Tr}(\mathbb{1}) = -4i. \tag{A.40}$$

Since $\epsilon^{0123} = +1$, we have $c = -4i$. □

More generally, traces with γ^5 and fewer than four other gamma matrices vanish:

$$\mathrm{Tr}(\gamma^5\gamma^{\mu_1}\cdots\gamma^{\mu_n}) = 0 \quad \text{for } n < 4\,. \tag{A.41}$$

With slashed vectors:

$$\boxed{\mathrm{Tr}(\gamma^5 \not{a}\not{b}\not{c}\not{d}) = -4i\epsilon^{\mu\nu\rho\sigma}a_\mu b_\nu c_\rho d_\sigma = -4i\epsilon(a,b,c,d)} \tag{A.42}$$

where $\epsilon(a,b,c,d) \equiv \epsilon^{\mu\nu\rho\sigma}a_\mu b_\nu c_\rho d_\sigma$ is the fully antisymmetric contraction.

Remark. [γ^5 in dimensional regularization] The treatment of γ^5 in $d \neq 4$ dimensions is subtle because γ^5 is intrinsically four-dimensional. Common prescriptions include:

- **Naive dimensional regularization (NDR):** Assume $\{\gamma^5, \gamma^\mu\} = 0$ holds in d dimensions. This is algebraically inconsistent but works for many calculations.
- **'t Hooft–Veltman scheme:** Split $\gamma^\mu = \hat{\gamma}^\mu + \tilde{\gamma}^\mu$ where $\hat{\gamma}^\mu$ is 4-dimensional and $\tilde{\gamma}^\mu$ lives in the extra $(d-4)$ dimensions. Then $\{\gamma^5, \hat{\gamma}^\mu\} = 0$ but $[\gamma^5, \tilde{\gamma}^\mu] = 0$.

For QED calculations without axial currents, these subtleties can often be avoided.

Gordon identity and related decompositions

The **Gordon identity** expresses a vector current in terms of a convection current and a spin current:

$$\boxed{\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{(p'+p)^\mu}{2m} + \frac{i\sigma^{\mu\nu}(p'-p)_\nu}{2m}\right]u(p)}\,, \tag{A.43}$$

where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$ and the spinors satisfy the Dirac equation.

Proof: Start from $\bar{u}(p')(\not{p}' - m) = 0$ and $(\not{p} - m)u(p) = 0$, which imply $\bar{u}(p')\not{p}' = m\bar{u}(p')$ and $\not{p}\,u(p) = m\,u(p)$. Then:

$$\begin{aligned}
\bar{u}(p')\gamma^\mu u(p) &= \frac{1}{2m}\bar{u}(p')(\not{p}'\gamma^\mu + \gamma^\mu\not{p})u(p) \\
&= \frac{1}{2m}\bar{u}(p')\left[p'^\mu + p^\mu - \frac{1}{2}[\gamma^\mu, \gamma^\nu](p'-p)_\nu\right]u(p) \\
&= \bar{u}(p')\left[\frac{(p'+p)^\mu}{2m} + \frac{i\sigma^{\mu\nu}(p'-p)_\nu}{2m}\right]u(p).
\end{aligned} \tag{A.44}$$

In the last step we used $[\gamma^\mu, \gamma^\nu] = -2i\sigma^{\mu\nu}$, which follows from $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$.

Generalized Gordon identities:

$$\bar{u}(p')\gamma^\mu\gamma^5 u(p) = \bar{u}(p')\left[\frac{(p'-p)^\mu}{2m}\gamma^5 + \frac{i\sigma^{\mu\nu}(p'+p)_\nu}{2m}\gamma^5\right]u(p)\,. \tag{A.45}$$

Fierz identities

The **Fierz identities** allow reordering of spinor bilinears. The key observation is that the 16 matrices

$$\Gamma_A \in \{\mathbb{1}, \gamma^\mu, \sigma^{\mu\nu}, \gamma^\mu\gamma^5, \gamma^5\} \quad (A = 1, \ldots, 16)\,, \tag{A.46}$$

form a complete basis for 4×4 matrices, satisfying the orthogonality relation $\mathrm{Tr}(\Gamma_A\Gamma_B) = 4\delta_{AB}$ (with appropriate index handling).

Derivation. Any 4×4 matrix M can be expanded as $M = \sum_A c_A\Gamma_A$ with $c_A = \frac{1}{4}\mathrm{Tr}(M\Gamma_A)$. In particular, applying this to the outer product $(\psi_4)_\alpha(\bar{\psi}_3)_\beta$ (viewed as a matrix in spinor space):

$$(\psi_4)_\alpha(\bar{\psi}_3)_\beta = \frac{1}{4}\sum_A (\Gamma_A)_{\alpha\beta}(\bar{\psi}_3\Gamma_A\psi_4)\,. \tag{A.47}$$

Contracting with $(\bar{\psi}_1)^\alpha(\Gamma_B)^\beta{}_\gamma(\psi_2)^\gamma$ yields the general Fierz rearrangement:

$$(\bar{\psi}_1\Gamma_B\psi_2)(\bar{\psi}_3\Gamma^A\psi_4) = \sum_C C_{BC}(\bar{\psi}_1\Gamma_C\psi_4)(\bar{\psi}_3\Gamma^C\psi_2)\,, \tag{A.48}$$

where the Fierz coefficients C_{BC} are determined by the algebra of gamma matrices.

The Fierz matrix for vector currents is particularly useful:

$$(\bar{\psi}_1\gamma^\mu\psi_2)(\bar{\psi}_3\gamma_\mu\psi_4) = -(\bar{\psi}_1\gamma^\mu\psi_4)(\bar{\psi}_3\gamma_\mu\psi_2) + \text{scalar and tensor terms} \tag{A.49}$$

Summary: most frequently used identities

For quick reference, here are the identities used most often in QED calculations.

Identity	Result
Contraction identities (in d = 4)	
$\gamma^\mu \gamma_\mu$	4
$\gamma^\mu \gamma^\nu \gamma_\mu$	$-2\gamma^\nu$
$\gamma^\mu \not{a} \gamma_\mu$	$-2\not{a}$
$\gamma^\mu \not{a} \not{b} \gamma_\mu$	$4(a \cdot b)$
Trace identities	
$\mathrm{Tr}(\mathbb{1})$	4
$\mathrm{Tr}(\gamma^\mu \gamma^\nu)$	$4g^{\mu\nu}$
$\mathrm{Tr}(\not{a} \not{b})$	$4(a \cdot b)$
$\mathrm{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma)$	$4(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho})$
$\mathrm{Tr}(\not{a} \not{b} \not{c} \not{d})$	$4[(a \cdot b)(c \cdot d) - (a \cdot c)(b \cdot d) + (a \cdot d)(b \cdot c)]$
Traces with γ^5	
$\mathrm{Tr}(\text{odd \# of } \gamma)$	0
$\mathrm{Tr}(\gamma^5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma)$	$-4i\epsilon^{\mu\nu\rho\sigma}$

B Choice of representation for Dirac matrices

The Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}$ defines the gamma matrices only up to a similarity transformation: if γ^μ satisfies the algebra, so does $S\gamma^\mu S^{-1}$ for any invertible matrix S. This freedom leads to several standard *representations* (or *bases*), each with distinct advantages. In these notes we adopt the **Weyl (chiral) representation**, which is optimal for relativistic and high-energy physics. This appendix explains the main representations and justifies our choice.

The Dirac (standard) representation

The Dirac representation was historically the first to be introduced and remains common in textbooks emphasizing non-relativistic limits. The matrices are:

$$\gamma^0_{\mathrm{D}} = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \qquad \gamma^i_{\mathrm{D}} = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \qquad \gamma^5_{\mathrm{D}} = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}, \tag{B.1}$$

where σ^i are the Pauli matrices. The key property is that γ^0 is diagonal.

Advantages:

- In the non-relativistic limit, the upper two components of a positive-energy Dirac spinor dominate ("large components"), while the lower two are suppressed by $O(v/c)$ ("small components"). This makes the connection to Pauli spinors transparent.
- The Dirac equation in this basis directly exhibits the non-relativistic reduction to the Pauli equation.

Disadvantages:

- The chiral projectors $P_{L,R} = \frac{1}{2}(1 \mp \gamma^5)$ mix upper and lower components, obscuring the separation of left- and right-handed degrees of freedom.
- Less natural for ultra-relativistic particles or massless fermions.

The Weyl (chiral) representation

The Weyl representation is defined by making γ^5 diagonal:

$$\boxed{\gamma^0_{\mathrm{W}} = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}, \qquad \gamma^i_{\mathrm{W}} = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \qquad \gamma^5_{\mathrm{W}} = \begin{pmatrix} -\mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix}.} \tag{B.2}$$

This can be written compactly using the sigma matrices $\sigma^\mu = (\mathbb{1}_2, \vec{\sigma})$ and $\bar{\sigma}^\mu = (\mathbb{1}_2, -\vec{\sigma})$:

$$\gamma^\mu_{\mathrm{W}} = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}. \tag{B.3}$$

The Dirac spinor decomposes into two-component Weyl spinors as:

$$\Psi = \begin{pmatrix} \chi_\alpha \\ \eta^{\dagger\dot{\alpha}} \end{pmatrix} \equiv \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \tag{B.4}$$

where $\psi_L = \chi_\alpha$ (upper, undotted indices) is left-handed and transforms under $(\frac{1}{2}, 0)$, while $\psi_R = \eta^{\dagger\dot{\alpha}}$ (lower, dotted indices) is right-handed and transforms under $(0, \frac{1}{2})$. The chiral projectors act simply:

$$P_L\Psi = \frac{1}{2}(1-\gamma^5)\Psi = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix}, \qquad P_R\Psi = \frac{1}{2}(1+\gamma^5)\Psi = \begin{pmatrix} 0 \\ \psi_R \end{pmatrix}. \tag{B.5}$$

Box B.1 Convention choices in the Weyl representation

There are two common conventions in the literature for the Weyl representation, differing in which chirality occupies the upper block:

Convention A (used in these notes, matching Chapters 9–10):

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} -\mathbb{1}_2 & 0 \\ 0 & +\mathbb{1}_2 \end{pmatrix}, \qquad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

Convention B (used in some QFT textbooks, e.g., Peskin & Schroeder):

$$\Psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} +\mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \qquad \gamma^\mu = \begin{pmatrix} 0 & \bar{\sigma}^\mu \\ \sigma^\mu & 0 \end{pmatrix}$$

The two conventions are related by the unitary transformation $\Psi_B = \gamma^0\Psi_A$ (which swaps upper and lower blocks). Physical observables are identical in both conventions, but intermediate expressions differ. When consulting external references, always verify which convention is being used.
Convention A is natural when building the Dirac spinor from Weyl spinors with standard index notation: the left-handed spinor χ_α carries undotted (lower) indices and sits in the upper block, while the right-handed spinor $\eta^{\dagger\dot{\alpha}}$ carries dotted (upper) indices and sits in the lower block.

Advantages of the Weyl representation:

1. **Chirality is manifest.** With γ^5 diagonal, the left- and right-handed components of a Dirac spinor are cleanly separated. This is essential for:
   - The electroweak sector of the Standard Model, where $\mathrm{SU}(2)_L$ couples only to left-handed fermions
   - Theories with chiral gauge symmetries
   - Supersymmetry, which is naturally formulated in terms of Weyl spinors
2. **Natural for massless and ultra-relativistic fermions.** In the massless limit, the Dirac equation decouples into two independent Weyl equations:

   $$i\bar{\sigma}^\mu\partial_\mu\psi_L = 0, \qquad i\sigma^\mu\partial_\mu\psi_R = 0\,. \tag{B.6}$$

   The mass term $m\bar{\Psi}\Psi = m(\psi_R^\dagger\psi_L + \psi_L^\dagger\psi_R)$ is the *only* term that couples left and right components. For high-energy processes where $E \gg m$, the chiral components propagate nearly independently.
3. **Direct connection to the Lorentz group.** The two-component Weyl spinors transform under definite representations of $\mathrm{SL}(2,\mathbb{C})$:

   $$\psi_L = \chi_\alpha \sim \left(\tfrac{1}{2},0\right) \text{ under } M, \qquad \psi_R = \eta^{\dagger\dot{\alpha}} \sim \left(0,\tfrac{1}{2}\right) \text{ under } (M^{-1})^\dagger\,. \tag{B.7}$$

   This makes the spinor transformation properties under boosts and rotations transparent (see Chapter 9).
4. **Simplified structure for QED and Standard Model calculations.** In the Weyl basis, the vertex $\bar{\Psi}\gamma^\mu\Psi$ decomposes as:

   $$\bar{\Psi}\gamma^\mu\Psi = \psi_R^\dagger\sigma^\mu\psi_R + \psi_L^\dagger\bar{\sigma}^\mu\psi_L\,. \tag{B.8}$$

   The off-diagonal form of γ^μ is compensated by the Dirac adjoint $\bar{\Psi} = (\psi_R^\dagger, \psi_L^\dagger)$, so the vector current is diagonal in chirality. For chiral currents (involving P_L or P_R), each term involves only one chirality, simplifying the bookkeeping in Feynman diagram calculations.

5. **Essential for supersymmetry.** The SUSY algebra naturally pairs a Weyl spinor with a complex scalar into a chiral supermultiplet. The superspace formalism and superpotential are most elegantly expressed using two-component notation.

Disadvantages:

- In the non-relativistic limit, neither the upper nor lower component dominates; both contribute comparably. The reduction to Pauli spinors is less direct.
- Some older textbooks and atomic physics literature use the Dirac representation, requiring translation.

The Majorana representation

The Majorana representation is defined by requiring all gamma matrices to be purely imaginary:

$$\gamma^0_{\mathrm{M}} = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \quad \gamma^1_{\mathrm{M}} = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix}, \quad \gamma^2_{\mathrm{M}} = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \quad \gamma^3_{\mathrm{M}} = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix}. \tag{B.9}$$

Advantages:

- The Dirac operator $i\partial\!\!\!/ - m$ is purely real, so the Dirac equation admits real solutions. These correspond to **Majorana fermions**, particles that are their own antiparticles.
- Essential for studying Majorana mass terms and see-saw mechanisms in neutrino physics.

Disadvantages:

- Chirality is not manifest (γ^5 is not diagonal).
- Less convenient for standard QED and Standard Model calculations.

Transformation between representations

The representations are related by unitary similarity transformations. For Convention A (left-handed upper, as used in these notes):

$$U_{\mathrm{D}\to\mathrm{W}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1}_2 & -\mathbb{1}_2 \\ \mathbb{1}_2 & \mathbb{1}_2 \end{pmatrix}, \tag{B.10}$$

we have $\gamma^\mu_{\rm W} = U_{\rm D\to W}\gamma^\mu_{\rm D} U^\dagger_{\rm D\to W}$. The spinors transform correspondingly: $\Psi_{\rm W} = U_{\rm D\to W}\Psi_{\rm D}$.

Remark. [Transformation to Convention B] For Convention B (right-handed upper, as in Peskin & Schroeder), the transformation matrix is instead:

$$U_{{\rm D\to W}_B} = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1}_2 & \mathbb{1}_2 \\ -\mathbb{1}_2 & \mathbb{1}_2 \end{pmatrix}. \tag{B.11}$$

The two conventions are related by $U_{{\rm D\to W}_A} = \gamma^0_{\rm D} \cdot U_{{\rm D\to W}_B}$, which swaps the upper and lower blocks.

Physical observables are representation-independent. Traces, contractions, and matrix elements computed in any representation yield identical results. The choice of representation is purely a matter of convenience for the problem at hand.

Summary: why we use the Weyl representation

Property	Dirac	Weyl	Majorana
γ^5 diagonal	No	**Yes**	No
γ^0 diagonal	Yes	No	No
All γ^μ imaginary	No	No	Yes
Chiral components separated	No	**Yes**	No
Non-relativistic limit clear	Yes	No	No
Massless limit natural	No	**Yes**	No
SUSY/Weyl spinor connection	Indirect	**Direct**	Indirect

For relativistic quantum field theory, the electroweak Standard Model, and supersymmetry, the Weyl representation is the natural choice. The diagonal γ^5 makes chirality explicit, the decomposition into Weyl spinors is immediate, and the massless limit is transparent. We adopt this convention throughout these notes.

Explicit identities in the Weyl basis

For reference, we collect useful identities specific to the Weyl representation with Convention A (left-handed upper).

Charge conjugation matrix:

$$C = i\gamma^2\gamma^0 = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{pmatrix} = \begin{pmatrix} -\epsilon & 0 \\ 0 & \epsilon \end{pmatrix}, \tag{B.12}$$

where $\epsilon = -i\sigma^2$ is the antisymmetric 2×2 matrix with $\epsilon^{12} = +1$.

Dirac adjoint:

$$\bar{\Psi} = \Psi^\dagger\gamma^0 = (\psi_L^\dagger, \psi_R^\dagger)\begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix} = (\psi_R^\dagger, \psi_L^\dagger)\,. \tag{B.13}$$

Mass term:

$$m\bar{\Psi}\Psi = m(\psi_R^\dagger\psi_L + \psi_L^\dagger\psi_R)\,. \tag{B.14}$$

The mass couples left to right; massless fermions have independent chiral components.

Vector current:

$$\bar{\Psi}\gamma^\mu\Psi = \psi_R^\dagger\sigma^\mu\psi_R + \psi_L^\dagger\bar{\sigma}^\mu\psi_L\,. \tag{B.15}$$

Axial current:

$$\bar{\Psi}\gamma^\mu\gamma^5\Psi = \psi_R^\dagger\sigma^\mu\psi_R - \psi_L^\dagger\bar{\sigma}^\mu\psi_L\,. \tag{B.16}$$

Left and right currents:

$$\bar{\Psi}\gamma^\mu P_L\Psi = \psi_L^\dagger\bar{\sigma}^\mu\psi_L, \qquad \bar{\Psi}\gamma^\mu P_R\Psi = \psi_R^\dagger\sigma^\mu\psi_R\,. \tag{B.17}$$

These are the currents that couple to the W and Z bosons in the electroweak theory: $\mathrm{SU}(2)_L$ gauge interactions involve only $\bar{\Psi}\gamma^\mu P_L\Psi$.

Remark. [Verification of the vector current] Using the explicit forms $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$ and $\bar{\Psi} = (\psi_R^\dagger, \psi_L^\dagger)$:

$$\bar{\Psi}\gamma^\mu\Psi = (\psi_R^\dagger, \psi_L^\dagger)\begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \psi_R^\dagger\sigma^\mu\psi_R + \psi_L^\dagger\bar{\sigma}^\mu\psi_L$$

This confirms consistency with the chiral currents above: $\bar{\Psi}\gamma^\mu\Psi = \bar{\Psi}\gamma^\mu P_L\Psi +$

$\bar{\Psi}\gamma^{\mu}P_R\Psi.$

C Weyl spinor identities

We collect here fundamental identities involving the matrices σ^μ and $\bar{\sigma}^\mu$:

$$\sigma^\mu_{\alpha\dot{\alpha}} = \varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\sigma}^{\mu\dot{\beta}\beta} \tag{C.1}$$

$$\sigma^\mu_{\alpha\dot{\alpha}}\bar{\sigma}^{\dot{\beta}\beta}_\mu = 2\delta^\beta_\alpha\delta^{\dot{\beta}}_{\dot{\alpha}} \tag{C.2}$$

$$[\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu]_\alpha{}^\beta = 2g^{\mu\nu}\delta^\beta_\alpha \tag{C.3}$$

$$[\bar{\sigma}^\mu\sigma^\nu + \bar{\sigma}^\nu\sigma^\mu]^{\dot{\alpha}}{}_{\dot{\beta}} = 2g^{\mu\nu}\delta^{\dot{\alpha}}_{\dot{\beta}}\,. \tag{C.4}$$

The matrix $p^\mu\sigma_\mu$ is Hermitian and its determinant is $|p^\mu\sigma_\mu| = p^2$. For $M \in SL(2,\mathbb{C})$, the transformation $Mp^\mu\sigma_\mu M^\dagger$ produces another four-vector p'^μ connected to p^μ by a Lorentz transformation:

$$M\sigma_\mu M^\dagger = \sigma_\nu\Lambda^\nu{}_\mu\,. \tag{C.5}$$

Object	Transformation	Index
χ_α (left-handed, $(\frac{1}{2},0)$)	M	undotted lower
χ^α	$(M^{-1})^\top$	undotted upper
$\bar{\chi}^{\dot{\alpha}}$ (right-handed, $(0,\frac{1}{2})$)	$(M^{-1})^\dagger$	dotted upper
$\bar{\chi}_{\dot{\alpha}}$	M^*	dotted lower

where $M = \exp\left(-\frac{i}{2}\vec{\theta}\cdot\vec{\sigma} - \frac{1}{2}\vec{\zeta}\cdot\vec{\sigma}\right)$ for rotation $\vec{\theta}$ and boost rapidity $\vec{\zeta}$, and $(M^{-1})^\dagger = \exp\left(-\frac{i}{2}\vec{\theta}\cdot\vec{\sigma} + \frac{1}{2}\vec{\zeta}\cdot\vec{\sigma}\right)$. Note that rotations act identically on both chiralities, while boosts act with opposite signs—this is the defining difference between $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ representations.

Contractions:

- $\chi\psi = \chi^\alpha\psi_\alpha$ (undotted: upper-lower)
- $\bar{\chi}\bar{\psi} = \bar{\chi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}$ (dotted: lower-upper)
- $\chi\sigma^\mu\bar{\psi} = \chi^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}$ (vector)
- $\bar{\chi}\bar{\sigma}^\mu\psi = \bar{\chi}_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\psi_\alpha$ (vector)

Hermitian conjugation: $(\chi\psi)^\dagger = \bar{\psi}\bar{\chi}$

D Mandelstam variables

Consider a generic $2 \to 2$ scattering process

$$a(p_1) + b(p_2) \longrightarrow c(p_3) + d(p_4), \tag{D.1}$$

where all momenta are on-shell: $p_i^2 = m_i^2$. Four-momentum conservation reads $p_1 + p_2 = p_3 + p_4$.

Definition

The three **Mandelstam variables** are defined as

$$\boxed{s \equiv (p_1 + p_2)^2, \qquad t \equiv (p_1 - p_3)^2, \qquad u \equiv (p_1 - p_4)^2.} \tag{D.2}$$

They are Lorentz scalars constructed from the external momenta alone, and therefore frame-independent.

Physical interpretation

Each variable is the squared invariant mass of the four-momentum flowing through one of the three possible channels:

- $s = (p_1 + p_2)^2 = (p_3 + p_4)^2$ is the squared centre-of-mass energy. A propagator carrying momentum $p_1 + p_2$ has $q^2 = s$ (**s-channel**: annihilation/creation).
- $t = (p_1 - p_3)^2 = (p_2 - p_4)^2$ is the squared four-momentum transfer. A propagator carrying momentum $p_1 - p_3$ has $q^2 = t$ (**t-channel**: exchange).
- $u = (p_1 - p_4)^2 = (p_2 - p_3)^2$ is the crossed momentum transfer. A propagator carrying momentum $p_1 - p_4$ has $q^2 = u$ (**u-channel**: crossed exchange).

Constraint

The three variables are not independent. Adding the definitions and using $p_i^2 = m_i^2$:

$$s + t + u = (p_1 + p_2)^2 + (p_1 - p_3)^2 + (p_1 - p_4)^2. \tag{D.3}$$

Expanding each square and using momentum conservation $p_2 = p_3 + p_4 - p_1$ to eliminate cross terms:

$$\boxed{s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2\,.} \tag{D.4}$$

For equal-mass particles ($m_i = m$) this reduces to $s + t + u = 4m^2$.

Proof. Expand each term:

$$s = p_1^2 + 2\,p_1\cdot p_2 + p_2^2 = m_1^2 + m_2^2 + 2\,p_1\cdot p_2\,, \tag{D.5}$$
$$t = p_1^2 - 2\,p_1\cdot p_3 + p_3^2 = m_1^2 + m_3^2 - 2\,p_1\cdot p_3\,, \tag{D.6}$$
$$u = p_1^2 - 2\,p_1\cdot p_4 + p_4^2 = m_1^2 + m_4^2 - 2\,p_1\cdot p_4\,. \tag{D.7}$$

Summing: $s + t + u = 3m_1^2 + m_2^2 + m_3^2 + m_4^2 + 2\,p_1\cdot(p_2 - p_3 - p_4)$. Momentum conservation gives $p_2 - p_3 - p_4 = -p_1$, so the last term becomes $2\,p_1\cdot(-p_1) = -2m_1^2$, yielding the result. □

Counting degrees of freedom

A $2 \to 2$ process with fixed external masses has $4\times 4 = 16$ momentum components, constrained by:

- 4 on-shell conditions ($p_i^2 = m_i^2$): removes 4,
- 4 momentum-conservation equations: removes 4,
- 6 Lorentz transformations (3 boosts + 3 rotations): removes 6.

This leaves $16 - 4 - 4 - 6 = 2$ independent Lorentz-invariant kinematic variables. Since $s + t + u$ is fixed, any two of $\{s, t, u\}$ form a complete set. A common choice is (s, t), with u determined by the constraint.

Centre-of-mass frame expressions

In the CM frame, $\vec{p}_1 + \vec{p}_2 = 0$, so $p_1 = (E_1, \vec{p})$ and $p_2 = (E_2, -\vec{p})$. Then:

$$s = (E_1 + E_2)^2 = E_{\mathrm{cm}}^2\,. \tag{D.8}$$

For the final state: $p_3 = (E_3, \vec{p}\,')$ and $p_4 = (E_4, -\vec{p}\,')$. Defining θ as the scattering angle between $\vec{p}$ and $\vec{p}\,'$:

$$t = m_1^2 + m_3^2 - 2(E_1E_3 - |\vec{p}\,||\vec{p}\,'|\cos\theta)\,, \tag{D.9}$$
$$u = m_1^2 + m_4^2 - 2(E_1E_4 + |\vec{p}\,||\vec{p}\,'|\cos\theta)\,. \tag{D.10}$$

Ultrarelativistic limit

When all masses are negligible ($E \gg m_i$), the kinematics simplifies dramatically. In the CM frame with $|\vec{p}\,| \approx |\vec{p}\,'| \approx E_{\rm cm}/2$:

$$\boxed{s = 4E^2\,, \qquad t = -\frac{s}{2}(1-\cos\theta)\,, \qquad u = -\frac{s}{2}(1+\cos\theta)\,,} \tag{D.11}$$

with the constraint $s+t+u=0$. The physical scattering region is

$$s \geq 0\,, \qquad t \leq 0\,, \qquad u \leq 0\,. \tag{D.12}$$

Scalar products in terms of s, t, u

All independent scalar products of external momenta can be expressed as:

$$\boxed{p_1{\cdot}p_2 = \tfrac{1}{2}(s-m_1^2-m_2^2)\,, \qquad p_1{\cdot}p_3 = \tfrac{1}{2}(m_1^2+m_3^2-t)\,, \qquad p_1{\cdot}p_4 = \tfrac{1}{2}(m_1^2+m_4^2-u)\,.} \tag{D.13}$$

The remaining products follow from momentum conservation: $p_2{\cdot}p_3 = \frac{1}{2}(m_2^2 + m_3^2-u)$, $p_2{\cdot}p_4 = \frac{1}{2}(m_2^2+m_4^2-t)$, $p_3{\cdot}p_4 = \frac{1}{2}(s-m_3^2-m_4^2)$. In the massless limit these reduce to $p_1{\cdot}p_2 = s/2$, $p_1{\cdot}p_3 = -t/2$, $p_1{\cdot}p_4 = -u/2$.

Physical regions

For the *physical* s-channel process $a\,b \to c\,d$, the ranges are:

$$s \geq (m_3+m_4)^2\,, \qquad t_- \leq t \leq t_+\,, \qquad u \text{ fixed by the constraint}\,, \tag{D.14}$$

where $t_\pm$ correspond to $\cos\theta = \pm 1$ (forward/backward scattering).

The same three variables also describe the *crossed* processes:

- s-channel: $a\,b \to c\,d$ (physical region: $s \geq (m_3+m_4)^2$, $t \leq 0$, $u \leq 0$)
- t-channel: $a\,\bar{c} \to \bar{b}\,d$ (physical region: $t \geq (m_1+m_3)^2$, $s \leq 0$, $u \leq 0$)
- u-channel: $a\,\bar{d} \to \bar{b}\,c$ (physical region: $u \geq (m_1+m_4)^2$, $s \leq 0$, $t \leq 0$)

This is the basis of **crossing symmetry**: the amplitude $\mathcal{M}(s,t,u)$ is a single analytic function of which the three physical processes occupy different kinematic regions.

Propagator denominators

In Feynman diagrams, the Mandelstam variables appear naturally in propagator denominators. For a particle of mass M exchanged in the three channels:

$$s\text{-channel: } \frac{i}{s-M^2}\,, \qquad t\text{-channel: } \frac{i}{t-M^2}\,, \qquad u\text{-channel: } \frac{i}{u-M^2}\,. \tag{D.15}$$

This is why amplitudes are naturally expressed as functions of s, t, u.

E Single-spin projectors and helicity

When computing polarized cross sections, one needs the outer product $u(\vec{p},\lambda)\bar{u}(\vec{p},\lambda)$ for a *single* helicity state, rather than the unpolarized spin sum $\sum_s u\bar{u} = (\not{p}+m)/(2m)$. The result is:

$$\boxed{u(\vec{p},\lambda)\,\bar{u}(\vec{p},\lambda) = \Lambda^+(\vec{p})\,\Pi^\lambda(\vec{p})} \tag{E.1}$$

where $\Lambda^+(p) = (\not{p}+m)/(2m)$ is the positive-energy projector and $\Pi^\lambda(\vec{p}) = \frac{1}{2}(1+\lambda\gamma^5\not{n}_p)$ is the helicity projector. We now prove this identity step by step.

The covariant spin four-vector

To write helicity in a Lorentz-covariant form, we introduce the **spin four-vector** n_p^μ, defined by three conditions:

$$n_p \cdot p = 0\,, \qquad n_p^2 = -1\,, \qquad n_p^\mu \xrightarrow{\text{rest frame}} (0,\hat{p})\,. \tag{E.2}$$

The first condition ensures that n_p^μ is a pure spatial direction in the rest frame; the second normalizes it. The unique Lorentz boost from the rest frame gives:

$$n_p^\mu = \left(\frac{|\vec{p}|}{m},\, \frac{E_p}{m}\hat{p}\right)\,. \tag{E.3}$$

Verification:

$$n_p \cdot p = \frac{|\vec{p}|}{m}E_p - \frac{E_p}{m}|\vec{p}| = 0\,, \quad \checkmark \tag{E.4}$$

$$n_p^2 = \frac{|\vec{p}|^2}{m^2} - \frac{E_p^2}{m^2} = \frac{|\vec{p}|^2 - E_p^2}{m^2} = \frac{-m^2}{m^2} = -1\,. \quad \checkmark \tag{E.5}$$

The eigenvalue equation $\gamma^5\not{n}_p u = \lambda u$

The operator $\gamma^5\not{n}_p$ is the covariant generalization of the helicity operator. To prove the eigenvalue equation, work in the rest frame where $p^\mu = (m,\vec{0})$ and $n_p^\mu = (0,\hat{p})$. Then:

$$\not{n}_p = n_{p,\mu}\gamma^\mu = -\vec{\gamma}\cdot\hat{p} = \gamma^0(-\gamma^0\vec{\gamma}\cdot\hat{p})\,. \tag{E.6}$$

The spin operator is $\vec{\Sigma} = \gamma^0 \vec{\gamma} \gamma^5$, so $\gamma^0 \vec{\gamma} = \vec{\Sigma} \gamma^5$ and:

$$\gamma^5 \not{n}_p = \gamma^5 \gamma^0 (-\gamma^0 \vec{\gamma} \cdot \hat{p}) = -\gamma^5 \gamma^0 \gamma^0 \vec{\gamma} \cdot \hat{p} = -\gamma^5 \vec{\gamma} \cdot \hat{p} \,. \tag{E.7}$$

Using $\gamma^5 \gamma^i = -\gamma^i \gamma^5$ and $\Sigma^i = \gamma^0 \gamma^i \gamma^5$, one can show that in the rest frame:

$$\gamma^5 \not{n}_p = \vec{\Sigma} \cdot \hat{p} = 2\hat{h} \,, \tag{E.8}$$

where $\hat{h} = \vec{\Sigma} \cdot \hat{p}/2$ is the helicity operator. Since $u(\vec{p}, \lambda)$ has helicity $\lambda/2$ (i.e. $\hat{h}\, u = (\lambda/2)\, u$):

$$\boxed{\gamma^5 \not{n}_p\, u(\vec{p}, \lambda) = \lambda\, u(\vec{p}, \lambda)} \tag{E.9}$$

Because this is a Lorentz-covariant equation, it holds in *any* frame, not just the rest frame. ✓

The helicity projector

Define $\Pi^\lambda(\vec{p}) = \frac{1}{2}(1 + \lambda \gamma^5 \not{n}_p)$. Using eq. (E.9):

$$\Pi^\lambda\, u(\vec{p}, \lambda') = \frac{1}{2}(1 + \lambda\lambda')\, u(\vec{p}, \lambda') = \delta_{\lambda\lambda'}\, u(\vec{p}, \lambda') \,. \tag{E.10}$$

So Π^λ selects the helicity-λ spinor and annihilates the other. It satisfies the projector properties $(\Pi^\lambda)^2 = \Pi^\lambda$ and $\Pi^+ + \Pi^- = \mathbb{1}$.

Proof of eq. (E.1)

Starting from the completeness relation:

$$\sum_{s=1}^{2} u(\vec{p}, s)\, \bar{u}(\vec{p}, s) = \frac{\not{p} + m}{2m} = \Lambda^+(\vec{p}) \,, \tag{E.11}$$

we apply Π^λ from the right. On the left-hand side, Π^λ acts on $\bar{u}$ from the right, but since $\bar{u}(\vec{p}, s)\Pi^\lambda = \overline{\Pi^\lambda u(\vec{p}, s)} = \delta_{s\lambda} \bar{u}(\vec{p}, \lambda)$, only the $s = \lambda$ term survives:

$$\sum_s u(\vec{p}, s)\, \bar{u}(\vec{p}, s)\, \Pi^\lambda = u(\vec{p}, \lambda)\, \bar{u}(\vec{p}, \lambda) \,. \tag{E.12}$$

On the right-hand side, we simply get $\Lambda^+ \Pi^\lambda$. Therefore:

$$u(\vec{p}, \lambda)\, \bar{u}(\vec{p}, \lambda) = \Lambda^+(\vec{p})\, \Pi^\lambda(\vec{p}) \,. \quad ✓ \tag{E.13}$$

Ultra-relativistic limit

In the ultra-relativistic limit $E \gg m$, the spin four-vector (E.3) becomes $n_p^\mu \approx p^\mu/m$ (since $|\vec{p}| \approx E$). Then $\gamma^5 \not{n}_p \approx \gamma^5 \not{p}/m$. Acting on $u(\vec{p})$, the Dirac equation gives $\not{p}\, u = m\, u$, so $\gamma^5 \not{n}_p\, u \approx \gamma^5 u$. The helicity projector therefore simplifies to:

$$\Pi^\lambda(\vec{p}) \xrightarrow{E \gg m} \frac{1}{2}(1 + \lambda \gamma^5)\,, \tag{E.14}$$

which is the **chirality projector**. This is the mathematical statement of the fact that helicity and chirality coincide for massless (or ultra-relativistic) fermions.

F Special functions for loop integrals

The evaluation of Feynman integrals in dimensional regularization makes repeated use of a small number of special functions whose origins predate quantum field theory by more than two centuries. We collect here their definitions, key properties, and the identities most frequently needed in these notes.

The Gamma function

The Gamma function was introduced by Leonhard Euler in 1729 in a letter to Christian Goldbach, motivated by the problem of extending the factorial $n!$ from the positive integers to arbitrary (real or complex) values of the argument. Euler's original definition used an infinite product; the integral representation below is due to Euler himself (1730s) and was later formalized by Adrien-Marie Legendre, who also introduced the notation Γ.

Beyond its role as a generalized factorial, the Gamma function appears throughout physics and mathematics: in the normalization of probability distributions (the χ^2 distribution involves $\Gamma(n/2)$), in the volume of hyperspheres ($\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$, cf. Appendix G), in the Stirling approximation for statistical mechanics, and—centrally for us—in the master formula for dimensionally regularized loop integrals, where the poles of Γ at non-positive integers encode the UV divergences of quantum field theory.

Definition. For $\mathrm{Re}(s) > 0$:

$$\Gamma(s) \equiv \int_0^\infty dt\, t^{s-1} e^{-t} . \tag{F.1}$$

This integral converges at $t \to \infty$ for all s, and at $t \to 0$ when $\mathrm{Re}(s) > 0$.

Fundamental recursion. Integration by parts gives

$$\Gamma(s+1) = s\Gamma(s) . \tag{F.2}$$

Proof. $\Gamma(s+1) = \int_0^\infty t^s e^{-t}\, dt = \left[-t^s e^{-t}\right]_0^\infty + s\int_0^\infty t^{s-1} e^{-t}\, dt = s\Gamma(s)$.
Since $\Gamma(1) = \int_0^\infty e^{-t}\, dt = 1$, repeated application gives

$$\Gamma(n+1) = n! \qquad \text{for } n = 0, 1, 2, \ldots \tag{F.3}$$

The Gamma function is thus the analytic continuation of the factorial to the complex plane.

Analytic continuation and poles. The recursion (F.2) can be rewritten as $\Gamma(s) = \Gamma(s+1)/s$. Since the right-hand side is defined for $\mathrm{Re}(s) > -1$ (except $s = 0$), this extends Γ to $\mathrm{Re}(s) > -1$. Iterating:

$$\Gamma(s) = \frac{\Gamma(s+n+1)}{s(s+1)\cdots(s+n)}, \tag{F.4}$$

which defines $\Gamma(s)$ for all $s \in \mathbb{C}$ except $s = 0, -1, -2, \ldots$, where it has simple poles. Near $s = -n$ (with $n = 0, 1, 2, \ldots$):

$$\Gamma(-n+\epsilon) = \frac{(-1)^n}{n!}\left[\frac{1}{\epsilon} - \gamma_E + H_n + O(\epsilon)\right], \tag{F.5}$$

where $H_n = \sum_{k=1}^{n} 1/k$ is the n-th harmonic number ($H_0 = 0$). This is the formula that generates the $1/\epsilon$ poles in dimensional regularization.

Expansion around positive integers. In loop calculations, we frequently need Γ near $s = \epsilon$ (i.e. near the pole at $s = 0$) and near $s = 1+\epsilon$, $s = 2+\epsilon$, etc. The key expansions are:

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + O(\epsilon), \tag{F.6}$$

$$\Gamma(1+\epsilon) = 1 - \gamma_E\,\epsilon + O(\epsilon^2), \tag{F.7}$$

$$\Gamma(2+\epsilon) = 1 + (1-\gamma_E)\epsilon + O(\epsilon^2), \tag{F.8}$$

where $\gamma_E \approx 0.5772$ is the **Euler–Mascheroni constant** (introduced by Euler in 1734 and later computed to higher precision by Lorenzo Mascheroni in 1790; it remains unknown whether γ_E is rational or irrational), defined by

$$\gamma_E \equiv \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right) = -\Gamma'(1). \tag{F.9}$$

The combination $1/\epsilon - \gamma_E + \ln(4\pi)$ appears so ubiquitously that it defines the $\overline{\mathrm{MS}}$ subtraction (cf. Chapter G).

Special values.

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}, \qquad \Gamma\left(\tfrac{3}{2}\right) = \tfrac{1}{2}\sqrt{\pi}, \qquad \Gamma\left(\tfrac{5}{2}\right) = \tfrac{3}{4}\sqrt{\pi}. \tag{F.10}$$

More generally, $\Gamma(n+\frac{1}{2}) = \frac{(2n)!}{4^n\,n!}\sqrt{\pi}$.

Reflection formula (Euler).

$$\Gamma(s)\,\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}. \tag{F.11}$$

Setting $s = 1/2$ gives $[\Gamma(1/2)]^2 = \pi$, hence $\Gamma(1/2) = \sqrt{\pi}$. This formula is also useful for relating $\Gamma(1-d/2)$ (which appears in the tadpole J_1) to more familiar quantities.

Duplication formula (Legendre).

$$\Gamma(s)\,\Gamma\!\left(s+\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2s-1}}\,\Gamma(2s)\,. \tag{F.12}$$

The Beta function

The Beta function was also studied by Euler (1730s) and Legendre, originally in connection with the evaluation of integrals of powers of trigonometric functions. The key result $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$, which reduces the Beta function to a ratio of Gamma functions, was proved by Euler and later refined by Legendre.

In physics, the Beta function appears whenever an angular or Feynman-parameter integral can be reduced to the standard form $\int_0^1 t^{a-1}(1-t)^{b-1}\,dt$. This happens in the radial part of loop integrals (after switching to hyperspherical coordinates), in phase-space integrations for particle decays and scattering cross sections, and in statistical mechanics (where it normalizes the Beta distribution). Historically, the "Veneziano amplitude" (1968)—one of the founding results of string theory—was written as a ratio of Beta functions, chosen precisely to have the crossing symmetry and Regge pole structure expected of strong-interaction amplitudes.

Definition. For $\mathrm{Re}(a) > 0$ and $\mathrm{Re}(b) > 0$:

$$B(a,b) \;\equiv\; \int_0^1 dt\; t^{a-1}(1-t)^{b-1}\,. \tag{F.13}$$

Relation to the Gamma function. The fundamental identity is

$$\boxed{B(a,b) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}\,.} \tag{F.14}$$

This is the result used in evaluating the radial part of loop integrals (cf. Step 3 in the derivation of J_n, Appendix G).

Proof. Write $\Gamma(a)\,\Gamma(b)$ as a double integral and change variables:

$$\begin{aligned}\Gamma(a)\,\Gamma(b) &= \int_0^\infty du\; u^{a-1}e^{-u}\int_0^\infty dv\; v^{b-1}e^{-v}\\ &= \int_0^\infty\!\!\int_0^\infty du\,dv\; u^{a-1}v^{b-1}\,e^{-(u+v)}\,.\end{aligned} \tag{F.15}$$

Substituting $u = st$, $v = s(1-t)$ with $s \in (0,\infty)$, $t \in (0,1)$, the Jacobian is $|\partial(u,v)/\partial(s,t)| = s$, and $u+v = s$:

$$\begin{aligned}
\Gamma(a)\,\Gamma(b) &= \int_0^\infty ds \int_0^1 dt\ (st)^{a-1}\big[s(1-t)\big]^{b-1} e^{-s}\cdot s \\
&= \int_0^\infty ds\ s^{a+b-1} e^{-s} \cdot \int_0^1 dt\ t^{a-1}(1-t)^{b-1} \\
&= \Gamma(a+b)\cdot B(a,b)\,.
\end{aligned}$$

□

Symmetry.

$$B(a,b) = B(b,a)\,, \tag{F.16}$$

which is obvious from the substitution $t \to 1-t$ in the definition.

Useful alternative forms. The substitution $t = u/(1+u)$ transforms (F.13) into

$$B(a,b) = \int_0^\infty du\ \frac{u^{a-1}}{(1+u)^{a+b}}\,, \tag{F.17}$$

which is the form that arises directly from the radial integral after the substitution $r^2 = \Delta u$ in the derivation of J_n.

Special values.

$$B(a,1) = \frac{1}{a}\,, \qquad B\left(\tfrac{1}{2},\tfrac{1}{2}\right) = \pi\,, \qquad B(n,m) = \frac{(n-1)!(m-1)!}{(n+m-1)!} \quad (n,m \text{ positive integers})\,. \tag{F.18}$$

The digamma function

The digamma function $\psi(s) = \Gamma'(s)/\Gamma(s)$ was studied systematically by Gauss and Legendre in the early 19th century, though particular values had appeared earlier in Euler's work on harmonic series. The name "digamma" comes from the notation ψ (or F), used historically as an analogue of the "di-" prefix applied to the Gamma function (its logarithmic derivative).

In quantum field theory, the digamma function is less prominent than Γ and B, but it enters whenever one needs the $O(\epsilon)$ terms in the expansion of Γ functions—that is, whenever finite parts of loop integrals matter (not just the divergent poles). This is the case in any precision calculation: the anomalous magnetic moment, the Lamb shift, running couplings at two loops, and matching calculations between effective field theories all require the digamma function through its connection $\psi(n) = -\gamma_E + H_{n-1}$ to harmonic numbers.

Definition.

$$\psi(s) \equiv \frac{d}{ds}\ln\Gamma(s) = \frac{\Gamma'(s)}{\Gamma(s)}\,. \tag{F.19}$$

The digamma function appears when expanding Γ functions around their arguments, which happens systematically in extracting finite parts of dimensionally regularized integrals.

Key values and identities.

$$\psi(1) = -\gamma_E\,, \tag{F.20}$$

$$\psi(n+1) = -\gamma_E + \sum_{k=1}^{n}\frac{1}{k} = -\gamma_E + H_n \qquad (n \geq 1 \text{ integer})\,, \tag{F.21}$$

$$\psi(s+1) = \psi(s) + \frac{1}{s}\,, \tag{F.22}$$

$$\psi(1-s) - \psi(s) = \pi\cot(\pi s) \qquad \text{(reflection formula)}\,. \tag{F.23}$$

The recursion (F.22) follows immediately from $\Gamma(s+1) = s\Gamma(s)$ by taking the logarithmic derivative.

Connection to ϵ-expansions. When we expand $\Gamma(n + a\epsilon)$ around $\epsilon = 0$ for integer n, the $O(\epsilon)$ term is proportional to $\psi(n)$:

$$\Gamma(n + a\epsilon) = \Gamma(n)\left[1 + a\epsilon\,\psi(n) + O(\epsilon^2)\right]. \tag{F.24}$$

For example, $\Gamma(1+\epsilon) = 1 + \epsilon\,\psi(1) + O(\epsilon^2) = 1 - \gamma_E\epsilon + O(\epsilon^2)$, recovering (F.7).

Summary: the functions at a glance

Box F.1 Special functions: quick reference

Function	Definition	Key identity
$\Gamma(s)$	$\int_0^\infty t^{s-1}e^{-t}\,dt$	$\Gamma(s+1) = s\Gamma(s)$; $\quad\Gamma(n+1) = n!$
$B(a,b)$	$\int_0^1 t^{a-1}(1-t)^{b-1}\,dt$	$B(a,b) = \dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$
$\psi(s)$	$\dfrac{d}{ds}\ln\Gamma(s)$	$\psi(1) = -\gamma_E$; $\quad\psi(s+1) = \psi(s) + 1/s$

Euler–Mascheroni constant: $\gamma_E = -\psi(1) = -\Gamma'(1) \approx 0.5772$

Poles of Γ: $\Gamma(-n+\epsilon) = \dfrac{(-1)^n}{n!}\left[\dfrac{1}{\epsilon} - \gamma_E + H_n + O(\epsilon)\right], \quad H_n = \sum_{k=1}^{n} 1/k$

G Feynman-integral technology

This appendix develops the standard techniques for evaluating one-loop Feynman integrals in dimensional regularization. These methods are used throughout these notes: for the tadpole and bubble in $\lambda\phi^4$ theory (Chapter 4), for the vacuum polarization, electron self-energy, and vertex correction in QED (Chapter 6), and for any other one-loop calculation the reader may encounter.

Feynman parametrization

The central idea is to combine products of propagator denominators into a single denominator raised to a power, at the cost of introducing auxiliary integrations (the *Feynman parameters*).

Two propagators. The simplest case is

$$\frac{1}{AB} = \int_0^1 dx \, \frac{1}{\big[xA + (1-x)B\big]^2}. \tag{G.1}$$

Proof. Writing $xA + (1-x)B = B + x(A-B)$, the integral over x is elementary:

$$\int_0^1 dx \, \frac{1}{[B + x(A-B)]^2} = \left[-\frac{1}{(A-B)\big[B + x(A-B)\big]}\right]_0^1 = -\frac{1}{A-B}\left(\frac{1}{A} - \frac{1}{B}\right) = \frac{1}{AB}.$$

□

Three propagators.

$$\frac{1}{ABC} = 2\int_0^1 dx \int_0^{1-x} dy \, \frac{1}{\big[xA + yB + zC\big]^3}, \qquad z \equiv 1 - x - y. \tag{G.2}$$

This follows by applying (G.1) twice: first combine A and B, then combine the result with C, and change variables so that the integration region becomes the triangular region $x + y + z = 1$ with $x, y, z \geq 0$ (i.e. $0 \leq x \leq 1$, $0 \leq y \leq 1 - x$, $z = 1 - x - y$).

General formula: n propagators.

$$\frac{1}{A_1 A_2 \cdots A_n} = (n-1)! \int_0^1 dx_1 \cdots dx_n \, \delta\Big(1 - \sum_{i=1}^{n} x_i\Big) \frac{1}{\big[\sum_{i=1}^n x_i A_i\big]^n}. \tag{G.3}$$

The proof is by induction. The δ-function constrains the parameters to the region $x_i \geq 0$, $\sum x_i = 1$ (a segment for $n = 2$, a triangle for $n = 3$, a tetrahedron for $n = 4$, etc.).

Example: the bubble integral (two propagators)

Consider the scalar two-point integral that arises, for instance, in the $\lambda\phi^4$ self-energy or in the QED vacuum polarization:

$$\mathcal{B}(p^2; m_1^2, m_2^2) \equiv \int \frac{d^d k}{(2\pi)^d} \frac{1}{\left[k^2 - m_1^2 + i\epsilon\right]\left[(k-p)^2 - m_2^2 + i\epsilon\right]}. \tag{G.4}$$

Step 1: Feynman parameter. Apply (G.1) with $A = k^2 - m_1^2 + i\epsilon$ and $B = (k-p)^2 - m_2^2 + i\epsilon$:

$$\mathcal{B}(p^2; m_1^2, m_2^2) = \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{1}{\left[x(k^2 - m_1^2) + (1-x)((k-p)^2 - m_2^2) + i\epsilon\right]^2}.$$

We now expand the combined denominator step by step. Writing $D \equiv x(k^2 - m_1^2) + (1-x)((k-p)^2 - m_2^2)$:

$$\begin{aligned} D &= x\,k^2 - x\,m_1^2 + (1-x)\left[k^2 - 2k\cdot p + p^2 - m_2^2\right] \\ &= \underbrace{\left[x + (1-x)\right]}_{=1} k^2 - 2(1-x)\,k\cdot p + (1-x)\,p^2 - x\,m_1^2 - (1-x)\,m_2^2 \\ &= k^2 - 2(1-x)\,k\cdot p + (1-x)\,p^2 - x\,m_1^2 - (1-x)\,m_2^2. \end{aligned} \tag{G.5}$$

Step 2: Complete the square. The first two terms of (G.5) can be rewritten by shifting the loop momentum. Define $\ell \equiv k - (1-x)p$. Then $k = \ell + (1-x)p$ and:

$$\begin{aligned} k^2 - 2(1-x)\,k\cdot p &= \left[\ell + (1-x)p\right]^2 - 2(1-x)\left[\ell + (1-x)p\right]\cdot p \\ &= \ell^2 + 2(1-x)\,\ell\cdot p + (1-x)^2 p^2 - 2(1-x)\,\ell\cdot p - 2(1-x)^2 p^2 \\ &= \ell^2 - (1-x)^2 p^2. \end{aligned} \tag{G.6}$$

Substituting back into (G.5):

$$\begin{aligned} D &= \ell^2 - (1-x)^2 p^2 + (1-x)\,p^2 - x\,m_1^2 - (1-x)\,m_2^2 \\ &= \ell^2 - \left[(1-x)^2 - (1-x)\right] p^2 - x\,m_1^2 - (1-x)\,m_2^2 \\ &= \ell^2 - \left[-(1-x)x\right] p^2 - x\,m_1^2 - (1-x)\,m_2^2 \\ &= \ell^2 - \left[x\,m_1^2 + (1-x)\,m_2^2 - x(1-x)\,p^2\right], \end{aligned} \tag{G.7}$$

where in the third line we used $(1-x)^2-(1-x)=(1-x)(1-x-1)=-x(1-x)$. Therefore

$$D \;=\; \ell^2-\Delta_{\text{bub}}(x)+i\epsilon\,,$$

with

$$\boxed{\Delta_{\text{bub}}(x) \;\equiv\; x\,m_1^2+(1-x)\,m_2^2-x(1-x)\,p^2\,.} \tag{G.8}$$

Note that Δ_{bub} is a simple quadratic in x that encodes all the dependence on masses and external momentum. Since the integration measure $d^dk = d^d\ell$ (translation invariance),

$$\boxed{\mathcal{B}(p^2;m_1^2,m_2^2) \;=\; \int_0^1 dx\; J_2\big(\Delta_{\text{bub}}(x)\big)\,,} \tag{G.9}$$

where $J_n(\Delta)$ is the master scalar integral defined in (G.22) below. For the common case of equal masses ($m_1=m_2=m$), the formula simplifies to $\Delta_{\text{bub}}(x) = m^2-x(1-x)\,p^2$.

Example: the triangle integral (three propagators)

A three-propagator integral arises in vertex corrections. The general structure is

$$I^{\{\cdots\}}(p',p) \;\equiv\; \int \frac{d^dk}{(2\pi)^d}\,\frac{\{\,1,\;k^\mu,\;k^\mu k^\nu,\;\ldots\,\}}{\big[k^2-\lambda^2+i\epsilon\big]\,\big[(p'-k)^2-m^2+i\epsilon\big]\,\big[(p-k)^2-m^2+i\epsilon\big]}\,. \tag{G.10}$$

Here $d=4-2\epsilon$ regulates UV divergences and λ is an IR regulator (e.g. a small photon mass). We keep the external momenta general at first.

Feynman parameters. Apply (G.2) with

$$\begin{aligned}
A &\equiv (p'-k)^2-m^2+i\epsilon,\\
B &\equiv (p-k)^2-m^2+i\epsilon,\\
C &\equiv k^2-\lambda^2+i\epsilon\,.
\end{aligned} \tag{G.11}$$

The combined denominator is

$$xA+yB+zC \;=\; k^2-2k\cdot(xp'+yp)+x(p'^2-m^2)+y(p^2-m^2)-z\lambda^2+i\epsilon\,. \tag{G.12}$$

On-shell specialization. If the external legs are on-shell, $p^2 = p'^2 = m^2$, the terms $x(p'^2 - m^2)$ and $y(p^2 - m^2)$ vanish, simplifying the denominator to

$$xA + yB + zC \;=\; k^2 - 2k\cdot(xp' + yp) - z\lambda^2 + i\epsilon\,. \tag{G.13}$$

Completing the square. Define the shifted momentum

$$\ell \;\equiv\; k - (xp' + yp)\,. \tag{G.14}$$

Then

$$k^2 - 2k\cdot(xp' + yp) \;=\; \ell^2 - (xp' + yp)^2\,, \tag{G.15}$$

so that

$$xA + yB + zC \;=\; \ell^2 - \Delta(x, y; q^2) + i\epsilon\,, \tag{G.16}$$

where (using on-shell conditions and $q \equiv p' - p$)

$$\begin{aligned}(xp' + yp)^2 &= x^2 p'^2 + y^2 p^2 + 2xy\, p'\cdot p \\ &= m^2(x^2 + y^2 + 2xy) - xy\, q^2 \\ &= m^2(x + y)^2 - xy\, q^2\,,\end{aligned} \tag{G.17}$$

and therefore

$$\Delta(x, y; q^2) \;=\; m^2(x + y)^2 \;-\; xy\, q^2 \;+\; z\lambda^2\,, \qquad z = 1 - x - y\,. \tag{G.18}$$

At this stage every loop integral becomes

$$I^{\{\cdots\}}(p', p) \;=\; 2\int_0^1 dx \int_0^{1-x} dy \int \frac{d^d\ell}{(2\pi)^d}\, \frac{\{\cdots\}}{\big[\ell^2 - \Delta(x, y; q^2) + i\epsilon\big]^3}\,, \tag{G.19}$$

where the numerator $\{\cdots\}$ is now a polynomial in ℓ plus terms built from p, p' (equivalently q and $p + p'$).

Tensor reduction

Dimensional regularization gives two crucial simplifications:

1. **Odd powers integrate to zero** (by symmetry $\ell \to -\ell$ of the integrand):

$$\int \frac{d^d\ell}{(2\pi)^d}\, \frac{\ell^\mu}{(\ell^2 - \Delta + i\epsilon)^n} \;=\; 0, \qquad \int \frac{d^d\ell}{(2\pi)^d}\, \frac{\ell^\mu \ell^\nu \ell^\rho}{(\ell^2 - \Delta + i\epsilon)^n} \;=\; 0\,. \tag{G.20}$$

2. **Even tensors reduce to the metric** (by Lorentz covariance, $g^{\mu\nu}$ is the only available symmetric tensor):

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^\mu \ell^\nu}{(\ell^2-\Delta+i\epsilon)^n} = \frac{g^{\mu\nu}}{d} \int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^2}{(\ell^2-\Delta+i\epsilon)^n}. \tag{G.21}$$

(Contract both sides with $g_{\mu\nu}$ to verify: $g_{\mu\nu}g^{\mu\nu} = d$.) Similarly, a rank-4 tensor reduces to symmetrized products of $g^{\mu\nu}$.

So the whole problem reduces to a small set of *scalar* integrals of the form

$$J_n(\Delta) \equiv \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{(\ell^2-\Delta+i\epsilon)^n}, \qquad n = 1,2,3,\ldots \tag{G.22}$$

and integrals with powers of ℓ^2 in the numerator.

The master integral $J_n(\Delta)$: complete derivation

We now derive the closed-form expression for $J_n(\Delta)$ in $d = 4-2\epsilon$ dimensions. The derivation proceeds in four steps.

Step 1: Wick rotation. Write $\ell^\mu = (\ell^0, \vec{\ell})$ in Minkowski space with signature $(+,-,-,-)$. The denominator is

$$\ell^2 - \Delta + i\epsilon = (\ell^0)^2 - |\vec{\ell}|^2 - \Delta + i\epsilon.$$

As a function of ℓ^0, the integrand has poles where $(\ell^0)^2 = |\vec{\ell}|^2 + \Delta - i\epsilon$, i.e. at $\ell^0 = \pm\sqrt{|\vec{\ell}|^2 + \Delta - i\epsilon}$. Writing $\omega^2 \equiv |\vec{\ell}|^2 + \Delta > 0$ and treating ϵ as a small positive number, we have $\ell^0 = \pm\sqrt{\omega^2 - i\epsilon} \approx \pm\omega(1 - i\epsilon/2\omega^2)$, so

$$\ell^0_+ \approx +\omega - i\epsilon' \quad \text{(IV quadrant)}, \qquad \ell^0_- \approx -\omega + i\epsilon' \quad \text{(II quadrant)}, \tag{G.23}$$

where $\epsilon' = \epsilon/(2\omega) > 0$. The $+i\epsilon$ prescription shifts the positive pole *below* the real axis and the negative pole *above* it. This means the first and third quadrants of the complex ℓ^0-plane are free of singularities, and we can rotate the integration contour counterclockwise by 90° (from the real axis to the imaginary axis) without crossing any pole. Setting

$$\ell^0 = i\ell^0_E, \qquad d\ell^0 = i\,d\ell^0_E, \tag{G.24}$$

we obtain $\ell^2 = (\ell^0)^2 - |\vec{\ell}|^2 = -(\ell^0_E)^2 - |\vec{\ell}|^2 = -\ell^2_E$, where $\ell^2_E \equiv (\ell^0_E)^2 + |\vec{\ell}|^2$ is the *Euclidean* norm. Since $d^d\ell = i\,d^d\ell_E$ (the factor of i from the ℓ^0 rotation, all spatial components unchanged), the integral becomes

$$J_n(\Delta) = i(-1)^n \int \frac{d^d\ell_E}{(2\pi)^d} \frac{1}{(\ell^2_E + \Delta)^n}. \tag{G.25}$$

The crucial sign: the Minkowski denominator $(\ell^2 - \Delta)^n = (-1)^n(\ell_E^2 + \Delta)^n$, and combined with the factor i from $d^d\ell = i\, d^d\ell_E$, we get the overall prefactor $i(-1)^n$.

Step 2: Hyperspherical coordinates. In d-dimensional Euclidean space, introduce the radial variable $r \equiv |\ell_E|$. The integration measure factorizes as

$$d^d\ell_E \; = \; \Omega_d \, r^{d-1}\, dr \,, \tag{G.26}$$

where $\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$ is the total solid angle (area of the unit $(d-1)$-sphere). Quick checks: $\Omega_2 = 2\pi$ (circle), $\Omega_3 = 4\pi$ (sphere), $\Omega_4 = 2\pi^2$.

Derivation of Ω_d. The trick is to evaluate the Gaussian integral $\int_{-\infty}^{+\infty} d^dx\, e^{-\vec{x}^2}$ in two ways. On one hand, since $\vec{x}^2 = x_1^2 + \cdots + x_d^2$, the integral factorizes into d independent one-dimensional Gaussians:

$$\int_{-\infty}^{+\infty} d^dx\, e^{-\vec{x}^2} = \prod_{i=1}^{d} \int_{-\infty}^{+\infty} dx_i\, e^{-x_i^2} = \left(\sqrt{\pi}\right)^d = \pi^{d/2} \,.$$

On the other hand, using hyperspherical coordinates $d^dx = \Omega_d\, r^{d-1}\, dr$ (where the angular integration has already been performed):

$$\int_{-\infty}^{+\infty} d^dx\, e^{-\vec{x}^2} = \Omega_d \int_0^{\infty} dr\, r^{d-1}\, e^{-r^2} \,.$$

The remaining radial integral is evaluated by the substitution $t = r^2$ ($dr = dt/(2\sqrt{t})$):

$$\int_0^{\infty} dr\, r^{d-1}\, e^{-r^2} = \frac{1}{2}\int_0^{\infty} dt\, t^{d/2-1}\, e^{-t} = \frac{1}{2}\Gamma\!\left(\frac{d}{2}\right),$$

where we recognized the definition of the Gamma function $\Gamma(s) = \int_0^\infty t^{s-1} e^{-t}\, dt$. Equating the two results: $\pi^{d/2} = \frac{1}{2}\Omega_d\, \Gamma(d/2)$, hence

$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} \,. \tag{G.27}$$

Step 3: The radial integral. After angular integration, we need

$$\mathcal{R} \; \equiv \; \int_0^{\infty} dr\, \frac{r^{d-1}}{(r^2 + \Delta)^n} \,. \tag{G.28}$$

Substitute $t = r^2$ (so $dr = dt/(2\sqrt{t}\,)$ and $r^{d-1} = t^{(d-1)/2}$):

$$\mathcal{R} \; = \; \frac{1}{2}\int_0^{\infty} dt\, \frac{t^{d/2-1}}{(t + \Delta)^n} \,.$$

Now let $u = t/\Delta$:

$$\mathcal{R} \;=\; \frac{\Delta^{d/2-n}}{2}\int_0^\infty du\,\frac{u^{d/2-1}}{(1+u)^n} \;=\; \frac{\Delta^{d/2-n}}{2}\,B\!\left(\frac{d}{2}, n-\frac{d}{2}\right),$$

where $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ is the Euler Beta function. This is valid for $\mathrm{Re}(d/2) > 0$ and $\mathrm{Re}(n-d/2) > 0$; the final answer is extended to all d by analytic continuation. Therefore:

$$\mathcal{R} \;=\; \frac{1}{2}\Delta^{d/2-n}\frac{\Gamma(d/2)\,\Gamma(n-d/2)}{\Gamma(n)}. \tag{G.29}$$

Step 4: Combining everything. Putting together the Wick-rotation prefactor (G.25), the angular integral $\Omega_d/(2\pi)^d$, and the radial result (G.29):

$$J_n(\Delta) \;=\; i(-1)^n\frac{\Omega_d}{(2\pi)^d}\mathcal{R} \;=\; i(-1)^n\frac{2\pi^{d/2}}{\Gamma(d/2)\,(2\pi)^d}\cdot\frac{1}{2}\Delta^{d/2-n}\frac{\Gamma(d/2)\,\Gamma(n-d/2)}{\Gamma(n)}.$$

The factors of $\Gamma(d/2)$ cancel, and $2\pi^{d/2}/[(2\pi)^d\cdot 2] = 1/[2(4\pi)^{d/2}]\cdot 2\cdot 1/2 = 1/(4\pi)^{d/2}$, giving the master formula:

$$\boxed{J_n(\Delta) \;=\; \frac{i(-1)^n}{(4\pi)^{d/2}}\frac{\Gamma\!\left(n-\frac{d}{2}\right)}{\Gamma(n)}\,(\Delta)^{\frac{d}{2}-n}, \qquad \mathrm{Re}(\Delta) > 0.} \tag{G.30}$$

The derivation assumed $\mathrm{Re}(n-d/2) > 0$ for UV convergence of the radial integral. For the physically relevant case $d = 4-2\epsilon$ and $n = 2$, this requires $\mathrm{Re}(\epsilon) > 0$—satisfied for any $\epsilon > 0$, no matter how small. The crucial step of **dimensional regularization** is now the following: the right-hand side of (G.30) is a closed-form expression built from Gamma functions and powers of Δ. These are well-defined analytic functions of the *continuous* variable d (or equivalently ϵ), even for values of d where the original integral diverges.

This is the idea of **analytic continuation**: we derived the formula in a region of d where the integral converges, but the *result*—as a function of d—has a natural extension to all $d \in \mathbb{C}$. We *define* the dimensionally regularized integral to be this analytic continuation. The UV divergence of the original $d = 4$ integral now manifests itself as a *pole* of the Gamma function $\Gamma(n-d/2)$ at $n-d/2 = 0, -1, -2, \ldots$ (cf. Appendix F). For example, J_2 at $d = 4$ has $n-d/2 = 0$, producing $\Gamma(\epsilon) = 1/\epsilon - \gamma_E + O(\epsilon)$: the $1/\epsilon$ pole is the regularized version of the logarithmic divergence $\ln\Lambda$.

The power of this approach is that intermediate expressions remain finite for $\epsilon > 0$, standard algebraic manipulations (shifts, symmetry arguments, Dirac algebra) remain valid in d dimensions, and divergences appear only at the very end as isolated $1/\epsilon$ poles that can be systematically cancelled by counterterms.

Special cases: J_1, J_2, J_3

Tadpole: $J_1(\Delta)$**.** Setting $n = 1$ in (G.30) (with $\Gamma(1) = 1$):

$$J_1(\Delta) \;=\; \frac{-i}{(4\pi)^{d/2}}\,\Gamma\!\left(1-\frac{d}{2}\right)\Delta^{d/2-1}\,. \tag{G.31}$$

This integral appears in the tadpole diagram of $\lambda\phi^4$ theory. In $d = 4 - 2\epsilon$, the Gamma function has a pole: $\Gamma(1-d/2) = \Gamma(-1+\epsilon) = -1/\epsilon + (\gamma_E - 1) + O(\epsilon)$, reflecting the quadratic UV divergence (in cutoff language) of the tadpole.

Bubble: $J_2(\Delta)$**.** Setting $n = 2$ (with $\Gamma(2) = 1$):

$$J_2(\Delta) \;=\; \frac{i}{(4\pi)^{d/2}}\,\Gamma\!\left(2-\frac{d}{2}\right)\Delta^{d/2-2}\,. \tag{G.32}$$

This is the integral that appears after Feynman parametrization of any two-propagator (bubble) diagram (cf. Eq. (G.9)).

Triangle: $J_3(\Delta)$**.** Setting $n = 3$ (with $\Gamma(3) = 2$):

$$J_3(\Delta) \;=\; -\frac{i}{2(4\pi)^{d/2}}\,\Gamma\!\left(3-\frac{d}{2}\right)\Delta^{d/2-3}\,. \tag{G.33}$$

For reference we also record the compact form:

$$\begin{aligned} J_2(\Delta) &= \frac{i}{(4\pi)^{d/2}}\,\Gamma\!\left(2-\frac{d}{2}\right)\Delta^{\frac{d}{2}-2}, \\ J_3(\Delta) &= -\frac{i}{2(4\pi)^{d/2}}\,\Gamma\!\left(3-\frac{d}{2}\right)\Delta^{\frac{d}{2}-3}\,. \end{aligned} \tag{G.34}$$

Numerators with ℓ^2

Use the algebraic identity

$$\ell^2 \;=\; (\ell^2 - \Delta) + \Delta\,, \tag{G.35}$$

to reduce

$$\begin{aligned} \int \frac{d^d\ell}{(2\pi)^d}\,\frac{\ell^2}{(\ell^2-\Delta+i\epsilon)^n} &= \int \frac{d^d\ell}{(2\pi)^d}\,\frac{1}{(\ell^2-\Delta+i\epsilon)^{n-1}} + \Delta\int \frac{d^d\ell}{(2\pi)^d}\,\frac{1}{(\ell^2-\Delta+i\epsilon)^n} \\ &= J_{n-1}(\Delta) + \Delta\,J_n(\Delta)\,. \end{aligned} \tag{G.36}$$

For example, with $n = 3$: $\int \frac{d^d\ell}{(2\pi)^d}\frac{\ell^2}{(\ell^2-\Delta+i\epsilon)^3} = J_2(\Delta) + \Delta\,J_3(\Delta)$. Combining (G.21) and (G.36), any rank-2 tensor integral is expressible in terms of J_{n-1} and J_n.

UV structure: expansion around $d = 4$

Set $d = 4 - 2\epsilon$ and introduce a renormalization scale μ so that the coupling constant remains dimensionless. Define the standard combination that collects the $\overline{\text{MS}}$ terms:

$$\Delta_{\text{UV}} \equiv \frac{1}{\epsilon} - \gamma_E + \ln(4\pi)\,. \tag{G.37}$$

Expansion of J_1 (tadpole). Using $\Gamma(-1+\epsilon) = -1/\epsilon + (\gamma_E - 1) + \mathcal{O}(\epsilon)$ and $\Delta^{1-\epsilon} = \Delta\,[1 - \epsilon \ln \Delta + \mathcal{O}(\epsilon^2)]$:

$$J_1(\Delta) = \frac{i}{(4\pi)^2}\,\Delta\left[\Delta_{\text{UV}} + 1 - \ln\left(\frac{\Delta}{\mu^2}\right) + \mathcal{O}(\epsilon)\right]. \tag{G.38}$$

The $1/\epsilon$ pole signals a quadratic UV divergence (in cutoff language, $J_1 \sim \Lambda^2$). This is the divergence encountered in the $\lambda\phi^4$ tadpole.

Expansion of J_2 (bubble). Using $\Gamma(\epsilon) = 1/\epsilon - \gamma_E + \mathcal{O}(\epsilon)$ and $\Delta^{-\epsilon} = 1 - \epsilon \ln \Delta + \mathcal{O}(\epsilon^2)$:

$$J_2(\Delta) = \frac{i}{(4\pi)^2}\left[\Delta_{\text{UV}} - \ln\left(\frac{\Delta}{\mu^2}\right) + \mathcal{O}(\epsilon)\right]. \tag{G.39}$$

The $1/\epsilon$ pole corresponds to a logarithmic UV divergence ($\sim \ln \Lambda$ in cutoff language). This is the divergence that appears in one-loop bubble-type integrals: the $\lambda\phi^4$ four-point (vertex) correction, the QED vacuum polarization, and the electron self-energy.

Expansion of J_3 (triangle). Since $\Gamma(1+\epsilon) = 1 + \mathcal{O}(\epsilon)$:

$$J_3(\Delta) = -\frac{i}{2(4\pi)^2}\,\frac{1}{\Delta}\,[1 + \mathcal{O}(\epsilon)]\,. \tag{G.40}$$

This is UV-finite: no $1/\epsilon$ pole. Triangle integrals (and higher-point functions) are UV-convergent at one loop in four dimensions.

General pattern. The integral $J_n(\Delta)$ is UV-divergent when $n \leq d/2$, i.e. when $\Gamma(n - d/2)$ has a pole. In $d = 4$ dimensions:

- $n = 1$ (tadpole): divergent (quadratic),
- $n = 2$ (bubble): divergent (logarithmic),
- $n \geq 3$ (triangle, box, . . .): UV-finite.

Box G.1 **Summary of one-loop master integrals**

Integral	Exact in d dimensions	Expansion ($d = 4 - 2\epsilon$)
$J_1(\Delta)$	$\dfrac{-i}{(4\pi)^{d/2}}\Gamma\Big(1-\dfrac{d}{2}\Big)\,\Delta^{d/2-1}$	$\dfrac{i\Delta}{(4\pi)^2}\Big[\Delta_{\rm UV}+1-\ln\dfrac{\Delta}{\mu^2}\Big]$
$J_2(\Delta)$	$\dfrac{i}{(4\pi)^{d/2}}\Gamma\Big(2-\dfrac{d}{2}\Big)\,\Delta^{d/2-2}$	$\dfrac{i}{(4\pi)^2}\Big[\Delta_{\rm UV}-\ln\dfrac{\Delta}{\mu^2}\Big]$
$J_3(\Delta)$	$-\dfrac{i}{2(4\pi)^{d/2}}\Gamma\Big(3-\dfrac{d}{2}\Big)\,\Delta^{d/2-3}$	$-\dfrac{i}{2(4\pi)^2}\dfrac{1}{\Delta}$

$\Delta_{\rm UV} \equiv 1/\epsilon - \gamma_E + \ln(4\pi)$. All ϵ-expansions are to leading order; $+\mathcal{O}(\epsilon)$ terms are suppressed.

Putting it together: the parameter integral

After completing the Dirac algebra (or other numerator algebra) and applying the tensor-reduction identities above, every one-loop diagram reduces to an integral over Feynman parameters of J_n functions. For a two-propagator diagram (bubble), the integration is over a single Feynman parameter $x \in [0,1]$. For a three-propagator diagram (triangle), the integration is over the triangular region

$$0 \le x \le 1, \qquad 0 \le y \le 1-x, \qquad z = 1-x-y\,. \tag{G.41}$$

Schematically, a typical one-loop contribution (with n propagators combined into J_n) takes the form

$$\text{(one-loop diagram)} \sim \int [dx_i]\,\Big[\mathcal{A}(x_i)\,J_{n-1}(\Delta) + \mathcal{B}(x_i)\,J_n(\Delta)\Big], \tag{G.42}$$

where $\int [dx_i]$ denotes the Feynman-parameter integral over the appropriate region ($0 \le x \le 1$ for a bubble, the triangle $0 \le x \le 1$, $0 \le y \le 1-x$ for a vertex, etc.), and $\mathcal{A}$, $\mathcal{B}$ are rational functions of the Feynman parameters and external kinematic invariants that come from the numerator algebra. The UV divergence is carried entirely by the lower-n master integral (e.g. J_2 for a triangle diagram, J_1 for a bubble), while the higher-n piece is UV-finite.

H Optical theorem and Cutkosky cutting rules

In Chapter 8 we showed, on general grounds (unitarity of S), that the imaginary part of the exact propagator Δ'_F encodes the multi-particle states into which the field can decay. The corresponding statement at the level of forward scattering amplitudes is the *optical theorem* (8.176). This appendix establishes the perturbative, diagrammatic translation of these results: the **Cutkosky cutting rules**, which allow one to compute $\operatorname{Im}\Pi(p^2)$ — and hence the spectral density $\rho(p^2)$ — by "cutting" Feynman diagrams along sets of internal propagators that go simultaneously on shell. We motivate, state, and apply the rules to two concrete examples in scalar field theory: the one-loop bubble and the two-loop sunset.

Motivation: where does $\operatorname{Im}\Pi$ come from?

The self-energy $\Pi(p^2)$ is, order by order in perturbation theory, a sum of 1PI Feynman diagrams with two external legs (Fig. H.1). Each such diagram is, individually, an integral over loop momenta of products of Feynman propagators

$$\frac{i}{k^2 - m^2 + i\epsilon}, \tag{H.1}$$

with an overall factor of i from the Feynman rules. At first glance this is a *real* integral with a small imaginary regulator ϵ: one might wonder how $\Pi(p^2)$ ever acquires a finite imaginary part as $\epsilon \to 0^+$.

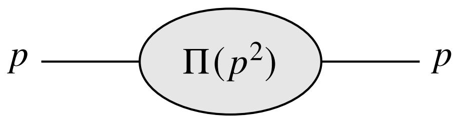

Figure H.1. Diagrammatic representation of the 1PI self-energy $\Pi(p^2)$: sum of all 1PI two-point graphs.

The answer lies in the $+i\epsilon$ prescription itself. By Sokhotski–Plemelj (8.169),

$$\frac{1}{k^2 - m^2 + i\epsilon} = \mathcal{P}\frac{1}{k^2 - m^2} - i\pi\,\delta(k^2 - m^2), \tag{H.2}$$

so each propagator carries a built-in δ-function contribution localized on its mass shell. When the external momentum p is large enough that several internal propagators can be *simultaneously* on shell with positive energies, the corresponding δ-functions produce an honest, finite imaginary part of the diagram. The set of internal lines that go on shell defines a *cut* of the diagram, and identifies the physical multi-particle intermediate state $|X\rangle$ being produced.

The Cutkosky rules turn this observation into a precise prescription.

Statement of the Cutkosky cutting rules

Let $\mathcal{D}$ be a 1PI two-point Feynman diagram contributing to $\Pi(p^2)$. To compute its imaginary part one proceeds as follows.

1. **Enumerate the cuts.** Identify every way of cutting $\mathcal{D}$ into two connected pieces by slicing through a set of internal propagators $\{k_1, \ldots, k_n\}$ such that the cut propagators can be simultaneously put on shell with positive energies summing to $\sqrt{p^2}$. Such a cut is *kinematically allowed* only if
$$p^2 \;\geq\; \left(m_1 + m_2 + \cdots + m_n\right)^2, \tag{H.3}$$
where m_i is the mass of the i-th cut line. Equivalently: each cut is in one-to-one correspondence with a physical multi-particle intermediate state $|X\rangle = |k_1, \ldots, k_n\rangle$ whose constituents are exactly the particles flowing through the cut lines..
2. **Apply the cut substitution.** For each cut, replace each cut propagator by its on-shell mass-shell projector
$$\frac{i}{k^2 - m^2 + i\epsilon} \;\longrightarrow\; 2\pi\,\delta(k^2 - m^2)\,\theta(k^0)\,, \tag{H.4}$$
enforcing that the cut particle is real, on shell, and propagating with positive energy.
3. **Complex-conjugate one side of the cut.** On one side of the cut — conventionally, the right-hand side — replace every uncut propagator and every coupling by its complex conjugate (in practice, flip the signs of all $+i\epsilon$ prescriptions and conjugate all complex couplings).
4. **Sum.** The imaginary part is obtained by summing the contributions of all kinematically allowed cuts of all 1PI diagrams at the given perturbative order:

$$2\,\mathrm{Im}\,\Pi(p^2) \;=\; \sum_{\text{cuts}} \big(\text{cut diagram}\big)\,. \tag{H.5}$$

This is the diagrammatic incarnation of the optical theorem (8.176): cutting the self-energy along a set of internal lines is the perturbative implementation of inserting a complete set of states $\sum_X |X\rangle\langle X|$ on the unitarity cut. The two "halves" of the cut diagram are precisely the two factors $\langle 0|\varphi(0)|X\rangle^*$ and $\langle X|\varphi(0)|0\rangle$ that build up the spectral density $\rho(p^2)$ in (8.130).

Why the $i\epsilon$ prescription matters: a one-line derivation

The mathematical heuristic underlying the rules is short. Consider a diagram contributing to $\Pi(p^2)$ with loop momentum k and a single propagator $i/(k^2 - m^2 + i\epsilon)$. Using (H.2), the diagram splits into a real principal-value part and an imaginary δ-function part. For a single propagator there is no kinematical constraint that forces the δ to be saturated, and the imaginary piece vanishes after the loop integration. But if the diagram contains *two or more* propagators that can be put on shell simultaneously — meaning the energy-momentum constraints of the diagram admit a solution where each of those propagators sits at its pole — then their δ-functions reinforce, and a finite imaginary part survives. The condition that the constraints admit such a solution is precisely the kinematical threshold $p^2 \geq (\sum m_i)^2$ in Step 1.

Example 1: the bubble diagram (one-loop self-energy)

Consider the simplest one-loop self-energy in a scalar theory with cubic coupling $g\varphi^2\chi$, where χ is a second scalar of mass μ and φ has mass m. The one-loop φ self-energy contains the *bubble* diagram with two internal χ propagators, Fig. H.2.

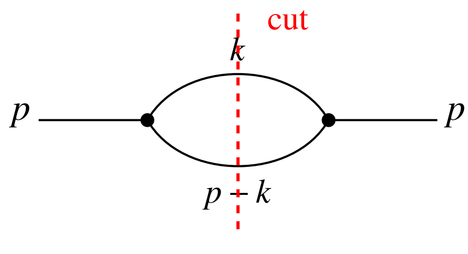

Figure H.2. One-loop bubble contribution to the φ self-energy with two internal χ propagators of mass μ. The dashed red line shows the (unique) Cutkosky cut, which sends both internal lines on shell.

The diagram contributes to $\Pi(p^2)$ as

$$-i\Pi_{\text{bubble}}(p^2) = (-ig)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - \mu^2 + i\epsilon} \frac{i}{(p-k)^2 - \mu^2 + i\epsilon}. \tag{H.6}$$

Cuts. The only kinematically allowed cut goes through both internal lines simultaneously. By Step 2,

$$\frac{i}{k^2-\mu^2+i\epsilon} \longrightarrow 2\pi\,\delta(k^2-\mu^2)\,\theta(k^0)\,, \tag{H.7}$$

$$\frac{i}{(p-k)^2-\mu^2+i\epsilon} \longrightarrow 2\pi\,\delta((p-k)^2-\mu^2)\,\theta(p^0-k^0)\,. \tag{H.8}$$

The cut diagram becomes (after also flipping the sign of $i\epsilon$ on the right of the cut, which has no propagators here),

$$2\,\mathrm{Im}\,\Pi_{\text{bubble}}(p^2) \;=\; g^2\int\frac{d^4k}{(2\pi)^2}\,\delta(k^2-\mu^2)\theta(k^0)\,\delta((p-k)^2-\mu^2)\theta(p^0-k^0)\,. \tag{H.9}$$

The two δ-functions are precisely the Lorentz-invariant two-body phase space $d\Pi_2$ for a final state of two on-shell χ's, in agreement with the optical theorem (8.176). *Kinematical threshold.* The cut is allowed only if $\sqrt{p^2} \geq 2\mu$, i.e. $p^2 \geq (2\mu)^2 = 4\mu^2$. Hence $\mathrm{Im}\,\Pi_{\text{bubble}}(p^2)$, and therefore the bubble's contribution to $\rho(p^2)$, switches on at the two-χ production threshold and is identically zero for $p^2 < 4\mu^2$. This is precisely the multi-particle threshold of the two-χ channel in the spectral density.

Example 2: the sunset diagram (two-loop self-energy in φ^4)

In a $\mathbb{Z}_2$-symmetric φ^4 theory, $\langle 0|\varphi(0)|n\rangle$ vanishes for any state with an even number of particles by parity. The lowest non-trivial multi-particle channel is therefore the three-particle channel, with threshold $3m$. The corresponding 1PI two-point diagram is the two-loop *sunset* (or "sunrise") diagram, Fig. H.3.

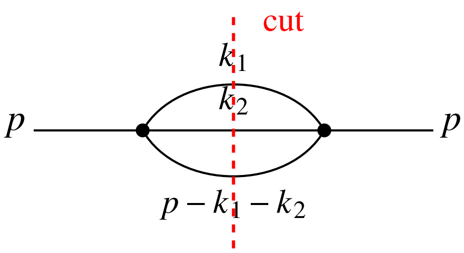

Figure H.3. Two-loop sunset contribution to the φ self-energy in φ^4 theory, with three internal φ propagators. The unique Cutkosky cut goes through all three internal lines, producing a three-particle on-shell intermediate state.

Cuts. The only kinematically allowed cut slices through all three internal φ lines simultaneously. By Step 2, each φ propagator is replaced by its on-shell mass-shell projector $2\pi\,\delta(k_i^2 - m^2)\theta(k_i^0)$, producing the Lorentz-invariant three-body phase space.

Kinematical threshold. The cut requires $\sqrt{p^2} \geq 3m$, i.e. $p^2 \geq 9m^2$. Hence the sunset diagram contributes to $\mathrm{Im}\,\Pi(p^2)$ and to $\rho_{\text{cont.}}(\sigma^2)$ only for $\sigma^2 \geq 9m^2$, opening the three-particle threshold that is the lowest continuum threshold of the spectral density in this theory — exactly as predicted by the non-perturbative analysis of Chapter 8.

Connection to the spectral density

Putting the pieces together, the diagrammatic procedure can be summarized as:

1. Compute the 1PI self-energy $\Pi(p^2)$ in perturbation theory to the desired order.
2. Take its imaginary part using the Cutkosky cutting rules.
3. Insert into the relation

$$\mathrm{Im}\,\Delta_F'(p^2) = \frac{\mathrm{Im}\,\Pi(p^2)}{\left|p^2 - m_0^2 - \Pi(p^2)\right|^2}, \tag{H.10}$$

   obtained from the Dyson form (8.155), and apply the inversion formula (8.173) to recover the spectral density $\rho(p^2)$.

The result is, by construction, consistent with the non-perturbative spectral structure: each kinematically allowed Cutkosky cut of a self-energy diagram corresponds to a multi-particle on-shell intermediate state, and contributes to $\rho_{\text{cont.}}(\sigma^2)$ only above the threshold $(\sum_i m_i)^2$ of that channel. Higher-order perturbative diagrams open higher-particle thresholds; non-perturbative effects (such as bound states) reveal themselves as additional isolated poles below the continuum, beyond the reach of any finite perturbative resummation.

I Derivation of the Gell-Mann–Low formula

The goal of this appendix is to derive, step by step and pedagogically, the master identity that connects the *Heisenberg-picture* Green's function of the full interacting theory to a vacuum expectation value of *interaction-picture* fields, which is the object that perturbation theory and Wick's theorem actually compute:

$$\boxed{G(x_1,\ldots,x_n) \;=\; \langle\Omega|T\big(\varphi(x_1)\cdots\varphi(x_n)\big)|\Omega\rangle \;=\; \frac{\langle 0|T\big\{S\,\varphi_I(x_1)\cdots\varphi_I(x_n)\big\}|0\rangle}{\langle 0|S|0\rangle}} \tag{I.1}$$

This is the *Gell-Mann–Low formula* (sometimes called the Gell-Mann–Low theorem). It is the bridge between the LSZ machinery and the diagrammatic expansion. We follow Peskin & Schroeder, Ch. 4.2–4.3 [6]; the equivalent derivation in Mandl & Shaw, Ch. 6 [1] adopts the same logic with slightly different notation.

Setup: three pictures and the reference time t_0

Recall the three standard pictures of quantum mechanics:

- **Schrödinger picture**: states $|\psi_S(t)\rangle$ carry the time dependence, operators O_S do not. Evolution: $i\partial_t|\psi_S\rangle = H|\psi_S\rangle$.
- **Heisenberg picture**: states $|\psi_H\rangle$ are time-independent, operators $O_H(t) = e^{iH(t-t_0)}O_S e^{-iH(t-t_0)}$ carry the full evolution under the *interacting* Hamiltonian $H = H_0 + H_{\text{int}}$.
- **Interaction picture**: operators $O_I(t) = e^{iH_0(t-t_0)}O_S e^{-iH_0(t-t_0)}$ evolve only with the free Hamiltonian H_0; the residual time evolution is carried by the states, $|\psi_I(t)\rangle = e^{iH_0(t-t_0)}|\psi_S(t)\rangle$.

The three pictures are defined to coincide at one reference time t_0:

$$\varphi_H(t_0,\vec{x}) \;=\; \varphi_I(t_0,\vec{x}) \;=\; \varphi_S(\vec{x})\,. \tag{I.2}$$

This identification at t_0 is the linchpin of the whole construction. We will eventually take t_0 to be arbitrary and the times $\pm T$ at the boundaries to $\pm\infty(1-i\epsilon)$.

The interacting vacuum $|\Omega\rangle$ (ground state of H) is, by definition, time-independent in the Heisenberg picture: $H|\Omega\rangle = E_\Omega|\Omega\rangle$. The free vacuum $|0\rangle$ is the ground state of H_0: $H_0|0\rangle = 0$ (we choose the zero-point energy so that $E_0^{(0)} = 0$).

Step 1: Relating φ_H and φ_I via the operator $U(t,t_0)$

Combining the definitions above,

$$\begin{aligned}\varphi_H(t,\vec{x}) &= e^{iH(t-t_0)}\,\varphi_S(\vec{x})\,e^{-iH(t-t_0)}\\ &= e^{iH(t-t_0)}\,e^{-iH_0(t-t_0)}\,\varphi_I(t,\vec{x})\,e^{iH_0(t-t_0)}\,e^{-iH(t-t_0)}\\ &= U^\dagger(t,t_0)\,\varphi_I(t,\vec{x})\,U(t,t_0)\,, \end{aligned} \tag{I.3}$$

where we have *defined*

$$U(t,t_0) \;\equiv\; e^{iH_0(t-t_0)}\,e^{-iH(t-t_0)}\,. \tag{I.4}$$

$U(t,t_0)$ is the *interaction-picture time-evolution operator*: it advances a state from t_0 to t in the interaction picture.

Key properties of $U(t,t')$.

1. $U(t_0,t_0) = 1$ (trivially from (I.4)).
2. $U(t,t_0)$ satisfies the Schrödinger-like equation

$$i\,\partial_t U(t,t_0) \;=\; H_I(t)\,U(t,t_0)\,, \tag{I.5}$$

where $H_I(t) \equiv e^{iH_0(t-t_0)}\,H_{\text{int}}\,e^{-iH_0(t-t_0)}$ is the interaction Hamiltonian in the interaction picture. Proof: differentiate (I.4) directly. The result is solved by the **Dyson series**:

$$U(t,t_0) \;=\; T\exp\left[-i\int_{t_0}^{t} dt'\,H_I(t')\right]. \tag{I.6}$$

3. *Composition law.* For three times t_1,t_2,t_3,

$$U(t_1,t_2)\,U(t_2,t_3) \;=\; U(t_1,t_3)\,, \tag{I.7}$$

where the general two-argument U is

$$U(t,t') \;\equiv\; e^{iH_0(t-t_0)}\,e^{-iH(t-t')}\,e^{-iH_0(t'-t_0)}\,. \tag{I.8}$$

One verifies that this reduces to (I.4) when $t' = t_0$, satisfies the same equation of motion (I.5), and obeys $U^\dagger(t,t') = U(t',t)$.

Iterating (I.7) we obtain the useful identity that will be crucial in the final step:

$$\begin{aligned}U(t_1,t_0)\,U^\dagger(t_2,t_0) &= U(t_1,t_0)\,U(t_0,t_2)\\ &= U(t_1,t_2)\,. \end{aligned} \tag{I.9}$$

Step 2: The interacting vacuum from the free vacuum (adiabatic trick)

We now construct $|\Omega\rangle$ starting from $|0\rangle$. The idea — which goes back to Gell-Mann and Low — is to project onto the ground state of H by "cooling" through imaginary time.

Expand the free vacuum on the basis of eigenstates of the full Hamiltonian, $H|n\rangle = E_n|n\rangle$ (with $|\Omega\rangle$ the unique ground state, $E_\Omega < E_n$ for $n \neq \Omega$):

$$|0\rangle = |\Omega\rangle\langle\Omega|0\rangle + \sum_{n\neq\Omega} |n\rangle\langle n|0\rangle\,. \tag{I.10}$$

Apply e^{-iHT} for some large $T > 0$:

$$e^{-iHT}|0\rangle = e^{-iE_\Omega T}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n\neq\Omega} e^{-iE_nT}|n\rangle\langle n|0\rangle\,. \tag{I.11}$$

On the real axis this is just a sum of oscillating phases. The trick is to rotate T slightly into the lower half plane, $T \to T(1-i\epsilon)$ with $\epsilon \to 0^+$ at the end. Then

$$e^{-iE_nT(1-i\epsilon)} = e^{-iE_nT}\, e^{-E_nT\epsilon}\,, \tag{I.12}$$

and, after factoring out the ground-state contribution, an excited state is suppressed relative to $|\Omega\rangle$ by

$$\exp\big[-(E_n - E_\Omega)T\epsilon\big]\,. \tag{I.13}$$

Since $E_n > E_\Omega$ for every excited state, all excited-state components disappear in the limit $T \to \infty$. We get

$$|\Omega\rangle \;=\; \lim_{T\to\infty(1-i\epsilon)} \frac{e^{-iHT}|0\rangle}{e^{-iE_\Omega T}\,\langle\Omega|0\rangle}\,. \tag{I.14}$$

This is the central "$i\epsilon$ rotation" identity. The numerator alone does not converge to $|\Omega\rangle$: the divergent phase factor $e^{-iE_\Omega T}$ and the overlap $\langle\Omega|0\rangle$ are subtracted explicitly so that the ratio is finite and unit-normalised.

Recasting in terms of U. We rewrite the right-hand side as the action of an interaction-picture evolution operator on the free vacuum. Replace $T \to t_0 + T$ in (I.14) (a harmless shift):

$$|\Omega\rangle \;=\; \lim_{T\to\infty(1-i\epsilon)} \frac{e^{-iH(t_0+T)}\,|0\rangle}{e^{-iE_\Omega(t_0+T)}\,\langle\Omega|0\rangle}\,. \tag{I.15}$$

Since $H_0|0\rangle = 0$, we may freely insert $e^{-iH_0(-T-t_0)}|0\rangle = |0\rangle$ on the right of the numerator:

$$\begin{aligned} e^{-iH(t_0+T)}|0\rangle &= e^{-iH(t_0-(-T))}\, e^{-iH_0(-T-t_0)}|0\rangle \\ &= U(t_0,-T)\,|0\rangle\,, \end{aligned} \tag{I.16}$$

where we used (I.8). Therefore

$$\boxed{|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{U(t_0,-T)\,|0\rangle}{e^{-iE_\Omega(t_0+T)}\,\langle\Omega|0\rangle}} \tag{I.17}$$

By an analogous argument applied to the bra,

$$\boxed{\langle\Omega| = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|\,U(T,t_0)}{e^{-iE_\Omega(T-t_0)}\,\langle 0|\Omega\rangle}} \tag{I.18}$$

Equations (I.17)–(I.18) are the key result of this step: they express the interacting vacuum entirely in terms of objects living in the interaction picture, plus c-number normalisation factors that will cancel in the next step.

Step 3: Putting it together for the n-point function

Consider the Heisenberg correlator

$$G(x_1,\ldots,x_n) = \langle\Omega|T\big(\varphi_H(x_1)\cdots\varphi_H(x_n)\big)|\Omega\rangle\,. \tag{I.19}$$

Without loss of generality (the time-ordering symbol takes care of permutations) assume the points are already time-ordered, $x_1^0 > x_2^0 > \cdots > x_n^0$. Drop the T symbol while we manipulate the operators, and use (I.3) on each field:

$$\begin{aligned} &\langle\Omega|\varphi_H(x_1)\varphi_H(x_2)\cdots\varphi_H(x_n)|\Omega\rangle \\ &= \langle\Omega|\,U^\dagger(x_1^0,t_0)\,\varphi_I(x_1)\,U(x_1^0,t_0) \\ &\qquad\quad \times U^\dagger(x_2^0,t_0)\,\varphi_I(x_2)\,U(x_2^0,t_0) \\ &\qquad\quad \times\cdots \\ &\qquad\quad \times U^\dagger(x_n^0,t_0)\,\varphi_I(x_n)\,U(x_n^0,t_0)\,|\Omega\rangle\,. \end{aligned} \tag{I.20}$$

Now collapse the U-pairs using the chain identity (I.9), $U(t_i^0,t_0)\,U^\dagger(t_{i+1}^0,t_0) = U(t_i^0,t_{i+1}^0)$:

$$\begin{aligned} &= \langle\Omega|\,U^\dagger(x_1^0,t_0) \\ &\qquad \times\varphi_I(x_1)\,U(x_1^0,x_2^0)\,\varphi_I(x_2)\,U(x_2^0,x_3^0)\cdots U(x_{n-1}^0,x_n^0)\,\varphi_I(x_n) \\ &\qquad \times U(x_n^0,t_0)\,|\Omega\rangle\,. \end{aligned} \tag{I.21}$$

The structure is now a chain of fields and U-operators sandwiched between $\langle\Omega|U^\dagger(x_1^0,t_0)$ on the left and $U(x_n^0,t_0)|\Omega\rangle$ on the right. Both extremes are absorbed by the vacuum identities of Step 2:

- Right end: from (I.17) and (I.9),

$$\begin{aligned} U(x_n^0,t_0)|\Omega\rangle &= \lim_{T\to\infty(1-i\epsilon)} \frac{U(x_n^0,t_0)\,U(t_0,-T)\,|0\rangle}{e^{-iE_\Omega(t_0+T)}\,\langle\Omega|0\rangle} \\ &= \lim \frac{U(x_n^0,-T)|0\rangle}{e^{-iE_\Omega(t_0+T)}\,\langle\Omega|0\rangle}\,. \end{aligned} \tag{I.22}$$

- Left end: similarly, from (I.18) and (I.9), using $U^\dagger(x_1^0,t_0) = U(t_0,x_1^0)$,

$$\begin{aligned} \langle\Omega|U^\dagger(x_1^0,t_0) &= \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|\,U(T,t_0)\,U(t_0,x_1^0)}{e^{-iE_\Omega(T-t_0)}\,\langle 0|\Omega\rangle} \\ &= \lim \frac{\langle 0|U(T,x_1^0)}{e^{-iE_\Omega(T-t_0)}\,\langle 0|\Omega\rangle}\,. \end{aligned} \tag{I.23}$$

Substituting both into (I.21), the numerator becomes

$$\langle 0|\,U(T,x_1^0)\,\varphi_I(x_1)\,U(x_1^0,x_2^0)\,\varphi_I(x_2)\cdots U(x_{n-1}^0,x_n^0)\,\varphi_I(x_n)\,U(x_n^0,-T)\,|0\rangle\,. \tag{I.24}$$

The denominator carries the c-number $e^{-iE_\Omega\cdot 2T}\,\langle\Omega|0\rangle\langle 0|\Omega\rangle$, which we will deal with below.

Recognising the time-ordered S-operator. With times satisfying $T > x_1^0 > x_2^0 > \cdots > x_n^0 > -T$, every operator in the string displayed in (I.24) is already in chronological order from left (latest) to right (earliest). The time-ordering symbol therefore preserves the string, and we can rewrite it as

$$\begin{aligned} &\langle 0|T\big\{\varphi_I(x_1)\cdots\varphi_I(x_n)\,U(T,x_1^0)U(x_1^0,x_2^0) \\ &\qquad \cdots U(x_n^0,-T)\big\}|0\rangle\,. \end{aligned} \tag{I.25}$$

Inside the T-symbol, the product of all the U's telescopes (composition law (I.7)) into a single one spanning the full range $[-T,T]$:

$$U(T,x_1^0)\,U(x_1^0,x_2^0)\cdots U(x_n^0,-T) \;=\; U(T,-T)\,. \tag{I.26}$$

Taking the limit $T\to\infty(1-i\epsilon)$ we define the *S-matrix operator in the interaction picture*:

$$S \;\equiv\; \lim_{T\to\infty(1-i\epsilon)} U(T,-T) \;=\; T\exp\!\left[-i\int_{-\infty}^{+\infty} dt\,H_I(t)\right]. \tag{I.27}$$

The numerator of (I.20), multiplied by its diverging prefactor, is therefore

$$\lim_{T\to\infty(1-i\epsilon)} \langle 0|T\{\varphi_I(x_1)\cdots\varphi_I(x_n)\,U(T,-T)\}\,|0\rangle = \langle 0|T\big\{\varphi_I(x_1)\cdots\varphi_I(x_n)\,S\big\}|0\rangle\,. \tag{I.28}$$

Step 4: The denominator and the cancellation of vacuum divergences

Going back to the full ratio: combining the Heisenberg bra and ket with their normalisation factors, we collected the c-number

$$\mathcal{N} \equiv e^{-iE_\Omega \cdot 2T} \langle\Omega|0\rangle \langle 0|\Omega\rangle \tag{I.29}$$

in the denominator. We now show that $\mathcal{N}$ is exactly cancelled by the vacuum-to-vacuum matrix element of S.

Apply the very same construction (I.17)–(I.18) to the trivial correlator $\langle\Omega|\Omega\rangle$ (with no field insertions):

$$1 = \langle\Omega|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|U(T,-T)|0\rangle}{\mathcal{N}} = \frac{\langle 0|S|0\rangle}{\mathcal{N}}. \tag{I.30}$$

Therefore $\mathcal{N} = \langle 0|S|0\rangle$. The diverging normalisation in the denominator of G is nothing but the vacuum-to-vacuum S-matrix element — which itself is a non-trivial divergent quantity (the sum of all disconnected "vacuum bubbles"), but it is exactly the factor needed to cancel them in the numerator.

Putting everything together: the Gell-Mann–Low formula

Combining (I.28) with $\mathcal{N} = \langle 0|S|0\rangle$ from (I.30):

$$\boxed{\begin{aligned} G(x_1,\ldots,x_n) &= \langle\Omega|T\big(\varphi_H(x_1)\cdots\varphi_H(x_n)\big)|\Omega\rangle \\ &= \frac{\langle 0|T\big\{\varphi_I(x_1)\cdots\varphi_I(x_n)\,S\big\}|0\rangle}{\langle 0|S|0\rangle}. \end{aligned}} \tag{I.31}$$

This is exactly equation (I.1), the formula used in the main text.

Physical and computational reading

The Gell-Mann–Low formula has both a computational and a conceptual content.

Computational content. The right-hand side is something the perturbative machinery handles directly:

- Expand the interaction-picture S-operator:

$$S = T\exp\left[-i\int H_I\,dt\right] = \sum_{k=0}^{\infty} \frac{(-i)^k}{k!} \int dt_1 \cdots dt_k\, T\{H_I(t_1)\cdots H_I(t_k)\}.$$

- Each term is a vacuum expectation of a time-ordered product of *free* (interaction-picture) fields, evaluated on the *free* vacuum $|0\rangle$.
- Wick's theorem reduces it to sums of products of free Feynman propagators Δ_F — i.e. to Feynman diagrams.

Conceptual content: cancellation of vacuum bubbles. The denominator $\langle 0|S|0\rangle$ does crucial bookkeeping. Order by order, the numerator $\langle 0|T\{\varphi_I \cdots \varphi_I S\}|0\rangle$ is a sum of two kinds of diagrams:

- diagrams with no vacuum bubbles, in which every connected component contains at least one external field $\varphi_I(x_i)$;
- disconnected vacuum pieces, in which an interaction sub-graph involves no external $\varphi_I(x_i)$ and just produces a "vacuum bubble".

The vacuum bubbles factorise out and give exactly the expansion of $\langle 0|S|0\rangle$:

$$\langle 0|T\{\varphi_I(x_1)\cdots\varphi_I(x_n)\,S\}|0\rangle \;=\; \Big[\sum(\text{connected w.r.t. at least one } x_i)\Big] \times \langle 0|S|0\rangle\,. \tag{I.32}$$

Hence dividing by $\langle 0|S|0\rangle$ removes the vacuum bubbles, leaving only diagrams in which every component is connected to at least one external point. This is the *linked-cluster theorem*, derived in detail in Peskin Ch. 4.4 and Mandl–Shaw Ch. 6.

The role of the $i\epsilon$ rotation. The slight tilt $T \to \infty(1-i\epsilon)$ is not optional. Without it, $e^{-iHT}|0\rangle$ would oscillate forever and never project onto $|\Omega\rangle$. The tilt is the same one that picks the *Feynman* contour in the propagator, $i/(p^2-m^2+i\epsilon)$, and the same one implicitly used in the LSZ asymptotic conditions to define in/out states. All of these are aspects of one and the same prescription: tilt the contour into the lower half plane, project onto the ground state, and analytically continue back.

Why the choice of t_0 is irrelevant. The intermediate steps depended on a reference time t_0 at which the pictures coincide. But t_0 never appears in the final formula (I.31): it cancels between the bra and ket sides, between U and $U^\dagger$, and between the numerator and the denominator. Physically this expresses the fact that the choice of reference time is gauge in this construction; what matters is the asymptotic structure $\pm T \to \pm\infty(1-i\epsilon)$ that selects the interacting vacuum.

Comparison with the LSZ side. The Gell-Mann–Low formula computes Green's functions $G(x_1,\ldots,x_n)$, which are off-shell objects in coordinate space. The LSZ formula then takes these Green's functions, Fourier transforms them, amputates external propagators, and puts the external momenta on shell, yielding the S-matrix elements $\langle f|S|i\rangle$. Schematically:

$$\langle 0|T\{\varphi_I\cdots\varphi_I S\}|0\rangle/\langle 0|S|0\rangle \xrightarrow{\text{Gell-Mann–Low}} G(x_1,\ldots,x_n) \xrightarrow{\text{LSZ}} S\text{-matrix.}$$

The first arrow takes us from a perturbatively computable expression to the interacting Green's function; the second arrow extracts the physical observable.

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