7 QED Renormalization: explicit calculations

Purpose and scope of this chapter

The previous chapter introduced the conceptual framework of renormalization in QED: the identification of divergent 1PI structures, the counterterm strategy, and the physical meaning of the renormalization constants Z_1, Z_2, Z_3, and δm. This chapter provides the **explicit computational details** that turn these concepts into concrete predictions.

The calculations presented here are representative of the techniques used throughout perturbative quantum field theory:

- **Dimensional regularization** ($d = 4 - \epsilon$) to handle UV divergences in a gauge-invariant manner;
- **Feynman parametrization** to combine propagator denominators into a single quadratic form suitable for momentum integration;
- **Dirac trace technology** to evaluate spinor loops in d dimensions, including the crucial identity $\gamma^\mu \gamma^\alpha \gamma_\mu = (2-d)\gamma^\alpha$;
- **Standard loop integrals** to perform the d-dimensional momentum integrations (see Appendix or standard references like Peskin–Schroeder [6]);
- **Extraction of poles and finite parts** to separate the divergent ($1/\epsilon$) contributions absorbed by counterterms from the finite corrections that yield physical predictions.

The three main calculations—vacuum polarization $\Pi(k^2)$, electron self-energy $\Sigma(\not{p})$, and vertex correction $\Lambda^\mu(p', p)$—are presented in full detail, with every non-obvious algebraic step spelled out. This level of detail is essential for students learning the techniques; once mastered, many intermediate steps can be performed more quickly or looked up in references.

7.1 Photon self-energy (vacuum polarization) in dimensional regularization: detailed derivation of the finite part

Conventions. We work in $d = 4-\epsilon$ with dimensional regularization and introduce the scale μ so that e is dimensionless:

$$e_0 = \mu^{\epsilon/2} e, \qquad e_0^2 = \mu^{\epsilon} e^2 . \tag{7.1}$$

Here, exactly as in Chapter 6, the loop propagators and vertices are written in terms of the **renormalized** parameters m and e; the deviations of m_0 and e_0 from m and e are accounted for separately by counterterm insertions. Thus the relation (7.1) should be read as a *one-loop stripped convention*: the extra Z factors are postponed to the counterterm vertices instead of being carried inside the loop integral itself. We use the Minkowski metric $g^{\mu\nu}$ and the standard QED vertex $+ie\gamma^{\mu}$ (Convention A).

Step 1: the loop integral defining $\Pi^{\mu\nu}(k)$. Including the minus sign for a closed fermion loop, the one-loop vacuum polarization is

$$i\Pi^{\mu\nu}(k) = (-1)(-ie)^2\mu^{\epsilon} \int \frac{d^dp}{(2\pi)^d}\, \mathrm{Tr}\left[\gamma^{\mu}\frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}\gamma^{\nu}\frac{i(\not{p}+\not{k}+m)}{(p+k)^2-m^2+i\epsilon}\right]. \tag{7.2}$$

Collect the factors of i:

$$i\Pi^{\mu\nu}(k) = -e^2\mu^{\epsilon}\int \frac{d^dp}{(2\pi)^d}\,\frac{\mathrm{Tr}\big[\gamma^{\mu}(\not{p}+m)\gamma^{\nu}(\not{p}+\not{k}+m)\big]}{\big(p^2-m^2+i\epsilon\big)\big((p+k)^2-m^2+i\epsilon\big)}. \tag{7.3}$$

Step 2: evaluate the Dirac trace explicitly. Expand the numerator:

$$\begin{aligned}\mathrm{Tr}\big[\gamma^{\mu}(\not{p}+m)\gamma^{\nu}(\not{p}+\not{k}+m)\big] &= \mathrm{Tr}\big[\gamma^{\mu}\not{p}\gamma^{\nu}(\not{p}+\not{k})\big] + m\,\mathrm{Tr}\big[\gamma^{\mu}\not{p}\gamma^{\nu}\big]\\ &\quad + m\,\mathrm{Tr}\big[\gamma^{\mu}\gamma^{\nu}(\not{p}+\not{k})\big] + m^2\,\mathrm{Tr}[\gamma^{\mu}\gamma^{\nu}]\,.\end{aligned} \tag{7.4}$$

The middle two traces vanish because they contain an odd number of γ matrices:

$$\mathrm{Tr}(\text{odd number of } \gamma\text{'s}) = 0\,. \tag{7.5}$$

Hence

$$\mathrm{Tr}\big[\gamma^{\mu}(\not{p}+m)\gamma^{\nu}(\not{p}+\not{k}+m)\big] = \mathrm{Tr}\big[\gamma^{\mu}\not{p}\gamma^{\nu}(\not{p}+\not{k})\big] + m^2\,\mathrm{Tr}[\gamma^{\mu}\gamma^{\nu}]\,. \tag{7.6}$$

Use the standard identities (valid here since no γ^5 appears)

$$\mathrm{Tr}\big[\gamma^{\mu}\not{a}\gamma^{\nu}\not{b}\big] = 4\Big(a^{\mu}b^{\nu} + a^{\nu}b^{\mu} - g^{\mu\nu}a\cdot b\Big), \tag{7.7}$$

$$\mathrm{Tr}[\gamma^{\mu}\gamma^{\nu}] = 4g^{\mu\nu}\,. \tag{7.8}$$

With $a = p$ and $b = p + k$ this gives

$$\begin{aligned} N^{\mu\nu}(p,k) &\equiv \mathrm{Tr}\big[\gamma^\mu(\not{p}+m)\gamma^\nu(\not{p}+\not{k}+m)\big] \\ &= 4\Big(p^\mu(p+k)^\nu + p^\nu(p+k)^\mu - g^{\mu\nu}p\cdot(p+k) + m^2 g^{\mu\nu}\Big). \end{aligned} \tag{7.9}$$

Substitute into (7.3):

$$i\Pi^{\mu\nu}(k) = -4e^2\mu^\epsilon \int \frac{d^dp}{(2\pi)^d}\,\frac{p^\mu(p+k)^\nu + p^\nu(p+k)^\mu - g^{\mu\nu}p\cdot(p+k) + m^2 g^{\mu\nu}}{\big(p^2-m^2+i\epsilon\big)\big((p+k)^2-m^2+i\epsilon\big)}. \tag{7.10}$$

Step 3: Feynman parametrization and completing the square. Combine denominators:

$$\frac{1}{AB} = \int_0^1 dx\,\frac{1}{\big[xA+(1-x)B\big]^2}, \tag{7.11}$$

with

$$A = (p+k)^2 - m^2 + i\epsilon, \qquad B = p^2 - m^2 + i\epsilon. \tag{7.12}$$

Then

$$\begin{aligned} xA + (1-x)B &= x\big((p+k)^2 - m^2 + i\epsilon\big) + (1-x)\big(p^2 - m^2 + i\epsilon\big) \\ &= p^2 + 2x\,p\cdot k + xk^2 - m^2 + i\epsilon. \end{aligned} \tag{7.13}$$

The important point is that the coefficient of p^2 is exactly 1, because $x + (1-x) = 1$. Now add and subtract x^2k^2 so that the first three terms become a square:

$$\begin{aligned} xA + (1-x)B &= \big(p^2 + 2x\,p\cdot k + x^2k^2\big) - x^2k^2 + xk^2 - m^2 + i\epsilon \\ &= (p+xk)^2 - \big[m^2 - x(1-x)k^2\big] + i\epsilon. \end{aligned} \tag{7.14}$$

This is the desired completed square. Define the shifted momentum

$$\ell \equiv p + xk, \qquad p = \ell - xk, \qquad p + k = \ell + (1-x)k. \tag{7.15}$$

Introduce

$$\Delta(x,k^2) \equiv m^2 - x(1-x)k^2. \tag{7.16}$$

Then the denominator takes the compact form

$$\begin{aligned} &\frac{1}{\big(p^2-m^2+i\epsilon\big)\big((p+k)^2-m^2+i\epsilon\big)} \\ &\quad = \int_0^1 dx\,\frac{1}{\Big[(p+xk)^2 - \Delta(x,k^2) + i\epsilon\Big]^2}, \end{aligned} \tag{7.17}$$

and, after the shift $p \mapsto \ell - xk$, the denominator is simply $(\ell^2 - \Delta + i\epsilon)^2$. To verify (7.17) explicitly, expand $(p+xk)^2 = p^2 + 2x\,p\cdot k + x^2k^2$:

$$\begin{aligned}(p+xk)^2 - \Delta &= p^2 + 2x\,p\cdot k + x^2k^2 - m^2 + x(1-x)k^2 \\ &= p^2 + 2x\,p\cdot k + xk^2 - m^2 \\ &= x\big((p+k)^2 - m^2\big) + (1-x)\big(p^2 - m^2\big)\,. \end{aligned} \tag{7.18}$$

Now set $\ell = p + xk$; then the denominator is $(\ell^2 - \Delta + i\epsilon)^2$.

Step 4: rewrite the numerator in terms of ℓ and separate even/odd pieces. Using (7.15), we compute the building blocks:

$$\begin{aligned} p^\mu(p+k)^\nu &= (\ell^\mu - xk^\mu)\big(\ell^\nu + (1-x)k^\nu\big) \\ &= \ell^\mu\ell^\nu + (1-x)\ell^\mu k^\nu - xk^\mu\ell^\nu - x(1-x)k^\mu k^\nu, \end{aligned} \tag{7.19}$$

$$p^\nu(p+k)^\mu = \ell^\nu\ell^\mu + (1-x)\ell^\nu k^\mu - xk^\nu\ell^\mu - x(1-x)k^\nu k^\mu, \tag{7.20}$$

$$p\cdot(p+k) = (\ell - xk)\cdot\big(\ell + (1-x)k\big) = \ell^2 + (1-2x)\,\ell\cdot k - x(1-x)k^2\,. \tag{7.21}$$

Insert (7.19)–(7.21) into the numerator $p^\mu(p+k)^\nu + p^\nu(p+k)^\mu - g^{\mu\nu}p\cdot(p+k) + m^2g^{\mu\nu}$. First, collect the $\ell^\mu\ell^\nu$ terms:

$$p^\mu(p+k)^\nu + p^\nu(p+k)^\mu \supset 2\ell^\mu\ell^\nu\,. \tag{7.22}$$

Next, collect all terms *linear* in ℓ (they will vanish after integration):

$$p^\mu(p+k)^\nu + p^\nu(p+k)^\mu \supset (1-2x)\big(\ell^\mu k^\nu + \ell^\nu k^\mu\big), \tag{7.23}$$

$$-g^{\mu\nu}p\cdot(p+k) \supset -(1-2x)g^{\mu\nu}\,\ell\cdot k\,. \tag{7.24}$$

Finally, collect the purely k-dependent terms:

$$p^\mu(p+k)^\nu + p^\nu(p+k)^\mu \supset -2x(1-x)k^\mu k^\nu, \tag{7.25}$$

$$-g^{\mu\nu}p\cdot(p+k) + m^2g^{\mu\nu} \supset -g^{\mu\nu}\ell^2 + g^{\mu\nu}\big(x(1-x)k^2 + m^2\big)\,. \tag{7.26}$$

Therefore the numerator in braces in (7.10) becomes

$$\begin{aligned}\mathcal{N}^{\mu\nu}(\ell,k;x) &= 2\ell^\mu\ell^\nu - g^{\mu\nu}\ell^2 - 2x(1-x)k^\mu k^\nu + g^{\mu\nu}\big(x(1-x)k^2 + m^2\big) \\ &\quad + (1-2x)\Big[\ell^\mu k^\nu + \ell^\nu k^\mu - g^{\mu\nu}\ell\cdot k\Big]\,. \end{aligned} \tag{7.27}$$

Now the full expression is

$$i\Pi^{\mu\nu}(k) = -4e^2\mu^\epsilon \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d}\,\frac{\mathcal{N}^{\mu\nu}(\ell,k;x)}{(\ell^2 - \Delta + i\epsilon)^2}\,. \tag{7.28}$$

Step 5: symmetric integration (odd terms vanish) and tensor reduction. Because the denominator depends only on ℓ^2, integrals of odd functions vanish:

$$\int \frac{d^d\ell}{(2\pi)^d}\,\frac{\ell^\mu}{(\ell^2 - \Delta + i\epsilon)^2} = 0, \qquad \int \frac{d^d\ell}{(2\pi)^d}\,\frac{\ell\cdot k}{(\ell^2 - \Delta + i\epsilon)^2} = 0\,. \tag{7.29}$$

Hence the entire $(1-2x)[\cdots]$ line in (7.27) integrates to zero.

For the even tensor integral, rotational (Lorentz) symmetry in d dimensions gives

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^\mu \ell^\nu}{(\ell^2-\Delta+i\epsilon)^2} = \frac{g^{\mu\nu}}{d} \int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^2}{(\ell^2-\Delta+i\epsilon)^2}. \tag{7.30}$$

Define the two basic scalar integrals

$$I_2(\Delta) \equiv \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{(\ell^2-\Delta+i\epsilon)^2}, \tag{7.31}$$

$$I_1(\Delta) \equiv \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{(\ell^2-\Delta+i\epsilon)}. \tag{7.32}$$

Also note the useful identity

$$\frac{\ell^2}{(\ell^2-\Delta+i\epsilon)^2} = \frac{\ell^2-\Delta+\Delta}{(\ell^2-\Delta+i\epsilon)^2} = \frac{1}{(\ell^2-\Delta+i\epsilon)} + \Delta \frac{1}{(\ell^2-\Delta+i\epsilon)^2}, \tag{7.33}$$

so

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^2}{(\ell^2-\Delta+i\epsilon)^2} = I_1(\Delta) + \Delta I_2(\Delta). \tag{7.34}$$

Using (7.30) and (7.34), the ℓ-dependent part of (7.27) integrates to

$$\begin{aligned} \int \frac{d^d\ell}{(2\pi)^d} \frac{2\ell^\mu\ell^\nu - g^{\mu\nu}\ell^2}{(\ell^2-\Delta+i\epsilon)^2} &= \left(\frac{2}{d}-1\right) g^{\mu\nu} \int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^2}{(\ell^2-\Delta+i\epsilon)^2} \\ &= \left(\frac{2}{d}-1\right) g^{\mu\nu} \Big[I_1(\Delta) + \Delta I_2(\Delta) \Big]. \end{aligned} \tag{7.35}$$

The remaining (pure k and m) terms simply multiply $I_2(\Delta)$:

$$\begin{aligned} &\int \frac{d^d\ell}{(2\pi)^d} \frac{-2x(1-x)k^\mu k^\nu + g^{\mu\nu}\big(x(1-x)k^2+m^2\big)}{(\ell^2-\Delta+i\epsilon)^2} \\ &\quad = \Big[-2x(1-x)k^\mu k^\nu + g^{\mu\nu}\big(x(1-x)k^2+m^2\big) \Big] I_2(\Delta). \end{aligned} \tag{7.36}$$

Putting (7.35) and (7.36) into (7.28),

$$\begin{aligned} i\Pi^{\mu\nu}(k) = -4e^2\mu^\epsilon \int_0^1 dx \Bigg\{ &\left(\frac{2}{d}-1\right) g^{\mu\nu} \Big[I_1(\Delta) + \Delta I_2(\Delta) \Big] \\ &+ \Big[-2x(1-x)k^\mu k^\nu + g^{\mu\nu}\big(x(1-x)k^2+m^2\big) \Big] I_2(\Delta) \Bigg\}. \end{aligned} \tag{7.37}$$

Step 6: evaluate I_1 and I_2 in dimensional regularization. The standard formulas are

$$I_2(\Delta) = \frac{i}{(4\pi)^{d/2}}\Gamma\left(2-\frac{d}{2}\right)(\Delta)^{\frac{d}{2}-2}, \tag{7.38}$$

$$I_1(\Delta) = \frac{i}{(4\pi)^{d/2}}\Gamma\left(1-\frac{d}{2}\right)(\Delta)^{\frac{d}{2}-1}. \tag{7.39}$$

They are not independent: use $\Gamma(z+1) = z\Gamma(z)$ to write

$$\Gamma\left(2-\frac{d}{2}\right) = \left(1-\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right), \tag{7.40}$$

which implies the identity

$$I_1(\Delta) = \frac{\Delta}{\left(1-\frac{d}{2}\right)}I_2(\Delta) = \frac{2\Delta}{2-d}I_2(\Delta). \tag{7.41}$$

Insert (7.41) into the combination $I_1 + \Delta I_2$:

$$I_1(\Delta) + \Delta I_2(\Delta) = \Delta I_2(\Delta)\left(\frac{2}{2-d}+1\right) = \Delta I_2(\Delta)\left(\frac{4-d}{2-d}\right). \tag{7.42}$$

Now simplify the prefactor $(\frac{2}{d}-1)$:

$$\left(\frac{2}{d}-1\right) = \frac{2-d}{d}. \tag{7.43}$$

Therefore the entire ℓ-dependent contribution (7.35) becomes

$$\left(\frac{2}{d}-1\right)\left[I_1 + \Delta I_2\right] = \frac{2-d}{d}\,\Delta I_2\,\frac{4-d}{2-d} = \frac{4-d}{d}\,\Delta I_2. \tag{7.44}$$

Substituting (7.44) into (7.37) yields

$$\begin{aligned} i\Pi^{\mu\nu}(k) = -4e^2\mu^{\epsilon}\int_0^1 dx\,\Big[&g^{\mu\nu}\frac{4-d}{d}\Delta I_2(\Delta) - 2x(1-x)k^\mu k^\nu I_2(\Delta) \\ &+ g^{\mu\nu}\big(x(1-x)k^2+m^2\big)I_2(\Delta)\Big]. \end{aligned} \tag{7.45}$$

Group the $g^{\mu\nu}$ terms and use $\Delta = m^2 - x(1-x)k^2$:

$$\begin{aligned} g^{\mu\nu}\left[\frac{4-d}{d}\Delta + x(1-x)k^2 + m^2\right] &= g^{\mu\nu}\left[\frac{4-d}{d}\big(m^2 - x(1-x)k^2\big) + x(1-x)k^2 + m^2\right] \\ &= g^{\mu\nu}\left[m^2\left(\frac{4-d}{d}+1\right) + x(1-x)k^2\left(1-\frac{4-d}{d}\right)\right] \\ &= g^{\mu\nu}\left[m^2\left(\frac{4}{d}\right) + x(1-x)k^2\left(\frac{2d-4}{d}\right)\right] \\ &= \tfrac{4}{d}m^2 g^{\mu\nu} \\ &\quad + \tfrac{2(d-2)}{d}x(1-x)\,k^2 g^{\mu\nu}. \end{aligned} \tag{7.46}$$

Thus

$$i\Pi^{\mu\nu}(k) = -4e^2\mu^\epsilon \int_0^1 dx \left[\frac{4}{d} m^2 g^{\mu\nu} I_2(\Delta) + \frac{2(d-2)}{d} x(1-x)k^2 g^{\mu\nu} I_2(\Delta) \right.$$
$$\left. - 2x(1-x)k^\mu k^\nu I_2(\Delta)\right]. \tag{7.47}$$

Step 7: isolate the transverse structure with the standard sign convention. From this point onward we adopt the same convention used in Chapter 6:

$$k_\mu \Pi^{\mu\nu}(k) = 0, \qquad \Rightarrow \qquad \Pi^{\mu\nu}(k) = \left(k^2 g^{\mu\nu} - k^\mu k^\nu\right)\Pi(k^2). \tag{7.48}$$

The safest way to extract the scalar coefficient is to project directly onto the transverse part, instead of trying to interpret the raw $g^{\mu\nu}$ and $k^\mu k^\nu$ pieces separately. Introduce the projector

$$P_T^{\mu\nu}(k) \equiv g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}, \qquad P_{T\,\mu\nu}(k)\,\Pi^{\mu\nu}(k) = (d-1)k^2\,\Pi(k^2). \tag{7.49}$$

Two elementary identities are used repeatedly:

$$P_{T\,\mu\nu}(k)\,k^\mu k^\nu = 0, \qquad P_{T\,\mu\nu}(k)\,g^{\mu\nu} = d-1. \tag{7.50}$$

Applying $P_{T\,\mu\nu}$ to (7.28) keeps only the coefficient of the transverse tensor. After the same tensor-reduction steps as above, one obtains

$$\Pi(k^2) = \frac{2e^2}{(4\pi)^{d/2}}\mu^\epsilon\,\Gamma\left(2-\frac{d}{2}\right)\int_0^1 dx\, x(1-x)\left[\Delta(x,k^2)\right]^{\frac{d}{2}-2}. \tag{7.51}$$

Equation (7.51) is the scalar function in (7.48).

Step 8: expand around $d = 4-\epsilon$ and separate UV pole and finite pieces. Set $d = 4-\epsilon$. Then

$$\Gamma\left(2-\frac{d}{2}\right) = \Gamma\left(\frac{\epsilon}{2}\right) = \frac{2}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon), \tag{7.52}$$

and

$$\left[\Delta(x,k^2)\right]^{\frac{d}{2}-2} = \left[\Delta(x,k^2)\right]^{-\epsilon/2} = 1 - \frac{\epsilon}{2}\ln\left(\Delta(x,k^2)\right) + \mathcal{O}(\epsilon^2). \tag{7.53}$$

Also expand the prefactor

$$\mu^\epsilon(4\pi)^{-d/2} = (4\pi)^{-2}\left[1 + \frac{\epsilon}{2}\ln(4\pi) + \epsilon\ln\mu + \mathcal{O}(\epsilon^2)\right]. \tag{7.54}$$

Insert (7.52)–(7.54) into (7.51) and keep terms up to $\mathcal{O}(\epsilon^0)$. The result can be written compactly as

$$\Pi(k^2) = \frac{e^2}{2\pi^2}\int_0^1 dx\, x(1-x)\left[\frac{2}{\epsilon} - \gamma_E + \ln(4\pi) - \ln\left(\frac{\Delta(x,k^2)}{\mu^2}\right)\right] + \mathcal{O}(\epsilon)\,. \tag{7.55}$$

This displays explicitly the UV divergence as a pole $2/\epsilon$.

Step 9: extract the convergent part by on-shell subtraction $\Pi_R(0) = 0$. With the convention (7.48), the quantity that enters the Dyson-resummed propagator as $1/(1-\Pi_R)$ is

$$\Pi_R(k^2) \equiv \Pi(0) - \Pi(k^2)\,. \tag{7.56}$$

Compute $\Pi(0)$ from (7.55): since $\Delta(x,0) = m^2$,

$$\Pi(0) = \frac{e^2}{2\pi^2}\int_0^1 dx\, x(1-x)\left[\frac{2}{\epsilon} - \gamma_E + \ln(4\pi) - \ln\left(\frac{m^2}{\mu^2}\right)\right] + \mathcal{O}(\epsilon)\,. \tag{7.57}$$

Subtracting (7.55) from (7.57), the pole and the scheme constants cancel *exactly*, leaving a manifestly finite integral:

$$\Pi_R(k^2) = \frac{e^2}{2\pi^2}\int_0^1 dx\, x(1-x)\,\ln\left(\frac{m^2 - x(1-x)k^2 - i\epsilon}{m^2}\right)\,. \tag{7.58}$$

Equivalently, in terms of $\alpha = e^2/(4\pi)$,

$$\Pi_R(k^2) = \frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\,\ln\left(\frac{m^2 - x(1-x)k^2 - i\epsilon}{m^2}\right)\,. \tag{7.59}$$

For space-like momentum $k^2 = -Q^2 < 0$ this becomes

$$\Pi_R(-Q^2) = \frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\,\ln\left(1 + \frac{x(1-x)Q^2}{m^2}\right), \tag{7.60}$$

which is manifestly positive, as expected from charge screening and the running of $\alpha(Q^2)$.

Step 10: reconstruct the renormalized tensor. The finite renormalized vacuum-polarization tensor is

$$\Pi_R^{\mu\nu}(k) = \left(k^2 g^{\mu\nu} - k^\mu k^\nu\right)\Pi_R(k^2)\,, \tag{7.61}$$

which is transverse and UV finite by construction.

Physical significance of vacuum polarization.

The calculation above reveals several deep aspects of quantum electrodynamics:

- **Dimensional regularization preserves the gauge-invariant structure.** Once the tensor is projected onto the transverse sector, the result reorganizes itself into $\Pi^{\mu\nu} \propto (k^2 g^{\mu\nu} - k^\mu k^\nu)$. This is one of the great virtues of dimensional regularization: it respects the Ward identity and therefore never generates a physical photon mass.
- **The photon remains massless.** The non-transverse piece (proportional to $m^2 g^{\mu\nu}$) cancels exactly, ensuring that no photon mass is generated. This is a consequence of gauge invariance (specifically, the Ward identity $k_\mu \Pi^{\mu\nu} = 0$) and would be violated by a naive cutoff regularization.
- **The pole in $1/\epsilon$ is universal.** The coefficient of the UV divergence is fixed by short-distance physics and is independent of the external momentum k^2. This is the essence of renormalizability: all the infinite parts can be absorbed into a finite number of parameters.
- **The finite part contains physical predictions.** The momentum-dependent function $\Pi_R(k^2)$ encodes the physical effect of virtual electron-positron pairs. It leads to the running of the fine structure constant $\alpha(Q^2)$ and contributes to precision observables like the Lamb shift and the muon anomalous magnetic moment.

7.2 Electron self-energy in dimensional regularization: detailed derivation of the finite part

Goal. Compute the one-loop (unrenormalized) electron self-energy $\Sigma(\not{p})$ in dimensional regularization $d = 4 - \epsilon$, isolate its UV pole, and write the *finite (convergent)* part after a clear subtraction. We work in Feynman gauge $\xi = 1$ for transparency.

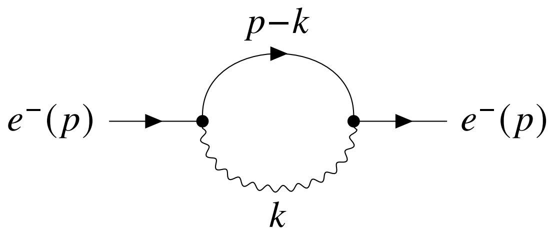
Conventions. We introduce μ so that e is dimensionless:

$$e_0 = \mu^{\epsilon/2} e, \qquad e_0^2 = \mu^\epsilon e^2 . \tag{7.62}$$

As in Chapter 6, the propagators and vertices appearing in the loop are built from the **renormalized** mass m and coupling e. The bare quantities m_0 and e_0 differ from them only through local counterterms, encoded in $(Z_2 - 1)$, δm, $(Z_1 - 1)$, and $(Z_3 - 1)$. In this sense, (7.62) is again the stripped one-loop convention in which the explicit Z factors are not inserted inside the loop integral. The free propagators are

$$S_0(\not{p}) = \frac{i}{\not{p} - m + i\epsilon}, \qquad D_{\mu\nu}(k) = \frac{-i g_{\mu\nu}}{k^2 + i\epsilon} \qquad (\xi = 1) . \tag{7.63}$$

Step 1: loop integral defining $\Sigma(\not p)$. The one-loop self-energy diagram is



(7.64)

and gives

$$-i\,\Sigma(\not p) = (-ie)^2\mu^\epsilon \int \frac{d^dk}{(2\pi)^d}\,\gamma^\mu \,\frac{i(\not p - \not k + m)}{(p-k)^2 - m^2 + i\epsilon}\,\gamma^\nu\,\frac{-ig_{\mu\nu}}{k^2 + i\epsilon}\,. \tag{7.65}$$

Simplifying factors of i yields

$$\Sigma(\not p) = -e^2\mu^\epsilon \int \frac{d^dk}{(2\pi)^d}\,\frac{\gamma^\mu(\not p - \not k + m)\gamma_\mu}{\big((p-k)^2 - m^2 + i\epsilon\big)\,(k^2 + i\epsilon)}\,. \tag{7.66}$$

Step 2: Dirac algebra in d dimensions. Use the identities (no γ^5 here)

$$\gamma^\mu\gamma^\alpha\gamma_\mu = (2-d)\gamma^\alpha, \qquad \gamma^\mu\gamma_\mu = d\,. \tag{7.67}$$

Therefore

$$\begin{aligned}\gamma^\mu(\not p - \not k + m)\gamma_\mu &= \gamma^\mu(\not p - \not k)\gamma_\mu + m\,\gamma^\mu\gamma_\mu \\ &= (2-d)(\not p - \not k) + dm\,.\end{aligned} \tag{7.68}$$

Insert into (7.66):

$$\Sigma(\not p) = -e^2\mu^\epsilon \int \frac{d^dk}{(2\pi)^d}\,\frac{(2-d)(\not p - \not k) + dm}{\big((p-k)^2 - m^2 + i\epsilon\big)\,(k^2 + i\epsilon)}\,. \tag{7.69}$$

Step 3: Feynman parametrization and completing the square. Combine denominators with

$$\frac{1}{ab} = \int_0^1 dx\,\frac{1}{\big[xa + (1-x)b\big]^2}\,, \tag{7.70}$$

where

$$a = (p-k)^2 - m^2 + i\epsilon, \qquad b = k^2 + i\epsilon\,. \tag{7.71}$$

Compute

$$\begin{aligned}xa + (1-x)b &= x\big((p-k)^2 - m^2 + i\epsilon\big) + (1-x)(k^2 + i\epsilon) \\ &= x\big(p^2 - 2p\cdot k + k^2 - m^2\big) + (1-x)k^2 + i\epsilon \\ &= x(p^2 - m^2) - 2x\,p\cdot k + k^2 + i\epsilon.\end{aligned} \tag{7.72}$$

Now shift the loop momentum to eliminate the mixed term. Define

$$\ell \equiv k - xp, \qquad k = \ell + xp, \qquad p - k = (1-x)p - \ell\,. \tag{7.73}$$

Then

$$k^2 = (\ell + xp)^2 = \ell^2 + 2x\,\ell\cdot p + x^2p^2, \tag{7.74}$$
$$-2x\,p\cdot k = -2x\,p\cdot(\ell + xp) = -2x\,p\cdot\ell - 2x^2p^2. \tag{7.75}$$

Insert (7.74) and (7.75) into (7.72):

$$\begin{aligned} xa + (1-x)b &= x(p^2 - m^2) + \left(\ell^2 + 2x\,\ell\cdot p + x^2p^2\right) + \left(-2x\,p\cdot\ell - 2x^2p^2\right) + i\epsilon \\ &= \ell^2 + x(p^2 - m^2) - x^2p^2 + i\epsilon \\ &= \ell^2 - \Delta(x, p^2) + i\epsilon\,, \end{aligned} \tag{7.76}$$

where

$$\Delta(x, p^2) \equiv xm^2 - x(1-x)p^2\,. \tag{7.77}$$

Thus

$$\frac{1}{\left((p-k)^2 - m^2 + i\epsilon\right)(k^2 + i\epsilon)} = \int_0^1 dx\, \frac{1}{\left(\ell^2 - \Delta + i\epsilon\right)^2}\,. \tag{7.78}$$

Step 4: rewrite the numerator in terms of ℓ and drop odd terms. Using $k = \ell + xp$,

$$\not{p} - \not{k} = \not{p} - (\not{\ell} + x\not{p}) = (1-x)\not{p} - \not{\ell}\,. \tag{7.79}$$

Insert into the numerator of (7.69):

$$(2-d)(\not{p} - \not{k}) + dm = (2-d)\left((1-x)\not{p} - \not{\ell}\right) + dm\,. \tag{7.80}$$

Therefore

$$\Sigma(\not{p}) = -e^2\mu^\epsilon \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d}\, \frac{(2-d)\left((1-x)\not{p} - \not{\ell}\right) + dm}{(\ell^2 - \Delta + i\epsilon)^2}\,. \tag{7.81}$$

By symmetric integration (the denominator depends only on ℓ^2),

$$\int \frac{d^d\ell}{(2\pi)^d}\, \frac{\not{\ell}}{(\ell^2 - \Delta + i\epsilon)^2} = 0, \tag{7.82}$$

so the $\not{\ell}$ term drops out:

$$\Sigma(\not{p}) = -e^2\mu^\epsilon \int_0^1 dx \left[(2-d)(1-x)\not{p} + dm\right] \int \frac{d^d\ell}{(2\pi)^d}\, \frac{1}{(\ell^2 - \Delta + i\epsilon)^2}\,. \tag{7.83}$$

Step 5: evaluate the basic scalar integral. Define

$$I_2(\Delta) \equiv \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{(\ell^2 - \Delta + i\epsilon)^2} . \tag{7.84}$$

Dimensional regularization gives

$$I_2(\Delta) = \frac{i}{(4\pi)^{d/2}} \Gamma\left(2 - \frac{d}{2}\right) (\Delta)^{\frac{d}{2}-2} . \tag{7.85}$$

Thus

$$\Sigma(\not{p}) = -e^2\mu^\epsilon \int_0^1 dx \left[(2-d)(1-x)\not{p} + dm\right] \frac{i}{(4\pi)^{d/2}} \Gamma\left(2 - \frac{d}{2}\right) (\Delta)^{\frac{d}{2}-2} . \tag{7.86}$$

Step 6: expand around $d = 4 - \epsilon$ and display the UV pole. Set $d = 4 - \epsilon$ so that

$$\Gamma\left(2 - \frac{d}{2}\right) = \Gamma\left(\frac{\epsilon}{2}\right) = \frac{2}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon) , \tag{7.87}$$

and

$$(\Delta)^{\frac{d}{2}-2} = (\Delta)^{-\epsilon/2} = 1 - \frac{\epsilon}{2} \ln\Delta + \mathcal{O}(\epsilon^2) . \tag{7.88}$$

Also,

$$\mu^\epsilon (4\pi)^{-d/2} = (4\pi)^{-2} \left[1 + \frac{\epsilon}{2}\ln(4\pi) + \epsilon\ln\mu + \mathcal{O}(\epsilon^2)\right] . \tag{7.89}$$

Finally expand the Dirac prefactor in (7.86):

$$(2-d)(1-x)\not{p} + dm = (-2+\epsilon)(1-x)\not{p} + (4-\epsilon)m = -2(1-x)\not{p} + 4m + \mathcal{O}(\epsilon) . \tag{7.90}$$

Keeping terms up to $\mathcal{O}(\epsilon^0)$ yields the standard compact form

$$\Sigma(\not{p}) = \frac{\alpha}{4\pi} \int_0^1 dx \left[-2(1-x)\not{p} + 4m\right] \left[\frac{2}{\epsilon} - \gamma_E + \ln(4\pi) - \ln\frac{\Delta(x,p^2)}{\mu^2}\right] + \mathcal{O}(\epsilon) , \tag{7.91}$$

where $\alpha = e^2/(4\pi)$ and $\Delta(x,p^2) = xm^2 - x(1-x)p^2$.

Step 7: identify the divergent structure and define a clear finite part. Write $\Sigma(\not{p})$ in the general decomposition

$$\Sigma(\not{p}) = \not{p}\,\Sigma_V(p^2) + m\,\Sigma_S(p^2) . \tag{7.92}$$

From (7.91), the UV pole is proportional to

$$\Sigma(\not{p})\Big|_{\mathrm{UV}} = \frac{\alpha}{4\pi}\frac{2}{\epsilon}\int_0^1 dx\,\Big[-2(1-x)\not{p}+4m\Big] = \frac{\alpha}{4\pi}\frac{2}{\epsilon}\Big[-\not{p}+4m\Big], \tag{7.93}$$

because

$$\int_0^1 dx\,(1-x) = \frac{1}{2}, \qquad \int_0^1 dx\; 1 = 1\,. \tag{7.94}$$

So the UV divergence is a pole in $1/\epsilon$ multiplying $(-\not{p}+4m)$.

To isolate the *finite (convergent) part* in the same spirit as the photon case, one must specify a subtraction condition. Two common choices are:
(A) $\overline{\mathrm{MS}}$ *finite part: subtract pole plus constants.* Define

$$\frac{1}{\bar{\epsilon}} \equiv \frac{1}{\epsilon} - \frac{\gamma_E}{2} + \frac{1}{2}\ln(4\pi), \tag{7.95}$$

and subtract the $2/\epsilon - \gamma_E + \ln(4\pi)$ part.
(B) On-shell finite part: subtract at the mass shell. Impose the on-shell conditions at $\not{p} = m$, which fix δm and Z_2. This corresponds to subtracting $\Sigma(\not{p})$ and its $\not{p}$-derivative at $\not{p} = m$.

Since you asked for the “convergent part” in direct analogy with the subtracted vacuum-polarization function introduced above, the simplest explicit finite object to display *before* discussing δm and Z_2 is the $\overline{\mathrm{MS}}$-subtracted (finite) self-energy:

$$\Sigma_{\overline{\mathrm{MS}}}(\not{p}) \equiv \Sigma(\not{p}) - \frac{\alpha}{4\pi}\int_0^1 dx\,\Big[-2(1-x)\not{p}+4m\Big]\left[\frac{2}{\epsilon}-\gamma_E+\ln(4\pi)\right] \tag{7.96}$$

$$= -\frac{\alpha}{4\pi}\int_0^1 dx\,\Big[-2(1-x)\not{p}+4m\Big]\,\ln\!\left(\frac{\Delta(x,p^2)}{\mu^2}\right). \tag{7.97}$$

Equation (7.97) is manifestly finite for generic p^2 and exhibits the full momentum dependence through $\Delta(x,p^2) = xm^2 - x(1-x)p^2$.

Step 8 (optional, but often useful): explicit separation into $\not{p}$ and m parts. From (7.97),

$$\Sigma_{\overline{\mathrm{MS}}}(\not{p}) = \not{p}\;\Sigma_{V,\overline{\mathrm{MS}}}(p^2) + m\;\Sigma_{S,\overline{\mathrm{MS}}}(p^2), \tag{7.98}$$

with

$$\Sigma_{V,\overline{\mathrm{MS}}}(p^2) = \frac{\alpha}{2\pi}\int_0^1 dx\;(1-x)\,\ln\!\left(\frac{\Delta(x,p^2)}{\mu^2}\right), \tag{7.99}$$

$$\Sigma_{S,\overline{\mathrm{MS}}}(p^2) = -\frac{\alpha}{\pi}\int_0^1 dx\;\ln\!\left(\frac{\Delta(x,p^2)}{\mu^2}\right). \tag{7.100}$$

Step 9: connect to the counterterm method (preview). In the counterterm approach, one defines the renormalized self-energy as

$$\Sigma_R(\not{p}) = \Sigma(\not{p}) + (Z_2 - 1)\not{p} - \delta m\,, \tag{7.101}$$

and chooses $(Z_2 - 1)$ and δm to cancel the UV pole (7.93) and enforce the chosen renormalization conditions (on-shell or $\overline{\mathrm{MS}}$). The expressions (7.97)–(7.100) are precisely the finite "convergent part" that remains after subtracting the UV-divergent pieces.

Physical significance of the electron self-energy.

The electron self-energy calculation reveals important aspects of QED renormalization:

- **Two distinct divergent structures.** The self-energy has a momentum-dependent part (proportional to $\not{p}$) and a momentum-independent part (proportional to m). Both require separate counterterms: the wavefunction renormalization Z_2 and the mass counterterm δm.
- **The physical electron mass.** The counterterm δm absorbs the infinite shift in the electron mass due to electromagnetic self-interactions. The "bare mass" m_0 in the Lagrangian is unphysical; only the renormalized mass m_R (defined by the pole of the propagator) is observable.
- **Gauge dependence of intermediate quantities.** The self-energy $\Sigma(\not{p})$ depends on the gauge parameter ξ (we computed in Feynman gauge $\xi = 1$). However, physical observables like the pole mass are gauge-independent—a powerful consistency check.
- **Connection to bound-state physics.** The finite momentum-dependent part of the self-energy contributes to atomic energy levels (the self-energy contribution to the Lamb shift), demonstrating that loop effects have real physical consequences.

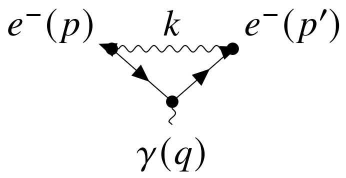
7.3 One-loop vertex correction in dimensional regularization: detailed derivation of the finite part

Goal. Compute the one-loop 1PI vertex function $\Lambda^\mu(p', p)$ in QED (Feynman gauge), display explicitly the UV pole in $d = 4 - \epsilon$, and write the *finite (convergent)* part after a clean subtraction (we use $\overline{\mathrm{MS}}$ for definiteness). We also show in detail how the UV pole is fixed by the Ward–Takahashi identity.

Diagram and definition. The proper vertex is

$$\Gamma^\mu(p', p) = \gamma^\mu + \Lambda^\mu(p', p), \qquad q \equiv p' - p\,. \tag{7.102}$$

Diagrammatically:



(7.103)

Conventions. Dimensional regularization: $d = 4-\epsilon$, and

$$e_0 = \mu^{\epsilon/2} e, \qquad e_0^2 = \mu^\epsilon e^2 . \tag{7.104}$$

The meaning is the same as in the previous two sections and in Chapter 6: the loop is written in terms of the **renormalized** mass m and coupling e, while the difference between bare and renormalized quantities is isolated in the counterterm vertex proportional to δZ_1 and in the other local counterterms. So (7.104) is not a new renormalization prescription, but simply the stripped one-loop convention used throughout this chapter. We work in Feynman gauge:

$$D_{\alpha\beta}(k) = \frac{-i g_{\alpha\beta}}{k^2 + i\epsilon} . \tag{7.105}$$

1. Loop integral for Λ^μ

Starting from the Feynman rules (including the two fermion propagators and one photon propagator), the one-loop 1PI correction is

$$\Lambda^\mu(p',p) = -e^2\mu^\epsilon \int \frac{d^d k}{(2\pi)^d} \frac{\gamma^\alpha(\not{p}'-\not{k}+m)\gamma^\mu(\not{p}-\not{k}+m)\gamma_\alpha}{(k^2+i\epsilon)\big((p'-k)^2-m^2+i\epsilon\big)\big((p-k)^2-m^2+i\epsilon\big)} . \tag{7.106}$$

Power counting: numerator $\sim k^2$, denominator $\sim k^6$, so $\int d^4k/k^4$ is logarithmically UV divergent $\Rightarrow$ a $1/\epsilon$ pole.

2. Three-denominator Feynman parametrization and completing the square

Introduce Feynman parameters $x, y, z \geq 0$ with $x+y+z=1$:

$$\frac{1}{abc} = 2\int_0^1 dx \int_0^{1-x} dy \, \frac{1}{\big[xa+yb+zc\big]^3}, \qquad z \equiv 1-x-y . \tag{7.107}$$

Here we choose

$$a = k^2 + i\epsilon, \qquad b = (p'-k)^2 - m^2 + i\epsilon, \qquad c = (p-k)^2 - m^2 + i\epsilon . \tag{7.108}$$

Expand the combination in the denominator:

$$\begin{aligned} xa + yb + zc &= xk^2 + y\big[(p'-k)^2 - m^2\big] + z\big[(p-k)^2 - m^2\big] + i\epsilon \\ &= (x+y+z)k^2 - 2k\cdot(yp' + zp) + yp'^2 + zp^2 - (y+z)m^2 + i\epsilon \\ &= k^2 - 2k\cdot(yp'+zp) + yp'^2 + zp^2 - (y+z)m^2 + i\epsilon . \end{aligned} \tag{7.109}$$

Define

$$r^\mu \equiv y p'^\mu + z p^\mu, \qquad \ell^\mu \equiv k^\mu - r^\mu, \qquad k^\mu = \ell^\mu + r^\mu . \tag{7.110}$$

Since $k^2 - 2k\cdot r = (k-r)^2 - r^2 = \ell^2 - r^2$, (7.109) becomes

$$\begin{aligned} xa + yb + zc &= \ell^2 - r^2 + y p'^2 + z p^2 - (y+z)m^2 + i\epsilon \\ &= \ell^2 - \Delta(x, y; p'^2, p^2, q^2) + i\epsilon . \end{aligned} \tag{7.111}$$

The function Δ is obtained by working out r^2:

$$\begin{aligned} r^2 &= (y p' + z p)^2 = y^2 p'^2 + z^2 p^2 + 2yz\, p'\cdot p \\ &= y^2 p'^2 + z^2 p^2 + \frac{yz}{2}\left(2p'^2 + 2p^2 - 2q^2\right) \\ &= y^2 p'^2 + z^2 p^2 + yz(p'^2 + p^2 - q^2) , \end{aligned} \tag{7.112}$$

where we used

$$q^2 = (p'-p)^2 = p'^2 + p^2 - 2p'\cdot p \qquad \Longrightarrow \qquad 2p'\cdot p = p'^2 + p^2 - q^2 . \tag{7.113}$$

Therefore

$$\Delta = (y+z)m^2 - y(1-y)p'^2 - z(1-z)p^2 + 2yz\, p'\cdot p \tag{7.114}$$

or, eliminating $p'\cdot p$ in favor of q^2,

$$\Delta \equiv (y+z)m^2 - xy\, p'^2 - xz\, p^2 - yz\, q^2, \qquad z \equiv 1 - x - y . \tag{7.115}$$

This is the corrected quadratic form. It is often useful to keep both (7.114) and (7.115) in mind: the first shows the origin of each term, while the second makes the symmetry in the Feynman parameters more transparent.

3. Rewrite the numerator after the shift and classify the terms

Introduce the shifted external combinations

$$A^\mu \equiv (1-y)p'^\mu - z p^\mu, \qquad B^\mu \equiv -y p'^\mu + (1-z) p^\mu, \tag{7.116}$$

so that

$$p' - k = A - \ell, \qquad p - k = B - \ell. \tag{7.117}$$

The numerator of (7.106) becomes

$$\begin{aligned} N^\mu(\ell; x, y) &\equiv \gamma^\alpha(\not{p}' - \not{k} + m)\gamma^\mu(\not{p} - \not{k} + m)\gamma_\alpha \\ &= \gamma^\alpha(\not{A} - \not{\ell} + m)\gamma^\mu(\not{B} - \not{\ell} + m)\gamma_\alpha \\ &= N_2^\mu + N_1^\mu + N_0^\mu , \end{aligned} \tag{7.118}$$

where

$$N_2^\mu = \gamma^\alpha \not{\ell} \gamma^\mu \not{\ell} \gamma_\alpha, \tag{7.119}$$
$$N_1^\mu = -\gamma^\alpha (\not{A} + m)\gamma^\mu \not{\ell} \gamma_\alpha - \gamma^\alpha \not{\ell} \gamma^\mu (\not{B} + m)\gamma_\alpha, \tag{7.120}$$
$$N_0^\mu = \gamma^\alpha (\not{A} + m)\gamma^\mu (\not{B} + m)\gamma_\alpha. \tag{7.121}$$

This split is didactically useful: N_2^μ contains the two powers of loop momentum and therefore controls the UV pole, N_1^μ is odd in ℓ and integrates to zero, and N_0^μ is independent of ℓ and contributes only to the finite part.

4. Symmetric integration and identification of the UV-divergent part

Using (7.107) and (7.111), the vertex becomes

$$\Lambda^\mu(p', p) = -2e^2\mu^\epsilon \int_0^1 dx \int_0^{1-x} dy \int \frac{d^d\ell}{(2\pi)^d} \frac{N_2^\mu + N_1^\mu + N_0^\mu}{(\ell^2 - \Delta + i\epsilon)^3}. \tag{7.122}$$

The term linear in ℓ drops out:

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{N_1^\mu}{(\ell^2 - \Delta + i\epsilon)^3} = 0, \tag{7.123}$$

because the denominator depends only on ℓ^2.

For the quadratic term, symmetric integration gives

$$\ell_\rho \ell_\sigma \longrightarrow \frac{g_{\rho\sigma}}{d}\ell^2 \qquad \text{inside the loop integral.} \tag{7.124}$$

Therefore

$$\begin{aligned} N_2^\mu &= \ell_\rho \ell_\sigma \gamma^\alpha \gamma^\rho \gamma^\mu \gamma^\sigma \gamma_\alpha \\ &\longrightarrow \frac{\ell^2}{d} \gamma^\alpha \gamma^\rho \gamma^\mu \gamma_\rho \gamma_\alpha \\ &= \frac{\ell^2}{d} \gamma^\alpha \left[(2-d)\gamma^\mu\right] \gamma_\alpha \\ &= \frac{(d-2)^2}{d} \ell^2 \gamma^\mu. \end{aligned} \tag{7.125}$$

The vertex correction can thus be written as

$$\Lambda^\mu(p', p) = -2e^2\mu^\epsilon \int_0^1 dx \int_0^{1-x} dy \left[\frac{(d-2)^2}{d} \gamma^\mu J_2(\Delta) + N_0^\mu I_3(\Delta)\right], \tag{7.126}$$

where

$$J_2(\Delta) \equiv \int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^2}{(\ell^2 - \Delta + i\epsilon)^3}, \tag{7.127}$$

$$I_3(\Delta) \equiv \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{(\ell^2 - \Delta + i\epsilon)^3}. \tag{7.128}$$

The integral J_2 is simplified by the algebraic identity

$$\frac{\ell^2}{(\ell^2-\Delta+i\epsilon)^3}=\frac{1}{(\ell^2-\Delta+i\epsilon)^2}+\Delta\frac{1}{(\ell^2-\Delta+i\epsilon)^3}\,. \tag{7.129}$$

Define also

$$I_2(\Delta)\equiv\int\frac{d^d\ell}{(2\pi)^d}\,\frac{1}{(\ell^2-\Delta+i\epsilon)^2}. \tag{7.130}$$

Then

$$J_2(\Delta)=I_2(\Delta)+\Delta\,I_3(\Delta). \tag{7.131}$$

This immediately shows where the UV pole comes from: $I_2(\Delta)$ is logarithmically divergent in $d=4-\epsilon$, while $I_3(\Delta)$ is already finite at $d=4$. Therefore the UV divergence is local and necessarily proportional to γ^μ.

5. UV pole from the Ward–Takahashi identity

The Ward–Takahashi identity for the proper vertex reads

$$q_\mu\Gamma^\mu(p',p)=S^{-1}(p')-S^{-1}(p),\qquad S^{-1}(p)=\not{p}-m-\Sigma(\not{p})\,. \tag{7.132}$$

To turn this into a derivative relation, set $p'=p+q$ and expand for small q:

$$S^{-1}(p+q)-S^{-1}(p)=q_\nu\frac{\partial S^{-1}(p)}{\partial p_\nu}+\mathcal{O}(q^2), \tag{7.133}$$

while

$$q_\mu\Gamma^\mu(p+q,p)=q_\mu\Gamma^\mu(p,p)+\mathcal{O}(q^2). \tag{7.134}$$

Comparing the coefficients of the linear term in q gives

$$\Gamma^\mu(p,p)=\frac{\partial S^{-1}(p)}{\partial p_\mu}=\gamma^\mu-\frac{\partial\Sigma(\not{p})}{\partial p_\mu}\,, \tag{7.135}$$

hence

$$\Lambda^\mu(p,p)=-\frac{\partial}{\partial p_\mu}\Sigma(\not{p})\,. \tag{7.136}$$

From the self-energy section,

$$\Sigma(\not{p})\Big|_{\rm UV}=\frac{\alpha}{4\pi}\frac{2}{\epsilon}\left[-\not{p}+4m\right]. \tag{7.137}$$

Differentiating with respect to p_μ is now trivial because

$$\frac{\partial\not{p}}{\partial p_\mu}=\gamma^\mu,\qquad\frac{\partial m}{\partial p_\mu}=0\,. \tag{7.138}$$

Therefore

$$\Lambda^{\mu}(p,p)\Big|_{\mathrm{UV}} = +\frac{\alpha}{4\pi}\frac{2}{\epsilon}\gamma^{\mu}\,. \tag{7.139}$$

This fixes both the sign and the coefficient of the local divergence. Since the UV pole comes from the large-ℓ region, it cannot depend on the external momenta: the same coefficient multiplies γ^{μ} for general (p', p).

6. $\overline{\mathrm{MS}}$ subtraction and the finite (convergent) part

The vertex counterterm has the form

$$\Gamma^{\mu}_{\mathrm{ct}} = +ie\,\mu^{\epsilon/2}\,\delta Z_1\,\gamma^{\mu}, \qquad \Rightarrow \qquad \Lambda^{\mu}_{R} = \Lambda^{\mu} - \delta Z_1\,\gamma^{\mu}\,. \tag{7.140}$$

In the $\overline{\mathrm{MS}}$ scheme one chooses

$$\delta Z_1^{\overline{\mathrm{MS}}} = \frac{\alpha}{4\pi}\frac{2}{\bar{\epsilon}}, \qquad \frac{1}{\bar{\epsilon}} \equiv \frac{1}{\epsilon} - \frac{\gamma_E}{2} + \frac{1}{2}\ln(4\pi)\,. \tag{7.141}$$

(With our conventions this equals $\delta Z_2^{\overline{\mathrm{MS}}}$, consistent with $Z_1 = Z_2$.) The equality $Z_1 = Z_2$ is therefore not just a mnemonic: in the derivation above it is the direct consequence of differentiating the self-energy pole and comparing with the vertex pole.

To write the finite remainder explicitly, note that

$$I_2(\Delta) = \frac{i}{(4\pi)^{d/2}}\Gamma\left(2-\frac{d}{2}\right)\Delta^{\frac{d}{2}-2}, \tag{7.142}$$

$$I_3(\Delta) = -\frac{i}{2(4\pi)^{d/2}}\Gamma\left(3-\frac{d}{2}\right)\Delta^{\frac{d}{2}-3}. \tag{7.143}$$

At $d = 4-\epsilon$, I_2 contains the UV pole while I_3 is finite. Since the full amplitude carries an overall factor μ^{ϵ}, the combination that is actually expanded is $\mu^{\epsilon}I_2(\Delta)$:

$$\mu^{\epsilon}I_2(\Delta) = \frac{i}{(4\pi)^2}\left[\frac{2}{\epsilon} - \gamma_E + \ln(4\pi) - \ln\frac{\Delta}{\mu^2}\right] + O(\epsilon), \tag{7.144}$$

$$I_3(\Delta) = -\frac{i}{2(4\pi)^2}\frac{1}{\Delta} + O(\epsilon). \tag{7.145}$$

Substituting (7.144) and (7.145) into (7.126), and then subtracting the local $\overline{\mathrm{MS}}$ counterterm, leaves a finite parameter integral. A compact way to write it is

$$\Lambda^{\mu}_{\overline{\mathrm{MS}}}(p',p) = -2e^2\int_0^1 dx \int_0^{1-x} dy\,\left[\gamma^{\mu}\,I_{2,\mathrm{fin}}(\Delta) + \left(\Delta\gamma^{\mu} + N_0^{\mu}\right)I_3(\Delta)\right]_{d=4}, \tag{7.146}$$

where

$$I_{2,\mathrm{fin}}(\Delta) \equiv \left[\mu^{\epsilon}I_2(\Delta)\right]_{\mathrm{finite}} = \frac{i}{(4\pi)^2}\left[-\ln\left(\frac{\Delta}{\mu^2}\right)\right]. \tag{7.147}$$

Equation (7.146) is the precise meaning of the "finite part": all $1/\epsilon$, γ_E, and $\ln 4\pi$ terms have been removed, while the full dependence on (p'^2, p^2, q^2) remains in Δ and in the Dirac structure N_0^μ.

Renormalized vertex to one loop. Finally,

$$\Gamma_R^\mu(p', p) = \gamma^\mu + \Lambda_{\overline{\rm MS}}^\mu(p', p) + O(e^4), \tag{7.148}$$

and the counterterm choice ensures UV finiteness.

Physical significance of the vertex correction.

The vertex correction is the most directly physical of the three one-loop diagrams:

- **The anomalous magnetic moment.** At zero momentum transfer ($q^2 = 0$), the vertex correction contributes to the magnetic form factor $F_2(0)$, giving the famous Schwinger correction $a_e = \alpha/(2\pi) \approx 0.00116$ to the electron g-factor.
- **Ward identity and $Z_1 = Z_2$.** The relation $\Lambda^\mu(p, p) = -\partial\Sigma/\partial p_\mu$ is not an accident: it follows from gauge invariance and ensures that the charge renormalization is universal. This is why we need only one coupling constant e after renormalization.
- **On-shell versus infrared divergences.** When external legs are put on-shell ($p^2 = p'^2 = m^2$), the vertex develops an infrared divergence as $q^2 \to 0$. This is physical—it signals the emission of soft photons and cancels against real photon emission in physical cross sections.

7.4 Numerical size of the one-loop QED corrections and phenomenological evidence

We summarize the typical *finite* sizes of the three one-loop QED corrections (vacuum polarization, electron self-energy, vertex correction), and we point to standard experimental plots/tables showing that these effects are real and must be included in precision physics.

(i) Photon self-energy (vacuum polarization) and the running of α

In momentum space, gauge invariance implies that the renormalized vacuum polarization tensor is transverse,

$$\Pi_R^{\mu\nu}(q) = \left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)\Pi_R(q^2), \tag{7.149}$$

and the transverse part of the photon propagator is dressed as

$$D^{\mu\nu}(q) = \frac{-i}{q^2 + i\epsilon}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\frac{1}{1 - \Pi_R(q^2)} + (\text{long. part}). \tag{7.150}$$

Equivalently, one introduces an *effective charge* (in a given scheme)

$$\alpha(q^2) \equiv \frac{\alpha}{1 - \Pi_R(q^2)}. \tag{7.151}$$

For a single lepton of mass m_ℓ, at large *space-like* momentum transfer $q^2 = -Q^2$ with $Q^2 \gg m_\ell^2$, the renormalized one-loop result has the exact Feynman-parameter representation

$$\Pi_R(-Q^2) = \frac{2\alpha}{\pi} \int_0^1 dx\, x(1-x)\, \ln\left(1 + \frac{x(1-x)Q^2}{m_\ell^2}\right), \tag{7.152}$$

which is manifestly positive. For $Q^2 \gg m_\ell^2$ this reduces to the standard asymptotic form

$$\Pi_R(-Q^2) \simeq \frac{\alpha}{3\pi}\left[\ln\left(\frac{Q^2}{m_\ell^2}\right) - \frac{5}{3}\right], \qquad (Q^2 \gg m_\ell^2), \tag{7.153}$$

so the relative correction to the photon exchange amplitude is typically $\mathcal{O}(\Pi_R)$, i.e. logarithmically enhanced at high Q.

For the *electron loop* ($m_\ell = m_e$) this gives the indicative numerical sizes (using $\alpha \simeq 1/137.036$ and $m_e \simeq 0.511$ MeV):

Q (GeV)	$\Pi_R(-Q^2)\big\rvert_{e\text{-loop}}$	size
0.1	6.9×10^{-3}	0.7%
1	1.04×10^{-2}	1.0%
10	1.40×10^{-2}	1.4%
$M_Z \simeq 91.2$	1.74×10^{-2}	1.7%

Why this is not just formal: plots and direct evidence. Direct measurements of the running of α (hence of vacuum polarization) exist both in the time-like and space-like regions:

- KLOE measures $\alpha(s)$ below 1 GeV in the time-like region via $e^+e^- \to \mu^+\mu^-\gamma$. [18]
- OPAL measures the running in the space-like region using small-angle Bhabha scattering at LEP. [19]
- MUonE is designed to extract the *space-like* hadronic vacuum polarization from μe elastic scattering. [20]

It is also useful to quote the *full* (leptonic + hadronic) effect at the Z pole: a standard compilation gives

$$\alpha^{-1}(0) \simeq 137.036 \qquad \longrightarrow \qquad \alpha^{-1}(M_Z) \simeq 128.95, \tag{7.154}$$

i.e. a $\sim 6\%$ increase of α from $q^2 = 0$ to $q^2 = M_Z^2$. [21]

(ii) Electron self-energy and the size of propagator corrections

The dressed renormalized electron propagator is

$$S_R(p) = \frac{i}{\not p - m - \Sigma_R(\not p)}\,. \tag{7.155}$$

A convenient way to quote a *finite* size without infrared subtleties is to evaluate the (one-loop, MS-like subtracted) self-energy at a *space-like* Euclidean point, e.g. $p^2 = -m^2$ with $\mu = m$. At that point

$$\Delta(x, -m^2) = xm^2 - x(1-x)(-m^2) = x(2-x)m^2, \tag{7.156}$$

so the parameter integrals can be done explicitly:

$$2\int_0^1 dx\,(1-x)\ln\big[x(2-x)\big] = -1, \qquad -4\int_0^1 dx\,\ln\big[x(2-x)\big] = 8 - 8\ln 2. \tag{7.157}$$

Hence

$$\Sigma_R(\not p)\Big|_{p^2=-m^2,\ \mu=m} = \frac{\alpha}{4\pi}\big[A\not p + Bm\big], \qquad A = -1, \qquad B = 8 - 8\ln 2 \simeq 2.45\,. \tag{7.158}$$

Numerically,

$$\begin{aligned} &\frac{\alpha}{4\pi} \simeq 5.80\times 10^{-4}, \\ &\Rightarrow \quad \Sigma_R(\not p) \simeq -(5.8\times 10^{-4})\not p + (1.43\times 10^{-3})\,m \\ &\qquad (p^2 = -m^2,\ \mu = m)\,, \end{aligned} \tag{7.159}$$

so the finite one-loop correction is at the level of $O(10^{-3})$ (as expected on dimensional grounds: $\alpha/\pi \sim 2.3\times 10^{-3}$).

Phenomenological evidence (bound-state self-energy). While $\Sigma_R(\not p)$ is not measured directly, its physical content appears prominently in bound-state QED: the *electron self-energy* provides a dominant contribution to the Lamb shift. [22]

(iii) Vertex correction and a clean numerical observable: $g-2$

The renormalized on-shell vertex can be parameterized by the Dirac and Pauli form factors,

$$\bar u(p')\,\Gamma^\mu(p',p)\,u(p) = \bar u(p')\left[\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2)\right]u(p), \qquad q = p' - p\,. \tag{7.160}$$

The most famous finite piece is the *anomalous magnetic moment*

$$a_e \equiv \frac{g-2}{2} = F_2(0)\,. \tag{7.161}$$

At one loop (Schwinger),

$$a_e^{(1)} = \frac{\alpha}{2\pi} \simeq 1.16141 \times 10^{-3} \,. \tag{7.162}$$

This is a *pure vertex effect* (in the on-shell definition) and is experimentally established at astonishing precision. [23, 24]

Bottom line: sizes and "why they matter"

Putting the above together, at one loop you should expect:

- **vacuum polarization:** percent-level effects in typical GeV-scale photon exchange (log-enhanced), and a $\sim 6\%$ shift between $\alpha(0)$ and $\alpha(M_Z)$ once all charged species (including hadrons) are accounted for;
- **electron self-energy:** per-mille-level finite corrections to the propagator structure, with direct physical manifestation in bound-state energy shifts (Lamb shift);
- **vertex:** a clean $O(10^{-3})$ correction already at one loop (Schwinger), with higher orders required to match the observed a_e at the 10^{-12} level.

One-loop $\overline{\mathrm{MS}}$ summary table

Quantity	Divergent one-loop part	$\overline{\mathrm{MS}}$ counterterm(s)	Renormalized one-loop result
Photon self-energy	$\Pi^{\mu\nu}(q) = \left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)\Pi(q^2)$, $\Pi(q^2)\big\vert_{\mathrm{div}} = \frac{2\alpha}{3\pi}\frac{1}{\bar{\epsilon}}$	$\delta Z_3^{\overline{\mathrm{MS}}} = Z_3 - 1 = \frac{2\alpha}{3\pi}\frac{1}{\bar{\epsilon}}$	$\Pi^{\mu\nu}_{R,\overline{\mathrm{MS}}}(q) = \left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)\Pi_{\overline{\mathrm{MS}}}(q^2)$, $\Pi_{\overline{\mathrm{MS}}}(q^2) = -\frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x) \times \ln\frac{\Delta_q(x)}{\mu^2}$
Electron self-energy	$\Sigma(\not{p})\big\vert_{\mathrm{div}} = \frac{\alpha}{4\pi}\frac{2}{\bar{\epsilon}}\left(-\not{p} + 4m\right)$	$\delta Z_2^{\overline{\mathrm{MS}}} = Z_2 - 1 = \frac{\alpha}{2\pi}\frac{1}{\bar{\epsilon}}$, $\delta m^{\overline{\mathrm{MS}}} = \frac{2\alpha}{\pi}\frac{m}{\bar{\epsilon}}$	$\Sigma_{\overline{\mathrm{MS}}}(\not{p}) = -\frac{\alpha}{4\pi}\int_0^1 dx \times \left[-2(1-x)\not{p} + 4m\right]\ln\frac{\Delta_p(x)}{\mu^2}$
Vertex correction	$\Lambda^\mu(p',p)\big\vert_{\mathrm{div}} = \frac{\alpha}{2\pi}\frac{1}{\bar{\epsilon}}\gamma^\mu$	$\delta Z_1^{\overline{\mathrm{MS}}} = Z_1 - 1 = \frac{\alpha}{2\pi}\frac{1}{\bar{\epsilon}}$, $Z_1^{\overline{\mathrm{MS}}} = Z_2^{\overline{\mathrm{MS}}}$	$\Gamma^\mu_R(p',p) = \gamma^\mu + \Lambda^\mu_{\overline{\mathrm{MS}}}(p',p) + O(e^4)$
$\frac{1}{\bar{\epsilon}} \equiv \frac{1}{\epsilon} - \frac{\gamma_E}{2} + \frac{1}{2}\ln(4\pi)$, $\Delta_q(x) = m^2 - x(1-x)q^2 - i\epsilon$, $\Delta_p(x) = xm^2 - x(1-x)p^2$, $e = Z_3^{1/2} e_0$, $Z_e = Z_3^{-1/2}$			