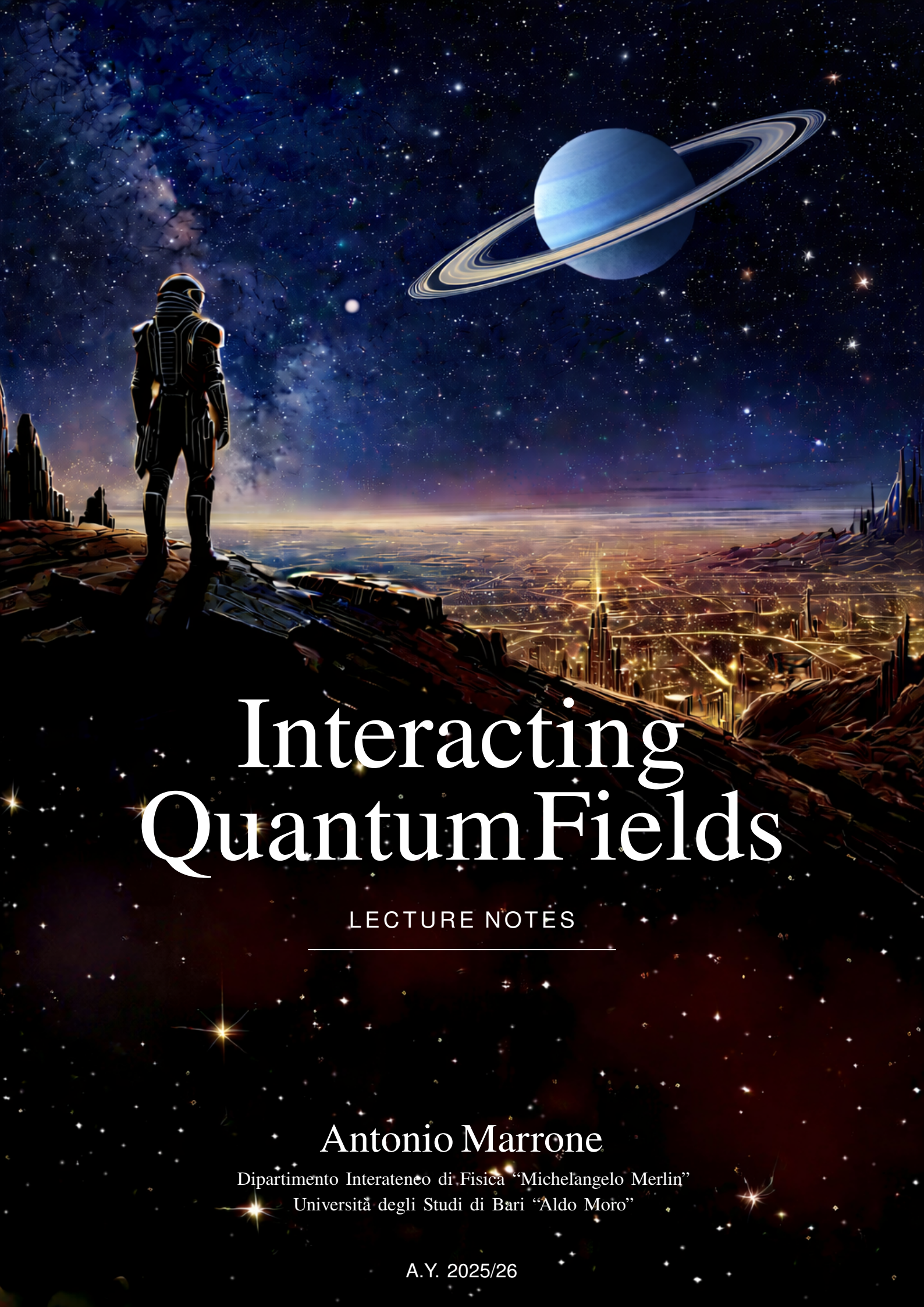

Interacting Quantum Fields

LECTURE NOTES

Antonio Marrone

Dipartimento Interateneo di Fisica "Michelangelo Merlin"
Università degli Studi di Bari "Aldo Moro"

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Lecture notes for the course “Interacting Quantum Fields”

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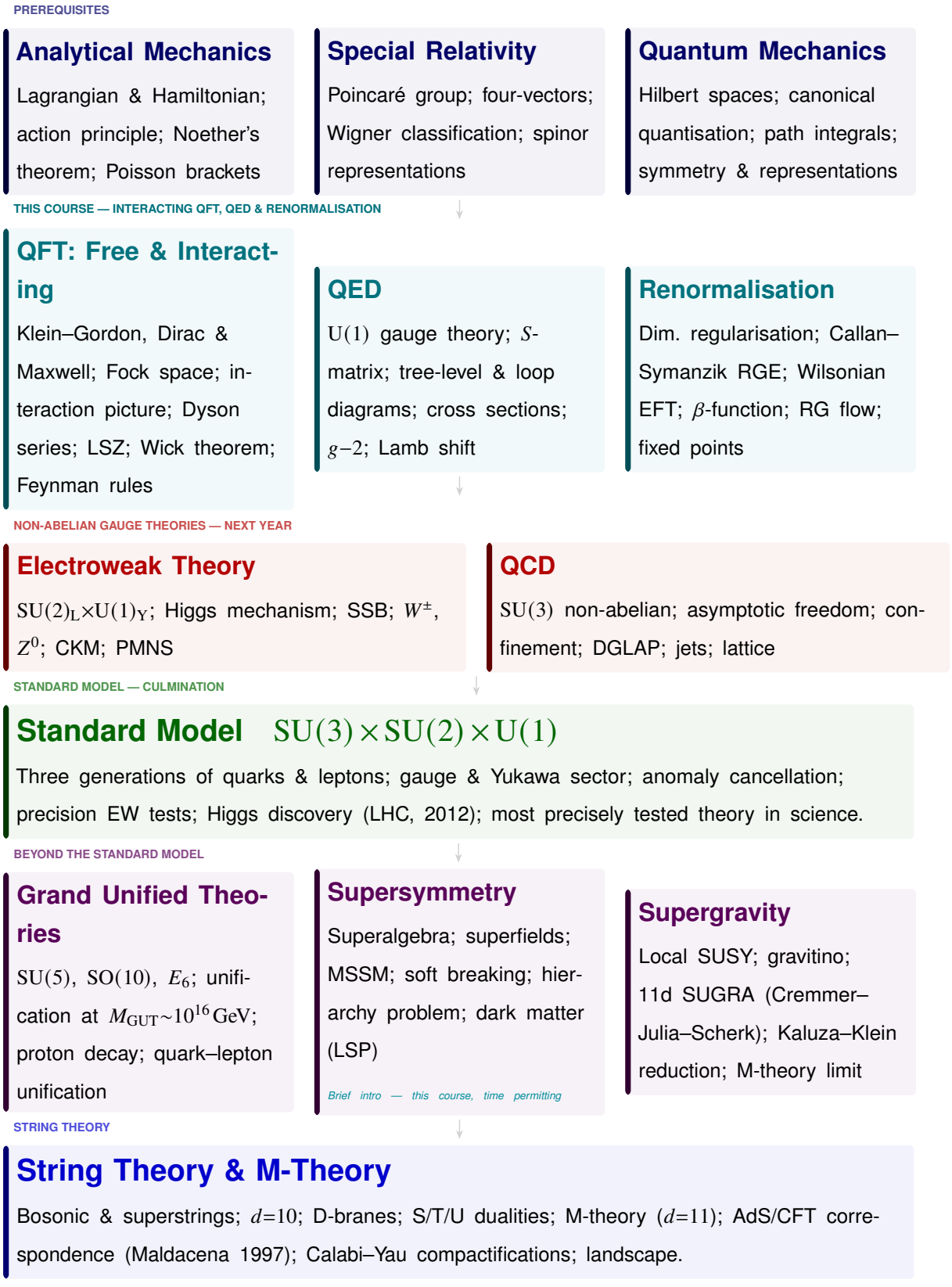
Preface: A Roadmap of Theoretical Physics


PREREQUISITES
Analytical Mechanics
Lagrangian & Hamiltonian; action principle; Noether's theorem; Poisson brackets
Special Relativity
Poincaré group; four-vectors; Wigner classification; spinor representations
Quantum Mechanics
Hilbert spaces; canonical quantisation; path integrals; symmetry & representations
THIS COURSE — INTERACTING QFT, QED & RENORMALISATION
QFT: Free & Interacting
Klein–Gordon, Dirac & Maxwell; Fock space; interaction picture; Dyson series; LSZ; Wick theorem; Feynman rules
QED
U(1) gauge theory; S-matrix; tree-level & loop diagrams; cross sections; g−2; Lamb shift
Renormalisation
Dim. regularisation; Callan–Symanzik RGE; Wilsonian EFT; β-function; RG flow; fixed points
NON-ABELIAN GAUGE THEORIES — NEXT YEAR
Electroweak Theory
SU(2)L×U(1)Y; Higgs mechanism; SSB; W±, Z0; CKM; PMNS
QCD
SU(3) non-abelian; asymptotic freedom; confinement; DGLAP; jets; lattice
STANDARD MODEL — CULMINATION
Standard Model SU(3)×SU(2)×U(1)
Three generations of quarks & leptons; gauge & Yukawa sector; anomaly cancellation; precision EW tests; Higgs discovery (LHC, 2012); most precisely tested theory in science.
BEYOND THE STANDARD MODEL
Grand Unified Theories
SU(5), SO(10), E6; unification at MGUT∼10^16 GeV; proton decay; quark–lepton unification
Supersymmetry
Superalgebra; superfields; MSSM; soft breaking; hierarchy problem; dark matter (LSP)
Brief intro — this course, time permitting
Supergravity
Local SUSY; gravitino; 11d SUGRA (Cremmer–Julia–Scherk); Kaluza–Klein reduction; M-theory limit
STRING THEORY
String Theory & M-Theory
Bosonic & superstrings; d=10; D-branes; S/T/U dualities; M-theory (d=11); AdS/CFT correspondence (Maldacena 1997); Calabi–Yau compactifications; landscape.

The Landscape of Theoretical Physics

Physics progresses through layers of abstraction. At each stage, new mathematical structures capture phenomena that the previous framework could not reach.

Classical mechanics (Newton, Euler, Lagrange, Hamilton) describes the motion of particles and rigid bodies. Its Lagrangian and Hamiltonian formulations introduce the *action principle* and *phase space*—the natural language for quantisation.

Special relativity (Einstein, Minkowski, Poincaré) merges space and time into spacetime and replaces the Galilean group with the Poincaré group $\mathrm{ISO}(1,3)$. Particle states are classified by the irreducible representations of this group (Wigner), labelled by mass m and spin s.

Quantum mechanics (Heisenberg, Schrödinger, Dirac, von Neumann) replaces classical observables with operators on a Hilbert space and introduces probability amplitudes. Canonical quantisation ($[q_i, p_j] = i\hbar\,\delta_{ij}$) and path integrals provide two complementary routes to the quantum theory.

Quantum field theory unifies quantum mechanics with special relativity. A field $\phi(x)$ is defined at every spacetime point, and its quantisation produces a many-particle theory in which particles can be created, annihilated, and scattered. The *free* theory describes non-interacting particles (Klein–Gordon, Dirac, Maxwell fields, Fock space, propagators). The *interacting* theory introduces couplings and is treated perturbatively via the Dyson series, Wick's theorem, and Feynman diagrams. **QED** ($\mathrm{U}(1)$ gauge theory) is the first complete example, predicting the anomalous magnetic moment of the electron to 12 significant figures. **Renormalisation** tames the ultraviolet divergences of loop diagrams and reveals that coupling constants *run* with energy—a profound insight formalised by the renormalisation group.

Non-abelian gauge theories ($\mathrm{SU}(N)$, Yang–Mills) generalise QED. The electroweak theory ($\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$, Glashow–Weinberg–Salam) unifies electromagnetic and weak interactions via the Higgs mechanism. QCD ($\mathrm{SU}(3)$) describes the strong interaction and exhibits asymptotic freedom (Gross, Wilczek, Politzer) and confinement. Together, they form the **Standard Model** $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$, which has passed every experimental test to date.

Beyond the Standard Model. Despite its success, the Standard Model leaves fundamental questions unanswered: the origin of neutrino masses, the nature of dark matter and dark energy, the hierarchy problem, the strong CP problem, and the unification of forces with gravity. Grand Unified Theories, supersymmetry, supergravity, and string theory offer partial answers, none yet confirmed experimentally.

This course. We develop the formalism of interacting quantum field theory from the ground up, apply it to QED as the paradigmatic example, and develop the theory of renormalisation. If time permits, we give a brief introduction to supersymmetry.

Open Problems in Particle Physics (2026)

Quantum Gravity & Unification

- No renormalisable quantum theory of gravity
- Hierarchy problem: why $M_W \ll M_{\rm Pl}$?
- Gauge coupling unification (GUT scale)
- String landscape: $\sim 10^{500}$ vacua, no selection principle
- Black hole information paradox

Flavour & Neutrino Physics

- Neutrino mass: Dirac or Majorana? See-saw scale?
- Absolute neutrino mass scale ($\sum m_\nu < 0.12\,\text{eV}$)
- Normal vs inverted mass ordering
- CP violation in lepton sector ($\delta_{\rm CP}$)
- Origin of three generations; flavour puzzle

Supersymmetry & BSM Signatures

- No SUSY particles at LHC ($\tilde{g} > 2.2\,\text{TeV}$)
- Muon $g-2$: $\sim 4\sigma$ vs. 2020 White Paper SM prediction; tension reduced ($\lesssim 2\sigma$) vs. updated lattice / hadronic input (2024–25)
- B-physics anomalies: partially resolved, residual tension
- Naturalness: is fine-tuning acceptable?

Dark Matter & Dark Energy

- DM: $\sim 27\%$ of energy budget; no SM candidate
- WIMP miracle under tension (post-LHC)
- Axion, sterile ν, primordial BH as alternatives
- Dark energy / cosmological constant: fine-tuned to $\sim 10^{-120}$ in Planck units

CP Violation & Baryogenesis

- SM CP violation insufficient for $\eta \sim 6 \times 10^{-10}$
- Strong CP: why $|\bar{\theta}| < 10^{-10}$? Axion unconfirmed
- Leptogenesis via heavy Majorana ν
- Electric dipole moments (nEDM, eEDM) as BSM probes

Early Universe & Phase Transitions

- EW phase transition: first or second order? (LISA)
- Inflation: mechanism, energy scale (CMB-S4, LiteBIRD)
- Hubble tension: $\sim 5\sigma$ discrepancy (2025)
- Gravitational wave background (PTA, LISA 2035)

Experimental status (early 2026). The LHC Run 3 dataset ($\sim 300\,\text{fb}^{-1}$ at 13.6 TeV) has substantially tightened exclusion limits on new coloured particles, compressed-spectrum SUSY, and leptoquarks, while confirming the Higgs boson properties to the 10% level across most couplings. The Fermilab Muon $g-2$ experiment has published its full dataset, sustaining a $\gtrsim 4\sigma$ tension with the White Paper SM prediction, though the lattice QCD estimate of the hadronic vacuum polarisation contribution (BMW collaboration and others) partially alleviates this discrepancy—the situation remains unresolved. The KATRIN experiment has set the world-best limit $m_\nu < 0.45\,\text{eV}$ (90% CL). The NANOGrav/IPTA pulsar timing array results (2023) provided strong evidence for a stochastic gravitational-wave background consistent with supermassive black-hole binary mergers, with BSM interpretations (cosmic strings, first-order phase transitions) not yet excluded. The Hubble tension—$H_0 \approx 67\,\text{km}\,\text{s}^{-1}\text{Mpc}^{-1}$ from Planck CMB versus $\approx 73\,\text{km}\,\text{s}^{-1}\text{Mpc}^{-1}$ from distance-ladder methods (SH0ES)—persists at $\sim 5\sigma$ and has no accepted SM explanation. The HL-LHC (Run 4, starting June 2030), LISA (launch 2035), CMB-S4, Hyper-Kamiokande, and next-generation direct DM detectors represent the primary experimental frontier.

Part I

Introduction to Quantum Field Theory

1 Free Fields: A Recap

Before developing the perturbative machinery of Feynman diagrams, it is essential to have a firm grasp of the formalism for free quantum fields. This chapter provides a self-contained summary of the main results for scalar, Dirac, and photon fields: Lagrangians and Hamiltonians, field decompositions in terms of creation and annihilation operators, (anti)commutation relations, and propagators. We use natural units $\hbar = c = 1$ throughout, and work with continuum normalization (as opposed to box normalization, for instance in [1]).

1.1 The real scalar field

The Lagrangian density for a real scalar field $\phi(x) = \phi^\dagger(x)$ of mass m is:

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\,\partial^\mu\phi - m^2\phi^2\right). \tag{1.1}$$

The canonical momentum conjugate to ϕ is:

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} = \dot{\phi}(x). \tag{1.2}$$

The Klein–Gordon equation

The equation of motion follows from the Euler–Lagrange equations:

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0. \tag{1.3}$$

Box 1.1 Klein–Gordon equation from the Lagrangian

Starting from $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi\,\partial^\mu\phi - m^2\phi^2)$, we compute:

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \partial^\mu\phi, \tag{1.4}$$

$$\frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi. \tag{1.5}$$

Substituting into the Euler–Lagrange equation:

$$\partial_\mu(\partial^\mu\phi) + m^2\phi = 0\,. \tag{1.6}$$

This is the **Klein–Gordon equation**:

$$\boxed{(\Box + m^2)\phi = 0} \tag{1.7}$$

where $\Box = \partial_\mu\partial^\mu = \partial_t^2 - \nabla^2$ is the d'Alembertian operator.

The Klein–Gordon equation is a relativistic wave equation: it is second-order in both time and space derivatives, ensuring Lorentz covariance. Each mode of the field oscillates as $e^{\pm i(E_p t - \vec{p}\cdot\vec{x})}$ with the relativistic dispersion relation $E_p = \sqrt{|\vec{p}|^2 + m^2}$.

The Hamiltonian

The Hamiltonian density is obtained by the Legendre transformation:

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} = \frac{1}{2}\left(\pi^2 + (\nabla\phi)^2 + m^2\phi^2\right). \tag{1.8}$$

Box 1.2 Hamiltonian density via Legendre transformation

The Legendre transformation relates the Lagrangian formulation (in terms of ϕ and $\dot{\phi}$) to the Hamiltonian formulation (in terms of ϕ and π). The Hamiltonian density is defined as:

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L}, \tag{1.9}$$

where $\dot{\phi}$ must be expressed in terms of π.
Step 1: From the definition of the conjugate momentum, $\pi = \partial\mathcal{L}/\partial\dot{\phi} = \dot{\phi}$, we immediately have:

$$\dot{\phi} = \pi\,. \tag{1.10}$$

Step 2: Substitute into the Lagrangian density. Recall:

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\,\partial^\mu\phi - m^2\phi^2\right) = \frac{1}{2}\left(\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2\right). \tag{1.11}$$

Step 3: Compute the Hamiltonian density:

$$\begin{aligned}\mathcal{H} &= \pi\dot{\phi} - \mathcal{L} \\ &= \pi\cdot\pi - \frac{1}{2}\left(\pi^2 - (\nabla\phi)^2 - m^2\phi^2\right) \\ &= \pi^2 - \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \\ &= \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \,. \end{aligned} \tag{1.12}$$

The three terms have clear physical interpretations:

- $\frac{1}{2}\pi^2$: kinetic energy density (associated with time evolution)
- $\frac{1}{2}(\nabla\phi)^2$: gradient energy density (cost of spatial variations)
- $\frac{1}{2}m^2\phi^2$: mass (potential) energy density

The total Hamiltonian is $H = \int d^3x\,\mathcal{H}$.

Mode expansion

The field operator $\phi(x)$ and its conjugate momentum $\pi(x)$ can be decomposed as:

$$\phi(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2E_p}}\left[a(\vec{p})\,e^{-ip\cdot x} + a^\dagger(\vec{p})\,e^{ip\cdot x}\right] \tag{1.13}$$

$$\pi(x) = \dot{\phi}(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2E_p}}(-iE_p)\left[a(\vec{p})\,e^{-ip\cdot x} - a^\dagger(\vec{p})\,e^{ip\cdot x}\right], \tag{1.14}$$

where $E_p = \sqrt{|\vec{p}|^2 + m^2}$ is the on-shell energy, and $p\cdot x = E_p t - \vec{p}\cdot\vec{x}$.

Normalization convention: We use the Bjorken–Drell (BD) normalization throughout these notes. A detailed comparison with other conventions appears below.

Commutation relations

Physical motivation

The transition from classical to quantum field theory requires promoting the classical fields to operators satisfying appropriate commutation relations. The choice of commutation relations is not arbitrary—it is dictated by fundamental physical requirements:

1. **Causality:** Field operators at spacelike-separated points must commute, ensuring that measurements at one point cannot instantaneously affect measurements

at another. This is automatically satisfied by the equal-time commutators below, which imply $[\phi(x), \phi(y)] = 0$ for $(x-y)^2 < 0$.

2. **Correspondence principle:** In the limit $\hbar \to 0$, the commutators reduce to Poisson brackets multiplied by $i\hbar$. The canonical commutation relations mirror the classical Poisson bracket $\{q, p\} = 1$.
3. **Positive-definite Hilbert space:** For bosonic fields (integer spin), commutation relations lead to the correct bosonic Fock-space structure with positive-definite norm. Using anticommutation relations would instead produce the wrong statistics and occupancy structure for bosons.
4. **Lorentz invariance:** The commutation relations must be compatible with Lorentz transformations, which is ensured by working with covariant fields.

Equal-time canonical commutation relations

The equal-time canonical commutation relations are:

$$[\phi(t,\vec{x}), \pi(t,\vec{x}')] = i\delta^{(3)}(\vec{x}-\vec{x}') \tag{1.15}$$

$$[\phi(t,\vec{x}), \phi(t,\vec{x}')] = [\pi(t,\vec{x}), \pi(t,\vec{x}')] = 0\,. \tag{1.16}$$

These are the field-theoretic analogue of the quantum mechanical commutator $[\hat{q}, \hat{p}] = i\hbar$ (with $\hbar = 1$), extended to a continuous infinity of degrees of freedom labeled by $\vec{x}$.

Let us verify that these commutation relations satisfy the four physical requirements:

1. **Causality:** The equal-time commutators

$$[\phi(t,\vec{x}), \phi(t,\vec{x}')] = 0 \quad \text{and} \quad [\pi(t,\vec{x}), \pi(t,\vec{x}')] = 0 \tag{1.17}$$

trivially vanish. More importantly, one can show that the full (unequal-time) commutator is

$$[\phi(x), \phi(y)] = i\Delta(x-y)\,, \tag{1.18}$$

where $\Delta(x-y)$ is the **Pauli–Jordan function** (see box below). This function vanishes for spacelike separations $(x-y)^2 < 0$, ensuring that field measurements at causally disconnected points do not interfere.
2. **Correspondence principle:** The classical Poisson bracket for fields is

$$\{\phi(t,\vec{x}), \pi(t,\vec{x}')\}_{\text{PB}} = \delta^{(3)}(\vec{x}-\vec{x}')\,. \tag{1.19}$$

The quantization rule $\{A, B\}_{\text{PB}} \to \frac{1}{i\hbar}[A, B]$ gives precisely $[\phi, \pi] = i\hbar\,\delta^{(3)}(\vec{x} - \vec{x}')$, which reduces to our expression with $\hbar = 1$.

3. **Positive-definite Hilbert space:** The commutation relation $[a, a^\dagger] = 1$ (for a single mode) implies that

$$\|a^\dagger |n\rangle\|^2 = \langle n| aa^\dagger |n\rangle = \langle n| (a^\dagger a + 1) |n\rangle = n + 1 > 0 . \tag{1.20}$$

All states $|n\rangle = (a^\dagger)^n |0\rangle / \sqrt{n!}$ have positive norm. Had we used anticommutators, we would find $(a^\dagger)^2 = 0$, and attempting to construct a Fock space leads to inconsistencies or negative norms.

4. **Lorentz invariance:** While the equal-time commutators are not manifestly covariant (they single out a time slice), the full theory is Lorentz invariant. The commutator $[\phi(x), \phi(y)]$ depends only on the invariant interval $(x-y)^2$, and the S-matrix constructed from these fields transforms correctly under the Poincaré group.

Box 1.3 The Pauli–Jordan function

The **Pauli–Jordan function** (also called the commutator function $\Delta(x-y)$) is defined as:

$$\begin{aligned} i\Delta(x) = [\phi(x), \phi(0)] &= \int \frac{d^3p}{(2\pi)^3 2E_p} \left(e^{-ip\cdot x} - e^{ip\cdot x}\right) \\ &= -2i \int \frac{d^3p}{(2\pi)^3 2E_p} \sin(p\cdot x) . \end{aligned} \tag{1.21}$$

Equivalently, in terms of the positive and negative frequency parts:

$$\Delta(x) = \Delta_+(x) - \Delta_-(x) , \tag{1.22}$$

where $i\Delta_+(x) = \langle 0| \phi(x)\phi(0) |0\rangle$ and $i\Delta_-(x) = \langle 0| \phi(0)\phi(x) |0\rangle$.

Key properties:

1. *Antisymmetry:* $\Delta(-x) = -\Delta(x)$, which follows from $[\phi(x), \phi(y)] = -[\phi(y), \phi(x)]$.
2. *Lorentz invariance:* $\Delta(x)$ depends only on $x^2 = t^2 - |\vec{x}|^2$ and $\mathrm{sign}(t)$, making it a Lorentz scalar (up to the sign of t).
3. *Causality (microcausality):* $\Delta(x) = 0$ for $x^2 < 0$ (spacelike separations). *Proof:* For spacelike x, we can choose a Lorentz frame where $t = 0$. Then:

$$\Delta(0, \vec{x}) = -2 \int \frac{d^3p}{(2\pi)^3 2E_p} \sin(-\vec{p}\cdot\vec{x}) = 2 \int \frac{d^3p}{(2\pi)^3 2E_p} \sin(\vec{p}\cdot\vec{x}) . \tag{1.23}$$

Under $\vec{p} \to -\vec{p}$, the measure d^3p and $E_p = \sqrt{|\vec{p}|^2 + m^2}$ are invariant, but $\sin(\vec{p}\cdot\vec{x}) \to -\sin(\vec{p}\cdot\vec{x})$. Therefore the integral equals minus itself, hence $\Delta(0, \vec{x}) = 0$. By Lorentz invariance, $\Delta(x) = 0$ for all spacelike x.

4. *Equal-time limit:* $\Delta(0,\vec{x}) = 0$ and $\partial_t \Delta(t,\vec{x})|_{t=0} = -\delta^{(3)}(\vec{x})$.
   Proof of the derivative: From the mode expansion,

$$\partial_t \Delta(t,\vec{x})\big|_{t=0} = -2\int \frac{d^3p}{(2\pi)^3 2E_p} E_p \cos(\vec{p}\cdot\vec{x})\big|_{t=0} = -\int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} = -\delta^{(3)}(\vec{x})\,. \tag{1.24}$$

5. *Klein–Gordon equation:* $(\Box + m^2)\Delta(x) = 0$ for $x \neq 0$.
   Proof: Each plane-wave mode $e^{\pm ip\cdot x}$ satisfies the Klein–Gordon equation when $p^0 = E_p$, so the integral does as well.

The Pauli–Jordan function is the key object ensuring **microscopic causality**: observables at spacelike-separated points commute, so no signal can propagate faster than light.

Creation and annihilation operator commutators

These are equivalent to the commutation relations for the creation and annihilation operators:

$$[a(\vec{p}), a^\dagger(\vec{p}')] = \delta^{(3)}(\vec{p}-\vec{p}') \tag{1.25}$$

$$[a(\vec{p}), a(\vec{p}')] = [a^\dagger(\vec{p}), a^\dagger(\vec{p}')] = 0\,. \tag{1.26}$$

Box 1.4 Commutation relations for a and $a^\dagger$ from canonical commutators

We derive the operator commutation relations by inverting the mode expansion. From Eq. (1.13), we can express the creation and annihilation operators in terms of the fields:

$$a(\vec{p}) = \int \frac{d^3x}{(2\pi)^{3/2}\sqrt{2E_p}} e^{ip\cdot x}\left(E_p\,\phi(x) + i\pi(x)\right)\Big|_{t=0}, \tag{1.27}$$

where we evaluate at $t = 0$ (the result is independent of time for free fields). This can be verified by substituting the mode expansion back.
To compute $[a(\vec{p}), a^\dagger(\vec{p}')]$, we use the linearity of the commutator:

$$[a(\vec{p}), a^\dagger(\vec{p}')] = \int \frac{d^3x\, d^3x'}{(2\pi)^3\sqrt{4E_pE_{p'}}} e^{i\vec{p}\cdot\vec{x} - i\vec{p}'\cdot\vec{x}'} \times \left[(E_p\phi + i\pi)_{\vec{x}}, (E_{p'}\phi - i\pi)_{\vec{x}'}\right]. \tag{1.28}$$

Expanding the commutator and using $[\phi,\phi] = [\pi,\pi] = 0$:

$$= \int \frac{d^3x\, d^3x'}{(2\pi)^3\sqrt{4E_pE_{p'}}} e^{i\vec{p}\cdot\vec{x} - i\vec{p}'\cdot\vec{x}'} \times \left(-iE_{p'}[\phi_{\vec{x}}, \pi_{\vec{x}'}] + iE_p[\pi_{\vec{x}}, \phi_{\vec{x}'}]\right). \tag{1.29}$$

Using $[\phi(t,\vec{x}), \pi(t,\vec{x}')] = i\delta^{(3)}(\vec{x} - \vec{x}')$:

$$= \int \frac{d^3x\, d^3x'}{(2\pi)^3\sqrt{4E_pE_{p'}}} e^{i\vec{p}\cdot\vec{x} - i\vec{p}'\cdot\vec{x}'} (E_{p'} + E_p)\, \delta^{(3)}(\vec{x} - \vec{x}')$$
$$= \int \frac{d^3x}{(2\pi)^3} \frac{E_p + E_{p'}}{\sqrt{4E_pE_{p'}}} e^{i(\vec{p} - \vec{p}')\cdot\vec{x}}. \tag{1.30}$$

The integral gives $(2\pi)^3\delta^{(3)}(\vec{p} - \vec{p}')$, and the delta function sets $E_{p'} = E_p$:

$$[a(\vec{p}), a^\dagger(\vec{p}')] = \frac{2E_p}{2E_p}\delta^{(3)}(\vec{p} - \vec{p}') = \delta^{(3)}(\vec{p} - \vec{p}'). \tag{1.31}$$

A similar calculation shows $[a(\vec{p}), a(\vec{p}')] = 0$: the relevant terms give $(E_{p'} - E_p)\delta^{(3)}(\vec{p} - \vec{p}') = 0$.

The Hamiltonian and normal ordering

Substituting the mode expansion into the Hamiltonian gives:

$$H = \frac{1}{2}\int d^3p\, E_p \left(a^\dagger(\vec{p})a(\vec{p}) + a(\vec{p})a^\dagger(\vec{p})\right). \tag{1.32}$$

Using the commutation relation $[a(\vec{p}), a^\dagger(\vec{p}')] = \delta^{(3)}(\vec{p} - \vec{p}')$:

$$H = \int d^3p\, E_p\, a^\dagger(\vec{p})a(\vec{p}) + \frac{1}{2}\delta^{(3)}(\vec{0})\int d^3p\, E_p. \tag{1.33}$$

The second term is the (infinite) zero-point energy. By **normal ordering** (denoted by colons), we define:

$$:H: = \int d^3p\, E_p\, a^\dagger(\vec{p})a(\vec{p}). \tag{1.34}$$

Box 1.5 Hamiltonian in terms of creation and annihilation operators

Starting from the Hamiltonian density $\mathcal{H} = \frac{1}{2}(\pi^2 + (\nabla\phi)^2 + m^2\phi^2)$, we compute $H = \int d^3x\,\mathcal{H}$ by substituting the mode expansions.

Step 1: Mode expansions at $t = 0$. For convenience, we work at $t = 0$:

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2E_p}}\left[a(\vec{p})\,e^{i\vec{p}\cdot\vec{x}} + a^\dagger(\vec{p})\,e^{-i\vec{p}\cdot\vec{x}}\right], \tag{1.35}$$

$$\pi(\vec{x}) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2E_p}}(-iE_p)\left[a(\vec{p})\,e^{i\vec{p}\cdot\vec{x}} - a^\dagger(\vec{p})\,e^{-i\vec{p}\cdot\vec{x}}\right]. \tag{1.36}$$

Step 2: Compute $\int d^3x\,\pi^2$. Squaring π and integrating:

$$\int d^3x\,\pi^2 = \int \frac{d^3p\,d^3q}{(2\pi)^3\sqrt{4E_pE_q}}(-E_pE_q) \times \left[a_p - a^\dagger_{-p}\right]\left[a_q - a^\dagger_{-q}\right]\int d^3x\,e^{i(\vec{p}+\vec{q})\cdot\vec{x}}, \tag{1.37}$$

where we use the shorthand $a_p \equiv a(\vec{p})$, $a_{-p} \equiv a(-\vec{p})$, etc.
The spatial integral gives $(2\pi)^3\delta^{(3)}(\vec{p}+\vec{q})$, which sets $\vec{q} = -\vec{p}$ (hence $E_q = E_p$):

$$\int d^3x\,\pi^2 = -\int d^3p\,\frac{E_p}{2}\left[a_p - a^\dagger_{-p}\right]\left[a_{-p} - a^\dagger_p\right]. \tag{1.38}$$

Expanding:

$$= \int d^3p\,\frac{E_p}{2}\left[-a_pa_{-p} + a_pa^\dagger_p + a^\dagger_{-p}a_{-p} - a^\dagger_{-p}a^\dagger_p\right]. \tag{1.39}$$

Step 3: Compute $\int d^3x\,[(\nabla\phi)^2 + m^2\phi^2]$. A similar calculation gives:

$$\int d^3x\,(\nabla\phi)^2 = \int d^3p\,\frac{|\vec{p}|^2}{2E_p}\left[a_pa_{-p} + a^\dagger_{-p}a^\dagger_p + a_pa^\dagger_p + a^\dagger_{-p}a_{-p}\right], \tag{1.40}$$

$$\int d^3x\,m^2\phi^2 = \int d^3p\,\frac{m^2}{2E_p}\left[a_pa_{-p} + a^\dagger_{-p}a^\dagger_p + a_pa^\dagger_p + a^\dagger_{-p}a_{-p}\right]. \tag{1.41}$$

Step 4: Combine all terms. Adding the gradient and mass terms:

$$\int d^3x\left[(\nabla\phi)^2 + m^2\phi^2\right] = \int d^3p\,\frac{m^2 + |\vec{p}|^2}{2E_p}\left[a_pa_{-p} + a^\dagger_{-p}a^\dagger_p\right] + \int d^3p\,\frac{m^2 + |\vec{p}|^2}{2E_p}\left[a_pa^\dagger_p + a^\dagger_{-p}a_{-p}\right]. \tag{1.42}$$

Using $E_p^2 = |\vec{p}|^2 + m^2$, both coefficients are equal to E_p.
Adding the π^2 term (which contributes $-E_p/2$ to the aa and $a^\dagger a^\dagger$ terms, and $+E_p/2$ to the $aa^\dagger$ and $a^\dagger a$ terms):

$$H = \frac{1}{2}\int d^3p \left[\left(-E_p + \frac{m^2+|\vec{p}|^2}{E_p}\right)\left(a_p a_{-p} + a_{-p}^\dagger a_p^\dagger\right) + E_p\left(a_p a_p^\dagger + a_p^\dagger a_p\right)\right]. \tag{1.43}$$

Step 5: The aa and $a^\dagger a^\dagger$ terms cancel. Using $E_p^2 = |\vec{p}|^2 + m^2$, the coefficient of the pair-creation/annihilation terms is

$$\frac{1}{2}\left(-E_p + \frac{m^2+|\vec{p}|^2}{E_p}\right) = \frac{-E_p^2 + m^2 + |\vec{p}|^2}{2E_p} = 0\,. \tag{1.44}$$

Therefore the terms proportional to $a_p a_{-p}$ and $a_{-p}^\dagger a_p^\dagger$ cancel *algebraically*, before normal ordering. Only the number-operator structure remains.
Final result:

$$H = \frac{1}{2}\int d^3p\, E_p \left(a^\dagger(\vec{p})a(\vec{p}) + a(\vec{p})a^\dagger(\vec{p})\right). \tag{1.45}$$

Using the commutation relation to normal order:

$$a(\vec{p})a^\dagger(\vec{p}) = a^\dagger(\vec{p})a(\vec{p}) + \delta^{(3)}(\vec{0})\,, \tag{1.46}$$

we obtain

$$H = \int d^3p\, E_p\, a^\dagger(\vec{p})a(\vec{p}) + \underbrace{\frac{1}{2}\int d^3p\, E_p\, \delta^{(3)}(\vec{0})}_{\text{vacuum energy } E_0}. \tag{1.47}$$

The vacuum energy E_0 is infinite (both from the $\delta^{(3)}(\vec{0})$ factor and the unbounded $\int d^3p\, E_p$), but is unobservable and removed by normal ordering.

Lorentz-invariant normalization: The simple choice $|p\rangle = a^\dagger(\vec{p})\,|0\rangle$ leads to $\langle p|q\rangle = \delta^{(3)}(\vec{p}-\vec{q})$, which is not Lorentz invariant. A Lorentz-invariant normalization is $|p\rangle \propto \sqrt{2E_p}\,a^\dagger(\vec{p})\,|0\rangle$.

Four-momentum operator

From Noether's theorem applied to spacetime translations, the conserved four-momentum is (see Section 1.6):

$$P^\mu = \int d^3x\, T^{0\mu}\,, \tag{1.48}$$

where $T^{\mu\nu}$ is the canonical energy-momentum tensor. For the real scalar field, the spatial components are:

$$P^i = \int d^3x\, \pi(\vec{x},t)\, \partial^i \phi(\vec{x},t) = -\int d^3x\, \pi(\vec{x},t)\, \partial_i \phi(\vec{x},t)\,, \tag{1.49}$$

where $\partial^i = -\partial_i$ in the mostly-minus signature $(+,-,-,-)$.

Substituting the mode expansion, one obtains the normal-ordered result:

$$\vec{P} = \int d^3p\, \vec{p}\, a^\dagger(\vec{p})\, a(\vec{p})\,, \tag{1.50}$$

confirming the physical interpretation: each quantum of momentum $\vec{p}$ contributes $\vec{p}$ to the total momentum.

P^μ as the generator of spacetime translations. The four-momentum operator generates translations:

$$[\phi(x), P^\mu] = +i\partial^\mu \phi(x)\,. \tag{1.51}$$

This means that under a finite translation $x \to x + a$, the field transforms as $\phi(x) \to e^{ia\cdot P}\phi(x)\, e^{-ia\cdot P} = \phi(x+a)$.[1]

Box 1.6 Proof that $[\phi(y), P^i] = +i\partial^i\phi(y)$

We prove the spatial component using only the equal-time canonical commutation relations (1.15).

[1] The exponentiation from the infinitesimal relation to the finite transformation follows from the **Hadamard lemma** (also known as the Baker–Campbell–Hausdorff formula for conjugation):

$$e^A B\, e^{-A} = B + [A,B] + \tfrac{1}{2!}[A,[A,B]] + \tfrac{1}{3!}[A,[A,[A,B]]] + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}\, \mathrm{ad}_A^n(B)\,,$$

where $\mathrm{ad}_A^n(B) \equiv [A, \mathrm{ad}_A^{n-1}(B)]$ with $\mathrm{ad}_A^0(B) = B$. This identity holds as a formal power series for *any* operators A and B; convergence is guaranteed whenever A is bounded or, more relevantly for QFT, when $A = ia_\mu P^\mu$ is anti-Hermitian (i.e. P^μ is Hermitian), so that e^A is unitary and the conjugation $e^A B\, e^{-A}$ is a well-defined automorphism of the operator algebra. Setting $A = ia_\mu P^\mu$ and using $[ia_\mu P^\mu, \phi] = -ia_\mu[\phi, P^\mu] = -ia_\mu(+i\partial^\mu\phi) = a_\mu\partial^\mu\phi$, the nested commutators reproduce the Taylor expansion: $e^{ia\cdot P}\phi(x)\, e^{-ia\cdot P} = \sum_{n=0}^{\infty} \frac{1}{n!}(a\cdot\partial)^n\phi(x) = \phi(x+a)$.

Starting from Eq. (1.49):

$$[\phi(\vec{y},t),\, P^i] = -\int d^3x\, [\phi(\vec{y},t),\, \pi(\vec{x},t)\,\partial_i\phi(\vec{x},t)]\,. \tag{1.52}$$

Step 1: Apply the identity $[A,BC] = [A,B]C + B[A,C]$:

$$[\phi(\vec{y},t),\, \pi(\vec{x},t)\,\partial_i\phi(\vec{x},t)] = \underbrace{[\phi(\vec{y},t),\, \pi(\vec{x},t)]}_{=i\delta^{(3)}(\vec{y}-\vec{x})}\,\partial_i\phi(\vec{x},t) + \pi(\vec{x},t)\,\underbrace{[\phi(\vec{y},t),\, \partial_i\phi(\vec{x},t)]}_{=0}\,. \tag{1.53}$$

The second term vanishes because $\partial_i^{(x)}$ acts on $\vec{x}$ and can be pulled out of the commutator: $[\phi(\vec{y},t),\partial_i\phi(\vec{x},t)] = \partial_i^{(x)}[\phi(\vec{y},t),\phi(\vec{x},t)] = 0$.
Step 2: Substituting:

$$[\phi(\vec{y},t),\, P^i] = -\int d^3x\, i\delta^{(3)}(\vec{y}-\vec{x})\,\partial_i\phi(\vec{x},t) = -i\,\partial_i\phi(\vec{y},t) = +i\partial^i\phi(\vec{y},t)\,, \tag{1.54}$$

where in the last step we used $\partial_i = -\partial^i$ for spatial indices with metric $(+,-,-,-)$. □

Extension to composite operators. The relation (1.51) extends to *any* local operator built from ϕ and its derivatives. In particular, P^μ generates translations of $\partial_\nu\phi$ as well:

$$[\partial_\nu\phi(x),\, P^\mu] = +i\partial^\mu\partial_\nu\phi(x)\,. \tag{1.55}$$

This follows directly by taking $\partial_\nu^{(x)}$ of both sides of Eq. (1.51): since P^μ is an integral over all space and does not depend on x, it commutes with $\partial_\nu^{(x)}$, giving

$$\partial_\nu^{(x)}[\phi(x),P^\mu] = [\partial_\nu\phi(x),P^\mu] = +i\partial_\nu\partial^\mu\phi(x)\,. \tag{1.56}$$

More generally, for any local operator $\mathcal{O}(x)$ constructed from products of fields and their derivatives at the point x, one can show by repeated application of the identity $[AB,C] = A[B,C] + [A,C]B$ that

$$[\mathcal{O}(x),\, P^\mu] = +i\partial^\mu\mathcal{O}(x)\,. \tag{1.57}$$

This is the statement that P^μ is the generator of spacetime translations for the entire operator algebra, not just for the fundamental field ϕ.

Normalization conventions: a detailed comparison

Different textbooks use different normalization conventions for the field expansion. The choice affects the commutation relations, state normalization, and the form of the completeness relation. We compare three common conventions.

Convention I: Bjorken–Drell (used in these notes)

Field expansion:

$$\phi(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2E_p}} \left[a(\vec{p})\, e^{-ip\cdot x} + a^\dagger(\vec{p})\, e^{ip\cdot x}\right] . \tag{1.58}$$

Commutation relations:

$$[a(\vec{p}), a^\dagger(\vec{p}')] = \delta^{(3)}(\vec{p} - \vec{p}') . \tag{1.59}$$

State normalization: $|\vec{p}\rangle = a^\dagger(\vec{p})\,|0\rangle$ gives

$$\langle \vec{p}|\vec{p}'\rangle = \delta^{(3)}(\vec{p} - \vec{p}') . \qquad \text{(not Lorentz invariant)} \tag{1.60}$$

Completeness relation:

$$\mathbb{1} = |0\rangle\langle 0| + \int d^3p\, |\vec{p}\rangle\langle \vec{p}| + \cdots . \tag{1.61}$$

Convention II: Covariant $(2E)$ normalization

Field expansion:

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3 2E_p} \left[a(\vec{p})\, e^{-ip\cdot x} + a^\dagger(\vec{p})\, e^{ip\cdot x}\right] . \tag{1.62}$$

Commutation relations:

$$[a(\vec{p}), a^\dagger(\vec{p}')] = (2\pi)^3\, 2E_p\, \delta^{(3)}(\vec{p} - \vec{p}') . \tag{1.63}$$

State normalization: $|\vec{p}\rangle = a^\dagger(\vec{p})\,|0\rangle$ gives the **Lorentz-invariant** normalization

$$\langle \vec{p}|\vec{p}'\rangle = (2\pi)^3\, 2E_p\, \delta^{(3)}(\vec{p} - \vec{p}') . \tag{1.64}$$

This is Lorentz invariant because the one-particle phase-space measure $d^3p/(2E_p)$ is invariant under boosts, as we now show.

Box 1.7 Lorentz invariance of $d^3p/(2E_p)$ and of $2E_p\,\delta^{(3)}$

The measure $d^3p/(2E_p)$. Start from the manifestly Lorentz-invariant combination

$$\int \frac{d^3p}{2E_{\vec{p}}}\, f(\vec{p}) = \int d^4p\, \delta(p^2-m^2)\, \theta(p^0)\, f(p)\,. \tag{1.65}$$

To verify the right-hand side, use

$$\delta(p^2-m^2) = \delta\big((p^0)^2 - E_{\vec{p}}^{\,2}\big) = \frac{1}{2\,|p^0|}\Big[\delta(p^0-E_{\vec{p}}) + \delta(p^0+E_{\vec{p}})\Big], \tag{1.66}$$

and note that $\theta(p^0)$ selects only the positive-energy root, giving $\int dp^0\, \delta(p^2-m^2)\,\theta(p^0) = 1/(2E_{\vec{p}})$.
Each factor on the right-hand side of (1.65) is Lorentz invariant:

- d^4p: because $|\det\Lambda| = 1$ for proper Lorentz transformations;
- $\delta(p^2-m^2)$: because p^2-m^2 is a Lorentz scalar, and the delta function of a scalar is a scalar;
- $\theta(p^0)$: because proper orthochronous transformations preserve the sign of p^0 for timelike vectors on the mass shell ($p^2 = m^2 > 0$).

Therefore $d^3p/(2E_{\vec{p}})$ is Lorentz invariant. □

Transformation of $\delta^{(3)}$ under boosts. By contrast, $\delta^{(3)}(\vec{p}-\vec{q})$ alone is *not* Lorentz invariant. Under a boost Λ, the three-momentum transforms with Jacobian

$$\left|\det\frac{\partial\vec{p}\,'}{\partial\vec{p}}\right| = \frac{E'}{E}\,, \tag{1.67}$$

so that $\delta^{(3)}(\vec{p}-\vec{q}) \to (E'/E)\,\delta^{(3)}(\vec{p}\,'-\vec{q}\,')$. The combination $2E_p\,\delta^{(3)}(\vec{p}-\vec{q})$ *is* invariant, because the factor E exactly compensates the Jacobian. This is why the covariant normalization $\langle\vec{p}|\vec{p}\,'\rangle = (2\pi)^3\,2E_p\,\delta^{(3)}(\vec{p}-\vec{p}\,')$ transforms as a Lorentz scalar.

Completeness relation:

$$\mathbb{1} = |0\rangle\langle 0| + \int \frac{d^3p}{(2\pi)^3 2E_p}\, |\vec{p}\rangle\langle\vec{p}| + \cdots\,. \tag{1.68}$$

Convention III: Box normalization (Mandl–Shaw)

In a finite volume $V = L^3$ with periodic boundary conditions, momenta are discrete: $\vec{p} = (2\pi/L)(n_x, n_y, n_z)$.

Field expansion:

$$\phi(x) = \sum_{\vec{p}} \frac{1}{\sqrt{2VE_p}} \left[a(\vec{p})\, e^{-ip\cdot x} + a^\dagger(\vec{p})\, e^{ip\cdot x} \right] . \tag{1.69}$$

Commutation relations:

$$[a(\vec{p}), a^\dagger(\vec{p}')] = \delta_{\vec{p},\vec{p}'} . \tag{1.70}$$

State normalization:

$$\langle \vec{p} | \vec{p}' \rangle = \delta_{\vec{p},\vec{p}'} . \qquad \text{(dimensionless)} \tag{1.71}$$

The continuum limit is obtained via

$$\sum_{\vec{p}} \to V \int \frac{d^3p}{(2\pi)^3} , \qquad \delta_{\vec{p},\vec{p}'} \to \frac{(2\pi)^3}{V} \delta^{(3)}(\vec{p} - \vec{p}') .$$

The completeness relation: definition or consequence?

A natural question arises: is the completeness relation an independent axiom, or does it follow from the other assumptions? The answer is that **the completeness relation is a consequence** of the commutation relations and the definition of the Fock space, not an additional postulate.

Box 1.8 Derivation of the completeness relation

The completeness relation in the one-particle sector follows from the commutation relations and the definition of states.

Setup: We have operators satisfying $[a(\vec{p}), a^\dagger(\vec{p}')] = f(\vec{p})\, \delta^{(3)}(\vec{p} - \vec{p}')$ for some function $f(\vec{p})$ (which depends on the convention), and one-particle states $|\vec{p}\rangle = a^\dagger(\vec{p})\, |0\rangle$.

Goal: Show that $\int d\mu(\vec{p})\, |\vec{p}\rangle \langle \vec{p}| = \mathbb{1}_{\text{1-particle}}$ for an appropriate measure $d\mu(\vec{p})$.

Derivation: Consider the action of the proposed completeness operator on an arbitrary one-particle state $|\vec{q}\rangle$:

$$\left(\int d\mu(\vec{p})\, |\vec{p}\rangle \langle \vec{p}| \right) |\vec{q}\rangle = \int d\mu(\vec{p})\, |\vec{p}\rangle \langle \vec{p} | \vec{q} \rangle . \tag{1.72}$$

The inner product is $\langle \vec{p} | \vec{q} \rangle = \langle 0 |\, a(\vec{p}) a^\dagger(\vec{q})\, |0\rangle = f(\vec{p})\, \delta^{(3)}(\vec{p} - \vec{q})$, so:

$$= \int d\mu(\vec{p})\, |\vec{p}\rangle\, f(\vec{p})\, \delta^{(3)}(\vec{p} - \vec{q}) = f(\vec{q})\, d\mu(\vec{q})\, |\vec{q}\rangle . \tag{1.73}$$

For this to equal $|\vec{q}\rangle$, we need $f(\vec{q})\, d\mu(\vec{q}) = 1$, i.e., the measure must be:

$$d\mu(\vec{p}) = \frac{d^3p}{f(\vec{p})}\,. \tag{1.74}$$

Application to conventions:

- *BD convention:* $f(\vec{p}) = 1$, so $d\mu = d^3p$, giving $\mathbb{1} = \int d^3p\, |\vec{p}\rangle\langle\vec{p}|$.
- *Covariant convention:* $f(\vec{p}) = (2\pi)^3 2E_p$, so $d\mu = d^3p/[(2\pi)^3 2E_p]$, giving $\mathbb{1} = \int \frac{d^3p}{(2\pi)^3 2E_p}\, |\vec{p}\rangle\langle\vec{p}|$.

Conclusion: The completeness relation is uniquely determined by the commutation relations. Once you fix the normalization factor $\mathcal{N}(\vec{p})$ in the field expansion (which determines the commutator), both the state normalization $\langle\vec{p}|\vec{q}\rangle$ and the measure in the completeness relation are fixed. There is no freedom to choose them independently.

Conversion between conventions

Let $a_{\rm BD}$ denote the annihilation operator in Convention I, and let $a_{\rm PS}$ denote the Peskin–Schroeder continuum convention, where the same mode expansion is written with measure $d^3p/(2\pi)^3$ and coefficient $1/\sqrt{2E_p}$. The key relations between operators in different conventions are:

$$a_{\rm PS}(\vec{p}) = (2\pi)^{3/2}\, a_{\rm BD}(\vec{p})\,, \tag{1.75}$$

$$a_{\rm cov}(\vec{p}) = (2\pi)^{3/2}\sqrt{2E_p}\, a_{\rm BD}(\vec{p})\,, \tag{1.76}$$

$$a_{\rm MS}(\vec{p}) = \sqrt{\frac{V}{(2\pi)^3}}\, a_{\rm PS}(\vec{p}) = \sqrt{\frac{V}{(2\pi)^3\, 2E_p}}\, a_{\rm cov}(\vec{p})\,. \tag{1.77}$$

Physical observables are convention-independent. When computing S-matrix elements and cross sections, different conventions lead to different intermediate expressions but the same final results for measurable quantities.

Advantages and drawbacks of each convention

While all conventions yield identical physical predictions, they differ in practical convenience depending on the context:

Convention I: Bjorken–Drell

+ Simple commutator: $[a, a^\dagger] = \delta^{(3)}(\vec{p} - \vec{p}\,')$ with no extra factors.
+ Direct analogy with non-relativistic quantum mechanics (harmonic oscillator algebra).

+ State counting is straightforward: $\langle \vec{p}|\vec{p}'\rangle = \delta^{(3)}(\vec{p}-\vec{p}')$.
– State normalization is *not* Lorentz invariant; must track transformation factors under boosts.
– The field expansion has an awkward $(2\pi)^{-3/2}$ factor.

Convention II: Covariant $(2E)$ normalization

+ Lorentz-invariant state normalization: $\langle \vec{p}|\vec{p}'\rangle = (2\pi)^3 2E_p\,\delta^{(3)}(\vec{p}-\vec{p}')$.
+ The completeness measure $d^3p/[(2\pi)^3 2E_p]$ is the Lorentz-invariant phase space.
+ Preferred in modern QFT textbooks (Peskin & Schroeder, Schwartz, Weinberg).
+ Cross-section formulas have manifestly covariant form.
– Commutator has energy-dependent factors: $[a, a^\dagger] = (2\pi)^3 2E_p\,\delta^{(3)}$.
– Less intuitive for beginners; the $2E_p$ factors can obscure the physics.

Convention III: Box normalization (Mandl–Shaw)

+ All quantities are dimensionless: $[a, a^\dagger] = \delta_{\vec{p},\vec{p}'}$, $\langle \vec{p}|\vec{p}'\rangle = \delta_{\vec{p},\vec{p}'}$.
+ No delta-function singularities; sums replace integrals.
+ Conceptually cleaner for counting states (e.g., in statistical mechanics).
+ Useful for lattice QFT and finite-volume calculations.
– Requires taking $V \to \infty$ limit at the end of calculations.
– Translation invariance is only approximate (recovered in the infinite-volume limit).
– Lorentz invariance is obscured by the finite box.

Recommendation: For learning QFT, the BD convention (used in these notes) offers the most direct connection to familiar quantum mechanics. For advanced calculations, especially those involving Lorentz transformations or crossing symmetry, the covariant convention is more natural. Box normalization is valuable for conceptual clarity and numerical work.

Consistency between field expansion, (anti)commutators, and state normalization

The normalization convention is not an arbitrary cosmetic choice: the factors in the field expansion, the (anti)commutation relations, and the normalization of one-particle states are tightly linked.

Box 1.9 Deriving the commutator from the field expansion

Consider a general mode expansion for a real scalar field:

$$\phi(x) = \int d^3p\, \mathcal{N}(\vec{p}) \left[a(\vec{p})\, e^{-ip\cdot x} + a^\dagger(\vec{p})\, e^{ip\cdot x} \right], \tag{1.78}$$

with normalization factor $\mathcal{N}(\vec{p})$ (assumed real for simplicity). The conjugate momentum is:

$$\pi(x) = \dot{\phi}(x) = \int d^3p\, \mathcal{N}(\vec{p})\, (-iE_p) \left[a(\vec{p})\, e^{-ip\cdot x} - a^\dagger(\vec{p})\, e^{ip\cdot x} \right]. \tag{1.79}$$

Goal: Determine $[a(\vec{p}), a^\dagger(\vec{q})]$ such that $[\phi(t,\vec{x}), \pi(t,\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y})$. Assume $[a(\vec{p}), a^\dagger(\vec{q})] = f(\vec{p})\,\delta^{(3)}(\vec{p}-\vec{q})$ for some function $f(\vec{p})$ to be determined. At equal times ($t = 0$):

$$[\phi(\vec{x}), \pi(\vec{y})] = \int d^3p\, d^3q\, \mathcal{N}(\vec{p})\mathcal{N}(\vec{q})(-iE_q) \times \left[\left(a_p e^{i\vec{p}\cdot\vec{x}} + a_p^\dagger e^{-i\vec{p}\cdot\vec{x}}\right), \left(a_q e^{i\vec{q}\cdot\vec{y}} - a_q^\dagger e^{-i\vec{q}\cdot\vec{y}}\right) \right]. \tag{1.80}$$

The only non-vanishing commutators are $[a_p, a_q^\dagger]$ and $[a_p^\dagger, a_q] = -[a_q, a_p^\dagger]$:

$$= \int d^3p\, d^3q\, \mathcal{N}(\vec{p})\mathcal{N}(\vec{q})(-iE_q)\, f(\vec{p})\, \delta^{(3)}(\vec{p}-\vec{q}) \times \left[-e^{i\vec{p}\cdot\vec{x}} e^{-i\vec{q}\cdot\vec{y}} - e^{-i\vec{p}\cdot\vec{x}} e^{i\vec{q}\cdot\vec{y}} \right]. \tag{1.81}$$

Integrating over $\vec{q}$ (setting $\vec{q} = \vec{p}$):

$$= \int d^3p\, \mathcal{N}(\vec{p})^2 E_p\, f(\vec{p})\, (i) \cdot 2\cos\big(\vec{p}\cdot(\vec{x}-\vec{y})\big). \tag{1.82}$$

Since the integrand is even in $\vec{p}$, we can write $2\cos(\vec{p}\cdot\vec{z}) = e^{i\vec{p}\cdot\vec{z}} + e^{-i\vec{p}\cdot\vec{z}} \to 2e^{i\vec{p}\cdot\vec{z}}$ under integration:

$$[\phi(\vec{x}), \pi(\vec{y})] = 2i \int d^3p\, \mathcal{N}(\vec{p})^2 E_p\, f(\vec{p})\, e^{i\vec{p}\cdot(\vec{x}-\vec{y})}. \tag{1.83}$$

Comparing with $i\delta^{(3)}(\vec{x}-\vec{y}) = i\int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{y})}$, we require:

$$\boxed{f(\vec{p}) = \frac{1}{2(2\pi)^3 E_p\, \mathcal{N}(\vec{p})^2}} \tag{1.84}$$

The one-particle state $|\vec{p}\rangle \equiv a^\dagger(\vec{p})|0\rangle$ then satisfies

$$\langle \vec{p}\,'|\vec{p}\rangle = f(\vec{p})\,\delta^{(3)}(\vec{p}-\vec{p}\,') = \frac{1}{2(2\pi)^3 E_p\,\mathcal{N}(\vec{p})^2}\,\delta^{(3)}(\vec{p}-\vec{p}\,')\,. \tag{1.85}$$

Thus, once $\mathcal{N}(\vec{p})$ is chosen, the mode commutators and state normalization are uniquely determined.

BD choice. In the Bjorken–Drell convention,

$$\mathcal{N}_{\rm BD}(\vec{p}) = \frac{1}{(2\pi)^{3/2}\sqrt{2E_p}}, \qquad \mathcal{N}_{\rm BD}^2 = \frac{1}{(2\pi)^3\cdot 2E_p}\,. \tag{1.86}$$

Substituting into (1.84):

$$f(\vec{p}) = \frac{1}{2(2\pi)^3 E_p\cdot\frac{1}{(2\pi)^3\cdot 2E_p}} = \frac{(2\pi)^3\cdot 2E_p}{2(2\pi)^3 E_p} = 1\,. \tag{1.87}$$

Therefore $[a(\vec{p}),a^\dagger(\vec{p}\,')] = \delta^{(3)}(\vec{p}-\vec{p}\,')$ and $\langle\vec{p}\,'|\vec{p}\rangle = \delta^{(3)}(\vec{p}-\vec{p}\,')$. This is the convention used throughout these notes.

Covariant choice. In the covariant convention,

$$\mathcal{N}_{\rm cov}(\vec{p}) = \frac{1}{(2\pi)^3\cdot 2E_p}, \qquad \mathcal{N}_{\rm cov}^2 = \frac{1}{(2\pi)^6\cdot 4E_p^2}\,. \tag{1.88}$$

Substituting into (1.84):

$$f(\vec{p}) = \frac{1}{2(2\pi)^3 E_p\cdot\frac{1}{(2\pi)^6\cdot 4E_p^2}} = \frac{(2\pi)^6\cdot 4E_p^2}{2(2\pi)^3 E_p} = (2\pi)^3\cdot 2E_p\,. \tag{1.89}$$

Therefore $[a(\vec{p}),a^\dagger(\vec{p}\,')] = (2\pi)^3\,2E_p\,\delta^{(3)}(\vec{p}-\vec{p}\,')$ and the state normalization is Lorentz-invariant:

$$\langle\vec{p}\,'|\vec{p}\rangle = (2\pi)^3\,2E_p\,\delta^{(3)}(\vec{p}-\vec{p}\,')\,. \tag{1.90}$$

Fermions. The same logic applies to Dirac fields, with anticommutators replacing commutators and with a normalization choice for the spinors u, v consistent with the chosen mode normalization. In BD conventions, the factors $\sqrt{m/E_p}$ in the mode expansion, together with $\bar{u}u = \delta_{ss'}$, $\bar{v}v = -\delta_{ss'}$, ensure $\{b,b^\dagger\} = \delta^{(3)}$ and $\{d,d^\dagger\} = \delta^{(3)}$.

The key point is that the triple $(\mathcal{N}, [a,a^\dagger], \langle p'|p\rangle)$ is not independent: choosing any one of them fixes the others, and the same pattern holds for fermions with anticommutators.

1.2 The complex scalar field

A complex scalar field $\phi(x) = (\phi_1(x) + i\phi_2(x))/\sqrt{2}$ consists of two real scalar fields. The crucial difference from the real case is that it carries a conserved charge, implying the existence of distinct particles and antiparticles.

The Lagrangian for a complex field $\phi(x) \neq \phi^\dagger(x)$ is:

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi \,. \tag{1.91}$$

Note the absence of the factor $1/2$ compared to the real scalar case.

Equation of motion

Box 1.10 Klein–Gordon equation for the complex scalar field

For a complex field, we treat ϕ and $\phi^\dagger$ as independent variables. The Euler–Lagrange equation for $\phi^\dagger$ is:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\dagger)} \right) - \frac{\partial \mathcal{L}}{\partial \phi^\dagger} = 0 \,. \tag{1.92}$$

Computing the derivatives from $\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi$:

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\dagger)} = \partial^\mu \phi \,, \tag{1.93}$$

$$\frac{\partial \mathcal{L}}{\partial \phi^\dagger} = -m^2 \phi \,. \tag{1.94}$$

Substituting:

$$\partial_\mu(\partial^\mu \phi) + m^2 \phi = 0 \quad \Longrightarrow \quad \boxed{(\Box + m^2)\phi = 0} \tag{1.95}$$

Similarly, varying with respect to ϕ gives the equation for $\phi^\dagger$:

$$(\Box + m^2)\phi^\dagger = 0 \,. \tag{1.96}$$

Both ϕ and $\phi^\dagger$ satisfy the same Klein–Gordon equation, as expected for a free field theory.

Canonical momenta and Hamiltonian

The canonical momenta are:

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \dot{\phi}^\dagger , \tag{1.97}$$

$$\pi^\dagger = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi^\dagger)} = \dot{\phi} . \tag{1.98}$$

Note that π is conjugate to ϕ (not to $\phi^\dagger$), and $\pi^\dagger$ is conjugate to $\phi^\dagger$.

Box 1.11 Hamiltonian density for the complex scalar field

The Hamiltonian density is obtained via the Legendre transformation:

$$\mathcal{H} = \pi \dot{\phi} + \pi^\dagger \dot{\phi}^\dagger - \mathcal{L} . \tag{1.99}$$

Step 1: Express time derivatives in terms of momenta. From the definitions:

$$\dot{\phi} = \pi^\dagger , \qquad \dot{\phi}^\dagger = \pi . \tag{1.100}$$

Step 2: Write the Lagrangian in terms of $\dot{\phi}$, $\dot{\phi}^\dagger$, and spatial derivatives:

$$\mathcal{L} = \dot{\phi}^\dagger \dot{\phi} - \nabla\phi^\dagger \cdot \nabla\phi - m^2 \phi^\dagger \phi . \tag{1.101}$$

Step 3: Compute the Hamiltonian density:

$$\begin{aligned} \mathcal{H} &= \pi \dot{\phi} + \pi^\dagger \dot{\phi}^\dagger - \mathcal{L} \\ &= \dot{\phi}^\dagger \cdot \pi^\dagger + \dot{\phi} \cdot \pi - \left(\dot{\phi}^\dagger \dot{\phi} - \nabla\phi^\dagger \cdot \nabla\phi - m^2 \phi^\dagger \phi \right) \\ &= \pi \pi^\dagger + \pi^\dagger \pi - \pi \pi^\dagger + \nabla\phi^\dagger \cdot \nabla\phi + m^2 \phi^\dagger \phi \\ &= \pi^\dagger \pi + \nabla\phi^\dagger \cdot \nabla\phi + m^2 \phi^\dagger \phi . \end{aligned} \tag{1.102}$$

The three terms represent:

- $\pi^\dagger \pi = \dot{\phi}^\dagger \dot{\phi}$: kinetic energy density
- $\nabla\phi^\dagger \cdot \nabla\phi$: gradient energy density
- $m^2 \phi^\dagger \phi$: mass energy density

The Hamiltonian density is:

$$\mathcal{H} = \pi^\dagger \pi + \nabla\phi^\dagger \cdot \nabla\phi + m^2 \phi^\dagger \phi . \tag{1.103}$$

Creation and annihilation operators

In terms of the operators for the two real fields:

$$a(\vec{k}) = \frac{1}{\sqrt{2}}\left(a_1(\vec{k}) + i a_2(\vec{k})\right), \tag{1.104}$$

$$b(\vec{k}) = \frac{1}{\sqrt{2}}\left(a_1(\vec{k}) - i a_2(\vec{k})\right). \tag{1.105}$$

Here $a(\vec{k})$ annihilates a particle (quantum of ϕ) and $b(\vec{k})$ annihilates an antiparticle (quantum of $\phi^\dagger$) when acting on kets.

The commutation relations are:

$$[a(\vec{k}), a^\dagger(\vec{k}')] = [b(\vec{k}), b^\dagger(\vec{k}')] = \delta^{(3)}(\vec{k} - \vec{k}'), \tag{1.106}$$

$$[a(\vec{k}), b(\vec{k}')] = [a(\vec{k}), b^\dagger(\vec{k}')] = 0. \tag{1.107}$$

All other commutators vanish.

Box 1.12 Normal-ordered Hamiltonian for the complex scalar field

The mode expansion for the complex scalar field is:

$$\phi(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2E_k}}\left[a(\vec{k})\, e^{-ik\cdot x} + b^\dagger(\vec{k})\, e^{ik\cdot x}\right], \tag{1.108}$$

$$\phi^\dagger(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2E_k}}\left[a^\dagger(\vec{k})\, e^{ik\cdot x} + b(\vec{k})\, e^{-ik\cdot x}\right]. \tag{1.109}$$

Note that ϕ contains a (annihilates particle) and $b^\dagger$ (creates antiparticle), while $\phi^\dagger$ contains $a^\dagger$ and b.

Substituting into $H = \int d^3x\, \mathcal{H}$ and following the same procedure as for the real scalar field, one finds:

$$H = \int d^3k\, E_k \left(a^\dagger(\vec{k})a(\vec{k}) + a(\vec{k})a^\dagger(\vec{k}) + b^\dagger(\vec{k})b(\vec{k}) + b(\vec{k})b^\dagger(\vec{k})\right)/2. \tag{1.110}$$

Using the commutation relations $[a, a^\dagger] = [b, b^\dagger] = \delta^{(3)}$:

$$H = \int d^3k\, E_k \left(a^\dagger(\vec{k})a(\vec{k}) + b^\dagger(\vec{k})b(\vec{k})\right) + \delta^{(3)}(\vec{0}) \int d^3k\, E_k. \tag{1.111}$$

The infinite vacuum energy is removed by normal ordering. The physical interpretation is clear:

- $a^\dagger a$ counts particles with energy E_k

- $b^\dagger b$ counts antiparticles with energy E_k
- Particles and antiparticles have the *same* mass and energy (as required by CPT symmetry)

The normal-ordered Hamiltonian is:

$$:H: = \int d^3k\, E_k \left(a^\dagger(\vec{k})a(\vec{k}) + b^\dagger(\vec{k})b(\vec{k})\right). \tag{1.112}$$

The conserved charge

The complex scalar field has a global $U(1)$ symmetry: $\phi \to e^{-i\alpha}\phi$, $\phi^\dagger \to e^{+i\alpha}\phi^\dagger$ (the field ϕ carries charge -1, so $\phi^\dagger$ creates positive-charge states—this convention makes the charge of the a-particle in the mode expansion below come out positive). By Noether's theorem, this leads to a conserved current:

$$j^\mu = i\left(\phi^\dagger \partial^\mu \phi - (\partial^\mu \phi^\dagger)\phi\right). \tag{1.113}$$

Box 1.13 Derivation of the Noether current and its conservation

Step 1: Noether current from the symmetry. Under an infinitesimal $U(1)$ transformation $\phi \to \phi - i\alpha\phi$, $\phi^\dagger \to \phi^\dagger + i\alpha\phi^\dagger$ (with characteristic variations $\Delta\phi = -i\phi$, $\Delta\phi^\dagger = +i\phi^\dagger$), the Lagrangian $\mathcal{L} = \partial_\mu\phi^\dagger\partial^\mu\phi - m^2\phi^\dagger\phi$ is invariant. The Noether current is:

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^\dagger)}\Delta\phi^\dagger = (\partial^\mu\phi^\dagger)(-i\phi) + (\partial^\mu\phi)(+i\phi^\dagger) = i\left(\phi^\dagger\partial^\mu\phi - (\partial^\mu\phi^\dagger)\phi\right). \tag{1.114}$$

Step 2: Proof of current conservation. Compute the divergence:

$$\begin{aligned}\partial_\mu j^\mu &= i\partial_\mu\left(\phi^\dagger\partial^\mu\phi - (\partial^\mu\phi^\dagger)\phi\right)\\ &= i\left((\partial_\mu\phi^\dagger)(\partial^\mu\phi) + \phi^\dagger(\partial_\mu\partial^\mu\phi) - (\partial_\mu\partial^\mu\phi^\dagger)\phi - (\partial^\mu\phi^\dagger)(\partial_\mu\phi)\right)\\ &= i\left(\phi^\dagger\Box\phi - (\Box\phi^\dagger)\phi\right).\end{aligned} \tag{1.115}$$

Using the Klein-Gordon equations $\Box\phi = -m^2\phi$ and $\Box\phi^\dagger = -m^2\phi^\dagger$:

$$\partial_\mu j^\mu = i\left(\phi^\dagger(-m^2\phi) - (-m^2\phi^\dagger)\phi\right) = i\left(-m^2\phi^\dagger\phi + m^2\phi^\dagger\phi\right) = 0. \tag{1.116}$$

The conserved charge is:

$$Q = \int d^3x\, j^0 = i\int d^3x\left(\phi^\dagger\dot{\phi} - \dot{\phi}^\dagger\phi\right) = i\int d^3x\left(\phi^\dagger\pi^\dagger - \pi\phi\right), \tag{1.117}$$

where we used $\pi = \dot{\phi}^\dagger$ and $\pi^\dagger = \dot{\phi}$.

Box 1.14 Derivation of the charge operator

Substituting the mode expansions for ϕ and $\phi^\dagger$ given just above into the charge:

$$Q = i \int d^3x \left(\phi^\dagger \dot{\phi} - \dot{\phi}^\dagger \phi\right). \tag{1.118}$$

Using $\dot{\phi} \sim -iE_k(a\, e^{-ikx} - b^\dagger e^{ikx})$ and the orthogonality of plane waves:

$$\begin{aligned} Q &= \int d^3k \left(a^\dagger(\vec{k})a(\vec{k}) - b(\vec{k})b^\dagger(\vec{k})\right) \\ &= \int d^3k \left(a^\dagger(\vec{k})a(\vec{k}) - b^\dagger(\vec{k})b(\vec{k})\right) - \delta^{(3)}(\vec{0}) \int d^3k\,. \end{aligned} \tag{1.119}$$

After normal ordering, the infinite constant is removed:

$$:Q: = \int d^3k \left(a^\dagger(\vec{k})a(\vec{k}) - b^\dagger(\vec{k})b(\vec{k})\right) = N_a - N_b\,. \tag{1.120}$$

The normal-ordered charge operator is:

$$:Q: = \int d^3k \left(a^\dagger(\vec{k})a(\vec{k}) - b^\dagger(\vec{k})b(\vec{k})\right) = N_a - N_b\,, \tag{1.121}$$

where N_a and N_b are the number operators for particles and antiparticles, respectively.

Physical interpretation:

- Particles (created by $a^\dagger$) carry charge $+1$
- Antiparticles (created by $b^\dagger$) carry charge -1
- The vacuum has zero charge: $Q\,|0\rangle = 0$
- Charge is *conserved*: $[Q, H] = 0$

Contrast with energy: While the Hamiltonian $H = \int d^3k\, E_k(a^\dagger a + b^\dagger b)$ counts particles and antiparticles with the *same sign* (both contribute positive energy), the charge $Q = N_a - N_b$ counts them with *opposite signs*. This reflects that energy is always positive, but charge can be positive or negative.

1.3 The Dirac field

The Lagrangian and Dirac equation

The Lagrangian for a free Dirac field ψ is:

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi\,, \tag{1.122}$$

where $\bar{\psi} = \psi^\dagger\gamma^0$ is the Dirac adjoint.

Box 1.15 Derivation of the Dirac equation from the Lagrangian

The Dirac Lagrangian can be written explicitly as:

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi = i\bar{\psi}_\alpha(\gamma^\mu)_{\alpha\beta}\partial_\mu\psi_\beta - m\bar{\psi}_\alpha\psi_\alpha\,, \tag{1.123}$$

where $\alpha,\beta = 1,\ldots,4$ are spinor indices.

Variation with respect to $\bar{\psi}$: Treating ψ and $\bar{\psi}$ as independent variables (justified because $\bar{\psi} = \psi^\dagger\gamma^0$ involves complex conjugation), the Euler–Lagrange equation for $\bar{\psi}$ is:

$$\frac{\partial\mathcal{L}}{\partial\bar{\psi}} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\right) = 0\,. \tag{1.124}$$

Since $\mathcal{L}$ contains no derivatives of $\bar{\psi}$, the second term vanishes:

$$\frac{\partial\mathcal{L}}{\partial\bar{\psi}_\alpha} = i(\gamma^\mu)_{\alpha\beta}\partial_\mu\psi_\beta - m\psi_\alpha = 0\,. \tag{1.125}$$

This gives the **Dirac equation**:

$$\boxed{(i\not{\partial} - m)\psi = 0} \qquad \text{or} \qquad (i\gamma^\mu\partial_\mu - m)\psi = 0\,. \tag{1.126}$$

Variation with respect to ψ: The Euler–Lagrange equation for ψ is:

$$\frac{\partial\mathcal{L}}{\partial\psi_\beta} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_\beta)}\right) = 0\,. \tag{1.127}$$

Computing:

$$\frac{\partial\mathcal{L}}{\partial\psi_\beta} = -m\bar{\psi}_\beta\,, \tag{1.128}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_\beta)} = i\bar{\psi}_\alpha(\gamma^\mu)_{\alpha\beta}\,. \tag{1.129}$$

Therefore:

$$-m\bar{\psi}_\beta - \partial_\mu\left(i\bar{\psi}_\alpha(\gamma^\mu)_{\alpha\beta}\right) = 0 \quad\Longrightarrow\quad -i\partial_\mu\bar{\psi}\gamma^\mu - m\bar{\psi} = 0\,. \tag{1.130}$$

This is the **adjoint Dirac equation**:

$$\bar{\psi}(i\overleftarrow{\partial\!\!\!/} + m) = 0 \qquad \text{i.e.} \qquad i(\partial_\mu \bar{\psi})\gamma^\mu + m\bar{\psi} = 0\,, \tag{1.131}$$

where $\overleftarrow{\partial}$ acts to the left.

Degrees of freedom

Now that we have the Dirac equation, we can understand the counting of degrees of freedom. The Dirac spinor ψ has four complex components, but not all represent independent physical degrees of freedom:

- **Off-shell:** The four complex components of ψ give 8 real degrees of freedom before imposing the Dirac equation.
- **On-shell:** The Dirac equation $(i\partial\!\!\!/ - m)\psi = 0$ is a *first-order* equation that relates components, effectively halving the degrees of freedom. For a plane-wave solution $\psi \sim u(\vec{p}, s)\, e^{-ip\cdot x}$, the constraint $(p\!\!\!/ - m)u = 0$ reduces 4 complex components to 2 independent ones (the two spin polarizations $s = \pm 1/2$).
- **Physical count:** A massive spin-1/2 field has four physical one-particle states on shell: particle/antiparticle, each with two spin polarizations. This matches the reduction from 8 real off-shell components to 4 on-shell physical degrees of freedom.
- **Comparison with Klein–Gordon:** The scalar field ϕ is second-order in time, so both ϕ and $\dot{\phi}$ are needed as initial data (2 real DOF for real scalar, 4 for complex). The Dirac field is first-order, so only ψ is needed—but ψ has 4 complex components, giving 8 real DOF off-shell, reduced to 4 on-shell.

Canonical momenta and constraints

The momentum conjugate to ψ is:

$$\pi_\psi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} = i\bar{\psi}\gamma^0 = i\psi^\dagger\,. \tag{1.132}$$

The momentum conjugate to $\bar{\psi}$ is:

$$\pi_{\bar{\psi}} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \bar{\psi})} = 0\,. \tag{1.133}$$

Box 1.16 Physical meaning of $\pi_{\bar{\psi}} = 0$

The vanishing of $\pi_{\bar{\psi}}$ is not a pathology but reflects the **first-order nature** of the Dirac equation:

1. **Constraint, not dynamical equation:** In the Hamiltonian formalism, $\pi_{\bar{\psi}} = 0$ is a **primary constraint** (in Dirac's terminology). It tells us that $\bar{\psi}$ and ψ are not independent phase-space variables—they are related by $\bar{\psi} = \psi^\dagger \gamma^0$.
2. **First-order vs. second-order:** The Klein–Gordon equation is second-order in time, so ϕ and $\dot{\phi} = \pi$ are independent. The Dirac equation is first-order, so ψ and $\dot{\psi}$ are *not* independent—$\dot{\psi}$ is determined by spatial derivatives of ψ via the equation of motion.
3. **Counting degrees of freedom:** For the complex scalar field, we have (ϕ, π) and $(\phi^\dagger, \pi^\dagger)$—four phase-space variables. For the Dirac field, the constraint $\pi_{\bar{\psi}} = 0$ and the relation $\pi_\psi = i\psi^\dagger$ mean we effectively have only ψ (4 complex components = 8 real) as independent, matching the scalar field count.
4. **Anticommutators:** The canonical quantization proceeds with $\{\psi_\alpha, \pi_{\psi,\beta}\} = \{\psi_\alpha, i\psi^\dagger_\beta\} = i\delta_{\alpha\beta}\delta^{(3)}(\vec{x} - \vec{y})$, which gives the standard anti-commutation relation $\{\psi_\alpha, \psi^\dagger_\beta\} = \delta_{\alpha\beta}\delta^{(3)}$.

The Hamiltonian

The Hamiltonian density can be written in two equivalent forms:

$$\mathcal{H} = \psi^\dagger(-i\vec{\alpha}\cdot\nabla + \beta m)\psi\,, \tag{1.134}$$

$$\mathcal{H} = \bar{\psi}(-i\vec{\gamma}\cdot\nabla + m)\psi\,, \tag{1.135}$$

where $\beta = \gamma^0$, $\vec{\alpha} = \gamma^0\vec{\gamma}$, and $\vec{\gamma} = (\gamma^1, \gamma^2, \gamma^3)$.

The matrices $\vec{\alpha}$ and β satisfy the anticommutation relations $\{\alpha^i, \alpha^j\} = 2\delta^{ij}$, $\{\alpha^i, \beta\} = 0$, and $\beta^2 = 1$, which are the defining relations of the Dirac algebra in the Hamiltonian formalism.

Box 1.17 Derivation of the Dirac Hamiltonian

The Hamiltonian density is obtained via the Legendre transformation. Since $\pi_{\bar{\psi}} = 0$:

$$\mathcal{H} = \pi_\psi \dot{\psi} - \mathcal{L} = i\psi^\dagger\dot{\psi} - \mathcal{L}\,. \tag{1.136}$$

Step 1: Write the Lagrangian explicitly:

$$\mathcal{L} = i\bar{\psi}\gamma^0\partial_0\psi + i\bar{\psi}\gamma^k\partial_k\psi - m\bar{\psi}\psi = i\psi^\dagger\dot{\psi} + i\bar{\psi}\gamma^k\partial_k\psi - m\bar{\psi}\psi\,, \tag{1.137}$$

where we used $\bar{\psi}\gamma^0 = \psi^\dagger\gamma^0\gamma^0 = \psi^\dagger$.
Step 2: Compute the Hamiltonian density:

$$\begin{aligned}\mathcal{H} &= i\psi^\dagger\dot{\psi} - \left(i\psi^\dagger\dot{\psi} + i\bar{\psi}\gamma^k\partial_k\psi - m\bar{\psi}\psi\right) \\ &= -i\bar{\psi}\gamma^k\partial_k\psi + m\bar{\psi}\psi \\ &= \bar{\psi}(-i\gamma^k\partial_k + m)\psi\,. \end{aligned} \tag{1.138}$$

Step 3: Express in terms of $\psi^\dagger$. Using $\bar{\psi} = \psi^\dagger\gamma^0$:

$$\mathcal{H} = \psi^\dagger\gamma^0(-i\gamma^k\partial_k + m)\psi = \psi^\dagger(-i\gamma^0\gamma^k\partial_k + m\gamma^0)\psi\,. \tag{1.139}$$

Defining $\alpha^k = \gamma^0\gamma^k$ and $\beta = \gamma^0$:

$$\mathcal{H} = \psi^\dagger(-i\alpha^k\partial_k + \beta m)\psi = \psi^\dagger(-i\vec{\alpha}\cdot\nabla + \beta m)\psi\,. \tag{1.140}$$

Alternative form using γ matrices:

$$\mathcal{H} = \psi^\dagger\gamma^0(-i\gamma^k\partial_k + m)\psi = \bar{\psi}(-i\vec{\gamma}\cdot\nabla + m)\psi\,. \tag{1.141}$$

Mode expansion

The field operators are expanded as:

$$\psi(x) = \sum_{s=\pm}\int \frac{d^3p}{(2\pi)^{3/2}}\sqrt{\frac{m}{E_p}}\left[b(\vec{p},s)\,u(\vec{p},s)\,e^{-ip\cdot x} + d^\dagger(\vec{p},s)\,v(\vec{p},s)\,e^{ip\cdot x}\right] \tag{1.142}$$

$$\bar{\psi}(x) = \sum_{s=\pm}\int \frac{d^3p}{(2\pi)^{3/2}}\sqrt{\frac{m}{E_p}}\left[b^\dagger(\vec{p},s)\,\bar{u}(\vec{p},s)\,e^{ip\cdot x} + d(\vec{p},s)\,\bar{v}(\vec{p},s)\,e^{-ip\cdot x}\right]. \tag{1.143}$$

Here:

- $b(\vec{p},s)$ annihilates an electron with momentum $\vec{p}$ and spin s when acting on kets
- $d^\dagger(\vec{p},s)$ creates a positron with momentum $\vec{p}$ and spin s
- $u(\vec{p},s)$ and $v(\vec{p},s)$ are the positive and negative energy spinors

When acting on bras, the roles of creation/annihilation are reversed.

Spinor properties

The spinors satisfy the Dirac equation in momentum space:

$$(\not{p} - m)u(\vec{p}, s) = 0, \qquad \bar{u}(\vec{p}, s)(\not{p} - m) = 0, \tag{1.144}$$

$$(\not{p} + m)v(\vec{p}, s) = 0, \qquad \bar{v}(\vec{p}, s)(\not{p} + m) = 0. \tag{1.145}$$

The orthogonality relations are:

$$\bar{u}(\vec{p}, s)u(\vec{p}, s') = \delta_{ss'}, \qquad \bar{v}(\vec{p}, s)v(\vec{p}, s') = -\delta_{ss'}, \tag{1.146}$$

$$u^\dagger(\vec{p}, s)u(\vec{p}, s') = \frac{E_p}{m}\delta_{ss'}, \qquad v^\dagger(\vec{p}, s)v(\vec{p}, s') = \frac{E_p}{m}\delta_{ss'}, \tag{1.147}$$

$$\bar{v}(\vec{p}, s)u(\vec{p}, s') = 0. \tag{1.148}$$

The completeness relations (spin sums) are:

$$\sum_s u_\alpha(\vec{p}, s)\bar{u}_\beta(\vec{p}, s) = \left(\frac{\not{p} + m}{2m}\right)_{\alpha\beta}, \tag{1.149}$$

$$\sum_s v_\alpha(\vec{p}, s)\bar{v}_\beta(\vec{p}, s) = \left(\frac{\not{p} - m}{2m}\right)_{\alpha\beta}. \tag{1.150}$$

Box 1.18 Proofs of spinor properties

1. Orthogonality $\bar{u}(\vec{p}, s)u(\vec{p}, s') = \delta_{ss'}$:
In the rest frame $\vec{p} = 0$, the spinors take a simple form. Let $u^{(s)}$ denote the rest-frame spinor for spin s. In the Dirac representation with BD normalization:

$$u^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad u^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad v^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad v^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{1.151}$$

With $\gamma^0 = \mathrm{diag}(1, 1, -1, -1)$ in the Dirac representation, we have $\bar{u}^{(s)} = u^{(s)\dagger}\gamma^0$:

$$\bar{u}^{(1)}u^{(1)} = (1, 0, 0, 0)\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 1, \qquad \bar{u}^{(1)}u^{(2)} = 0. \tag{1.152}$$

For a general frame, boost the spinors: the combination $\bar{u}u$ is Lorentz invariant (a scalar), so $\bar{u}(\vec{p}, s)u(\vec{p}, s') = \delta_{ss'}$ in all frames.

2. Orthogonality $\bar{v}(\vec{p}, s)v(\vec{p}, s') = -\delta_{ss'}$:

Similarly, in the rest frame:

$$\bar{v}^{(1)}v^{(1)} = (0,0,-1,0)\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = -1\,. \tag{1.153}$$

The minus sign arises from γ^0 having -1 entries in the lower block.

3. Mixed orthogonality $\bar{v}(\vec{p},s)u(\vec{p},s') = 0$:
From the Dirac equations $(\not{p} - m)u = 0$ and $\bar{v}(\not{p} + m) = 0$:

$$\bar{v}(\not{p} + m)u = \bar{v}\not{p}u + m\bar{v}u = 0\,. \tag{1.154}$$

But also $\not{p}u = mu$, so $\bar{v}mu + m\bar{v}u = 2m\bar{v}u = 0$, hence $\bar{v}u = 0$.

4. Relation between $u^\dagger u$ and $\bar{u}u$:
Note first that $u^\dagger u$ is *not* a Lorentz scalar — it is the timelike component of the conserved current $\bar{u}\gamma^\mu u$, since $\bar{u}\gamma^0 u = u^\dagger\gamma^0\gamma^0 u = u^\dagger u$. We therefore cannot just "invoke Lorentz covariance" to deduce its value from the rest frame; we must derive it directly. The cleanest route uses the well-known identity

$$\bar{u}(p,s)\,\gamma^\mu\,u(p,s') = \frac{p^\mu}{m}\,\bar{u}(p,s)\,u(p,s') = \frac{p^\mu}{m}\,\delta_{ss'}\,, \tag{1.155}$$

which is itself a consequence of the Gordon decomposition together with the spin-summed Dirac equation $(\not{p} - m)u = 0$ (a self-contained proof is given below). Taking $\mu = 0$ in (1.155):

$$u^\dagger(\vec{p},s)\,u(\vec{p},s') \;=\; \bar{u}(p,s)\,\gamma^0\,u(p,s') \;=\; \frac{p^0}{m}\,\delta_{ss'} \;=\; \frac{E_p}{m}\,\delta_{ss'}\,, \tag{1.156}$$

which is the desired result.

Sketch of (1.155). Using $\not{p}\,u = mu$ and $\bar{u}\not{p} = m\bar{u}$, write $2m\,\bar{u}\gamma^\mu u = \bar{u}(\not{p}\gamma^\mu + \gamma^\mu\not{p})u = \bar{u}\{\not{p},\gamma^\mu\}u = 2p^\mu\,\bar{u}u$. Hence $\bar{u}\gamma^\mu u = (p^\mu/m)\bar{u}u$, which on the diagonal $s = s'$ gives (1.155) after using $\bar{u}u = 1$.

5. Completeness relation $\sum_s u(\vec{p},s)\bar{u}(\vec{p},s) = (\not{p} + m)/(2m)$:
The sum $\sum_s u\bar{u}$ is a 4×4 matrix. It must be built from available objects: $\mathbb{1}$, γ^μ, and p^μ. The most general Lorentz-covariant form is:

$$\sum_s u(\vec{p},s)\bar{u}(\vec{p},s) = A\not{p} + B\,\mathbb{1}\,. \tag{1.157}$$

To find A and B, use two constraints:

- *Dirac equation:* $(\not{p}-m)u=0$ implies $(\not{p}-m)\sum_s u\bar{u}=0$, so $(A\not{p}^2+B\not{p})-m(A\not{p}+B)=0$. Using $\not{p}^2=p^2=m^2$: $(Am^2-mB)+(B-mA)\not{p}=0$. This gives $B=mA$.
- *Trace:* $\mathrm{Tr}(\sum_s u\bar{u})=\sum_s \bar{u}u=2$ (summing over 2 spin states). Also, $\mathrm{Tr}(A\not{p}+B)=4B$ (since $\mathrm{Tr}(\not{p})=0$). So $4B=2$, giving $B=1/2$ and $A=1/(2m)$.

Therefore:

$$\sum_s u(\vec{p},s)\bar{u}(\vec{p},s)=\frac{\not{p}+m}{2m}. \tag{1.158}$$

6. Completeness for v:
The same argument with $(\not{p}+m)v=0$ and $\sum_s \bar{v}v=-2$ gives:

$$\sum_s v(\vec{p},s)\bar{v}(\vec{p},s)=\frac{\not{p}-m}{2m}. \tag{1.159}$$

Explicit construction of spinors in the Dirac representation

We now construct the explicit form of the spinors $u(\vec{p},s)$ and $v(\vec{p},s)$ in the **Dirac (standard) representation** of the gamma matrices, following the **Bjorken–Drell (BD) normalization** conventions.

The Dirac representation of gamma matrices

In the Dirac (or standard) representation, the gamma matrices are:

$$\gamma^0=\begin{pmatrix}\mathbb{1}_2 & 0\\ 0 & -\mathbb{1}_2\end{pmatrix},\qquad \gamma^i=\begin{pmatrix}0 & \sigma^i\\ -\sigma^i & 0\end{pmatrix}, \tag{1.160}$$

where σ^i are the Pauli matrices:

$$\sigma^1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma^2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad \sigma^3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{1.161}$$

The matrices $\alpha^i=\gamma^0\gamma^i$ and $\beta=\gamma^0$ appearing in the Hamiltonian are:

$$\alpha^i=\begin{pmatrix}0 & \sigma^i\\ \sigma^i & 0\end{pmatrix},\qquad \beta=\begin{pmatrix}\mathbb{1}_2 & 0\\ 0 & -\mathbb{1}_2\end{pmatrix}. \tag{1.162}$$

The fifth gamma matrix is:

$$\gamma^5\equiv i\gamma^0\gamma^1\gamma^2\gamma^3=\begin{pmatrix}0 & \mathbb{1}_2\\ \mathbb{1}_2 & 0\end{pmatrix}. \tag{1.163}$$

Rest-frame spinors

In the rest frame ($\vec{p}=0$, $p^\mu=(m,\vec{0})$), the Dirac equation $(\not{p}-m)u=0$ becomes:

$$(\gamma^0 m - m)u = m(\gamma^0 - 1)u = 0. \tag{1.164}$$

In the Dirac representation, $\gamma^0 - 1 = \mathrm{diag}(0,0,-2,-2)$, so the solutions have vanishing lower components.

With the BD normalization $\bar{u}u = 1$, the rest-frame spinors are:

$$u^{(1)}(0) = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \qquad u^{(2)}(0) = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}. \tag{1.165}$$

For the negative-energy spinors, $(\not{p}+m)v=0$ gives $(\gamma^0+1)v=0$, so the solutions have vanishing upper components:

$$v^{(1)}(0) = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \qquad v^{(2)}(0) = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}. \tag{1.166}$$

Spin interpretation: The labels (1) and (2) correspond to spin-up and spin-down along the z-axis. In the rest frame, the spin operator is $\vec{S} = \frac{1}{2}\vec{\Sigma}$ with:

$$\Sigma^i = \begin{pmatrix}\sigma^i & 0\\ 0 & \sigma^i\end{pmatrix}, \qquad S^3 u^{(1)}(0) = +\tfrac{1}{2}u^{(1)}(0), \quad S^3 u^{(2)}(0) = -\tfrac{1}{2}u^{(2)}(0). \tag{1.167}$$

Boosted spinors: general momentum

To obtain spinors for arbitrary momentum $\vec{p}$, we apply a Lorentz boost to the rest-frame spinors. The boost is implemented by the spinor transformation:

$$u(\vec{p},s) = \frac{\not{p}+m}{\sqrt{2m(E_p+m)}}\,u^{(s)}(0). \tag{1.168}$$

Box 1.19 Derivation of the boost formula

A Lorentz boost along direction $\hat{n}$ with rapidity η is represented on spinors by:

$$S(\Lambda) = \exp\!\left(\frac{\eta}{2}\,\vec{\alpha}\cdot\hat{n}\right) = \cosh\frac{\eta}{2} + \vec{\alpha}\cdot\hat{n}\,\sinh\frac{\eta}{2}, \tag{1.169}$$

where $\tanh\eta = |\vec{p}|/E_p$, so $\cosh\eta = E_p/m$ and $\sinh\eta = |\vec{p}|/m$.

Using the identities $\cosh(\eta/2) = \sqrt{(E_p+m)/(2m)}$ and $\sinh(\eta/2) = |\vec{p}|/\sqrt{2m(E_p+m)}$:

$$S(\Lambda) = \frac{E_p + m + \vec{\alpha}\cdot\vec{p}}{\sqrt{2m(E_p+m)}}. \tag{1.170}$$

Since $\vec{\alpha}\cdot\vec{p} = \gamma^0\vec{\gamma}\cdot\vec{p}$, and using $\{\gamma^0,\gamma^i\}=0$ together with $\gamma^0 u^{(s)}(0) = +u^{(s)}(0)$ in the rest frame, one finds

$$\vec{\alpha}\cdot\vec{p}\,u^{(s)}(0) = \gamma^0\vec{\gamma}\cdot\vec{p}\,u^{(s)}(0) = -\vec{\gamma}\cdot\vec{p}\,\gamma^0 u^{(s)}(0) = -\vec{\gamma}\cdot\vec{p}\,u^{(s)}(0)\,.$$

Therefore

$$S(\Lambda)\,u^{(s)}(0) = \frac{E_p + m - \vec{\gamma}\cdot\vec{p}}{\sqrt{2m(E_p+m)}}\,u^{(s)}(0) = \frac{\gamma^0 E_p + m - \vec{\gamma}\cdot\vec{p}}{\sqrt{2m(E_p+m)}}\,u^{(s)}(0), \tag{1.171}$$

where in the last step we used $E_p u^{(s)}(0) = \gamma^0 E_p u^{(s)}(0)$ (since $\gamma^0 u(0) = u(0)$). Recognizing $\gamma^0 E_p - \vec{\gamma}\cdot\vec{p} = \gamma^\mu p_\mu = \not{p}$ (with $p_\mu = (E_p, -\vec{p})$ in the BD metric):

$$u(\vec{p},s) = \frac{\not{p} + m}{\sqrt{2m(E_p+m)}}\,u^{(s)}(0). \tag{1.172}$$

Explicit form. Writing $u = \begin{pmatrix}\xi\\ \eta\end{pmatrix}$ with two-component spinors ξ and η, and using:

$$\not{p} + m = \begin{pmatrix} E_p + m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -E_p + m\end{pmatrix}, \tag{1.173}$$

we obtain for $u^{(s)}(0) = \begin{pmatrix}\xi^{(s)}\\ 0\end{pmatrix}$:

$$\boxed{u(\vec{p},s) = \sqrt{\frac{E_p+m}{2m}}\begin{pmatrix}\xi^{(s)} \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\,\xi^{(s)}\end{pmatrix}} \tag{1.174}$$

where $\xi^{(1)} = \begin{pmatrix}1\\0\end{pmatrix}$ and $\xi^{(2)} = \begin{pmatrix}0\\1\end{pmatrix}$ are the two-component Pauli spinors.

For the v spinors, starting from $v^{(s)}(0) = \begin{pmatrix} 0 \\ \eta^{(s)} \end{pmatrix}$ with $\eta^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\eta^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$:

$$\boxed{v(\vec{p},s) = \sqrt{\frac{E_p+m}{2m}} \begin{pmatrix} \dfrac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\eta^{(s)} \\ \eta^{(s)} \end{pmatrix}} \tag{1.175}$$

Explicit spinors for motion along z-axis

For $\vec{p} = (0,0,p_z)$ with $p_z > 0$, we have $\vec{\sigma}\cdot\vec{p} = p_z\sigma^3 = p_z\,\mathrm{diag}(1,-1)$.

Particle spinors:

$$u(\vec{p},\uparrow) = \sqrt{\frac{E_p+m}{2m}} \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E_p+m} \\ 0 \end{pmatrix}, \qquad u(\vec{p},\downarrow) = \sqrt{\frac{E_p+m}{2m}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{-p_z}{E_p+m} \end{pmatrix}. \tag{1.176}$$

Antiparticle spinors:

$$v(\vec{p},\uparrow) = \sqrt{\frac{E_p+m}{2m}} \begin{pmatrix} \frac{p_z}{E_p+m} \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad v(\vec{p},\downarrow) = \sqrt{\frac{E_p+m}{2m}} \begin{pmatrix} 0 \\ \frac{-p_z}{E_p+m} \\ 0 \\ 1 \end{pmatrix}. \tag{1.177}$$

Verification of normalization

Let us verify that these spinors satisfy the BD normalization $\bar{u}u = 1$.

Direct calculation:

$$\begin{aligned} \bar{u}(\vec{p},s)u(\vec{p},s) = u^\dagger\gamma^0 u &= \frac{E_p+m}{2m}\left(\xi^{(s)\dagger},\ \xi^{(s)\dagger}\frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\right)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \xi^{(s)} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\xi^{(s)} \end{pmatrix} \\ &= \frac{E_p+m}{2m}\left[\xi^{(s)\dagger}\xi^{(s)} - \frac{(\vec{\sigma}\cdot\vec{p})^2}{(E_p+m)^2}\xi^{(s)\dagger}\xi^{(s)}\right]. \end{aligned} \tag{1.178}$$

Using $(\vec{\sigma}\cdot\vec{p})^2 = |\vec{p}|^2 = E_p^2 - m^2 = (E_p-m)(E_p+m)$ and $\xi^{(s)\dagger}\xi^{(s)} = 1$:

$$\bar{u}u = \frac{E_p+m}{2m}\left[1 - \frac{E_p-m}{E_p+m}\right] = \frac{E_p+m}{2m}\cdot\frac{2m}{E_p+m} = 1. \quad \checkmark \tag{1.179}$$

Similarly, one can verify $\bar{v}v = -1$, where the minus sign arises from the -1 entries in the lower block of γ^0.

Relation between u and v via charge conjugation

The antiparticle spinor v can be obtained from u through the *charge conjugation matrix*

$$C = i\gamma^0\gamma^2, \tag{1.180}$$

which is the unique (up to phase) 4×4 matrix satisfying $C^{-1}\gamma^\mu C = -(\gamma^\mu)^\top$, or equivalently the defining property

$$C(\gamma^\mu)^\top = -\gamma^\mu C, \qquad C^\dagger = C^{-1} = -C. \tag{1.181}$$

A more detailed discussion of C and its properties is given in Chapter 9. We prove that $C\bar{u}(p,s)^\top$ satisfies the antiparticle Dirac equation and is therefore a v spinor.

Start from the particle equation $(\not{p} - m)\,u = 0$. Taking the Dirac adjoint gives $\bar{u}\,(\not{p} - m) = 0$. Transposing:

$$(\not{p}^\top - m)\,\bar{u}^\top = 0. \tag{1.182}$$

Multiplying from the left by C and using the defining property $C(\gamma^\mu)^\top = -\gamma^\mu C$ above, which implies $C\not{p}^\top = -\not{p}\,C$:

$$C(\not{p}^\top - m)\,\bar{u}^\top = (-\not{p}\,C - mC)\,\bar{u}^\top = -(\not{p} + m)\,C\bar{u}^\top = 0. \tag{1.183}$$

Therefore $C\bar{u}(p,s)^\top$ satisfies $(\not{p} + m)\psi = 0$, which is precisely the Dirac equation for antiparticle spinors (1.145). With an appropriate phase convention for the two-component spinors $\eta^{(s)}$ relative to $\xi^{(s)}$, one can identify:

$$\boxed{v(p,s) = C\bar{u}(p,s)^\top} \tag{1.184}$$

Non-relativistic and ultra-relativistic limits

Non-relativistic limit ($|\vec{p}| \ll m$, so $E_p \approx m$):

$$u(\vec{p},s) \approx \begin{pmatrix} \xi^{(s)} \\ \frac{\vec{\sigma}\cdot\vec{p}}{2m}\,\xi^{(s)} \end{pmatrix}. \tag{1.185}$$

The lower ("small") components are suppressed by $|\vec{p}|/m \ll 1$. This is the regime where the Dirac equation reduces to the Pauli equation.

Ultra-relativistic limit ($|\vec{p}| \gg m$, so $E_p \approx |\vec{p}|$):

$$u(\vec{p},s) \approx \sqrt{\frac{E_p}{2m}} \begin{pmatrix} \xi^{(s)} \\ (\hat{p}\cdot\vec{\sigma})\,\xi^{(s)} \end{pmatrix}, \tag{1.186}$$

where $\hat{p} = \vec{p}/|\vec{p}|$. The upper and lower components become comparable in magnitude. In this limit, chirality (eigenvalue of γ^5) becomes approximately equal to helicity (projection of spin on momentum).

Helicity spinors

For many applications, especially in high-energy physics, it is convenient to use **helicity spinors**—eigenstates of the helicity operator:

$$h = \frac{\vec{\Sigma}\cdot\vec{p}}{2|\vec{p}|} = \frac{1}{2|\vec{p}|}\begin{pmatrix}\vec{\sigma}\cdot\vec{p} & 0\\ 0 & \vec{\sigma}\cdot\vec{p}\end{pmatrix}. \tag{1.187}$$

The helicity eigenvalues are $\lambda = \pm 1/2$, corresponding to spin parallel (+) or antiparallel (−) to the momentum direction.

For motion along the z-axis, the helicity eigenstates coincide with the spin eigenstates constructed above. For general $\vec{p}$, one must rotate the two-component spinors $\xi^{(s)}$ to the direction of $\vec{p}$.

If $\vec{p}$ has polar angles (θ,ϕ), the helicity eigenspinors are:

$$\xi^{(+)}(\hat{p}) = \begin{pmatrix}\cos\frac{\theta}{2}\\ e^{i\phi}\sin\frac{\theta}{2}\end{pmatrix}, \qquad \xi^{(-)}(\hat{p}) = \begin{pmatrix}-e^{-i\phi}\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}\end{pmatrix}. \tag{1.188}$$

These satisfy $(\vec{\sigma}\cdot\hat{p})\,\xi^{(\pm)} = \pm\xi^{(\pm)}$.

Anticommutation relations

The Dirac field obeys **anticommutation** relations (required by the spin-statistics theorem):

$$\{b(\vec{p},s), b^\dagger(\vec{p}',s')\} = \{d(\vec{p},s), d^\dagger(\vec{p}',s')\} = \delta_{ss'}\delta^{(3)}(\vec{p}-\vec{p}'), \tag{1.189}$$

$$\{b(\vec{p},s), b(\vec{p}',s')\} = \{d(\vec{p},s), d(\vec{p}',s')\} = 0, \tag{1.190}$$

$$\{b(\vec{p},s), d(\vec{p}',s')\} = \{b(\vec{p},s), d^\dagger(\vec{p}',s')\} = 0. \tag{1.191}$$

These imply the equal-time anticommutation relations for the fields:

$$\{\psi_\alpha(t,\vec{x}), \psi_\beta^\dagger(t,\vec{x}')\} = \delta_{\alpha\beta}\delta^{(3)}(\vec{x}-\vec{x}'), \tag{1.192}$$

$$\{\psi_\alpha(t,\vec{x}), \psi_\beta(t,\vec{x}')\} = \{\psi_\alpha^\dagger(t,\vec{x}), \psi_\beta^\dagger(t,\vec{x}')\} = 0. \tag{1.193}$$

Box 1.20 Field anticommutators from operator anticommutators

We derive $\{\psi_\alpha(t,\vec{x}), \psi_\beta^\dagger(t,\vec{x}')\} = \delta_{\alpha\beta}\delta^{(3)}(\vec{x}-\vec{x}')$ from the mode expansion.

At equal times ($t = 0$ for simplicity), the mode expansion gives:

$$\psi_\alpha(\vec{x}) = \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{m}{E_p}} \left[b_{\vec{p},s}\, u_\alpha(\vec{p},s)\, e^{i\vec{p}\cdot\vec{x}} + d^\dagger_{\vec{p},s}\, v_\alpha(\vec{p},s)\, e^{-i\vec{p}\cdot\vec{x}} \right], \tag{1.194}$$

$$\psi^\dagger_\beta(\vec{x}') = \sum_{s'} \int \frac{d^3p'}{(2\pi)^{3/2}} \sqrt{\frac{m}{E_{p'}}} \Big[b^\dagger_{\vec{p}',s'}\, u^*_\beta(\vec{p}',s')\, e^{-i\vec{p}'\cdot\vec{x}'} + d_{\vec{p}',s'}\, v^*_\beta(\vec{p}',s')\, e^{i\vec{p}'\cdot\vec{x}'} \Big]. \tag{1.195}$$

The anticommutator $\{\psi_\alpha, \psi^\dagger_\beta\}$ has four terms. Using $\{b, b^\dagger\} = \{d, d^\dagger\} = \delta_{ss'}\delta^{(3)}$ and $\{b, d\} = \{b, d^\dagger\} = 0$:

$$\{\psi_\alpha(\vec{x}), \psi^\dagger_\beta(\vec{x}')\} = \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{m}{E_p} \Big[u_\alpha(\vec{p},s) u^*_\beta(\vec{p},s)\, e^{i\vec{p}\cdot(\vec{x}-\vec{x}')} + v_\alpha(\vec{p},s) v^*_\beta(\vec{p},s)\, e^{-i\vec{p}\cdot(\vec{x}-\vec{x}')} \Big]. \tag{1.196}$$

Now use the relation $u^* = \gamma^0 \bar{u}^\top$ (in standard representations), which gives:

$$\sum_s u_\alpha u^*_\beta = \sum_s u_\alpha (\gamma^0)_{\beta\gamma} \bar{u}_\gamma = \left[(\not{p} + m)\gamma^0 / (2m) \right]_{\alpha\beta}. \tag{1.197}$$

Similarly for v. Using the identity (valid for on-shell p):

$$\frac{(\not{p} + m)\gamma^0 + (\not{p} - m)\gamma^0}{2m} = \frac{2p^0\gamma^0\gamma^0}{2m} = \frac{E_p}{m}, \tag{1.198}$$

where we used $\not{p}\gamma^0 + \gamma^0\not{p} = 2p^0$.
After changing $\vec{p} \to -\vec{p}$ in the second term (which is symmetric under this):

$$\{\psi_\alpha(\vec{x}), \psi^\dagger_\beta(\vec{x}')\} = \delta_{\alpha\beta} \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{x}')} = \delta_{\alpha\beta}\, \delta^{(3)}(\vec{x} - \vec{x}'). \tag{1.199}$$

The proof that $\{\psi, \psi\} = 0$ follows similarly: the bb and $d^\dagger d^\dagger$ terms vanish by the operator anticommutators, and the cross terms $bd^\dagger$ and $d^\dagger b$ also vanish.

Four-momentum operator

The four-momentum operator is:

$$P^\mu = \sum_s \int d^3p\, p^\mu \left[b^\dagger(\vec{p},s) b(\vec{p},s) + d^\dagger(\vec{p},s) d(\vec{p},s) \right] . \tag{1.200}$$

Physical interpretation:

- $P^0 = H$ is the Hamiltonian (total energy). The normal-ordered form ensures $\langle 0|\, H\, |0\rangle = 0$.
- $\vec{P} = (P^1, P^2, P^3)$ is the total three-momentum.
- $b^\dagger b$ counts electrons: $N_e = \sum_s \int d^3p\, b^\dagger(\vec{p},s) b(\vec{p},s)$.
- $d^\dagger d$ counts positrons: $N_{\bar{e}} = \sum_s \int d^3p\, d^\dagger(\vec{p},s) d(\vec{p},s)$.
- Both particles and antiparticles contribute *positively* to the energy and momentum. This is a consequence of normal ordering and the anticommutation relations.

P^μ as the generator of spacetime translations. The four-momentum operator generates infinitesimal spacetime translations via the commutator:

$$[\psi(x), P^\mu] = +i\partial^\mu \psi(x) . \tag{1.201}$$

This means that under a translation $x^\mu \to x^\mu + a^\mu$, the field transforms as:

$$\psi(x) \to e^{ia\cdot P} \psi(x) e^{-ia\cdot P} = \psi(x+a) . \tag{1.202}$$

This is the quantum field theory analogue of the classical mechanics relation $\{f, p\} = \partial f/\partial q$, where the momentum generates translations in position space.

Box 1.21 Proof that $[\psi(x), P^\mu] = +i\partial^\mu \psi(x)$

We prove this for $P^0 = H$ (time translations); the spatial components follow analogously.

Method 1: Direct calculation from the mode expansion. Recall the Heisenberg-picture mode expansion

$$\psi(x) = \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{m}{E_p}} \left[b_{\vec{p},s}\, u(p,s)\, e^{-ip\cdot x} + d^\dagger_{\vec{p},s}\, v(p,s)\, e^{+ip\cdot x} \right] ,$$

together with the normal-ordered Hamiltonian $H = \sum_s \int d^3k\, E_k \left[b^\dagger_{\vec{k},s} b_{\vec{k},s} + d^\dagger_{\vec{k},s} d_{\vec{k},s} \right]$. Using the canonical anticommutator identity for fermionic operators,

$$[A, BC] = \{A, B\} C - B \{A, C\} , \tag{1.203}$$

we evaluate the relevant brackets:

$$[b_{\vec{p},s},\, b^\dagger_{\vec{k},s'} b_{\vec{k},s'}] = \{b_{\vec{p},s}, b^\dagger_{\vec{k},s'}\}\, b_{\vec{k},s'} - b^\dagger_{\vec{k},s'}\, \{b_{\vec{p},s}, b_{\vec{k},s'}\}$$
$$= \delta_{ss'}\delta^{(3)}(\vec{p}-\vec{k})\, b_{\vec{k},s'} - 0 = +\delta_{ss'}\delta^{(3)}(\vec{p}-\vec{k})\, b_{\vec{p},s}\,, \tag{1.204}$$

$$[d^\dagger_{\vec{p},s},\, d^\dagger_{\vec{k},s'} d_{\vec{k},s'}] = \{d^\dagger_{\vec{p},s}, d^\dagger_{\vec{k},s'}\}\, d_{\vec{k},s'} - d^\dagger_{\vec{k},s'}\, \{d^\dagger_{\vec{p},s}, d_{\vec{k},s'}\}$$
$$= 0 - \delta_{ss'}\delta^{(3)}(\vec{p}-\vec{k})\, d^\dagger_{\vec{k},s'} = -\delta_{ss'}\delta^{(3)}(\vec{p}-\vec{k})\, d^\dagger_{\vec{p},s}\,. \tag{1.205}$$

Substituting and collapsing the delta functions:

$$[\psi(x), H] = \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\tfrac{m}{E_p}}\, E_p \left[+b_{\vec{p},s}\, u\, e^{-ip\cdot x} - d^\dagger_{\vec{p},s}\, v\, e^{+ip\cdot x}\right]. \tag{1.206}$$

On the other hand, differentiating the mode expansion w.r.t. x^0 brings down $(-iE_p)$ on the b term and $(+iE_p)$ on the $d^\dagger$ term, so

$$i\partial_0\psi(x) = \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\tfrac{m}{E_p}}\, E_p \left[+b_{\vec{p},s}\, u\, e^{-ip\cdot x} - d^\dagger_{\vec{p},s}\, v\, e^{+ip\cdot x}\right]. \tag{1.207}$$

Comparing (1.206) and (1.207): $[\psi, H] = +i\partial_0\psi = +i\partial^0\psi$, since $\partial_0 = \partial^0$ in Bjorken–Drell metric.

Method 2: From the Heisenberg equation. In the Heisenberg picture, operators evolve as $\frac{dO}{dt} = i[H, O]$, equivalently

$$[O, H] \;=\; -[H, O] \;=\; -\frac{1}{i}\frac{dO}{dt} \;=\; i\frac{dO}{dt}\,. \tag{1.208}$$

Applied to $\psi(x)$, for which $dO/dt = \partial_0\psi$ (no explicit Schrödinger-picture time dependence), this gives directly $[\psi, H] = +i\partial_0\psi = +i\partial^0\psi$.

For spatial translations, the same logic (now using $P^i = -\int d^3x\, \pi\, \partial_i\psi$ and the canonical equal-time anticommutator) gives $[\psi, P^i] = +i\partial^i\psi$.

The conserved charge

The Dirac Lagrangian is invariant under the global $U(1)$ transformation $\psi \to e^{-i\alpha}\psi$, $\bar\psi \to e^{+i\alpha}\bar\psi$ (the field ψ carries charge -1: it annihilates positive-charge particles,

and $\bar{\psi}$ creates them; this reproduces the standard $j^\mu = \bar{\psi}\gamma^\mu\psi$ and $Q = N_b - N_d$ below). The associated Noether current is:

$$j^\mu = \bar{\psi}\gamma^\mu\psi\,. \tag{1.209}$$

Box 1.22 Derivation of the Noether current and its conservation

Step 1: Noether current from the symmetry. Under the infinitesimal transformation $\psi \to \psi - i\alpha\psi$, $\bar{\psi} \to \bar{\psi} + i\alpha\bar{\psi}$ (characteristic variations $\Delta\psi = -i\psi$, $\Delta\bar{\psi} = +i\bar{\psi}$), the Lagrangian $\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$ is invariant. The canonical momentum conjugate to ψ is $\pi_\psi = i\bar{\psi}\gamma^0 = i\psi^\dagger$. The Noether current is:

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\Delta\psi + \Delta\bar{\psi}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})} = (i\bar{\psi}\gamma^\mu)(-i\psi) + 0 = -i^2\bar{\psi}\gamma^\mu\psi = \bar{\psi}\gamma^\mu\psi\,, \tag{1.210}$$

where we used that the Dirac Lagrangian has no $\partial_\mu\bar{\psi}$ term.

Step 2: Proof of current conservation. Compute the divergence:

$$\partial_\mu j^\mu = \partial_\mu(\bar{\psi}\gamma^\mu\psi) = (\partial_\mu\bar{\psi})\gamma^\mu\psi + \bar{\psi}\gamma^\mu(\partial_\mu\psi)\,. \tag{1.211}$$

The Dirac equation gives $i\gamma^\mu\partial_\mu\psi = m\psi$, hence $\gamma^\mu\partial_\mu\psi = -im\psi$. The adjoint Dirac equation gives $i\partial_\mu\bar{\psi}\gamma^\mu = -m\bar{\psi}$, hence $\partial_\mu\bar{\psi}\gamma^\mu = im\bar{\psi}$. Substituting:

$$\partial_\mu j^\mu = (im\bar{\psi})\psi + \bar{\psi}(-im\psi) = im\bar{\psi}\psi - im\bar{\psi}\psi = 0\,. \tag{1.212}$$

The conserved charge is:

$$Q = \int d^3x\, j^0 = \int d^3x\, \psi^\dagger\psi\,. \tag{1.213}$$

Box 1.23 Derivation of the Dirac charge operator

Using the mode expansion (1.142) at $t = 0$ (so that $\psi \sim b\,u\,e^{+i\vec{p}\cdot\vec{x}} + d^\dagger\,v\,e^{-i\vec{p}\cdot\vec{x}}$):

$$\begin{aligned}\psi^\dagger\psi = \sum_{s,s'}\int \frac{d^3p\,d^3p'}{(2\pi)^3}\frac{m}{\sqrt{E_pE_{p'}}}\Big\{ & b^\dagger_{\vec{p},s}b_{\vec{p}',s'}\,u^\dagger_s(\vec{p})u_{s'}(\vec{p}')\,e^{-i(\vec{p}-\vec{p}')\cdot\vec{x}} \\ & + b^\dagger_{\vec{p},s}d^\dagger_{\vec{p}',s'}\,u^\dagger_s(\vec{p})v_{s'}(\vec{p}')\,e^{-i(\vec{p}+\vec{p}')\cdot\vec{x}} \\ & + d_{\vec{p},s}b_{\vec{p}',s'}\,v^\dagger_s(\vec{p})u_{s'}(\vec{p}')\,e^{+i(\vec{p}+\vec{p}')\cdot\vec{x}} \\ & + d_{\vec{p},s}d^\dagger_{\vec{p}',s'}\,v^\dagger_s(\vec{p})v_{s'}(\vec{p}')\,e^{+i(\vec{p}-\vec{p}')\cdot\vec{x}}\Big\}\,.\end{aligned} \tag{1.214}$$

Integrating over $\vec{x}$: the *diagonal* pieces ($b^\dagger b$ and $dd^\dagger$) yield $\delta^{(3)}(\vec{p}-\vec{p}')$, while the *mixed* pieces ($b^\dagger d^\dagger$ and db) yield $\delta^{(3)}(\vec{p}+\vec{p}')$ — this is the crucial point

that determines which spinor orthogonality must be used:

- For the diagonal pieces we need spinor scalar products at the *same* 3-momentum:

$$u_s^\dagger(\vec{p})\,u_{s'}(\vec{p}) = v_s^\dagger(\vec{p})\,v_{s'}(\vec{p}) = \frac{E_p}{m}\,\delta_{ss'}\,. \tag{1.215}$$

- For the mixed pieces the delta function forces $\vec{p}' = -\vec{p}$, so we need spinor scalar products at *opposite* 3-momenta, and the relevant relation is

$$u_s^\dagger(\vec{p})\,v_{s'}(-\vec{p}) = v_s^\dagger(\vec{p})\,u_{s'}(-\vec{p}) = 0\,. \tag{1.216}$$

The orthogonality (1.216) (not the naive "$u^\dagger v = 0$" at the same momentum, which is not the relation needed here) is precisely what kills the mixed terms. Both (1.215) and (1.216) follow directly from the explicit BD-normalised spinors.
With these relations,

$$\begin{aligned} Q &= \sum_s \int d^3p\,\left(b^\dagger_{\vec{p},s} b_{\vec{p},s} + d_{\vec{p},s} d^\dagger_{\vec{p},s}\right) \\ &= \sum_s \int d^3p\,\left(b^\dagger_{\vec{p},s} b_{\vec{p},s} - d^\dagger_{\vec{p},s} d_{\vec{p},s}\right) + (\text{infinite constant})\,, \end{aligned} \tag{1.217}$$

where we used the anticommutator $\{d, d^\dagger\} = \delta_{ss'}\delta^{(3)}(\vec{p}-\vec{p}')$.

The normal-ordered charge operator is:

$$:Q: = \sum_s \int d^3p\,\left(b^\dagger(\vec{p},s)b(\vec{p},s) - d^\dagger(\vec{p},s)d(\vec{p},s)\right) = N_e - N_{\bar{e}}\,, \tag{1.218}$$

where N_e and $N_{\bar{e}}$ are the number operators for electrons and positrons.
Physical interpretation:

- Electrons (created by $b^\dagger$) carry charge $+1$ (in units where the electron charge is $+1$; in SI, multiply by e)
- Positrons (created by $d^\dagger$) carry charge -1
- The vacuum has zero charge: $Q\,|0\rangle = 0$
- Charge is conserved: $[Q, H] = 0$
- The electric charge operator is $Q_{\text{em}} = -e \cdot Q = e(N_{\bar{e}} - N_e)$, where $e > 0$ is the elementary charge magnitude

Contrast with four-momentum: While P^μ sums over particles and antiparticles with the same sign (both contribute positive energy and momentum in their direction

of motion), the charge $Q = N_e - N_{\bar{e}}$ counts them with opposite signs. This is the key difference between "gravitational charge" (mass-energy) and electric charge.

1.4 The photon field

The electromagnetic field is the most familiar gauge field, yet the most subtle to quantize. The difficulty arises because the four-potential A^μ contains redundant degrees of freedom: a massless spin-1 particle has only **two physical polarizations**, but A^μ has four components. This redundancy is encoded in the *gauge invariance* of electromagnetism.

The Lagrangian and equations of motion

The free electromagnetic Lagrangian is:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{1.219}$$

where the **field strength tensor** is defined as:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu . \tag{1.220}$$

The four-potential $A^\mu = (A^0, \vec{A}) = (\Phi, \vec{A})$ contains the scalar potential Φ and the vector potential $\vec{A}$. The electric and magnetic fields are:

$$\vec{E} = -\nabla\Phi - \frac{\partial \vec{A}}{\partial t}, \qquad \vec{B} = \nabla \times \vec{A} . \tag{1.221}$$

In components, with metric $(+,-,-,-)$ and $A^\mu = (\Phi, \vec{A})$, one finds $F^{0i} = \partial^0 A^i - \partial^i A^0 = \partial_t A^i + \partial_i \Phi = -E^i$ and $B^i = -\frac{1}{2}\epsilon^{ijk} F_{jk} = \frac{1}{2}\epsilon^{ijk} F^{jk}$. Hence the identifications are

$$F^{0i} = -E^i , \qquad F^{ij} = -\epsilon^{ijk} B^k . \tag{1.222}$$

The field strength tensor in matrix form is:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}. \tag{1.223}$$

The Lagrangian can be written in terms of the electric and magnetic fields:

$$\mathcal{L} = \frac{1}{2}(\vec{E}^2 - \vec{B}^2) . \tag{1.224}$$

Box 1.24 Derivation of Maxwell's equations from the Lagrangian

The Euler–Lagrange equations for the field A_ν are:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0\,. \tag{1.225}$$

Step 1: Expand the Lagrangian:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)\,. \tag{1.226}$$

Step 2: Compute $\partial\mathcal{L}/\partial(\partial_\mu A_\nu)$:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} &= -\frac{1}{4}\cdot 2(\partial^\mu A^\nu - \partial^\nu A^\mu) - \frac{1}{4}\cdot 2(-1)(\partial^\nu A^\mu - \partial^\mu A^\nu) \\ &= -(\partial^\mu A^\nu - \partial^\nu A^\mu) = -F^{\mu\nu}\,.\end{aligned} \tag{1.227}$$

Step 3: Since $\partial\mathcal{L}/\partial A_\nu = 0$ (no mass term), the equation of motion is:

$$\partial_\mu F^{\mu\nu} = 0\,. \tag{1.228}$$

These are the **source-free Maxwell equations**:

- $\nu = 0$: $\partial_i F^{i0} = \nabla\cdot\vec{E} = 0$ (Gauss's law, no charges)
- $\nu = j$: $\partial_0 F^{0j} + \partial_i F^{ij} = -\dot{E}^j + (\nabla\times\vec{B})^j = 0$ (Ampère's law, no currents)

The other two Maxwell equations ($\nabla\cdot\vec{B} = 0$ and Faraday's law) follow identically from the definition $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ (the Bianchi identity $\partial_{[\lambda}F_{\mu\nu]} = 0$).

Gauge invariance: classical analysis

The electromagnetic Lagrangian possesses a fundamental symmetry: it is invariant under **gauge transformations**:

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu\chi(x)\,, \tag{1.229}$$

for any scalar function $\chi(x)$.

Box 1.25 Proof of gauge invariance

Under the transformation (1.229), the field strength transforms as:

$$\begin{aligned} F'_{\mu\nu} &= \partial_\mu A'_\nu - \partial_\nu A'_\mu \\ &= \partial_\mu (A_\nu + \partial_\nu \chi) - \partial_\nu (A_\mu + \partial_\mu \chi) \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu + \partial_\mu \partial_\nu \chi - \partial_\nu \partial_\mu \chi \\ &= F_{\mu\nu}, \end{aligned} \tag{1.230}$$

since partial derivatives commute. Therefore the Lagrangian $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ is gauge invariant.

Physical meaning of gauge invariance:

- The electric and magnetic fields $\vec{E}$ and $\vec{B}$ are *physical observables* and are gauge invariant.
- The four-potential A_μ is *not* directly observable; different gauge choices give the same physics.
- Gauge invariance implies that A_μ contains *redundant* degrees of freedom.
- A massless vector field in 4D has $4 - 2 = 2$ physical degrees of freedom (two transverse polarizations).

Counting degrees of freedom: The four-potential A_μ has 4 components. Gauge invariance allows us to impose one condition (gauge fixing), reducing to 3. The constraint $\partial_\mu F^{\mu 0} = 0$ (Gauss's law) is not a dynamical equation but a constraint, removing one more degree of freedom. This leaves **2 physical degrees of freedom**, corresponding to the two transverse polarizations of the photon.

Gauge fixing: summary of choices

To quantize the theory, we must *fix the gauge*, i.e., choose a specific representative from each equivalence class of gauge-related potentials. Different gauge choices are useful in different contexts.

Gauge	Condition	Covariant?	Ghosts?
Coulomb (radiation)	$\nabla \cdot \vec{A} = 0$	No	No
Lorenz	$\partial_\mu A^\mu = 0$	Yes	No in QED
Feynman ($\xi = 1$)	$\partial_\mu A^\mu = 0$ + gauge-fixing term	Yes	No in QED
Landau ($\xi = 0$)	$\partial_\mu A^\mu = 0$ (strict)	Yes	No in QED
Axial	$n_\mu A^\mu = 0$	Partially	No
Temporal	$A^0 = 0$	No	No

Coulomb (radiation) gauge

The Coulomb gauge imposes:

$$\nabla \cdot \vec{A} = 0 \qquad \text{(transversality condition)}\,. \tag{1.231}$$

This condition can always be achieved: given any $\vec{A}$, we can find χ such that $\nabla \cdot (\vec{A} + \nabla\chi) = 0$ by solving the Poisson equation $\nabla^2\chi = -\nabla \cdot \vec{A}$.

In this gauge, Gauss's law $\nabla \cdot \vec{E} = 0$ becomes:

$$\nabla^2 A^0 + \frac{\partial}{\partial t}(\nabla \cdot \vec{A}) = \nabla^2 A^0 = 0\,. \tag{1.232}$$

For the free field with appropriate boundary conditions, this gives $A^0 = 0$.

Key features:

- Only the two transverse components of $\vec{A}$ are dynamical.
- Manifest Lorentz covariance is lost, but physical results are Lorentz invariant.
- No unphysical (ghost) states appear in the quantum theory.
- Most transparent for understanding the physical degrees of freedom.

Lorenz gauge and covariant gauges

The Lorenz gauge[2] imposes:

$$\partial_\mu A^\mu = 0 \qquad \text{(Lorenz condition)}\,. \tag{1.233}$$

This condition is manifestly Lorentz covariant. However, it does not completely fix the gauge: if A_μ satisfies (1.233), so does $A_\mu + \partial_\mu\chi$ for any χ satisfying $\Box\chi = 0$ (residual gauge freedom).

The equation of motion $\partial_\mu F^{\mu\nu} = 0$ can be rewritten as:

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \Box A^\nu - \partial^\nu(\partial_\mu A^\mu) = 0\,. \tag{1.234}$$

[2]Named after Ludvig Lorenz, not Hendrik Lorentz.

In the Lorenz gauge, $\partial_\mu A^\mu = 0$, so the second term vanishes and the equation of motion simplifies to:

$$\Box A^\mu = 0\,, \tag{1.235}$$

i.e., each component of A^μ satisfies the massless wave equation.

For practical calculations, one adds a **gauge-fixing term** to the Lagrangian:

$$\mathcal{L}_{\text{gf}} = -\frac{1}{2\xi}(\partial_\mu A^\mu)^2\,, \tag{1.236}$$

where ξ is the *gauge parameter*. Common choices:

- $\xi = 1$: **Feynman gauge** — simplest propagator: $D_{\mu\nu}(k) = -g_{\mu\nu}/(k^2 + i\epsilon)$
- $\xi = 0$: **Landau gauge** — propagator is transverse: $D_{\mu\nu}(k) = (-g_{\mu\nu} + k_\mu k_\nu/k^2)/(k^2 + i\epsilon)$
- General ξ: $D_{\mu\nu}(k) = \left(-g_{\mu\nu} + (1-\xi)k_\mu k_\nu/k^2\right)/(k^2 + i\epsilon)$

Box 1.26 Why does $\xi \to 0$ make sense?

At first sight, the limit $\xi \to 0$ seems problematic: the gauge-fixing term $-\frac{1}{2\xi}(\partial_\mu A^\mu)^2$ appears to diverge!

The resolution is that this term acts as a *penalty function* that suppresses field configurations violating the Lorenz condition. Consider the contribution to the action:

$$S_{\text{gf}} = -\frac{1}{2\xi}\int d^4x\,(\partial_\mu A^\mu)^2\,. \tag{1.237}$$

In the path integral formulation: The path integral weight is e^{iS}. For any configuration with $\partial_\mu A^\mu \neq 0$, the action $S_{\text{gf}} \to -\infty$ as $\xi \to 0$, making the weight $e^{iS_{\text{gf}}}$ oscillate infinitely fast and average to zero. Only configurations with $\partial_\mu A^\mu = 0$ *exactly* contribute.

Analogy with constrained mechanics: This is analogous to enforcing a constraint $f(q) = 0$ by adding a potential $V = \frac{1}{2\xi}f(q)^2$ and taking $\xi \to 0$. In the limit, the system is forced onto the constraint surface.

In practice: The propagator for general ξ is:

$$D_{\mu\nu}(k) = \frac{1}{k^2 + i\epsilon}\left(-g_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2}\right). \tag{1.238}$$

Taking $\xi \to 0$ gives the transverse (Landau) propagator, which automatically satisfies $k^\mu D_{\mu\nu}(k) = 0$. The limit is smooth and well-defined at the level of the propagator and Feynman rules.

Physical observables are gauge-parameter independent: The Ward-Takahashi identities[3] ensure that S-matrix elements do not depend on ξ.

Axial and temporal gauges

The **axial gauge** fixes $n_\mu A^\mu = 0$ for a fixed vector n^μ. The **temporal gauge** is the special case $A^0 = 0$.

These gauges avoid ghost fields but have more complicated propagators with spurious singularities that require careful treatment.

Plane wave expansion and polarization vectors

In the Coulomb gauge ($\nabla \cdot \vec{A} = 0$, $A^0 = 0$), the equation of motion for $\vec{A}$ is:

$$\Box \vec{A} = \left(\frac{\partial^2}{\partial t^2} - \nabla^2 \right) \vec{A} = 0 \,. \tag{1.239}$$

The general solution is a superposition of plane waves:

$$\vec{A}(x) = \int \frac{d^3k}{(2\pi)^3} \, \vec{\tilde{A}}(\vec{k}) \, e^{i(\vec{k}\cdot\vec{x} - \omega t)} + \text{c.c.}\,, \tag{1.240}$$

where $\omega = |\vec{k}|$ (dispersion relation for massless particles).

The transversality condition $\nabla \cdot \vec{A} = 0$ becomes $\vec{k} \cdot \vec{\tilde{A}}(\vec{k}) = 0$ in Fourier space. This means $\vec{\tilde{A}}$ must be *perpendicular* to the direction of propagation $\vec{k}$.

Polarization vectors

For each wave vector $\vec{k}$, we introduce two orthonormal **polarization vectors** $\vec{\varepsilon}(\vec{k}, \lambda)$ with $\lambda = 1, 2$, satisfying:

$$\vec{k} \cdot \vec{\varepsilon}(\vec{k}, \lambda) = 0 \qquad \text{(transversality)}\,, \tag{1.241}$$

$$\vec{\varepsilon}(\vec{k}, \lambda) \cdot \vec{\varepsilon}(\vec{k}, \lambda') = \delta_{\lambda\lambda'} \qquad \text{(orthonormality)}\,, \tag{1.242}$$

$$\vec{\varepsilon}(\vec{k}, +) \times \vec{\varepsilon}^{\,*}(\vec{k}, +) = -i\,\hat{k}\,, \qquad \vec{\varepsilon}(\vec{k}, -) \times \vec{\varepsilon}^{\,*}(\vec{k}, -) = +i\,\hat{k} \quad \text{for circular polarization}\,. \tag{1.243}$$

Linear polarization basis: For $\vec{k} = k\hat{z}$, a common choice is:

$$\vec{\varepsilon}(\vec{k}, 1) = \hat{x}\,, \qquad \vec{\varepsilon}(\vec{k}, 2) = \hat{y}\,. \tag{1.244}$$

[3] The Ward-Takahashi identities are relations between Green's functions that follow from gauge invariance. They are the quantum analogue of current conservation $\partial_\mu j^\mu = 0$. In momentum space, they state that contracting an amplitude with the photon momentum k^μ gives zero (up to terms involving external charged particles). These identities will be derived in detail when we discuss QED and renormalization.

Circular polarization basis:

$$\vec{\varepsilon}(\vec{k},\pm) = \frac{1}{\sqrt{2}}(\hat{x} \pm i\hat{y})\,. \tag{1.245}$$

Circular polarizations correspond to definite helicity: $\lambda = +$ (right-handed) and $\lambda = -$ (left-handed).

Completeness relation (polarization sum): The two transverse polarization vectors span the plane perpendicular to $\vec{k}$:

$$\sum_{\lambda=1}^{2} \varepsilon_i(\vec{k},\lambda)\varepsilon_j(\vec{k},\lambda) = \delta_{ij} - \frac{k_i k_j}{|\vec{k}|^2} \equiv \delta^{\perp}_{ij}(\vec{k})\,. \tag{1.246}$$

The right-hand side is the **transverse projector**: it projects any vector onto the plane perpendicular to $\vec{k}$.

Canonical quantization in the Coulomb gauge

In the Coulomb gauge, the canonical momentum conjugate to A^i is:

$$\pi^i = \frac{\partial \mathcal{L}}{\partial \dot{A}^i} = \dot{A}^i = -E^i\,. \tag{1.247}$$

Note that $\pi^0 = \partial\mathcal{L}/\partial\dot{A}^0 = 0$ identically — A^0 is not a dynamical variable. This is a **primary constraint** in Dirac's terminology.

The transverse delta function

We cannot impose the naive commutation relation $[A^i(\vec{x}), \pi^j(\vec{x}')] = i\delta^{ij}\delta^{(3)}(\vec{x}-\vec{x}')$ because it would violate the constraint $\nabla\cdot\vec{A} = 0$. Instead, we use the **transverse delta function**:

$$\delta^{\perp}_{ij}(\vec{x}-\vec{x}') = \int \frac{d^3k}{(2\pi)^3}\left(\delta_{ij} - \frac{k_i k_j}{|\vec{k}|^2}\right) e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\,. \tag{1.248}$$

This is the Fourier transform of the transverse projector (1.246).

Box 1.27 Properties of the transverse delta function

1. Transversality:

$$\partial_i \delta^{\perp}_{ij}(\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\, ik_i\left(\delta_{ij} - \frac{k_i k_j}{|\vec{k}|^2}\right) e^{i\vec{k}\cdot\vec{x}} = 0\,, \tag{1.249}$$

since $k_i(\delta_{ij} - k_i k_j/|\vec{k}|^2) = k_j - k_j = 0$.

2. Projection property: For any vector field $\vec{V}(\vec{x})$:

$$\int d^3x'\, \delta^{\perp}_{ij}(\vec{x}-\vec{x}')V^j(\vec{x}') = V^{\perp}_i(\vec{x})\,, \tag{1.250}$$

where $\vec{V}^{\perp}$ is the transverse part of $\vec{V}$ (satisfying $\nabla\cdot\vec{V}^{\perp}=0$).

3. Decomposition:

$$\delta^{\perp}_{ij}(\vec{x}) = \delta_{ij}\delta^{(3)}(\vec{x}) - \delta^{\parallel}_{ij}(\vec{x})\,, \tag{1.251}$$

where $\delta^{\parallel}_{ij}(\vec{x}) = -\partial_i\partial_j(4\pi|\vec{x}|)^{-1}$ is the longitudinal delta function.

Equal-time commutation relations

The correct equal-time commutation relations in the Coulomb gauge are:

$$[A^i(t,\vec{x}),\pi^j(t,\vec{x}')] = i\delta^{\perp}_{ij}(\vec{x}-\vec{x}')\,, \tag{1.252}$$

$$[A^i(t,\vec{x}),A^j(t,\vec{x}')] = [\pi^i(t,\vec{x}),\pi^j(t,\vec{x}')] = 0\,. \tag{1.253}$$

These commutators are consistent with the constraint $\nabla\cdot\vec{A}=0$:

$$[\nabla\cdot\vec{A}(t,\vec{x}),\pi^j(t,\vec{x}')] = i\partial_i\delta^{\perp}_{ij}(\vec{x}-\vec{x}') = 0\,. \tag{1.254}$$

Mode expansion and creation/annihilation operators

The quantized field in the Coulomb gauge is:

$$\vec{A}(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \sum_{\lambda=1}^{2} \vec{\varepsilon}(\vec{k},\lambda)\left[a(\vec{k},\lambda)\,e^{-ik\cdot x} + a^{\dagger}(\vec{k},\lambda)\,e^{ik\cdot x}\right], \tag{1.255}$$

where $\omega_k = |\vec{k}|$ and $k\cdot x = \omega_k t - \vec{k}\cdot\vec{x}$.

The creation and annihilation operators satisfy:

$$[a(\vec{k},\lambda),a^{\dagger}(\vec{k}',\lambda')] = \delta_{\lambda\lambda'}\delta^{(3)}(\vec{k}-\vec{k}')\,, \tag{1.256}$$

$$[a(\vec{k},\lambda),a(\vec{k}',\lambda')] = [a^{\dagger}(\vec{k},\lambda),a^{\dagger}(\vec{k}',\lambda')] = 0\,. \tag{1.257}$$

Box 1.28 Derivation of field commutators from operator commutators

Using the mode expansion (1.255):

$$[A^i(t,\vec{x}),\pi^j(t,\vec{x}')] = [A^i,-\dot{A}^j]$$
$$= \sum_{\lambda,\lambda'}\int \frac{d^3k\,d^3k'}{(2\pi)^3\sqrt{4\omega_k\omega_{k'}}}\varepsilon^i(\vec{k},\lambda)\varepsilon^j(\vec{k}',\lambda')$$
$$\times(i\omega_{k'})[a(\vec{k},\lambda),a^\dagger(\vec{k}',\lambda')]e^{i\vec{k}\cdot\vec{x}}e^{-i\vec{k}'\cdot\vec{x}'}+\text{h.c.} \tag{1.258}$$

Using (1.256) and the polarization sum (1.246):

$$[A^i(t,\vec{x}),\pi^j(t,\vec{x}')] = i\int\frac{d^3k}{(2\pi)^3}\left(\delta^{ij}-\frac{k^ik^j}{|\vec{k}|^2}\right)e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} = i\delta^{ij}_\perp(\vec{x}-\vec{x}')\,. \tag{1.259}$$

The Hamiltonian

The Hamiltonian in the Coulomb gauge is:

$$H = \frac{1}{2}\int d^3x\,(\vec{E}^2+\vec{B}^2) = \frac{1}{2}\int d^3x\,(\dot{\vec{A}}^2+(\nabla\times\vec{A})^2)\,. \tag{1.260}$$

In terms of creation and annihilation operators (the $+\frac{1}{2}$ written below is schematic: in the continuum normalisation the zero-point piece is properly $\frac{1}{2}\delta^{(3)}(\vec{0})$ per mode, an infinite vacuum-energy density that becomes $\frac{1}{2}$ per mode only in a finite quantisation volume V, where $\delta^{(3)}(\vec{0})\to V/(2\pi)^3$):

$$H = \sum_{\lambda=1}^{2}\int d^3k\,\omega_k\left(a^\dagger(\vec{k},\lambda)a(\vec{k},\lambda)+\tfrac{1}{2}\delta^{(3)}(\vec{0})\right). \tag{1.261}$$

After normal ordering (subtracting the infinite vacuum energy):

$$:H: = \sum_{\lambda=1}^{2}\int d^3k\,\omega_k\,a^\dagger(\vec{k},\lambda)a(\vec{k},\lambda)\,. \tag{1.262}$$

A photon state $|\vec{k},\lambda\rangle = a^\dagger(\vec{k},\lambda)|0\rangle$ has energy $\omega_k = |\vec{k}|$ and momentum $\vec{k}$.

The role of A^0 and the instantaneous Coulomb interaction

In the Coulomb gauge for the *free* field, we set $A^0 = 0$. However, in the *interacting* theory (QED), the situation is more subtle.

When matter (charged particles) is present, Gauss's law becomes:

$$\nabla \cdot \vec{E} = \rho \,, \tag{1.263}$$

where ρ is the charge density. Using $\vec{E} = -\nabla A^0 - \dot{\vec{A}}$ and the Coulomb gauge $\nabla \cdot \vec{A} = 0$:

$$-\nabla^2 A^0 = \rho \qquad \Rightarrow \qquad A^0(\vec{x}) = \int d^3x' \, \frac{\rho(\vec{x}')}{4\pi|\vec{x} - \vec{x}'|} \,. \tag{1.264}$$

Key observations:

- A^0 is *not* an independent dynamical variable; it is completely determined by the instantaneous charge distribution ρ.
- There is no time derivative in (1.264): A^0 responds *instantaneously* to changes in ρ.
- This does not violate causality! The physical fields $\vec{E}$ and $\vec{B}$ propagate causally; the apparent non-locality is an artifact of the Coulomb gauge.

The contribution of A^0 to the Hamiltonian gives the **instantaneous Coulomb interaction**:

$$H_{\text{Coul}} = \frac{1}{2} \int d^3x \, (\nabla A^0)^2 = \frac{1}{2} \int d^3x \, d^3x' \, \frac{\rho(\vec{x})\rho(\vec{x}')}{4\pi|\vec{x} - \vec{x}'|} \,. \tag{1.265}$$

For QED with electrons, the physical electric charge is $q_e = -e$ (with $e > 0$ the elementary charge), so the charge density is $\rho = q_e \psi^\dagger \psi = -e \psi^\dagger \psi$. Substituting:

$$H_{\text{Coul}} = \frac{q_e^2}{2} \int d^3x \, d^3x' \, \frac{\psi^\dagger(\vec{x})\psi(\vec{x})\psi^\dagger(\vec{x}')\psi(\vec{x}')}{4\pi|\vec{x} - \vec{x}'|} = \frac{e^2}{2} \int d^3x \, d^3x' \, \frac{\psi^\dagger(\vec{x})\psi(\vec{x})\psi^\dagger(\vec{x}')\psi(\vec{x}')}{4\pi|\vec{x} - \vec{x}'|} \,. \tag{1.266}$$

The Coulomb energy is even in the charge, so it does not depend on the sign of q_e — but having the charge density expressed with the correct sign is essential whenever ρ couples linearly to an external potential or to other fields (e.g. the photon field via $j^\mu A_\mu$).

This is the familiar Coulomb interaction between charges. In covariant gauges, this interaction is instead mediated by the exchange of longitudinal and timelike photons, which cancel against each other to give the same physical result.

Covariant quantization: Gupta–Bleuler formalism

In the covariant approach, we quantize all four components of A^μ and impose the Lorenz condition as a constraint on physical states.

The modified Lagrangian

We add the gauge-fixing term (1.236) to obtain:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 . \tag{1.267}$$

The equation of motion becomes:

$$\Box A^\mu - \left(1 - \frac{1}{\xi}\right)\partial^\mu(\partial_\nu A^\nu) = 0 . \tag{1.268}$$

In the Feynman gauge ($\xi = 1$), this simplifies to $\Box A^\mu = 0$ for each component.

Covariant commutation relations

We impose the covariant commutation relations:

$$[A^\mu(t,\vec{x}), \dot{A}^\nu(t,\vec{x}')] = -ig^{\mu\nu}\delta^{(3)}(\vec{x}-\vec{x}') . \tag{1.269}$$

Note the metric $g^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$: the timelike component has the *opposite* sign!

Mode expansion with four polarizations

The covariant mode expansion is:

$$A^\mu(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \sum_{\lambda=0}^{3} \varepsilon^\mu(\vec{k},\lambda)\left[a(\vec{k},\lambda)\, e^{-ik\cdot x} + a^\dagger(\vec{k},\lambda)\, e^{ik\cdot x}\right], \tag{1.270}$$

with *four* polarization vectors:

- $\lambda = 1,2$: transverse (physical) polarizations, $\varepsilon^\mu = (0,\vec{\varepsilon})$
- $\lambda = 0$: timelike (scalar) polarization, $\varepsilon^\mu = (1,0,0,0)$
- $\lambda = 3$: longitudinal polarization, $\varepsilon^\mu = (0,\hat{k})$

The polarization vectors satisfy

$$g_{\mu\nu}\,\varepsilon^\mu(\vec{k},\lambda)\,\varepsilon^\nu(\vec{k},\lambda') \;=\; \varepsilon^\mu(\vec{k},\lambda)\,\varepsilon_\mu(\vec{k},\lambda') \;=\; \eta_{\lambda\lambda'} , \tag{1.271}$$

where the index μ is raised/lowered with the *Minkowski* metric $g^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$, and the result $\eta_{\lambda\lambda'}$ is a 4×4 *symbol* that turns out to be numerically equal to $\mathrm{diag}(1,-1,-1,-1)$. This is not a new "polarisation metric" but a consequence of choosing the four basis polarisations aligned with the metric basis (one timelike + three spatial unit vectors): for the timelike $\lambda = 0$ one has $\varepsilon^0\varepsilon_0 = +1$ and $\vec{\varepsilon} = 0$, hence $+1$; for the transverse and longitudinal $\lambda = 1,2,3$ one has $\varepsilon^0 = 0$ and $|\vec{\varepsilon}| = 1$, hence -1.

The problem of negative-norm states

The commutation relation (1.269) implies:

$$[a(\vec{k},\lambda), a^\dagger(\vec{k}',\lambda')] = -\eta_{\lambda\lambda'}\delta^{(3)}(\vec{k}-\vec{k}')\,. \tag{1.272}$$

For the timelike polarization ($\lambda = 0$):

$$[a(\vec{k},0), a^\dagger(\vec{k}',0)] = -\delta^{(3)}(\vec{k}-\vec{k}')\,. \tag{1.273}$$

This leads to states with **negative norm**:

$$\| a^\dagger(\vec{k},0)\,|0\rangle \,\|^2 = \langle 0|\, a(\vec{k},0) a^\dagger(\vec{k},0)\,|0\rangle = -\delta^{(3)}(0) < 0\,. \tag{1.274}$$

Such states would give negative probabilities — clearly unphysical!

The Gupta–Bleuler condition

The resolution is that *not all states in the Fock space are physical*. We impose the **Gupta–Bleuler condition**: physical states $|\text{phys}\rangle$ must satisfy:

$$\partial_\mu A^{\mu(+)}(x)\,|\text{phys}\rangle = 0\,, \tag{1.275}$$

where $A^{\mu(+)}$ is the positive-frequency (annihilation) part of A^μ.

In terms of mode operators, this becomes:

$$\left[a(\vec{k},0) - a(\vec{k},3)\right]|\text{phys}\rangle = 0 \qquad \forall\,\vec{k}\,. \tag{1.276}$$

Box 1.29 Consequences of the Gupta–Bleuler condition

1. The vacuum is physical:
Since $a(\vec{k},\lambda)\,|0\rangle = 0$ for all λ, we have:

$$\left[a(\vec{k},0) - a(\vec{k},3)\right]|0\rangle = 0 - 0 = 0\,. \tag{1.277}$$

The vacuum satisfies the Gupta–Bleuler condition (1.276).

2. Transverse photon states are physical:
Consider a single transverse photon state $|\vec{k},\lambda_T\rangle = a^\dagger(\vec{k},\lambda_T)\,|0\rangle$ with $\lambda_T = 1$ or 2. We need to check:

$$\left[a(\vec{k}',0) - a(\vec{k}',3)\right] a^\dagger(\vec{k},\lambda_T)\,|0\rangle = 0\,. \tag{1.278}$$

Since $[a(\vec{k}',\lambda), a^\dagger(\vec{k},\lambda')] = -\eta_{\lambda\lambda'}\delta^{(3)}(\vec{k}'-\vec{k})$ and the transverse polarizations ($\lambda_T = 1,2$) are orthogonal to the scalar ($\lambda = 0$) and longitudinal ($\lambda = 3$) polarizations:

$$[a(\vec{k}',0), a^\dagger(\vec{k},\lambda_T)] = 0\,, \qquad [a(\vec{k}',3), a^\dagger(\vec{k},\lambda_T)] = 0\,. \tag{1.279}$$

Therefore $\left[a(\vec{k}',0) - a(\vec{k}',3)\right]$ commutes with $a^\dagger(\vec{k},\lambda_T)$, and acting on the vacuum gives zero. Multi-photon states with only transverse photons are also physical.

3. Scalar and longitudinal photons appear only in zero-norm combinations:

Consider creating a scalar photon: $|\vec{k},0\rangle = a^\dagger(\vec{k},0)\,|0\rangle$. Acting with the Gupta–Bleuler operator:

$$\begin{aligned}\left[a(\vec{k}',0) - a(\vec{k}',3)\right]a^\dagger(\vec{k},0)\,|0\rangle &= [a(\vec{k}',0), a^\dagger(\vec{k},0)]\,|0\rangle \\ &= -\delta^{(3)}(\vec{k}'-\vec{k})\,|0\rangle \neq 0\,. \end{aligned} \tag{1.280}$$

So $|\vec{k},0\rangle$ alone is *not* physical. Similarly, $|\vec{k},3\rangle = a^\dagger(\vec{k},3)\,|0\rangle$ alone is not physical.

However, the *combination*:

$$|\vec{k},S\rangle \equiv \left(a^\dagger(\vec{k},0) + a^\dagger(\vec{k},3)\right)|0\rangle \tag{1.281}$$

is physical:

$$\begin{aligned}\left[a(\vec{k}',0) - a(\vec{k}',3)\right]\left(a^\dagger(\vec{k},0) + a^\dagger(\vec{k},3)\right)|0\rangle &= \left(-\delta^{(3)}(\vec{k}'-\vec{k}) + \delta^{(3)}(\vec{k}'-\vec{k})\right)|0\rangle \\ &= 0\,, \end{aligned} \tag{1.282}$$

where we used $[a(\vec{k}',0), a^\dagger(\vec{k},0)] = -\delta^{(3)}$ and $[a(\vec{k}',3), a^\dagger(\vec{k},3)] = +\delta^{(3)}$.

But this state has **zero norm**. Expanding the four-term product and noting that the cross terms $\langle 0|\,a(\vec{k},\lambda)a^\dagger(\vec{k},\lambda')\,|0\rangle$ vanish for $\lambda \neq \lambda'$ (because $[a(\vec{k},\lambda), a^\dagger(\vec{k},\lambda')] \propto \eta_{\lambda\lambda'}$ is diagonal):

$$\begin{aligned}\langle \vec{k},S|\vec{k},S\rangle &= \langle 0|\left(a(\vec{k},0) + a(\vec{k},3)\right)\left(a^\dagger(\vec{k},0) + a^\dagger(\vec{k},3)\right)|0\rangle \\ &= \langle 0|\,a(\vec{k},0)a^\dagger(\vec{k},0)\,|0\rangle \;+\; \langle 0|\,a(\vec{k},3)a^\dagger(\vec{k},3)\,|0\rangle \\ &= \langle 0|\,[a(\vec{k},0), a^\dagger(\vec{k},0)]\,|0\rangle \;+\; \langle 0|\,[a(\vec{k},3), a^\dagger(\vec{k},3)]\,|0\rangle \quad (\text{using } a\,|0\rangle = 0) \\ &= -\delta^{(3)}(\vec{0}) + \delta^{(3)}(\vec{0}) = 0\,. \end{aligned} \tag{1.283}$$

The negative norm of the scalar photon *exactly cancels* the positive norm of the longitudinal photon!

4. Structure of the physical Hilbert space:

The space of states satisfying the Gupta–Bleuler condition has the structure:

$$\mathcal{H}_{\text{phys}} = \mathcal{H}_T \oplus \mathcal{H}_0\,, \tag{1.284}$$

where:

- $\mathcal{H}_T$ = states with only transverse photons (positive norm)
- $\mathcal{H}_0$ = states with scalar+longitudinal combinations (zero norm)

States in $\mathcal{H}_0$ are orthogonal to all physical states (including themselves). They form the *null space* or *spurious states*.
The **true physical Hilbert space** is the quotient:

$$\mathcal{H}_{\text{true}} = \mathcal{H}_{\text{phys}}/\mathcal{H}_0\,, \tag{1.285}$$

where we identify states differing by null vectors: $|\psi\rangle \sim |\psi\rangle + |\chi\rangle$ for any $|\chi\rangle \in \mathcal{H}_0$.
On $\mathcal{H}_{\text{true}}$, the inner product is **positive-definite**, and the space is equivalent to the Fock space of transverse photons only (as in Coulomb gauge quantization).

Equivalence with Coulomb gauge

The Gupta–Bleuler formalism is physically equivalent to quantization in the Coulomb gauge:

- Both give the same S-matrix elements for physical processes.
- In Coulomb gauge, only physical degrees of freedom appear from the start.
- In covariant gauges, unphysical degrees of freedom are present but decouple from physical observables.

The advantage of covariant quantization is that Lorentz invariance is manifest at every step, making it easier to verify that physical results are Lorentz invariant.

Polarization sums in calculations

Coulomb gauge

In the Coulomb gauge, the sum over physical polarizations gives:

$$\sum_{\lambda=1}^{2} \varepsilon_i(\vec{k},\lambda)\varepsilon_j(\vec{k},\lambda) = \delta_{ij} - \frac{k_i k_j}{|\vec{k}|^2} = \delta_{ij}^{\perp}(\hat{k})\,. \tag{1.286}$$

Covariant gauges

In covariant calculations, the polarization sum for external photons is:

$$\sum_{\lambda=1}^{2} \varepsilon_\mu(k,\lambda)\varepsilon_\nu^*(k,\lambda) = -g_{\mu\nu} + \frac{k_\mu \bar{k}_\nu + \bar{k}_\mu k_\nu}{k\cdot\bar{k}}\,, \tag{1.287}$$

where $\bar{k}^\mu$ is an auxiliary vector.

For practical calculations, we often use the simpler replacement:

$$\sum_{\lambda=1}^{2} \varepsilon_\mu(k,\lambda)\varepsilon_\nu^*(k,\lambda) \to -g_{\mu\nu}\,. \tag{1.288}$$

The **Ward identity** guarantees that the extra terms (proportional to k_μ or k_ν) do not contribute to physical amplitudes when contracted with conserved currents.

1.5 Feynman propagators

The **Feynman propagator** represents the amplitude for a particle to propagate from spacetime point y to point x. More precisely, it is the vacuum expectation value of the time-ordered product of field operators. The propagator encodes both particle and antiparticle propagation in a single object.

Scalar field propagator

For a real scalar field (where $\phi = \phi^\dagger$), the Feynman propagator is defined as the vacuum expectation value of the time-ordered product of two field operators:

$$i\Delta_F(x-y) \equiv \langle 0|\,T\big(\phi(x)\phi(y)\big)\,|0\rangle\,. \tag{1.289}$$

(For a complex scalar one would instead pair ϕ with $\phi^\dagger$; the present derivation is the real case but the structure carries over.)

The time-ordered product is:

$$T\big(\phi(x)\phi(y)\big) = \theta(x_0-y_0)\,\phi(x)\phi(y) + \theta(y_0-x_0)\,\phi(y)\phi(x)\,. \tag{1.290}$$

Derivation of the momentum-space form

Substituting the mode expansion and using the commutation relations:

$$i\Delta_F(x-y) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\left[\theta(x_0-y_0)\,e^{-ik\cdot(x-y)} + \theta(y_0-x_0)\,e^{ik\cdot(x-y)}\right]. \tag{1.291}$$

To convert this to a four-dimensional integral, we use contour integration. Consider:

$$\begin{aligned}&\frac{1}{2\omega_k}\left[\theta(x_0-y_0)\,e^{-i\omega_k(x_0-y_0)} + \theta(y_0-x_0)\,e^{i\omega_k(x_0-y_0)}\right]\\ &\quad = -\int_C \frac{dk_0}{2\pi i}\,\frac{e^{-ik_0(x_0-y_0)}}{(k_0-\omega_k)(k_0+\omega_k)}\end{aligned} \tag{1.292}$$

where C is one of the two contours shown in Figure 1.1.

The contour is chosen to ensure convergence of the exponential $e^{-ik_0(x_0-y_0)}$ at infinity:

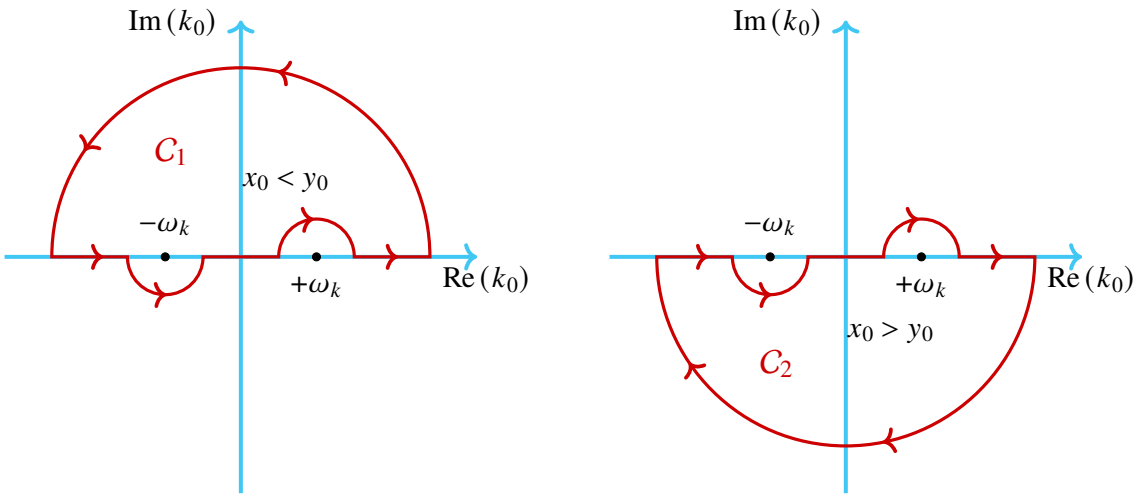


Figure 1.1. Integration contours for the scalar field propagator. *Left:* contour C_1, closed in the upper half-plane (used when $x_0 < y_0$). *Right:* contour C_2, closed in the lower half-plane (used when $x_0 > y_0$). The poles at $k_0 = \pm\omega_k$ lie on the real axis.

- If $x_0 < y_0$: we close in the *upper* half-plane (C_1), since $\lim_{k_0 \to +i\infty} e^{-ik_0(x_0-y_0)} \to 0$ when $(x_0 - y_0) < 0$. The enclosed pole is at $k_0 = -\omega_k$.
- If $x_0 > y_0$: we close in the *lower* half-plane (C_2), since $\lim_{k_0 \to -i\infty} e^{-ik_0(x_0-y_0)} \to 0$ when $(x_0 - y_0) > 0$. The enclosed pole is at $k_0 = +\omega_k$.

Evaluating the residues explicitly (note that C_2 is traversed clockwise, contributing an extra minus sign):

$$
\begin{aligned}
-\int_C \frac{dk_0}{2\pi i} \frac{e^{-ik_0(x_0-y_0)}}{(k_0-\omega_k)(k_0+\omega_k)}
&= \theta(x_0-y_0)\,(-1)\cdot 2\pi i \cdot \frac{-1}{2\pi i} \lim_{k_0\to\omega_k} \frac{(k_0-\omega_k)\,e^{-ik_0(x_0-y_0)}}{(k_0-\omega_k)(k_0+\omega_k)} \\
&\quad + \theta(y_0-x_0)\cdot 2\pi i \cdot \frac{-1}{2\pi i} \lim_{k_0\to-\omega_k} \frac{(k_0+\omega_k)\,e^{-ik_0(x_0-y_0)}}{(k_0-\omega_k)(k_0+\omega_k)} \\
&= \theta(x_0-y_0)\,\frac{e^{-i\omega_k(x_0-y_0)}}{2\omega_k} + \theta(y_0-x_0)\,\frac{e^{i\omega_k(x_0-y_0)}}{2\omega_k}\,,
\end{aligned}
\tag{1.293}
$$

which reproduces the time-ordered expression (1.291).

The $i\epsilon$ **prescription** encodes this contour choice by displacing the poles slightly off the real axis:

$$
k_0^2 - \omega_k^2 + i\epsilon = 0 \quad\Longrightarrow\quad k_0 = +\omega_k - i\epsilon\,, \qquad k_0 = -\omega_k + i\epsilon\,, \tag{1.294}
$$

as illustrated in Figure 1.2.
This is equivalent to adding $+i\epsilon$ to the denominator:

$$
\frac{1}{k_0^2 - \omega_k^2 + i\epsilon} = \frac{1}{k^2 - m^2 + i\epsilon}\,. \tag{1.295}
$$

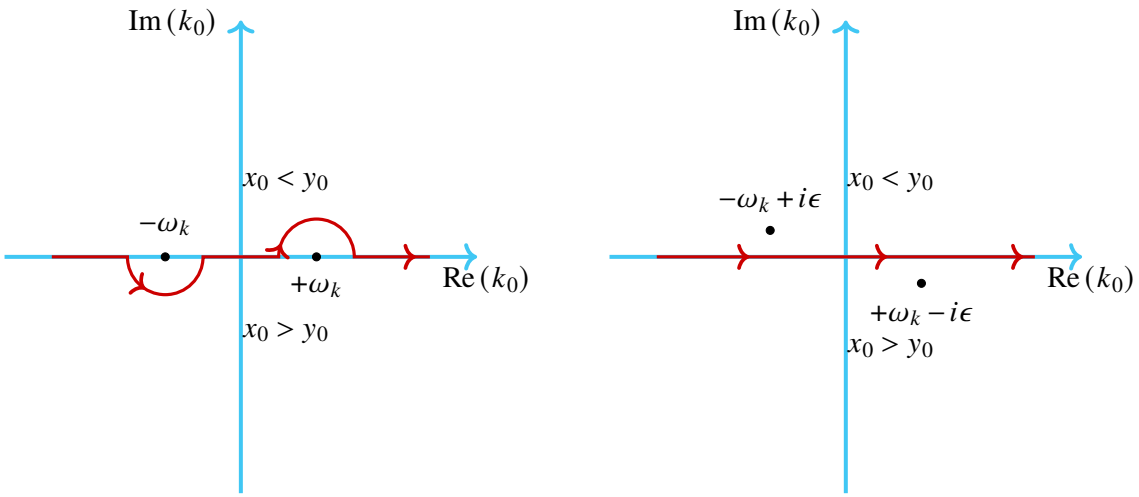


Figure 1.2. The $i\epsilon$ prescription. *Left:* original configuration with poles on the real axis and contour detouring around them via small semicircles. *Right:* equivalent configuration with poles displaced to $-\omega_k + i\epsilon$ and $+\omega_k - i\epsilon$, and integration along the entire real axis.

The final result

The Feynman propagator in momentum space is (we display it directly in the convention $i\Delta_F$, consistent with (1.289) and with Chapter 8):

$$\boxed{i\Delta_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i\, e^{-ik\cdot(x-y)}}{k^2 - m^2 + i\epsilon}} \tag{1.296}$$

Important: In the Fourier representation, k_0 is no longer constrained to equal $\omega_k = \sqrt{|\vec{k}|^2 + m^2}$; it is an independent integration variable. The on-shell condition is encoded in the pole structure of the denominator.

The propagator as a Green's function

The Feynman propagator satisfies:

$$(\Box_x + m^2)\, i\Delta_F(x-y) = -i\delta^{(4)}(x-y)\,. \tag{1.297}$$

Quick proof from the Fourier representation. Acting with $\Box_x + m^2$ on the momentum-space form (1.296):

$$\begin{aligned}(\Box_x + m^2)\, i\Delta_F(x-y) &= \int \frac{d^4k}{(2\pi)^4} \frac{i\,(-k^2 + m^2)\, e^{-ik\cdot(x-y)}}{k^2 - m^2 + i\epsilon} \\ &= -\int \frac{d^4k}{(2\pi)^4}\, i\, e^{-ik\cdot(x-y)} = -i\,\delta^{(4)}(x-y)\,.\end{aligned} \tag{1.298}$$

Proof via time derivatives of the T-product. A more instructive derivation works directly with the operator definition. Consider ∂_0 acting on $T(\phi(x)\phi(y))$:

$$\frac{\partial}{\partial x_0}T(\phi(x)\phi(y)) = \delta(x_0-y_0)\,[\phi(x),\phi(y)] + T(\partial_0\phi(x)\,\phi(y))\,. \tag{1.299}$$

The commutator vanishes for equal-time scalar fields: $[\phi(t,\vec{x}),\phi(t,\vec{y})]=0$. Taking a second time derivative:

$$\frac{\partial^2}{\partial x_0^2}T(\phi(x)\phi(y)) = \delta(x_0-y_0)\,[\dot\phi(x),\phi(y)] + T(\ddot\phi(x)\,\phi(y))\,. \tag{1.300}$$

Since $\dot\phi=\pi$ for the real scalar field, the equal-time canonical commutation relation gives $[\pi(t,\vec{x}),\phi(t,\vec{y})] = -i\delta^{(3)}(\vec{x}-\vec{y})$. Moreover, spatial derivatives pass through the θ functions unaffected: $\nabla_x^2 T(\phi(x)\phi(y)) = T(\nabla_x^2\phi(x)\,\phi(y))$. Combining:

$$\Box_x T(\phi(x)\phi(y)) = -i\delta^{(4)}(x-y) + T(\Box_x\phi(x)\,\phi(y))\,. \tag{1.301}$$

Taking the vacuum expectation value and using the Klein–Gordon equation $(\Box + m^2)\phi = 0$:

$$(\Box_x + m^2)\,\langle 0|\,T(\phi(x)\phi(y))\,|0\rangle = -i\delta^{(4)}(x-y)\,, \tag{1.302}$$

which is precisely $(\Box_x+m^2)\,i\Delta_F(x-y) = -i\delta^{(4)}(x-y)$, confirming (1.297).

Dirac field propagator

The Feynman propagator for the Dirac field is:

$$iS_{F,\alpha\beta}(x-y) = \langle 0|\,T(\psi_\alpha(x)\bar\psi_\beta(y))\,|0\rangle\,, \tag{1.303}$$

where $\alpha,\beta = 1,\ldots,4$ are spinor indices. In matrix notation (suppressing spinor indices):

$$iS_F(x-y) = \langle 0|\,T(\psi(x)\bar\psi(y))\,|0\rangle\,, \tag{1.304}$$

where $S_F(x-y)$ is a 4×4 matrix in spinor space.

Note the minus sign in the time-ordered product for fermions:

$$T(\psi(x)\bar\psi(y)) = \theta(x_0-y_0)\psi(x)\bar\psi(y) - \theta(y_0-x_0)\bar\psi(y)\psi(x)\,. \tag{1.305}$$

Physical interpretation

To understand what the Dirac propagator describes, it is useful to recall the role of ψ and $\bar\psi$ in creating and annihilating particles. From the mode expansion (1.142), $\psi \sim b\,u + d^\dagger v$ and $\bar\psi \sim b^\dagger\bar u + d\,\bar v$, so:

	Particles (e^-)	**Antiparticles** (e^+)
ψ: annihilates from $\lvert i\rangle$ creates in $\langle f\rvert$	b	$d^\dagger$
$\bar\psi$: creates in $\lvert i\rangle$ annihilates from $\langle f\rvert$	$b^\dagger$	d

The two terms in the time-ordered product therefore describe:

- $\theta(x_0-y_0)\,\psi(x)\bar\psi(y)$: the operator $\bar\psi(y)$ creates an electron at y, and $\psi(x)$ destroys it at x (particle propagation forward in time)
- $\theta(y_0-x_0)\,\bar\psi(y)\psi(x)$: the operator $\psi(x)$ creates a positron at x, and $\bar\psi(y)$ destroys it at y (antiparticle propagation forward in time, or equivalently, particle propagation backward in time)

The Feynman propagator treats both processes simultaneously in a single object.

Derivation

Step 1: Mode expansion. Using the field expansions (1.142) and keeping only the terms that give nonzero vacuum expectation values (only $b\,b^\dagger$ survives in $\langle 0|\psi\bar\psi\,|0\rangle$ and only $d\,d^\dagger$ in $\langle 0|\bar\psi\psi\,|0\rangle$):

$$\langle 0|\psi_\alpha(x)\bar\psi_\beta(y)\,|0\rangle = \sum_s \int \frac{d^3p}{(2\pi)^3}\frac{m}{E_p}\,u_\alpha(\vec p,s)\bar u_\beta(\vec p,s)\,e^{-ip\cdot(x-y)}\,, \tag{1.306}$$

$$\langle 0|\bar\psi_\beta(y)\psi_\alpha(x)\,|0\rangle = \sum_s \int \frac{d^3p}{(2\pi)^3}\frac{m}{E_p}\,v_\alpha(\vec p,s)\bar v_\beta(\vec p,s)\,e^{+ip\cdot(x-y)}\,. \tag{1.307}$$

Step 2: Spin sums. Using the completeness relations (1.150):

$$iS_F(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{m}{E_p}\left[\theta(x_0-y_0)\,e^{-ip\cdot(x-y)}\frac{\not p+m}{2m} - \theta(y_0-x_0)\,e^{ip\cdot(x-y)}\frac{\not p-m}{2m}\right]. \tag{1.308}$$

Step 3: Relation to the scalar propagator. To simplify (1.308), introduce the auxiliary functions

$$i\Delta_+(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\,e^{-ip\cdot x}\,, \qquad i\Delta_-(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\,e^{+ip\cdot x}\,. \tag{1.309}$$

Acting with $i\not\partial_x + m$ on $i\Delta_+(x-y)$ produces the factor $(\not p+m)/(2E_p)$ from the derivative of $e^{-ip\cdot x}$, and similarly $i\not\partial_x + m$ acting on $i\Delta_-(x-y)$ produces $(-\not p+$

$m)/(2E_p)$ from $e^{+ip\cdot x}$. One can verify that, after accounting for the factor m/E_p coming from the BD spinor normalization, equation (1.308) is equivalent to:

$$S_F(x-y) = (i\partial\!\!\!/_x + m)\left[\theta(x_0-y_0)\Delta_+(x-y) + \theta(y_0-x_0)\Delta_-(x-y)\right] . \quad (1.310)$$

The expression in brackets is precisely the scalar Feynman propagator (1.291): $i\Delta_F(x-y) = i[\theta(x_0-y_0)\Delta_+(x-y) + \theta(y_0-x_0)\Delta_-(x-y)]$. The derivative $\partial\!\!\!/_x$ can be pulled through the θ functions because the additional terms proportional to $\delta(x_0-y_0)$ cancel: the Pauli–Jordan function $\Delta(x) \equiv \Delta_+(x) - \Delta_-(x)$ is proportional to the equal-time commutator $[\phi(x), \phi^\dagger(y)]|_{x_0=y_0}$ and vanishes at equal times.

Therefore, we arrive at the fundamental relation:

$$S_F(x-y) = (i\partial\!\!\!/_x + m)\Delta_F(x-y) . \quad (1.311)$$

Step 4: Momentum-space form. Acting with $i\partial\!\!\!/_x + m$ on the Fourier representation of the scalar propagator (1.296), the derivative $i\partial_\mu$ brings down p_μ. Expressed in the same i-factor convention as $i\Delta_F$ for the scalar, i.e. $iS_F(x-y) \equiv \langle 0|T(\psi(x)\bar{\psi}(y))|0\rangle$:

$$\boxed{iS_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i(p\!\!\!/ + m)}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)}} \quad (1.312)$$

A useful shorthand notation in momentum space is

$$iS_F(p) = \frac{i}{p\!\!\!/ - m + i\epsilon} , \quad (1.313)$$

where the denominator is understood as $p\!\!\!/ - m = \gamma^\mu p_\mu - m$, and the equivalence with (1.312) follows from rationalising: multiplying numerator and denominator by $p\!\!\!/ + m$ and using $p\!\!\!/\,p\!\!\!/ = p^2$ gives $i(p\!\!\!/ + m)/(p^2 - m^2 + i\epsilon)$. Either form ($iS_F$ with the i on the right, or $S_F = 1/(p\!\!\!/ - m + i\epsilon)$ understood as "the Dirac propagator without the conventional Feynman i") is in use in the literature; in these notes we follow the chapter-8 convention and keep the i explicitly on both sides.

Photon propagator

The photon propagator describes the propagation of a virtual photon between two spacetime points. Its structure is more subtle than the scalar or Dirac propagators due to the gauge invariance of electromagnetism.

Definition

The Feynman propagator for the photon is defined as:

$$iD_{F,\mu\nu}(x-y) = \langle 0|T(A_\mu(x)A_\nu(y))|0\rangle , \quad (1.314)$$

where the time-ordered product for bosons is (no minus sign):

$$T(A_\mu(x)A_\nu(y)) = \theta(x_0-y_0)A_\mu(x)A_\nu(y) + \theta(y_0-x_0)A_\nu(y)A_\mu(x)\,. \quad (1.315)$$

The propagator $D_{F,\mu\nu}$ is a 4×4 matrix in Lorentz space (indices $\mu,\nu = 0,1,2,3$), in contrast to the Dirac propagator which is a matrix in spinor space.

Gauge dependence

Unlike the scalar and Dirac propagators, the photon propagator *depends on the choice of gauge*. This is because the gauge-fixing procedure modifies the Lagrangian and hence the two-point function.

General covariant gauge (R_ξ gauge):

With the gauge-fixing term $\mathcal{L}_{\rm gf} = -\frac{1}{2\xi}(\partial_\mu A^\mu)^2$, the propagator in momentum space is:

$$D_{F,\mu\nu}(k) = \frac{1}{k^2+i\epsilon}\left(-g_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2}\right). \quad (1.316)$$

Special cases:

- **Feynman gauge** ($\xi = 1$): $D_{F,\mu\nu}(k) = \dfrac{-g_{\mu\nu}}{k^2+i\epsilon}$ (simplest form)
- **Landau gauge** ($\xi = 0$): $D_{F,\mu\nu}(k) = \dfrac{1}{k^2+i\epsilon}\left(-g_{\mu\nu} + \dfrac{k_\mu k_\nu}{k^2}\right)$ (transverse)

Derivation in Feynman gauge

Box 1.30 Derivation of the photon propagator

In the Feynman gauge ($\xi = 1$), the equation of motion is $\Box A^\mu = 0$, and each component of A^μ behaves like an independent massless scalar field (with metric complications for A^0).

Step 1: Mode expansion. In covariant quantization:

$$A^\mu(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \sum_{\lambda=0}^{3} \varepsilon^\mu(\vec{k},\lambda)\left[a(\vec{k},\lambda)\,e^{-ik\cdot x} + a^\dagger(\vec{k},\lambda)\,e^{ik\cdot x}\right], \quad (1.317)$$

with $\omega_k = |\vec{k}|$ and commutation relations $[a(\vec{k},\lambda), a^\dagger(\vec{k}',\lambda')] = -\eta_{\lambda\lambda'}\delta^{(3)}(\vec{k}-\vec{k}')$.

Step 2: Polarization completeness. For the covariant polarization vectors $\varepsilon^\mu(\vec{k},\lambda)$ chosen as in Section 1.4 (one timelike, two transverse, one longitudinal), one verifies directly that the polarisation completeness relation reads

$$\sum_{\lambda=0}^{3} \eta_{\lambda\lambda}\, \varepsilon^{\mu}(\vec{k},\lambda)\, \varepsilon^{\nu}(\vec{k},\lambda) = g^{\mu\nu}, \tag{1.318}$$

i.e. the polarisation 4-vectors form an orthonormal basis (with metric $\eta_{\lambda\lambda}$) under the Minkowski inner product. The $-g^{\mu\nu}$ that will appear in the propagator comes *not* from this sum but from the commutator $[a(\vec{k},\lambda), a^{\dagger}(\vec{k}',\lambda')] = -\eta_{\lambda\lambda'}\delta^{(3)}(\vec{k}-\vec{k}')$ (note the minus sign), and the net minus is the source of the negative-norm states discussed in the Gupta–Bleuler analysis.

Step 3: Two-point function. Following the same steps as for the scalar field, with the photon mode expansion and commutation relations, the vacuum expectation value picks up exactly one factor of $-\eta_{\lambda\lambda}$ from the commutator, which combines with the polarisation completeness (1.318) to give

$$\langle 0|\, T(A^{\mu}(x)A^{\nu}(y))\, |0\rangle = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\,(-g^{\mu\nu}) \times \Big[\theta(x_0-y_0)e^{-ik\cdot(x-y)} + \theta(y_0-x_0)e^{ik\cdot(x-y)}\Big]. \tag{1.319}$$

Step 4: Fourier representation. Using the same contour arguments as for the scalar field, the Feynman propagator is:

$$iD_F^{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{-ig^{\mu\nu}}{k^2+i\epsilon}\, e^{-ik\cdot(x-y)}\,. \tag{1.320}$$

The result

The photon Feynman propagator in position space, in the convention $iD_F^{\mu\nu}(x-y) = \langle 0|T(A^{\mu}(x)A^{\nu}(y))|0\rangle$ (consistent with $i\Delta_F$ for the scalar and iS_F for the Dirac propagator), is

$$\boxed{iD_F^{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{-i\,g^{\mu\nu}}{k^2+i\epsilon}\, e^{-ik\cdot(x-y)} \qquad \text{(Feynman gauge)}} \tag{1.321}$$

In momentum space:

$$\boxed{iD_F^{\mu\nu}(k) = \frac{-i\,g^{\mu\nu}}{k^2+i\epsilon}} \tag{1.322}$$

Properties

- **Green's function:** In Feynman gauge, $iD_F^{\mu\nu}$ satisfies

$$\Box_x \, iD_F^{\mu\nu}(x-y) = +i\, g^{\mu\nu}\, \delta^{(4)}(x-y)\,, \tag{1.323}$$

  which follows directly from the Fourier representation: acting with $\Box_x$ brings down $(-k^2)$ in the integrand, $(-k^2)\cdot(-ig^{\mu\nu})/(k^2+i\epsilon) \to ig^{\mu\nu}$ as $\epsilon \to 0$, and the remaining Fourier integral gives $\delta^{(4)}(x-y)$; equivalently, dropping the i on both sides gives $\Box_x D_F^{\mu\nu} = +g^{\mu\nu}\delta^{(4)}(x-y)$, i.e. $D_F^{\mu\nu}$ is the Green's function of the d'Alembertian in Feynman gauge with the $+g^{\mu\nu}$ source convention.
- **Masslessness:** The pole is at $k^2 = 0$, reflecting the zero mass of the photon. There is no mass parameter in the propagator.
- **Tensor structure:** The factor $-g_{\mu\nu}$ ensures that the propagator connects all four components of A^μ. In Feynman gauge, all components propagate equally.
- **Physical observables:** Although the propagator depends on ξ, physical S-matrix elements are gauge-independent due to the Ward-Takahashi identities. The terms proportional to $k_\mu k_\nu$ vanish when contracted with conserved currents.
- **Coulomb gauge:** In the Coulomb gauge, only the transverse components propagate. The propagator is:

$$D_{ij}(\vec{k},\omega) = \frac{\delta_{ij} - \hat{k}_i\hat{k}_j}{\omega^2 - |\vec{k}|^2 + i\epsilon}\,, \tag{1.324}$$

  while $D_{0\mu} = D_{\mu 0} = 0$ *after* the instantaneous Coulomb piece (the Φ field, sourced by the charge density ρ) has been separated off and folded back into the Hamiltonian as H_{Coul}, eq. (1.265): only the transverse photon then propagates dynamically. Without this separation the Coulomb gauge propagator also carries a D_{00} piece encoding instantaneous Coulomb exchange; the statement "$D_{0\mu} = 0$" is correct for the *transverse*-only propagator of the radiation modes, not for the full $A^\mu A^\nu$ correlator.

Summary: comparison of propagators

Field	Propagator (momentum space)	Spin
Real scalar	$\dfrac{i}{k^2 - m^2 + i\epsilon}$	0
Dirac fermion	$\dfrac{i(\not{k} + m)}{k^2 - m^2 + i\epsilon}$	1/2
Photon (Feynman)	$\dfrac{-ig_{\mu\nu}}{k^2 + i\epsilon}$	1

1.6 Noether's theorem and conserved currents

One of the most profound results in theoretical physics is **Noether's theorem**, which establishes a deep connection between continuous symmetries of the action and conservation laws. We follow the treatment of Coleman [2].

General statement and proof

Consider a field theory with Lagrangian density $\mathcal{L}(\phi_r, \partial_\mu \phi_r)$, where ϕ_r denotes a collection of fields (the index r labels different field components). The action is:

$$S = \int d^4x \, \mathcal{L}(\phi_r, \partial_\mu \phi_r) \,. \tag{1.325}$$

Convention. Throughout this section we use the following convention to avoid ambiguity: $\Delta\phi_r$ denotes the *characteristic* (parameter-independent) variation of the field under the symmetry, and $\delta\phi_r \equiv \epsilon \, \Delta\phi_r$ denotes the *full* infinitesimal variation including the small parameter ϵ. Thus $\delta\phi_r$ is linear in ϵ and $\Delta\phi_r$ is not.

Noether's Theorem: *If the action is invariant under a continuous transformation of the fields*

$$\phi_r(x) \to \phi_r(x) + \epsilon \, \Delta\phi_r(x) \,, \qquad \delta\phi_r \equiv \epsilon \, \Delta\phi_r \,, \tag{1.326}$$

where ϵ is an infinitesimal parameter, then there exists a conserved current $j^\mu(x)$ satisfying

$$\partial_\mu j^\mu = 0 \,, \tag{1.327}$$

and a corresponding conserved charge

$$Q = \int d^3x \, j^0(x) \,, \qquad \frac{dQ}{dt} = 0 \,. \tag{1.328}$$

Proof

We present a detailed proof following the elegant approach of Coleman.

Step 1: Variation of the Lagrangian. Under the transformation (1.326), the Lagrangian varies as

$$\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi_r} \delta\phi_r + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)} \partial_\mu(\delta\phi_r) \,, \tag{1.329}$$

where here and below $\delta\phi_r = \epsilon \, \Delta\phi_r$ is the full ϵ-linear variation.

Using the Euler–Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial \phi_r} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)} \right) , \tag{1.330}$$

we can rewrite the first term:

$$\begin{aligned}\delta\mathcal{L} &= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_r)}\right)\delta\phi_r + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_r)}\partial_\mu(\delta\phi_r) \\ &= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_r)}\delta\phi_r\right),\end{aligned} \tag{1.331}$$

where we used the product rule in reverse.

Step 2: Symmetry condition. For the action to be invariant, the Lagrangian must change at most by a total divergence:

$$\delta\mathcal{L} = \epsilon\,\partial_\mu K^\mu\,, \tag{1.332}$$

for some four-vector K^μ. (If $\delta\mathcal{L} = 0$, we simply take $K^\mu = 0$.)

Step 3: Construction of the conserved current. Combining equations (1.331) and (1.332):

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_r)}\delta\phi_r\right) = \epsilon\,\partial_\mu K^\mu\,. \tag{1.333}$$

Substituting $\delta\phi_r = \epsilon\,\Delta\phi_r$ on the LHS and dividing by ϵ (which is arbitrary):

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_r)}\Delta\phi_r - K^\mu\right) = 0\,. \tag{1.334}$$

We identify the **Noether current**:

$$\boxed{j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_r)}\Delta\phi_r - K^\mu} \tag{1.335}$$

which satisfies $\partial_\mu j^\mu = 0$.

Step 4: The conserved charge. Define:

$$Q = \int d^3x\, j^0(x)\,. \tag{1.336}$$

To show that Q is conserved:

$$\frac{dQ}{dt} = \int d^3x\,\partial_0 j^0 = -\int d^3x\,\nabla\cdot\vec{j} = -\oint_{\text{surface}}\vec{j}\cdot d\vec{S} = 0\,, \tag{1.337}$$

where we used $\partial_\mu j^\mu = 0$ and assumed that the fields fall off sufficiently fast at spatial infinity, so the surface integral vanishes.

Important remarks

1. The current j^μ is not unique: one can always add terms of the form $\partial_\nu A^{\mu\nu}$ with $A^{\mu\nu} = -A^{\nu\mu}$, since $\partial_\mu \partial_\nu A^{\mu\nu} = 0$ automatically. The charge Q, however, is unique (up to normalization).
2. **On-shell vs. off-shell:** The proof above uses the Euler–Lagrange equations (1.330), meaning it holds *on-shell* (when the fields satisfy the classical equations of motion). Off-shell, the current is not conserved in general.
   In the **quantum theory**, the situation is more subtle:
   - The Euler–Lagrange equations become *operator equations* (Heisenberg equations of motion), valid inside correlation functions up to *contact terms*.
     What are contact terms? Contact terms are delta-function contributions that arise when two spacetime points coincide. They originate from the non-commutativity of field operators at equal times. For example, when differentiating a time-ordered product:
     $$\partial_0^x T(\phi(x)\phi(y)) = T(\partial_0\phi(x)\,\phi(y)) + \delta(x_0 - y_0)\,[\phi(x), \phi(y)]\,. \quad (1.338)$$
     The second term is a contact term: it is nonzero only when $x_0 = y_0$, i.e., when the points are in "contact" in time. More generally, contact terms are proportional to $\delta^{(4)}(x - x_i)$ or its derivatives.
   - Noether's theorem then implies $\partial_\mu j^\mu = 0$ as an *operator identity*, meaning:
     $$\langle 0|\, T\big(\partial_\mu j^\mu(x)\,\phi(x_1)\cdots\phi(x_n)\big)\,|0\rangle = \text{contact terms}\,, \quad (1.339)$$
     where the contact terms arise when x coincides with one of the x_i. These contact terms encode how the symmetry transformation acts on the fields and lead to the *Ward-Takahashi identities*, which constrain the structure of Green's functions and are crucial for proving renormalizability and gauge invariance of physical observables.
   - The conserved charge Q is a well-defined operator satisfying $[Q, H] = 0$, and generates the symmetry transformation via the commutator
     $$[Q, \phi] = -i\,\Delta\phi\,, \quad (1.340)$$
     where $\Delta\phi$ is the parameter-independent characteristic variation of ϕ under the symmetry (cf. the convention $\delta\phi = \epsilon\,\Delta\phi$ from (1.326)). Note that the infinitesimal parameter ϵ does *not* appear inside this operator identity: it is absorbed into the relation between $\delta\phi$ and $\Delta\phi$. Multiplying by ϵ recovers the finite-parameter form $[\epsilon Q, \phi] = -i\,\delta\phi$.
3. For spacetime symmetries (translations, rotations, boosts), both the coordinates and the fields transform, which requires a slight generalization of the above derivation.

Example 1: Internal U(1) symmetry and charge conservation

Consider the complex scalar field with Lagrangian:

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi \,. \tag{1.341}$$

This Lagrangian is invariant under the global $U(1)$ transformation (same convention as in the earlier complex-scalar section, where ϕ carries charge -1):

$$\phi \to e^{-i\alpha}\phi\,, \qquad \phi^\dagger \to e^{+i\alpha}\phi^\dagger\,. \tag{1.342}$$

For infinitesimal α, the characteristic variations (cf. the $\delta\phi = \epsilon\Delta\phi$ convention introduced in (1.326)) are

$$\Delta\phi = -i\phi\,, \qquad \Delta\phi^\dagger = +i\phi^\dagger\,, \tag{1.343}$$

so that with $\epsilon \equiv \alpha$ we have $\delta\phi = -i\alpha\phi$, $\delta\phi^\dagger = +i\alpha\phi^\dagger$.

Verification of symmetry:

$$\begin{aligned}
\delta\mathcal{L} &= \partial_\mu(\delta\phi^\dagger)\partial^\mu\phi + \partial_\mu\phi^\dagger\partial^\mu(\delta\phi) - m^2(\delta\phi^\dagger\phi + \phi^\dagger\delta\phi) \\
&= \partial_\mu(+i\alpha\phi^\dagger)\partial^\mu\phi + \partial_\mu\phi^\dagger\partial^\mu(-i\alpha\phi) - m^2(+i\alpha\phi^\dagger\phi - i\alpha\phi^\dagger\phi) \\
&= +i\alpha\partial_\mu\phi^\dagger\partial^\mu\phi - i\alpha\partial_\mu\phi^\dagger\partial^\mu\phi = 0\,.
\end{aligned} \tag{1.344}$$

The Lagrangian is strictly invariant, so $K^\mu = 0$.

The Noether current: Using $\Delta\phi = -i\phi$ and $\Delta\phi^\dagger = +i\phi^\dagger$ in the Noether formula (1.335):

$$\begin{aligned}
j^\mu &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(-i\phi) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^\dagger)}(+i\phi^\dagger) \\
&= (\partial^\mu\phi^\dagger)(-i\phi) + (\partial^\mu\phi)(+i\phi^\dagger) \\
&= i\left(\phi^\dagger\partial^\mu\phi - \phi\,\partial^\mu\phi^\dagger\right).
\end{aligned} \tag{1.345}$$

We can write this more symmetrically as:

$$\boxed{j^\mu = i\phi^\dagger \overleftrightarrow{\partial^\mu} \phi \equiv i(\phi^\dagger\partial^\mu\phi - (\partial^\mu\phi^\dagger)\phi)} \tag{1.346}$$

The conserved charge:

$$Q = \int d^3x\, j^0 = i\int d^3x\,(\phi^\dagger\dot{\phi} - \dot{\phi}^\dagger\phi)\,. \tag{1.347}$$

Using the mode expansion from Section 1.2:

$$Q = \int d^3k\,\left(a^\dagger(\vec{k})a(\vec{k}) - b^\dagger(\vec{k})b(\vec{k})\right) = N_a - N_b\,. \tag{1.348}$$

This is the **charge operator**: it counts the number of particles minus the number of antiparticles. This is the origin of electric charge conservation in QED.

Example 2: Spacetime translations and the energy-momentum tensor

Consider an infinitesimal spacetime translation:

$$x^\mu \longrightarrow x'^\mu = x^\mu + \epsilon^\mu \,. \tag{1.349}$$

The field at the new point takes the old value: $\phi'(x') = \phi(x)$. Therefore:

$$\phi'(x) = \phi(x-\epsilon) = \phi(x) - \epsilon^\mu \partial_\mu \phi(x) \,. \tag{1.350}$$

The field variation is:

$$\delta\phi = \phi'(x) - \phi(x) = -\epsilon^\mu \partial_\mu \phi \,. \tag{1.351}$$

Variation of the Lagrangian: Since $\mathcal{L}$ depends on x only through the fields,

$$\delta\mathcal{L} = -\epsilon^\mu \partial_\mu \mathcal{L} = \partial_\mu(-\epsilon^\mu \mathcal{L}) = \partial_\mu(-\epsilon^\nu g^\mu{}_\nu \mathcal{L}) \,. \tag{1.352}$$

The four translation directions are independent symmetries, parametrised by the four components ϵ^ν, so the role of ϵ in our general convention (1.326) is played here by each ϵ^ν separately. The characteristic variation (with $\delta\phi = \epsilon^\nu \Delta_\nu \phi$) and the corresponding K-vector (with $K^\mu = \epsilon^\nu K^\mu{}_\nu$) are therefore

$$\Delta_\nu \phi = -\partial_\nu \phi \,, \qquad K^\mu{}_\nu = -g^\mu{}_\nu \mathcal{L} = -\delta^\mu_\nu \mathcal{L} \,. \tag{1.353}$$

The Noether current. Plugging these characteristic variations into the Noether formula (1.335),

$$j^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)} \Delta_\nu \phi_r - K^\mu{}_\nu = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)} \partial_\nu \phi_r + \delta^\mu_\nu \mathcal{L} \,,$$

one gets one conserved current per translation direction ν. By long-standing convention one defines the **canonical energy-momentum tensor** as $T^\mu{}_\nu \equiv -j^\mu{}_\nu$ — the overall sign is chosen so that T^{00} comes out as the (positive) energy density of the field, see the explicit example below. Raising the second index with $g^{\nu\rho}$,

$$\boxed{T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)} \partial^\nu \phi_r - g^{\mu\nu} \mathcal{L}} \tag{1.354}$$

The minus sign relative to the raw Noether current $j^\mu{}_\nu$ is purely a convention; the conservation statement $\partial_\mu T^{\mu\nu} = 0$ is unaffected. Equivalently, one could absorb this sign into the definition of the variation as $\delta\phi = +\epsilon^\mu \partial_\mu \phi$ (the "active" translation), in which case the Noether current of (1.335) would directly give (1.354).

The conservation law is:

$$\partial_\mu T^{\mu\nu} = 0 \,. \tag{1.355}$$

The conserved charges: The four conserved quantities are the components of the four-momentum:

$$P^\nu = \int d^3x\, T^{0\nu}\,. \tag{1.356}$$

Explicitly:

- $P^0 = \int d^3x\, T^{00} = \int d^3x\,(\pi_r\dot{\phi}_r - \mathcal{L}) = \int d^3x\,\mathcal{H} = H$ is the **total energy**
- $P^i = \int d^3x\, T^{0i} = \int d^3x\,\pi_r\partial^i\phi_r$ is the **total momentum**

Example: real scalar field. For $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2)$:

$$T^{00} = \dot{\phi}\dot{\phi} - \mathcal{L} = \frac{1}{2}(\dot{\phi}^2 + (\nabla\phi)^2 + m^2\phi^2) = \mathcal{H}\,, \tag{1.357}$$

$$T^{0i} = \dot{\phi}\partial^i\phi\,. \tag{1.358}$$

Example 3: Lorentz transformations and angular momentum

Under an infinitesimal Lorentz transformation:

$$x^\mu \to x'^\mu = x^\mu + \omega^\mu{}_\nu x^\nu\,, \tag{1.359}$$

where $\omega^{\mu\nu} = -\omega^{\nu\mu}$ is antisymmetric (6 independent parameters: 3 rotations + 3 boosts).

The field transforms as:

$$\phi_r(x) \to \phi'_r(x) = \phi_r(\Lambda^{-1}x) + \frac{1}{2}\omega_{\mu\nu}(S^{\mu\nu})_r{}^s\phi_s(\Lambda^{-1}x)\,, \tag{1.360}$$

where $(S^{\mu\nu})_r{}^s$ are the generators of Lorentz transformations in the field representation (spin matrices). For a scalar field, $S^{\mu\nu} = 0$. For a Dirac spinor, $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$.

The conserved current: The calculation is more involved. The result is the **angular momentum tensor**:

$$M^{\mu\rho\sigma} = x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_r)}(S^{\rho\sigma})_r{}^s\phi_s\,. \tag{1.361}$$

The first two terms give the **orbital angular momentum**, while the last term gives the **spin angular momentum**.

The conservation law is:

$$\partial_\mu M^{\mu\rho\sigma} = 0\,. \tag{1.362}$$

The conserved charges:

$$M^{\rho\sigma} = \int d^3x\, M^{0\rho\sigma}\,. \tag{1.363}$$

For spatial indices $\rho, \sigma = i, j$:

$$J^k = \frac{1}{2}\epsilon^{ijk} M^{ij}\,, \tag{1.364}$$

is the angular momentum vector.
For $\rho = 0, \sigma = i$:

$$K^i = M^{0i}\,, \tag{1.365}$$

are the boost generators.

Example 4: Dirac field and the vector current

The Dirac Lagrangian:

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi\,, \tag{1.366}$$

is invariant under the global $U(1)$ transformation:

$$\psi \to e^{-i\alpha}\psi\,, \qquad \bar{\psi} \to e^{+i\alpha}\bar{\psi}\,. \tag{1.367}$$

Following the same procedure as for the complex scalar:

$$\delta\psi = -i\alpha\psi\,, \qquad \delta\bar{\psi} = +i\alpha\bar{\psi}\,. \tag{1.368}$$

The Noether current:

$$\begin{aligned} j^\mu &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}(-i\psi) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}(+i\bar{\psi}) \\ &= \bar{\psi}(i\gamma^\mu)(-i\psi) + 0\cdot(i\bar{\psi}) \\ &= \bar{\psi}\gamma^\mu\psi\,. \end{aligned} \tag{1.369}$$

$$\boxed{j^\mu = \bar{\psi}\gamma^\mu\psi} \tag{1.370}$$

This current does *not* contain the charge factor. The QED interaction Lagrangian is (a detailed discussion of the sign conventions for the minimal coupling $\partial_\mu \to D_\mu = \partial_\mu + iqA_\mu$ is given in Chapter 6):

$$\mathscr{L}_{\text{int}} = -q\, j^\mu A_\mu = -q\,\bar{\psi}\gamma^\mu\psi\, A_\mu\,, \tag{1.371}$$

where q is the charge of the fermion. For the electron, $q = q_e = -e$ with $e > 0$, giving:

$$\mathscr{L}_{\text{int}} = +e\,\bar{\psi}\gamma^\mu\psi\, A_\mu\,. \qquad (e > 0) \tag{1.372}$$

The conserved charge:

$$Q = \int d^3x\, j^0 = \int d^3x\, \psi^\dagger \psi \,. \tag{1.373}$$

Using the mode expansion:

$$Q = \sum_s \int d^3p \left(b^\dagger(\vec{p},s) b(\vec{p},s) - d^\dagger(\vec{p},s) d(\vec{p},s) \right) = N_{e^-} - N_{e^+} \,. \tag{1.374}$$

This is the **fermion number**: it counts particles minus antiparticles. The **electric charge** is $Q_{\rm em} = q_e Q = -e(N_{e^-} - N_{e^+})$, which is negative for a state with more electrons than positrons, as expected.

Summary: symmetries and conservation laws

The following table summarizes the key results:

Symmetry	**Conserved current**	**Conserved charge**
Time translation	$T^{0\mu}$ (energy-momentum)	H (energy)
Space translation	T^{0i} (momentum density)	P^i (momentum)
Rotation	M^{0ij}	J^k (angular momentum)
Boost	M^{00i}	K^i (boost generator)
$U(1)$ phase	$j^\mu = \bar{\psi}\gamma^\mu\psi$	Q (fermion number)

2 Pictures of Quantum Dynamics

In quantum mechanics, physical predictions—expectation values, transition probabilities, scattering amplitudes—must be independent of how we choose to describe the system mathematically. This freedom leads to several equivalent "pictures" of quantum dynamics, each distributing time dependence differently between states $|\psi\rangle$ and operators $\hat{A}$. The **Schrödinger picture** places all time dependence in the states, the **Heisenberg picture** transfers it entirely to the operators, and the **interaction (Dirac) picture** provides a hybrid approach essential for perturbation theory. Understanding the equivalence and transformations between these pictures is a prerequisite for perturbation theory and for grasping how scattering processes are computed in quantum field theory.

Throughout this chapter we work with a Hilbert space $\mathcal{H}$ and a (possibly time-dependent) Hermitian Hamiltonian $H(t) = H(t)^\dagger$, which ensures real energy eigenvalues and unitary time evolution. We fix a reference time t_0 (often taken to be zero) and follow the conventions of standard references [3, 4, 5, 6, 7]. We keep $\hbar$ explicit in this chapter to make the picture transformations transparent; in later perturbative QFT formulas we return to natural units, $\hbar = c = 1$.

2.1 The Schrödinger picture

The Schrödinger picture is historically the first formulation of quantum dynamics, introduced by Erwin Schrödinger in 1926. Its defining feature is that the quantum state $|\psi_S(t)\rangle$ evolves in time according to a first-order differential equation—the Schrödinger equation—while operators representing physical observables remain constant (unless they have explicit time dependence). This picture is conceptually natural: we think of the "wavefunction" as describing the state of the system, and we watch it spread, oscillate, or evolve as time progresses.

The Schrödinger equation. In this picture, states carry the time dependence $|\psi_S(t)\rangle$, while operators A_S are time-independent unless they have explicit time dependence $A_S(t)$. Time evolution of states is governed by

$$i\hbar \frac{d}{dt} |\psi_S(t)\rangle = H(t) |\psi_S(t)\rangle . \tag{2.1}$$

Box 2.1 A note on d/dt vs $\partial/\partial t$ in quantum mechanics

In the literature, the Schrödinger equation is written with either the total derivative d/dt or the partial derivative $\partial/\partial t$. The same ambiguity appears in the Heisenberg equation of motion.

The distinction:

- $\dfrac{\partial}{\partial t}$ denotes the *explicit* time derivative: the derivative with respect to t appearing explicitly in the functional form, holding all other variables fixed.
- $\dfrac{d}{dt}$ denotes the *total* time derivative: it includes both explicit time dependence and any implicit time dependence through other variables.

For the state vector $|\psi(t)\rangle$: The state depends on time only through its argument t; there are no other variables. Hence d/dt and $\partial/\partial t$ are equivalent, and both notations are correct.

For operators $A(t)$: An operator may have explicit time dependence $A_S(t)$ (e.g., a time-varying external field) and, in the Heisenberg picture, also implicit time dependence through the transformation $A_H(t) = U^\dagger(t,t_0)A_S U(t,t_0)$. The Heisenberg equation

$$i\hbar\frac{d}{dt}A_H(t) = [A_H(t), H_H(t)] + i\hbar\left(\frac{\partial A_S}{\partial t}\right)_H \tag{2.2}$$

distinguishes between the total derivative (left-hand side) and the explicit time derivative (last term on the right).

Convention in these notes: We use d/dt for the Schrödinger equation and the Heisenberg equation (total time derivative), and reserve $\partial/\partial t$ for explicit time dependence when the distinction matters.

The evolution operator. We define the evolution operator $U(t,t_0)$ by

$$|\psi_S(t)\rangle = U(t,t_0)\,|\psi_S(t_0)\rangle, \qquad U(t_0,t_0) = \mathbb{1}\,. \tag{2.3}$$

Differentiating with respect to t and comparing with the Schrödinger equation shows that $U(t,t_0)$ satisfies

$$i\hbar\frac{\partial}{\partial t}U(t,t_0) = H(t)\,U(t,t_0), \qquad U(t_0,t_0) = \mathbb{1}\,. \tag{2.4}$$

The Hermiticity of the Hamiltonian ensures that $U(t,t_0)$ is unitary. Indeed, computing the time derivative of $U^\dagger U$ using $\partial_t U = -\frac{i}{\hbar}HU$ and $\partial_t U^\dagger = \frac{i}{\hbar}U^\dagger H$, one finds

$$\frac{\partial}{\partial t}(U^\dagger U) = \frac{i}{\hbar}U^\dagger HU - \frac{i}{\hbar}U^\dagger HU = 0\,, \tag{2.5}$$

so $U^\dagger U$ is constant in time. Since $U(t_0,t_0) = \mathbb{1}$, we have $U(t,t_0)^\dagger = U(t,t_0)^{-1}$ for all t.

The evolution operator satisfies a **composition property**:

$$U(t_2,t_0) = U(t_2,t_1)\,U(t_1,t_0)\,. \tag{2.6}$$

Proof. By definition, $U(t,t_0)$ maps $|\psi_S(t_0)\rangle$ into $|\psi_S(t)\rangle$. Therefore

$$|\psi_S(t_1)\rangle = U(t_1,t_0)\,|\psi_S(t_0)\rangle\,.$$

Using t_1 as the initial time, the same definition gives

$$|\psi_S(t_2)\rangle = U(t_2,t_1)\,|\psi_S(t_1)\rangle = U(t_2,t_1)\,U(t_1,t_0)\,|\psi_S(t_0)\rangle\,.$$

On the other hand, evolving directly from t_0 to t_2,

$$|\psi_S(t_2)\rangle = U(t_2,t_0)\,|\psi_S(t_0)\rangle\,.$$

Since the initial state $|\psi_S(t_0)\rangle$ is arbitrary, the operator identity (2.6) follows. □

As an immediate corollary, setting $t_2 = t_0$ in (2.6) yields $\mathbb{1} = U(t_0,t_1)\,U(t_1,t_0)$, whence

$$U(t_0,t) = U(t,t_0)^{-1} = U(t,t_0)^\dagger\,. \tag{2.7}$$

Expectation values. For an operator A_S with no explicit time dependence, the expectation value at time t is

$$\langle A\rangle(t) = \langle\psi_S(t)|A_S|\psi_S(t)\rangle = \langle\psi_S(t_0)|\,U(t,t_0)^\dagger A_S\,U(t,t_0)\,|\psi_S(t_0)\rangle\,. \tag{2.8}$$

2.2 The Heisenberg picture

The Heisenberg picture, developed by Werner Heisenberg in 1925 (actually slightly before Schrödinger's formulation), shifts all time dependence from states to operators. This is reminiscent of classical Hamiltonian mechanics, where we track how observables change in time along a phase-space trajectory. The Heisenberg picture is particularly powerful for studying algebraic structures (the Heisenberg equation mirrors the classical Poisson-bracket formulation), for quantum field theory (where field operators $\phi(x,t)$ carry spacetime dependence), and for proving conservation laws.

Definition. In the Heisenberg picture, states are time-independent:

$$|\psi_H\rangle \equiv |\psi_S(t_0)\rangle\,, \tag{2.9}$$

and the time dependence is transferred to operators by defining

$$A_H(t) \equiv U(t,t_0)^\dagger\, A_S\, U(t,t_0)\,. \tag{2.10}$$

Expectation values then agree with the Schrödinger picture: $\langle A\rangle(t) = \langle\psi_H|A_H(t)|\psi_H\rangle$.

Heisenberg equation of motion. Differentiating the definition of $A_H(t)$ and using the Schrödinger equation for the time-evolution operator, $\partial_t U = -\frac{i}{\hbar}H_S U$ and $\partial_t U^\dagger = \frac{i}{\hbar}U^\dagger H_S$ (where H_S is the Hamiltonian in the Schrödinger picture), one obtains

$$\frac{d}{dt}A_H(t) = \frac{i}{\hbar}\,U^\dagger[H_S, A_S]\,U + U^\dagger\Big(\frac{\partial A_S}{\partial t}\Big)U\,. \tag{2.11}$$

Defining the Heisenberg-picture Hamiltonian $H_H(t) \equiv U^\dagger H_S(t)\,U$ and using the identity $U^\dagger[X,Y]U = [U^\dagger XU,\, U^\dagger YU]$, this becomes

$$i\hbar\,\frac{d}{dt}A_H(t) = [A_H(t), H_H(t)] + i\hbar\left(\frac{\partial A}{\partial t}\right)_H\,. \tag{2.12}$$

If A_S has no explicit time dependence, the last term vanishes and one has

$$i\hbar\,\frac{d}{dt}A_H(t) = [A_H(t), H_H(t)]\,. \tag{2.13}$$

2.3 The interaction (Dirac) picture

The *interaction picture* (also called the Dirac picture) is a hybrid between the Schrödinger and Heisenberg pictures: part of the time evolution is carried by states, part by operators. This split is tailored to perturbation theory and is the workhorse of quantum field theory.

In quantum field theory, the full Hamiltonian $H_S(t) = H_0(t) + V(t)$ consists of a "free" part H_0 (describing non-interacting particles, for which we can solve exactly) and an interaction part V (describing scattering, pair creation, etc.). The idea of the interaction picture is to absorb the *known* free evolution into the operators, leaving only the *unknown* interaction to drive the evolution of states. The interaction-picture evolution operator $U_I(t,t_0)$ can then be expanded perturbatively in powers of V—this is the Dyson series.

Setup. Fix a reference time t_0 and decompose the full Schrödinger-picture Hamiltonian as

$$H_S(t) = H_0(t) + V(t)\,, \tag{2.14}$$

with $H_S(t)$, $H_0(t)$, and $V(t)$ self-adjoint for each time t.

Box 2.2 Notation for H_0 and V

Strictly speaking, H_0 and V are operators that exist in every picture, and we should write $(H_0)_S$, V_S for the Schrödinger picture and $(H_0)_I$, V_I for the interaction picture.
However, in practice:

- **Convention:** We write H_0 and V (without subscript) to denote the *Schrödinger-picture* operators, which are the "original" operators appearing in the Hamiltonian.
- The subscript I in V_I explicitly indicates the interaction-picture version: $V_I(t) = U_0^\dagger(t,t_0)\,V(t)\,U_0(t,t_0)$.
- For the free Hamiltonian, we have $(H_0)_I(t) = U_0^\dagger(t,t_0)H_0(t)U_0(t,t_0)$. If H_0 is time-independent, then $[H_0, U_0] = 0$ and therefore $(H_0)_I = H_0$. If H_0 is time-dependent, one must keep the transformed operator $(H_0)_I(t)$ explicitly.
- In contrast, $V_I(t) \neq V(t)$ in general, because the free evolution generated by H_0 acts nontrivially on the interaction.

Summary:

	Schrödinger	**Interaction**
Free Ham.	$H_0(t) \equiv (H_0)_S(t)$	$(H_0)_I(t) = U_0^\dagger H_0(t) U_0$
Interaction	$V(t) \equiv V_S(t)$	$V_I(t) = U_0^\dagger V(t) U_0$

Throughout these notes, H_0 and V without subscript denote Schrödinger-picture operators.

Define the full evolution operator $U(t,t_0)$ by

$$i\hbar\frac{\partial}{\partial t}U(t,t_0) = H_S(t)\,U(t,t_0), \qquad U(t_0,t_0) = \mathbb{1}, \tag{2.15}$$

and the "free" evolution operator $U_0(t,t_0)$ by

$$i\hbar\frac{\partial}{\partial t}U_0(t,t_0) = H_0(t)\,U_0(t,t_0), \qquad U_0(t_0,t_0) = \mathbb{1}. \tag{2.16}$$

Both are unitary by virtue of Hermiticity.

Interaction-picture states. The interaction-picture state is obtained by "removing" the free evolution from the Schrödinger-picture state:

$$|\psi_I(t)\rangle \equiv U_0^\dagger(t,t_0)\,|\psi_S(t)\rangle\,. \tag{2.17}$$

Equivalently, $|\psi_S(t)\rangle = U_0(t,t_0)|\psi_I(t)\rangle$. Note that $|\psi_I(t_0)\rangle = |\psi_S(t_0)\rangle$: all pictures coincide at the reference time.

Interaction-picture operators. For any Schrödinger-picture operator $A_S(t)$, define

$$A_I(t) \equiv U_0^\dagger(t,t_0)\, A_S(t)\, U_0(t,t_0)\,. \tag{2.18}$$

In particular, the interaction Hamiltonian in the interaction picture is

$$V_I(t) \equiv U_0^\dagger(t,t_0)\, V(t)\, U_0(t,t_0)\,. \tag{2.19}$$

Picture independence of expectation values. Expectation values are the same in all pictures:

$$\begin{aligned}\langle\psi_I(t)|A_I(t)|\psi_I(t)\rangle &= \langle\psi_S(t)|U_0\,U_0^\dagger A_S U_0\,U_0^\dagger|\psi_S(t)\rangle \\ &= \langle\psi_S(t)|A_S(t)|\psi_S(t)\rangle\,.\end{aligned} \tag{2.20}$$

Box 2.3 Equation of motion for interaction-picture states

Differentiate the definition $|\psi_I(t)\rangle = U_0^\dagger(t,t_0)|\psi_S(t)\rangle$:

$$i\hbar\frac{d}{dt}|\psi_I(t)\rangle = i\hbar\frac{dU_0^\dagger}{dt}|\psi_S\rangle + U_0^\dagger\, i\hbar\frac{d}{dt}|\psi_S\rangle\,. \tag{2.21}$$

From $i\hbar\,\partial_t U_0 = H_0(t)U_0$, we get $i\hbar\,\partial_t U_0^\dagger = -U_0^\dagger H_0(t)$. Using the Schrödinger equation $i\hbar\,\partial_t|\psi_S\rangle = H_S|\psi_S\rangle$:

$$\begin{aligned}i\hbar\frac{d}{dt}|\psi_I(t)\rangle &= -U_0^\dagger H_0(t)|\psi_S\rangle + U_0^\dagger H_S(t)|\psi_S\rangle \\ &= U_0^\dagger(H_S(t)-H_0(t))|\psi_S\rangle \\ &= U_0^\dagger V(t)|\psi_S\rangle\,.\end{aligned} \tag{2.22}$$

Writing $|\psi_S\rangle = U_0|\psi_I\rangle$:

$$i\hbar\frac{d}{dt}|\psi_I(t)\rangle = U_0^\dagger V(t) U_0\,|\psi_I\rangle = V_I(t)\,|\psi_I(t)\rangle\,. \tag{2.23}$$

Equation of motion for interaction-picture states. The result of the derivation is:

$$\boxed{i\hbar\frac{d}{dt}|\psi_I(t)\rangle = V_I(t)\,|\psi_I(t)\rangle} \tag{2.24}$$

States evolve with the *interaction alone*—the free Hamiltonian has been absorbed into the operators.

Box 2.4 Derivation of the equation of motion for interaction-picture operators

Differentiate the definition $A_I(t) = U_0^\dagger(t,t_0)\,A_S(t)\,U_0(t,t_0)$:

$$\frac{dA_I}{dt} = \frac{dU_0^\dagger}{dt}A_S(t)U_0 + U_0^\dagger\frac{\partial A_S}{\partial t}U_0 + U_0^\dagger A_S(t)\frac{dU_0}{dt}. \tag{2.25}$$

Using $i\hbar\,\partial_t U_0 = H_0(t)U_0$ and $i\hbar\,\partial_t U_0^\dagger = -U_0^\dagger H_0(t)$:

$$\frac{dA_I}{dt} = \frac{i}{\hbar}U_0^\dagger H_0(t)A_S(t)U_0 + \left(\frac{\partial A_S}{\partial t}\right)_I - \frac{i}{\hbar}U_0^\dagger A_S(t)H_0(t)U_0\,. \tag{2.26}$$

Inserting $\mathbb{1} = U_0U_0^\dagger$ between $H_0(t)$ and $A_S(t)$, and rearranging to match the Heisenberg-style $[A,H]$ ordering used elsewhere in this chapter:

$$i\hbar\frac{dA_I}{dt} = [A_I, H_{0,I}] + i\hbar\left(\frac{\partial A_S}{\partial t}\right)_I, \tag{2.27}$$

where $H_{0,I}(t) = U_0^\dagger H_0(t)U_0$.

Equation of motion for interaction-picture operators.

$$\boxed{i\hbar\frac{d}{dt}A_I(t) = [A_I(t), H_{0,I}(t)] + i\hbar\left(\frac{\partial A_S}{\partial t}\right)_I} \tag{2.28}$$

Operators evolve with the transformed free Hamiltonian $H_{0,I}$, plus any explicit time dependence already present in $A_S(t)$. This mirrors the Heisenberg equation (2.12), except that the relevant Hamiltonian is the *free* one. Once the free problem has been solved, the dynamical evolution of A_I is exact. Moreover, as discussed in the box "Notation for H_0 and V", when H_0 is time-independent (the standard situation in QFT) one has $[H_0, U_0] = 0$ and therefore $H_{0,I} = H_0$: the free Hamiltonian is the same in the Schrödinger and interaction pictures. In this case the boxed equation simplifies to

$$i\hbar\frac{d}{dt}A_I(t) = [A_I(t), H_0] + i\hbar\left(\frac{\partial A_S}{\partial t}\right)_I, \tag{2.29}$$

and we shall use this lighter notation from now on.

The interaction-picture evolution operator and factorization. Define $U_I(t,t_0)$ by

$$|\psi_I(t)\rangle = U_I(t,t_0)\,|\psi_I(t_0)\rangle, \qquad U_I(t_0,t_0) = \mathbb{1}\,. \tag{2.30}$$

Then U_I satisfies

$$i\hbar\frac{\partial}{\partial t}U_I(t,t_0) = V_I(t)\,U_I(t,t_0), \qquad U_I(t_0,t_0) = \mathbb{1}\,. \tag{2.31}$$

The full evolution operator factorizes as

$$\boxed{U(t,t_0) = U_0(t,t_0)\, U_I(t,t_0)} \tag{2.32}$$

Equivalently, $U_I(t,t_0) = U_0^\dagger(t,t_0)\, U(t,t_0)$.

Physical interpretation: The full evolution U is split into the exactly-known free evolution U_0 (which generates propagators) and the interaction evolution U_I (which generates vertices in Feynman diagrams).

Connecting the Heisenberg and interaction pictures. Using the factorization (2.32) and the definition of Heisenberg-picture operators $A_H = U^\dagger A_S U$:

$$A_H(t) = U_I^\dagger(t,t_0)\, A_I(t)\, U_I(t,t_0)\,. \tag{2.33}$$

This provides an efficient bridge between the interaction picture (where A_I evolves with the simple free Hamiltonian) and Heisenberg observables.

Summary: the interaction picture at a glance.

States	evolve with V_I only
Operators	evolve with $H_{0,I}$, plus explicit time dependence
Transformation	$\lvert\psi_I\rangle = U_0^\dagger \lvert\psi_S\rangle, \quad A_I = U_0^\dagger A_S U_0$
Evolution operator	$U = U_0\, U_I$

Remark. The interaction picture is a special case of a general framework in which *any* picture is specified by a choice of unitary transformation $\Omega(t)$. The Schrödinger picture corresponds to $\Omega = \mathbb{1}$, the Heisenberg picture to $\Omega = U^\dagger$, and the interaction picture to $\Omega = U_0^\dagger$. This unified viewpoint is developed in Section 2.4.

2.4 A unified framework: the general picture

The Schrödinger and Heisenberg pictures are two extreme choices: in the former, states carry all the time dependence; in the latter, operators do. In fact, infinitely many intermediate pictures exist, each distributing time dependence differently between states and operators. The key insight is that *any* such picture is connected to the Schrödinger picture by a unitary transformation, and the equations of motion in each picture follow from a single master formula.

General setup. Let $\Omega(t)$ be an arbitrary time-dependent unitary operator, $\Omega(t)^\dagger \Omega(t) = \mathbb{1}$. Define the "$\Omega$-picture" by

$$\lvert\psi_\Omega(t)\rangle \equiv \Omega(t)\, \lvert\psi_S(t)\rangle\,, \qquad A_\Omega(t) \equiv \Omega(t)\, A_S(t)\, \Omega^\dagger(t)\,. \tag{2.34}$$

Expectation values are automatically picture-independent:

$$\langle\psi_\Omega|A_\Omega|\psi_\Omega\rangle = \langle\psi_S|\,\Omega^\dagger\,\Omega\,A_S(t)\,\Omega^\dagger\,\Omega\,|\psi_S\rangle = \langle\psi_S|A_S(t)|\psi_S\rangle\,.$$

Note also that the transformed Hamiltonian operator

$$H_\Omega(t) = \Omega(t)H_S(t)\Omega^\dagger(t)$$

has the *same functional form* in terms of Ω-picture operators as H_S has in terms of Schrödinger-picture operators: one can verify this by expanding H_S as a power series in p_S, q_S and repeatedly inserting $\Omega^\dagger\Omega = \mathbb{1}$ to convert each p_S, q_S into p_Ω, q_Ω. For a time-dependent change of picture, however, the actual generator of state evolution is the combination $H_\Omega + i\hbar\,\dot\Omega\,\Omega^\dagger$ appearing in (2.36), not H_Ω alone.

Equation of motion for states. Differentiating $|\psi_\Omega\rangle = \Omega|\psi_S\rangle$ and using the Schrödinger equation (2.1):

$$\begin{aligned} i\hbar\frac{d}{dt}|\psi_\Omega(t)\rangle &= i\hbar\,\dot\Omega\,|\psi_S\rangle + \Omega\, i\hbar\frac{d}{dt}|\psi_S\rangle \\ &= i\hbar\,\dot\Omega\,\Omega^\dagger\,|\psi_\Omega\rangle + \Omega\,H_S(t)\,\Omega^\dagger\,|\psi_\Omega\rangle\,. \end{aligned} \tag{2.35}$$

Therefore the general evolution equation for states reads

$$\boxed{i\hbar\frac{d}{dt}|\psi_\Omega(t)\rangle = \left(H_\Omega + i\hbar\,\dot\Omega\,\Omega^\dagger\right)|\psi_\Omega(t)\rangle} \tag{2.36}$$

Equation of motion for operators. Differentiating $A_\Omega = \Omega\,A_S(t)\,\Omega^\dagger$ and using $\frac{d}{dt}(\Omega\Omega^\dagger) = 0$, which implies $\dot\Omega\,\Omega^\dagger = -\Omega\,\dot\Omega^\dagger$:

$$\begin{aligned} i\hbar\frac{d}{dt}A_\Omega &= i\hbar\,\dot\Omega\,A_S(t)\,\Omega^\dagger + i\hbar\,\Omega\left(\frac{\partial A_S}{\partial t}\right)\Omega^\dagger + i\hbar\,\Omega\,A_S(t)\,\dot\Omega^\dagger \\ &= i\hbar\left(\frac{\partial A_S}{\partial t}\right)_\Omega + i\hbar\left[\dot\Omega\,\Omega^\dagger,\,A_\Omega\right], \end{aligned} \tag{2.37}$$

where in the last step we inserted $\Omega^\dagger\Omega$ and recognised the commutator. The general evolution equation for operators is

$$\boxed{i\hbar\frac{d}{dt}A_\Omega(t) = \left[A_\Omega,\,-i\hbar\,\dot\Omega\,\Omega^\dagger\right] + i\hbar\left(\frac{\partial A_S}{\partial t}\right)_\Omega} \tag{2.38}$$

Recovering the Schrödinger picture. Choose $\Omega(t) = \mathbb{1}$. Then $\dot\Omega = 0$ and equation (2.36) becomes $i\hbar\frac{d}{dt}|\psi_S\rangle = H_S|\psi_S\rangle$, the Schrödinger equation. The operator equation (2.38) gives $\frac{d}{dt}A_S = \partial_t A_S$: operators have no dynamical time dependence.

Recovering the Heisenberg picture. Choose $\Omega(t) = U(t,t_0)^\dagger$, the adjoint of the full evolution operator. Since $i\hbar\,\partial_t U = H_S U$ implies $i\hbar\,\partial_t U^\dagger = -U^\dagger H_S$, we have $i\hbar\,\dot\Omega = -U^\dagger H_S$ and therefore

$$i\hbar\,\dot\Omega\,\Omega^\dagger = -U^\dagger H_S\,U = -H_H\,.$$

The combination $H_\Omega + i\hbar\,\dot\Omega\,\Omega^\dagger = H_H - H_H = 0$: the state equation (2.36) yields $\frac{d}{dt}|\psi_H\rangle = 0$, confirming that Heisenberg-picture states are frozen. The operator equation (2.38) gives $i\hbar\frac{d}{dt}A_H = [A_H, H_H] + i\hbar(\partial_t A_S)_H$, which is the Heisenberg equation (2.12).

Recovering the interaction picture. Setting $\Omega(t) = U_0(t,t_0)^\dagger$ gives $i\hbar\,\dot\Omega\,\Omega^\dagger = -H_{0,I}(t)$, so:

- *States:* $H_\Omega + i\hbar\,\dot\Omega\,\Omega^\dagger = H_I - H_{0,I} = V_I$: states evolve with the interaction alone, as derived in Section 2.3.
- *Operators:* $-i\hbar\,\dot\Omega\,\Omega^\dagger = H_{0,I}$: operators evolve with the free Hamiltonian, plus any explicit time dependence of the original Schrödinger-picture operator.

This unified viewpoint makes transparent that the choice of picture amounts to choosing $\Omega(t)$, and that all pictures give the same physics—the same matrix elements, the same transition probabilities.

Remark: from quantum mechanics to quantum field theory. The formulas above were derived for ordinary quantum-mechanical operators, but the same structure carries over to QFT essentially unchanged. In canonical field theory the replacement rule is

$$q_i \longrightarrow \phi_k(\vec{x},t)\,, \qquad p_i \longrightarrow \pi_k(\vec{x},t) = \frac{\partial \mathcal{L}}{\partial \dot\phi_k}\,, \tag{2.39}$$

i.e. the discrete index i becomes the pair $(k,\vec{x})$, with k a discrete field label and $\vec{x}$ a continuous spatial argument. Equal-time canonical relations become

$$[\phi_k(\vec{x},t),\,\pi_l(\vec{y},t)] = i\hbar\,\delta_{kl}\,\delta^{(3)}(\vec{x}-\vec{y})\,, \tag{2.40}$$

and the Hamiltonian $H(p,q)$ becomes a spatial integral of a Hamiltonian density,

$$H[\phi,\pi] = \int d^3x\,\mathcal{H}(\phi,\pi)\,, \qquad V(t) = \int d^3x\,\mathcal{H}_{\rm int}(\phi,\pi)\,. \tag{2.41}$$

In the interaction picture the local interaction Hamiltonian density acquires its free time evolution. With $U_0 = U_0(t,t_0)$ (and writing $V(t)$ in case the Schrödinger-picture interaction also has explicit time dependence),

$$V_I(t) = U_0^\dagger(t,t_0)\,V(t)\,U_0(t,t_0) = \int d^3x\,\mathscr{H}_I(\vec{x},t)\,, \qquad \mathscr{H}_I(\vec{x},t) = U_0^\dagger(t,t_0)\,\mathcal{H}_{\rm int}\big(\phi(\vec{x}),\pi(\vec{x});t\big)\,U_0($$

(2.42)

Substituting into the Dyson series (3.10) and grouping the temporal and spatial integrations into a single $\int d^4x$ converts the perturbative expansion into a sum over T-products of local Hamiltonian densities,

$$U_I = T\exp\left[-\frac{i}{\hbar}\int d^4x\,\mathscr{H}_I(x)\right], \tag{2.43}$$

which is the form used throughout QFT and the starting point of the covariant Feynman rules. The structural lessons of this chapter—picture independence, the factorisation $U = U_0U_I$, the Dyson series, the existence of an S-matrix—all carry over without modification; only the specific dynamical content of $\mathscr{H}_I$ changes from problem to problem.

2.5 Exponential representations of the evolution operators

The evolution operators introduced above can be written as exponentials, ordinary or time-ordered depending on whether the Hamiltonian commutes with itself at different times. These exponential forms follow directly from solving the Schrödinger equation for the evolution operator.

Box 2.5 Origin of the exponential representation

The evolution operator $U(t,t_0)$ satisfies the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}U(t,t_0) = H(t)\,U(t,t_0)\,, \qquad U(t_0,t_0) = \mathbb{1}\,.$$

For a **time-independent** Hamiltonian H, this first-order ODE can be integrated directly:

$$U(t,t_0) = e^{-\frac{i}{\hbar}H(t-t_0)}\,.$$

For a **time-dependent** Hamiltonian that commutes with itself at different times, $[H(t_1),H(t_2)] = 0$, the solution generalizes to

$$U(t,t_0) = \exp\left[-\frac{i}{\hbar}\int_{t_0}^{t} dt'\,H(t')\right].$$

However, when $[H(t_1),H(t_2)] \neq 0$, the naive exponential of the integral is *not* correct—operator ordering matters. The proper solution involves the **time-ordered exponential**, which is defined through the Dyson series. The detailed derivation is presented in Chapter 3 (The Dyson Series).

General case (time-dependent, noncommuting Hamiltonian). If $[H(t_1), H(t_2)] \neq 0$ in general, the full evolution operator is given by the time-ordered exponential

$$U(t,t_0) = T\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t} dt'\, H(t')\right], \tag{2.44}$$

where T denotes **time ordering**: operators at later times appear to the left. This formula is written for forward evolution, $t > t_0$; backward evolution follows from unitarity and the composition law, or equivalently from anti-time ordering. This compact notation is shorthand for the infinite Dyson series derived in the next chapter. Similarly,

$$U_0(t,t_0) = T\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t} dt'\, H_0(t')\right], \tag{2.45}$$

$$U_I(t,t_0) = T\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t} dt'\, V_I(t')\right]. \tag{2.46}$$

Commuting case. If $[H(t_1), H(t_2)] = 0$ for all t_1, t_2, time ordering is unnecessary:

$$U(t,t_0) = \exp\left[-\frac{i}{\hbar}\int_{t_0}^{t} dt'\, H(t')\right]. \tag{2.47}$$

In particular, if H is time-independent,

$$U(t,t_0) = \exp\left[-\frac{i}{\hbar}H(t-t_0)\right]. \tag{2.48}$$

Time-independent case. When $H = H_0 + V$ with all terms time-independent, the evolution operators simplify to

$$U(t,t_0) = e^{-\frac{i}{\hbar}(H_0+V)(t-t_0)}, \qquad U_0(t,t_0) = e^{-\frac{i}{\hbar}H_0(t-t_0)}, \tag{2.49}$$

and the interaction-picture quantities become

$$V_I(t) = e^{\frac{i}{\hbar}H_0(t-t_0)}\, V\, e^{-\frac{i}{\hbar}H_0(t-t_0)}, \qquad U_I(t,t_0) = e^{\frac{i}{\hbar}H_0(t-t_0)}\, U(t,t_0)\,. \tag{2.50}$$

Note that $V_I(t)$ is still time-dependent unless $[H_0, V] = 0$. In the special commuting case $[H_0, V] = 0$, one has $V_I(t) = V$ and $U_I(t,t_0) = e^{-\frac{i}{\hbar}V(t-t_0)}$.

2.6 Summary: comparison of pictures

The following table summarizes the three pictures of quantum dynamics. In all cases, the full Hamiltonian is $H_S(t) = H_0(t) + V(t)$ and physical observables (expectation values, transition amplitudes) are picture-independent.

	Schrödinger	**Heisenberg**	
Transformation Ω	$\mathbb{1}$	$U^\dagger$	
State	$\lvert\psi_S(t)\rangle$	$\lvert\psi_H\rangle = \lvert\psi_S(t_0)\rangle$	
Operator	$A_S(t)$	$A_H = U^\dagger A_S(t) U$	
State equation	$i\hbar \dfrac{d}{dt}\lvert\psi_S\rangle = H_S\lvert\psi_S\rangle$	$\dfrac{d}{dt}\lvert\psi_H\rangle = 0$	
Operator equation	$\dfrac{dA_S}{dt} = \dfrac{\partial A_S}{\partial t}$	$i\hbar \dfrac{dA_H}{dt} = [A_H, H_H] + i\hbar \left(\dfrac{\partial A_S}{\partial t}\right)_H$	
Evolution operator	$U(t,t_0) = e^{-iH_S(t-t_0)/\hbar}$ (only if H_S is time-indep.)	—	
States evolve with	full H_S	nothing (frozen)	
Operators evolve with	explicit time dependence only	full H_H	f

Key relations:

- Factorization: $U(t,t_0) = U_0(t,t_0)\, U_I(t,t_0)$
- Heisenberg from interaction: $A_H(t) = U_I^\dagger(t,t_0)\, A_I(t)\, U_I(t,t_0)$
- For time-independent H_0: $U_0(t,t_0) = e^{-iH_0(t-t_0)/\hbar}$
- Interaction-picture operators, valid only when H_0 is time-independent and A_S has no residual explicit time dependence:

$$A_I(t) = e^{+iH_0(t-t_0)/\hbar}\, A_S\, e^{-iH_0(t-t_0)/\hbar}.$$

  In the general case keep the operator form $A_I(t) = U_0^\dagger(t,t_0)\, A_S(t)\, U_0(t,t_0)$ as in (2.18).

3 The Dyson Series

In 1949, Freeman Dyson published two landmark papers [8] that unified the perturbative approaches of Feynman, Schwinger, and Tomonaga and established the modern framework for computing S-matrix elements in quantum electrodynamics. The key mathematical tool is the Dyson series—a systematic expansion of the interaction-picture evolution operator $U_I(t,t_0)$ in powers of the interaction-picture operator $V_I(t)$.

The physical content of the Dyson series is profound. The n-th term in the series involves n powers of V_I, integrated over all possible orderings of the n interaction times, corresponding to processes with n vertices in Feynman diagrams. The time-ordering operator T ensures that operators at later times appear to the left; together with locality and microcausality, this is compatible with Lorentz-covariant scattering amplitudes. When the interaction density involves products of field operators, Wick's theorem converts the time-ordered products into sums of normal-ordered products plus contractions, with each contraction corresponding to a propagator line in a Feynman diagram. Thus the Dyson series is the bridge connecting the abstract operator formalism of QFT to the concrete computational rules that make predictions possible.

Natural units. From this chapter onward, we adopt **natural units** $\hbar = c = 1$, which is standard in particle physics and simplifies all formulas considerably.

3.1 Derivation of the series

Setup. Fix t_0 and split the Schrödinger-picture Hamiltonian as $H_S(t) = H_0(t) + V(t)$. The interaction-picture evolution operator $U_I(t,t_0)$ satisfies

$$i\frac{\partial}{\partial t}U_I(t,t_0) = V_I(t)\,U_I(t,t_0), \qquad U_I(t_0,t_0) = \mathbb{1}\,, \tag{3.1}$$

where $V_I(t) = U_0^\dagger(t,t_0)\,V(t)\,U_0(t,t_0)$.

Integral equation. Integrating the differential equation from t_0 to t with the initial condition gives the Volterra integral equation

$$U_I(t,t_0) = \mathbb{1} - i\int_{t_0}^{t} dt_1\, V_I(t_1)\,U_I(t_1,t_0)\,. \tag{3.2}$$

Picard iteration. Define iterates starting from $U_I^{(0)} = \mathbb{1}$ and recursively

$$U_I^{(n+1)}(t,t_0) = \mathbb{1} - i\int_{t_0}^{t} dt_1\, V_I(t_1)\, U_I^{(n)}(t_1,t_0)\,. \tag{3.3}$$

The first few iterates are:

$$U_I^{(1)} = \mathbb{1} - i\int_{t_0}^{t} dt_1\, V_I(t_1), \tag{3.4}$$

$$U_I^{(2)} = \mathbb{1} - i\int_{t_0}^{t} dt_1\, V_I(t_1) + (-i)^2\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, V_I(t_1)V_I(t_2)\,. \tag{3.5}$$

The general n-th term. Continuing the iteration, the formal solution is

$$U_I(t,t_0) = \mathbb{1} + \sum_{n=1}^{\infty}(-i)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n\, V_I(t_1)V_I(t_2)\cdots V_I(t_n)\,. \tag{3.6}$$

The nested integration limits enforce $t_0 \le t_n \le \cdots \le t_2 \le t_1 \le t$, so the operator product automatically appears with later times on the left.

3.2 Time ordering and the $1/n!$ factor

The time-ordering operator. Define the time-ordering operator T by

$$T\{V_I(t_1)V_I(t_2)\} = \begin{cases} V_I(t_1)V_I(t_2), & t_1 > t_2, \\ V_I(t_2)V_I(t_1), & t_2 > t_1\,, \end{cases} \tag{3.7}$$

and analogously for n factors: T permutes the operators so that the largest time argument stands leftmost. Equal-time cases can be assigned any fixed convention in the present integrals, since they form a set of measure zero. For elementary fermionic fields, time ordering is graded and includes the usual minus sign for odd permutations; the interaction Hamiltonian density itself is bosonic, so no additional sign appears in the products of V_I written here.

Geometric identity. The hypercube $C = [t_0,t]^n$ can be partitioned, up to measure-zero boundaries, into $n!$ regions corresponding to the $n!$ strict orderings of the times. The nested domain $t_0 \le t_n \le \cdots \le t_1 \le t$ is precisely one such region. On each ordering region, the time-ordered integrand rearranges the factors into decreasing time order; after relabeling dummy integration variables, every region

gives the same contribution as the nested one. Therefore integrating over the full hypercube with time ordering gives $n!$ times the nested integral:

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n \, V_I(t_1) \cdots V_I(t_n) = \frac{1}{n!} \int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t} dt_n \, T\{V_I(t_1) \cdots V_I(t_n)\} . \quad (3.8)$$

Explicit example for $n = 2$. Consider the square $C = [t_0, t]^2$. It decomposes into two triangular regions: $\mathcal{R}_{12} = \{t_1 \geq t_2\}$ and $\mathcal{R}_{21} = \{t_2 \geq t_1\}$. The nested integral covers $\mathcal{R}_{12}$. Since $T\{V_I(t_1)V_I(t_2)\}$ equals $V_I(t_1)V_I(t_2)$ on $\mathcal{R}_{12}$ and $V_I(t_2)V_I(t_1)$ on $\mathcal{R}_{21}$, and since these two contributions are equal after relabeling, one has

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \, V_I(t_1)V_I(t_2) = \frac{1}{2!} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \, T\{V_I(t_1)V_I(t_2)\} . \quad (3.9)$$

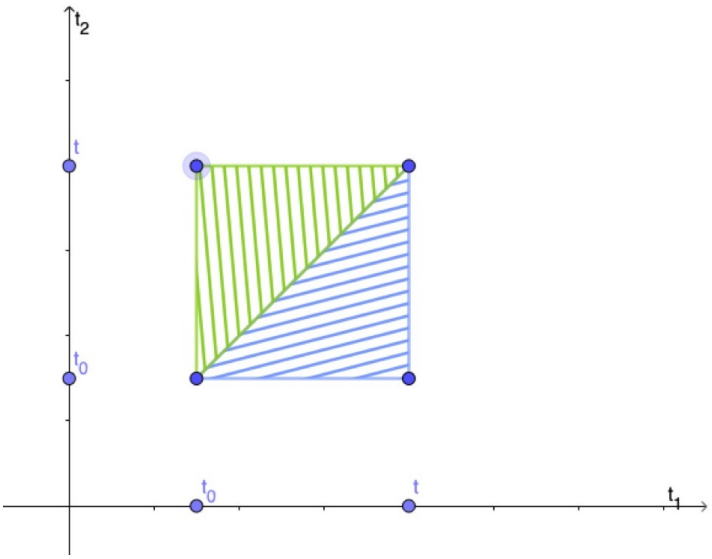


Figure 3.1. Decomposition of the integration square $C = [t_0, t]^2$ into two triangular regions $\mathcal{R}_{12}$ (blue, $t_1 \geq t_2$) and $\mathcal{R}_{21}$ (green, $t_2 \geq t_1$), separated by the diagonal $t_1 = t_2$.

3.3 The time-ordered exponential

For forward evolution $t > t_0$, substituting the identity relating nested integrals to time-ordered integrals gives

$$U_I(t, t_0) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t} dt_n \, T\{V_I(t_1) \cdots V_I(t_n)\} . \quad (3.10)$$

This is precisely the series expansion of the time-ordered exponential:

$$\boxed{U_I(t,t_0) = T\exp\left[-i\int_{t_0}^{t} dt'\, V_I(t')\right]} \tag{3.11}$$

For $t < t_0$, the inverse evolution is obtained from unitarity and composition, equivalently by using anti-time ordering.

Verification. Differentiating the series term by term with respect to t (using the fundamental theorem of calculus for the outermost integral) confirms that the Dyson series satisfies

$$\frac{d}{dt}U_I(t,t_0) = -i\,V_I(t)\,U_I(t,t_0), \qquad U_I(t_0,t_0) = \mathbb{1}\,. \tag{3.12}$$

Unitarity. If $V_I(t)$ is self-adjoint, the solution is unitary. Using the evolution equation and its adjoint, one finds $\partial_t(U_I^\dagger U_I) = 0$, so $U_I^\dagger U_I = \mathbb{1}$ for all t.

3.4 Definition of the S-matrix

The perturbation formalism developed above is particularly suitable for describing **scattering processes**. In a collision, particles approach each other, interact, and then separate. We now define the S-matrix, which encodes all the information about such transitions.

Physical setup. Consider a scattering event. Long before the collision ($t \to -\infty$), the particles are far apart and propagate as free particles—they constitute the **initial state** $|i\rangle$. After the collision ($t \to +\infty$), the resulting particles are again far apart and propagate freely—they constitute the **final state**. The equation of motion (3.1) determines how the initial state $|\Phi(-\infty)\rangle = |i\rangle$ evolves into the state $|\Phi(\infty)\rangle$ at $t = +\infty$.

Definition. The S-matrix (scattering matrix) is defined as the operator that relates the final state to the initial state:

$$\boxed{|\Phi(\infty)\rangle = S\,|\Phi(-\infty)\rangle = S\,|i\rangle} \tag{3.13}$$

Formally, and with the large-time limit understood through an adiabatic or equivalent LSZ prescription, $|\Phi(t)\rangle = U_I(t,t_0)|\Phi(t_0)\rangle$ implies

$$S \;=\; U_I(+\infty,-\infty) \;\equiv\; \lim_{\substack{t\to+\infty(1-i\epsilon)\\ t_0\to-\infty(1-i\epsilon)}} U_I(t,t_0)\,, \tag{3.14}$$

where the slight tilt of the asymptotic times into the lower complex plane, $\pm\infty(1-i\epsilon)$ with $\epsilon \to 0^+$, is the standard *adiabatic prescription*: it damps oscillations exponentially as $|t| \to \infty$ and selects the interacting ground state and isolated

one-particle states. The same $i\epsilon$ prescription underlies the Feynman propagator and is operationally equivalent to the adiabatic switching $V \to V\,g(t)$ discussed below; in the LSZ framework (Chapter 8) it returns as the choice of contour for the Fourier transform of the propagator.

The Dyson expansion of the S-matrix. From the time-ordered exponential representation (3.11), the S-matrix has the expansion

$$\begin{aligned} S &= T\exp\left[-i\int_{-\infty}^{+\infty} dt'\,V_I(t')\right] \\ &= \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{+\infty}dt_1\cdots\int_{-\infty}^{+\infty}dt_n\,T\{V_I(t_1)\cdots V_I(t_n)\}. \end{aligned} \tag{3.15}$$

This is the **Dyson expansion of the S-matrix**. It forms the starting point for perturbative calculations in quantum field theory.

Covariant form. Writing the interaction in terms of the interaction Hamiltonian density $\mathscr{H}_I(x)$ via

$$V_I(t) = \int d^3x\,\mathscr{H}_I(t,\mathbf{x})\,,$$

the Dyson expansion takes the manifestly Lorentz-covariant form, provided $\mathscr{H}_I(x)$ is a local Lorentz-scalar interaction density:

$$\boxed{S = \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int d^4x_1\cdots\int d^4x_n\,T\{\mathscr{H}_I(x_1)\cdots\mathscr{H}_I(x_n)\}} \tag{3.16}$$

where the integrations are over all spacetime.

Box 3.1 Why time ordering is compatible with Lorentz covariance

In the covariant form (3.16), the integration measure d^4x and the domain (all of spacetime) are manifestly Lorentz-invariant. The only potential obstacle is the time-ordered product $T\{\mathscr{H}_I(x_1)\cdots\mathscr{H}_I(x_n)\}$: since "which event comes first" depends on the observer, T could introduce frame dependence.
This does not happen for local bosonic interaction densities satisfying **microcausality**. Consider two spacetime points x_1 and x_2:

- *Timelike separation,* $(x_1-x_2)^2>0$: all Lorentz observers agree on the temporal ordering, so $T\{\cdots\}$ is unambiguous.
- *Spacelike separation,* $(x_1-x_2)^2<0$: different frames may disagree on the ordering. However, the microcausality condition requires that local bosonic densities commute at spacelike separation,

$$[\mathscr{H}_I(x_1),\mathscr{H}_I(x_2)]=0 \quad \text{for } (x_1-x_2)^2<0\,,$$

so the order is irrelevant and $T\{\cdots\}$ gives the same result in every frame.

In contrast, the original nested-integral form of the Dyson series (equation (3.6)) singles out a particular time coordinate and is *not* manifestly covariant. The passage to unrestricted integrals with time ordering is what makes the Lorentz invariance of the S-matrix transparent.

Transition amplitudes. The state $|\Phi(\infty)\rangle$ contains all possible outcomes of the collision. Expanding symbolically in a complete set of final states $|f\rangle$:

$$|\Phi(\infty)\rangle = \sum_f |f\rangle\langle f|\Phi(\infty)\rangle = \sum_f |f\rangle\, S_{fi}\,, \tag{3.17}$$

where the **S-matrix element**

$$S_{fi} = \langle f|S|i\rangle \tag{3.18}$$

is the probability amplitude for the transition $|i\rangle \to |f\rangle$. For continuum scattering states, the sum in (3.17) stands for the appropriate mixture of sums and phase-space integrals.

In practice one separates the trivial "no-scattering" identity contribution by writing

$$S \;=\; \mathbb{1} + iT\,, \tag{3.19}$$

which defines the **transition operator** T (often called the T-matrix). For momentum eigenstates, conservation of total four-momentum is then factored out explicitly:

$$\langle f|T|i\rangle \;=\; (2\pi)^4\,\delta^{(4)}(P_f - P_i)\,\mathcal{M}_{fi}\,, \tag{3.20}$$

with $\mathcal{M}_{fi}$ the *invariant amplitude* (or "reduced" matrix element). The connected, fully amputated, on-shell version of $\mathcal{M}_{fi}$ is what the LSZ reduction formula computes from n-point Green functions (Chapter 8); physical probabilities, rates, and cross sections are then extracted from $|\mathcal{M}_{fi}|^2$ via the standard invariant phase-space formulae (developed in the chapter on cross sections), rather than naively from $|S_{fi}|^2$.

Properties of the S-matrix. From the unitarity of U_I, the S-matrix satisfies:

$$S^\dagger S = S S^\dagger = \mathbb{1}\,. \tag{3.21}$$

For a discrete orthonormal basis this implies probability conservation in the form

$$\sum_f |S_{fi}|^2 = \sum_f \langle i|S^\dagger|f\rangle\langle f|S|i\rangle = \langle i|S^\dagger S|i\rangle = 1\,. \tag{3.22}$$

The adiabatic hypothesis. A subtlety arises in interpreting the initial and final states. In the perturbation formalism, $|i\rangle$ and $|f\rangle$ are eigenstates of the *free* Hamiltonian H_0—they describe "bare" particles without interactions. But physical particles, even when far apart, are surrounded by their self-interactions (an electron by its photon cloud, for example).

This issue is addressed by the **adiabatic hypothesis**: we replace the interaction $V(t)$ by $V(t)g(t)$, where $g(t)$ is a switching function satisfying $g(t) = 1$ for $|t| \leq T$ and $g(t) \to 0$ smoothly as $t \to \pm\infty$. This is a formal device used to define asymptotic in/out states and control the large-time limit. In this picture:

- At $t \ll -T$, the interaction is "switched off," and the system is described by bare-particle states.
- During $-T \leq t \leq T$, the full interaction is active and produces the scattering process.
- At $t \gg T$, the interaction switches off again.

The physical assumption is that observable scattering results are independent of the detailed switching procedure in the remote past and future. One takes the limit $T \to \infty$ at the end. In realistic relativistic QFTs, especially in gauge theories with infrared subtleties, this statement must be understood with care and is often replaced or refined by the LSZ framework.

In many tree-level calculations one often suppresses the explicit switching function and works directly with $V(t)$, with the understanding that adiabatic switching (or an equivalent LSZ prescription) is needed to define asymptotic in/out states and make the S-matrix well defined.

3.5 Cross sections in the Bjorken–Drell conventions

This section develops the master formula for the differential cross section of a generic scattering process $A + B \to 1 + 2 + \cdots + n$ within the Bjorken–Drell (BD) conventions [9, 10]. The derivation proceeds systematically from the normalization of states through the definition of the invariant amplitude to the final expression suitable for practical calculations. For a modern treatment, see also Peskin and Schroeder [6]. A concise explanation of how the field expansion, (anti)commutators, and state normalization are tied together is given in Section 1.1.

BD normalization of states. The BD conventions employ non-covariant normalization for one-particle states:

$$\langle \mathbf{p}', s' | \mathbf{p}, s \rangle = \delta^{(3)}(\mathbf{p} - \mathbf{p}')\, \delta_{ss'}, \qquad E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}\,. \tag{3.23}$$

This normalization is not Lorentz invariant, but the final cross-section formula can be expressed in terms of the invariant phase-space measure $d^3p/[(2\pi)^3 2E_{\mathbf{p}}]$.

Field expansions and commutation relations. The BD normalization propagates into the mode expansions of quantum fields. For a real scalar field,

$$\phi(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2E_{\mathbf{p}}}} \left(a(\mathbf{p}) e^{-ip\cdot x} + a^\dagger(\mathbf{p}) e^{+ip\cdot x} \right), \qquad p^0 = E_{\mathbf{p}}\,, \tag{3.24}$$

with the commutation relation $[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = \delta^{(3)}(\mathbf{p} - \mathbf{p}')$ ensuring that the state $|\mathbf{p}\rangle = a^\dagger(\mathbf{p})|0\rangle$ satisfies the BD normalization. For a Dirac field, the expansion reads

$$\psi(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{m}{E_{\mathbf{p}}}} \sum_s \left(b_s(\mathbf{p}) u_s(\mathbf{p}) e^{-ip\cdot x} + d_s^\dagger(\mathbf{p}) v_s(\mathbf{p}) e^{+ip\cdot x} \right), \tag{3.25}$$

with anticommutation relations

$$\{ b_s(\mathbf{p}), b_{s'}^\dagger(\mathbf{p}') \} = \{ d_s(\mathbf{p}), d_{s'}^\dagger(\mathbf{p}') \} = \delta^{(3)}(\mathbf{p} - \mathbf{p}')\, \delta_{ss'}\,. \tag{3.26}$$

The BD spinor normalizations are $\bar{u}_s(\mathbf{p}) u_{s'}(\mathbf{p}) = \delta_{ss'}$ and $u_s^\dagger(\mathbf{p}) u_{s'}(\mathbf{p}) = (E_{\mathbf{p}}/m)\, \delta_{ss'}$, with analogous relations for the v-spinors. For a massive vector field,

$$A_\mu(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2E_{\mathbf{p}}}} \sum_\lambda \left(a_\lambda(\mathbf{p}) \epsilon_\mu(\mathbf{p}, \lambda) e^{-ip\cdot x} + a_\lambda^\dagger(\mathbf{p}) \epsilon_\mu^*(\mathbf{p}, \lambda) e^{+ip\cdot x} \right), \tag{3.27}$$

where the polarization sum satisfies $\sum_\lambda \epsilon_\mu(\mathbf{p}, \lambda) \epsilon_\nu^*(\mathbf{p}, \lambda) = -g_{\mu\nu} + p_\mu p_\nu / m^2$.

The S-matrix and the invariant amplitude. For a scattering process $A(p_A) + B(p_B) \to 1(p_1) + \cdots + n(p_n)$, we define the invariant amplitude $\mathcal{M}$ through the decomposition

$$S_{fi} \equiv \langle f|S|i\rangle = \langle f|i\rangle + i(2\pi)^4 \delta^{(4)}\left(p_A + p_B - \sum_{k=1}^{n} p_k\right) \mathcal{M}_{fi}\,. \tag{3.28}$$

The first term represents the possibility that no interaction occurs, while the second contains the dynamics encoded in $\mathcal{M}$. The momentum-conserving delta function is factored out explicitly, leaving $\mathcal{M}$ as a function only of the kinematic invariants.

Transition probability and the $\delta^{(4)}(0)$ regularization. Squaring the transition amplitude yields

$$|S_{fi} - \delta_{fi}|^2 = (2\pi)^8 \delta^{(4)}(P_i - P_f)^2\, |\mathcal{M}|^2\,. \tag{3.29}$$

The square of the delta function is mathematically ill-defined and must be regularized. Introducing a spacetime box of volume V and duration T, one interprets

$$\delta^{(4)}(P)^2 = \delta^{(4)}(P)\,\delta^{(4)}(0), \qquad \delta^{(4)}(0) = \frac{VT}{(2\pi)^4}\,, \tag{3.30}$$

so that the transition probability becomes

$$|S_{fi} - \delta_{fi}|^2 = (2\pi)^4 \delta^{(4)}(P_i - P_f)\,(VT)\,|\mathcal{M}|^2\,. \tag{3.31}$$

The factors of V and T will cancel against normalization factors, yielding a finite cross section.

Box 3.2 Derivation of $\delta^{(4)}(0) = VT/(2\pi)^4$

The four-dimensional delta function has the integral representation

$$\delta^{(4)}(P) = \int \frac{d^4x}{(2\pi)^4}\, e^{iP\cdot x}\,. \tag{3.32}$$

Setting $P = 0$:

$$\delta^{(4)}(0) = \int \frac{d^4x}{(2\pi)^4} = \frac{1}{(2\pi)^4}\int d^4x\,. \tag{3.33}$$

In infinite spacetime this integral diverges, reflecting the fact that $\delta(0)$ is undefined as a distribution.

Box regularization. We regularize by confining the system to a finite spacetime region: a spatial box of volume $V = L^3$ and a time interval T. Then

$$\int d^4x = \int_0^T dt \int_V d^3x = VT\,, \tag{3.34}$$

and therefore

$$\boxed{\delta^{(4)}(0) = \frac{VT}{(2\pi)^4}} \tag{3.35}$$

Physical interpretation. This regularization has a clear physical meaning:

- The delta function $\delta^{(4)}(P_i - P_f)$ enforces energy-momentum conservation.
- The factor $\delta^{(4)}(0) = VT/(2\pi)^4$ represents the "phase space volume" available for the conserved total momentum—it counts the number of ways the system can satisfy $P_i = P_f$ within the finite spacetime region.
- As $V, T \to \infty$, individual transition probabilities diverge, but observable quantities like cross sections (probability per unit time per unit flux) remain finite after proper normalization.

The one-dimensional analogue. For a single momentum component, $\delta(p) = \int dx\, e^{ipx}/(2\pi)$, so $\delta(0) = L/(2\pi)$ in a box of length L. The four-dimensional case is simply the product of four such factors.

Phase space and the transition rate. To compute physical transition probabilities from the BD-normalized amplitudes, we must carefully account for the normalization of states.

The Mandl–Shaw convention: $V = (2\pi)^3$. In the BD convention, the continuous-momentum states satisfy $\langle \mathbf{p}'|\mathbf{p}\rangle = \delta^{(3)}(\mathbf{p} - \mathbf{p}')$. To define probabilities we need a finite normalization volume V. Following Mandl and Shaw [1], we set $V = (2\pi)^3$. This is not a physical restriction—V cancels in all observables—but it greatly simplifies the algebra, since the density of states becomes

$$\frac{V\, d^3p}{(2\pi)^3} = d^3p\,, \tag{3.36}$$

and the replacement $\delta^{(3)}(\mathbf{0}) \to V/(2\pi)^3 = 1$ removes all explicit factors of V from intermediate expressions.

Normalization factors per external leg. With $V = (2\pi)^3$, the BD field expansions (3.24)–(3.27) give a specific normalization factor for each external particle in a Feynman amplitude. These factors depend on the *spin* of the field:

Particle type	Factor per leg	Source
Scalar, photon	$\dfrac{1}{\sqrt{2E}}$	$\dfrac{1}{\sqrt{2EV}} = \dfrac{1}{\sqrt{2E\,(2\pi)^3}}$
Dirac fermion (BD)	$\sqrt{\dfrac{m}{E}}$	$\dfrac{1}{\sqrt{V}}\sqrt{\dfrac{m}{E}} = \dfrac{1}{(2\pi)^{3/2}}\sqrt{\dfrac{m}{E}}$

The crucial difference: a BD fermion carries $\sqrt{m/E}$, not $1/\sqrt{2E}$. The ratio is

$$\frac{\sqrt{m/E}}{1/\sqrt{2E}} = \sqrt{2m}\,. \tag{3.37}$$

This factor $\sqrt{2m}$ per external fermion leg is the price of using BD spinors ($\bar{u}u = 1$) instead of covariantly normalized ones ($\bar{u}u = 2m$).

Squared amplitude in the box. For a process with n_b external bosons and n_f external fermions, the squared amplitude between box-normalized states is:

$$|\mathcal{M}|^2_{\text{box}} = |\mathcal{M}_{\text{BD}}|^2 \times \underbrace{\prod_{i\in\text{bosons}} \frac{1}{2E_i}}_{n_b \text{ factors}} \times \underbrace{\prod_{j\in\text{fermions}} \frac{m_j}{E_j}}_{n_f \text{ factors}}\,. \tag{3.38}$$

Each fermion factor can be rewritten as $m_j/E_j = 2m_j/(2E_j)$. Factoring out $1/(2E)$ from every leg:

$$|\mathcal{M}|^2_{\text{box}} = \left(\prod_{i\in\text{fermions}} 2m_i\right) \times |\mathcal{M}_{\text{BD}}|^2 \times \prod_{\text{all legs}} \frac{1}{2E}\,. \tag{3.39}$$

The extra factor $\prod(2m_i)$ is present for fermions in BD normalization and absent for scalars/photons.

Phase space and the transition rate. Each final-state particle contributes a density of states d^3p_k (with $V = (2\pi)^3$, eq. (3.36)) times the normalization factor $1/(2E_k)$:

$$d^3p_k \times \frac{1}{2E_k} = \frac{d^3p_k}{2E_k}\,, \tag{3.40}$$

which is manifestly Lorentz invariant (the restriction of d^4p to the mass shell).

The total number of transitions into the phase-space element is therefore

$$dN = (2\pi)^4\delta^{(4)}(P_i - P_f)\,T\,\frac{|\mathcal{M}_{\text{BD}}|^2}{4E_AE_B}\left(\prod_{i\in\text{fermions}} 2m_i\right)\prod_{k=1}^{n}\frac{d^3p_k}{2E_k}\,, \tag{3.41}$$

where the factor T comes from $\delta^{(4)}(0) = T/(2\pi)^4 \times (2\pi)^3 = T$ with $V = (2\pi)^3$, and the $1/(4E_AE_B)$ from the two initial-state normalization factors.

Dividing (3.41) by T gives the transition rate dN/T. The cross section is defined as this rate divided by the incident flux. With $V = (2\pi)^3$ the factor T cancels and no explicit V remains.

The incident flux. With one particle of each type in the volume $V = (2\pi)^3$, the incident flux is

$$\text{flux} = \frac{|\mathbf{v}_A - \mathbf{v}_B|}{V} = \frac{|\mathbf{v}_A - \mathbf{v}_B|}{(2\pi)^3}, \qquad \mathbf{v}_i = \frac{\mathbf{p}_i}{E_i}. \tag{3.42}$$

The differential cross section is then:

$$d\sigma = \frac{dN/T}{\text{flux}} = \frac{(2\pi)^3}{|\mathbf{v}_A - \mathbf{v}_B|} \cdot \frac{1}{4E_AE_B} \left(\prod_{i \in \text{ferm.}} 2m_i \right) |\mathcal{M}_{\text{BD}}|^2 \cdot (2\pi)^4 \delta^{(4)} \prod_{k=1}^{n} \frac{d^3p_k}{2E_k}. \tag{3.43}$$

Where do the energies E_A, E_B come from? They arise from the normalisation of the **initial-state particles**: each particle in the box contributes a factor $1/(2E)$ (for bosons) or $m/E = 2m/(2E)$ (for BD fermions) to the squared amplitude (eq. (3.38)). The product of these two initial-state factors gives $1/(2E_A \cdot 2E_B) = 1/(4E_AE_B)$ in (3.41). So the combination that appears in the cross section is:

$$\frac{1}{4E_AE_B} \cdot \frac{1}{\text{flux}} = \frac{1}{4E_AE_B} \cdot \frac{V}{|\mathbf{v}_A - \mathbf{v}_B|} = \frac{1}{4E_AE_B\,|\mathbf{v}_A - \mathbf{v}_B|/(2\pi)^3}. \tag{3.44}$$

We see that E_AE_B is *not* part of the flux: it comes from the initial-state normalisation. The physical flux is just $|\mathbf{v}_A - \mathbf{v}_B|/V$.

The Lorentz-invariant phase-space element. It is convenient to define

$$d\Phi_n(P_i; p_1, \ldots, p_n) \equiv (2\pi)^4 \delta^{(4)}\left(P_i - \sum_{k=1}^{n} p_k \right) \prod_{k=1}^{n} \frac{d^3p_k}{(2\pi)^3\, 2E_k}, \tag{3.45}$$

which transforms as a scalar under Lorentz transformations (each factor $d^3p/(2E)$ is the restriction of d^4p to the forward mass shell, and the delta function is Lorentz-invariant since $|\det \Lambda| = 1$).

Comparing (3.43) with this definition, the factor $(2\pi)^4\delta^{(4)} \prod d^3p_k/(2E_k)$ in (3.43) differs from $d\Phi_n$ by $(2\pi)^{3n}$. Adding the $(2\pi)^3$ from $1/\text{flux}$, we collect a total of $(2\pi)^{3n+3}$ which must be absorbed somewhere. Explicitly, for $n = 2$:

$$\frac{(2\pi)^3}{|\mathbf{v}|} \cdot (2\pi)^4 \delta^{(4)} \prod_{k=1}^{2} \frac{d^3p_k}{2E_k} = \frac{(2\pi)^3}{|\mathbf{v}|} \cdot (2\pi)^6 \cdot d\Phi_2 = \frac{(2\pi)^9}{|\mathbf{v}|}\, d\Phi_2. \tag{3.46}$$

Where do the $(2\pi)^9$ go? They are absorbed into the *box-to-stripped amplitude conversion* that was already performed in passing from $|\mathcal{M}|^2_{\text{box}}$ to $|\mathcal{M}_{\text{BD}}|^2$. Specifically:

- Each external particle's box-normalised wavefunction carries a factor $1/\sqrt{(2\pi)^3}$ (from $V = (2\pi)^3$, see eq. (3.36)). For a $2 \to 2$ process with 4 external legs, $|\mathcal{M}|^2_{\text{box}}$ therefore contains $(2\pi)^{-12}$ relative to the *stripped* amplitude $|\mathcal{M}_{\text{BD}}|^2$ that appears in the master formula (the latter is computed with spinors $\bar{u}u = 1$ and *no* explicit wavefunction prefactors).
- The replacement $\delta^{(3)}(\mathbf{0}) \to V/(2\pi)^3 = 1$ in the transition rate dN/T removes a further factor $(2\pi)^3$, and the box volume $V = (2\pi)^3$ in the flux denominator removes another $(2\pi)^{-3}$.

The net effect: the $(2\pi)^9$ that appeared above is matched, term by term, by the combined effect of $V = (2\pi)^3$ in the density of states, in the flux, and in the $\delta^{(3)}(0)$ replacement – i.e. by the conversion of the box-normalised matrix element $\mathcal{M}_{\text{box}}$ into the stripped BD amplitude $\mathcal{M}_{\text{BD}}$. *$\mathcal{M}_{BD}$ itself contains no (2π) wavefunction factors – they have already been peeled off when one writes eq. (3.38). Crucially, the (2π) factors do **not** enter the kinematic flux factor $\mathcal{F}$.*

Convention choice for the rest of these notes. From this point on we keep the BD convention consistently. The master formula will be written as

$$d\sigma = \frac{1}{\mathcal{F}} \Big(\prod_{i \in \text{ferm.}} 2m_i \Big) |\mathcal{M}_{\text{BD}}|^2 \, d\Phi_n \,,$$

where $|\mathcal{M}_{\text{BD}}|^2$ uses BD-normalised spinors ($\bar{u}u = 1$), and the kinematic factor $\mathcal{F} = 4E_A E_B |\mathbf{v}|$ carries no (2π) powers. (Readers who prefer the covariant convention ($\bar{u}u = 2m$) can convert at any time via $\mathcal{M}_{\text{cov}} = \prod_{\text{ferm.}} \sqrt{2m_i}\, \mathcal{M}_{\text{BD}}$, which absorbs the explicit $\prod 2m_i$ factor and yields the equivalent form $d\sigma = |\mathcal{M}_{\text{cov}}|^2 d\Phi_n / \mathcal{F}$.)

Box 3.3 Dimensional analysis in natural units ($\hbar = c = 1$)

Every factor has a well-defined mass dimension:

Quantity	Dimension	Why
$\lvert\mathbf{v}_A - \mathbf{v}_B\rvert$	0	velocity is dimensionless
$E_A E_B$	+2	product of two energies
$\mathcal{F} = 4E_A E_B \lvert\mathbf{v}\rvert$	+2	energy^2
$\lvert\mathcal{M}_{\text{BD}}\rvert^2 \prod 2m_i$	0	dimensionless for $2 \to 2$
$d\Phi_2$	0	$(E^{-4})(E^{+2})^2 = E^0$
$d\sigma$	-2	area = length^2 = energy^{-2} ✓

$\mathcal{F}$ has dimensions of energy^2 and carries *no* (2π) factors. The two powers of energy come from the initial-state normalisation $(1/(2E_A) \cdot 1/(2E_B)$ in the box amplitude), not from the flux.

The invariant flux factor. The kinematic factor introduced above is

$$\mathcal{F} = 4E_A E_B \, |\mathbf{v}_A - \mathbf{v}_B| \,, \tag{3.47}$$

with $4E_AE_B$ coming from the initial-state normalisation and $|\mathbf{v}_A - \mathbf{v}_B|$ from the physical flux.

We now show that this can be written as a Lorentz invariant. Since $\mathbf{v}_i = \mathbf{p}_i/E_i$:

$$E_A E_B \, |\mathbf{v}_A - \mathbf{v}_B| = E_A E_B \left| \frac{\mathbf{p}_A}{E_A} - \frac{\mathbf{p}_B}{E_B} \right| = |E_B \, \mathbf{p}_A - E_A \, \mathbf{p}_B| \,. \tag{3.48}$$

Squaring:

$$(E_A E_B)^2 \, |\mathbf{v}_A - \mathbf{v}_B|^2 = E_B^2 \, |\mathbf{p}_A|^2 + E_A^2 \, |\mathbf{p}_B|^2 - 2E_A E_B \, (\mathbf{p}_A \cdot \mathbf{p}_B) \,. \tag{3.49}$$

Now, $p_A \cdot p_B = E_A E_B - \mathbf{p}_A \cdot \mathbf{p}_B$, so $\mathbf{p}_A \cdot \mathbf{p}_B = E_A E_B - p_A \cdot p_B$. Also, $|\mathbf{p}_i|^2 = E_i^2 - m_i^2$. Substituting:

$$\begin{aligned} &= E_B^2(E_A^2 - m_A^2) + E_A^2(E_B^2 - m_B^2) - 2E_A E_B(E_A E_B - p_A \cdot p_B) \\ &= 2E_A^2 E_B^2 - E_B^2 m_A^2 - E_A^2 m_B^2 - 2E_A^2 E_B^2 + 2E_A E_B(p_A \cdot p_B) \\ &= 2E_A E_B(p_A \cdot p_B) - m_A^2 E_B^2 - m_B^2 E_A^2 \,. \end{aligned} \tag{3.50}$$

We want to compare this with $(p_A \cdot p_B)^2 - m_A^2 m_B^2$. Expanding:

$$\begin{aligned} (p_A \cdot p_B)^2 &= (E_A E_B - \mathbf{p}_A \cdot \mathbf{p}_B)^2 \\ &= E_A^2 E_B^2 - 2E_A E_B(\mathbf{p}_A \cdot \mathbf{p}_B) + (\mathbf{p}_A \cdot \mathbf{p}_B)^2 \,. \end{aligned} \tag{3.51}$$

Therefore:

$$(p_A \cdot p_B)^2 - m_A^2 m_B^2 = E_A^2 E_B^2 - 2E_A E_B(\mathbf{p}_A \cdot \mathbf{p}_B) + (\mathbf{p}_A \cdot \mathbf{p}_B)^2 - m_A^2 m_B^2 \,. \tag{3.52}$$

This expression is *not* equal to (3.50) in general, because of the $(\mathbf{p}_A \cdot \mathbf{p}_B)^2$ term. The identity $E_A E_B |\mathbf{v}_A - \mathbf{v}_B| = \sqrt{(p_A \cdot p_B)^2 - m_A^2 m_B^2}$ holds exactly only when $\mathbf{p}_A$ and $\mathbf{p}_B$ are *collinear* (parallel or antiparallel), which is always the case in the CM frame ($\mathbf{p}_B = -\mathbf{p}_A$) and in any collinear frame. In that case $(\mathbf{p}_A \cdot \mathbf{p}_B)^2 = |\mathbf{p}_A|^2 |\mathbf{p}_B|^2$, and one can verify that (3.52) reduces to (3.50).

The **invariant flux factor** is defined as the Lorentz-invariant quantity that equals $4E_AE_B|\mathbf{v}_A - \mathbf{v}_B|$ in any collinear frame:

$$\boxed{\mathcal{F} \equiv 4\sqrt{(p_A \cdot p_B)^2 - m_A^2 m_B^2}} \,. \tag{3.53}$$

In the **center-of-mass frame**, $\mathbf{p}_A = -\mathbf{p}_B \equiv \vec{p}$, so $|\mathbf{v}_A - \mathbf{v}_B| = |\vec{p}\,|(1/E_A + 1/E_B) = |\vec{p}\,|(E_A + E_B)/(E_A E_B)$, and therefore:

$$\mathcal{F} = 4E_A E_B \cdot \frac{|\vec{p}\,|(E_A + E_B)}{E_A E_B} = 4|\vec{p}\,|(E_A + E_B) = 4|\vec{p}\,|\sqrt{s} \,. \tag{3.54}$$

The BD master formula for the cross section. Combining the transition rate (3.41) with the flux, the cross section is:

$$\boxed{d\sigma(AB \to 1\cdots n) = \frac{1}{\mathcal{F}} \left(\prod_{i\in\text{fermions}} 2m_i \right) |\mathcal{M}_{\text{BD}}|^2 \, d\Phi_n} \,. \tag{3.55}$$

Here $\mathcal{M}_{\text{BD}}$ is the amplitude computed with BD spinors ($\bar{u}u = 1$). The product $\prod(2m_i)$ runs over *all* external fermion legs (initial and final). For processes involving only scalars or photons it equals 1.

Box 3.4 Origin and cancellation of the $\prod(2m_i)$ factor

The factor $\prod(2m_i)$ arises because the BD Dirac field expansion (3.25) has $\sqrt{m/E}$ per external fermion, whereas the covariant normalization ($\bar{u}u = 2m$) would give $1/\sqrt{2E}$. Their ratio is $\sqrt{2m}$, so at the amplitude-squared level each fermion contributes a factor $2m$.

When computing $\overline{|\mathcal{M}_{\text{BD}}|^2}$ using the BD spin sums $\sum_s u_s\bar{u}_s = (\not{p} + m)/(2m)$, each fermion–antifermion pair produces a factor $1/(2m)^2$ in the leptonic tensors. For a process with n_f external fermions (always paired), the product of these factors is $1/\prod(2m_i)$.

The $(2m)$ factors in the cross section formula and the $1/(2m)$ factors in the spin sums **cancel exactly**:

$$\prod_{i\in\text{fermions}} 2m_i \times \overline{|\mathcal{M}_{\text{BD}}|^2}\Big|_{\text{contains } \prod 1/(2m_i)} = (\text{pure traces, no mass prefactors}) \,. \tag{3.56}$$

This cancellation is guaranteed: the cross section is a physical observable, independent of the spinor normalization convention. Equivalently, defining the covariantly normalized amplitude $\mathcal{M}_{\text{cov}} = \prod \sqrt{2m_i}\, \mathcal{M}_{\text{BD}}$, the master formula reduces to the standard form $d\sigma = |\mathcal{M}_{\text{cov}}|^2 \, d\Phi_n/\mathcal{F}$.

Spin and color averages. The kinematic prefactors derived above are universal, determined solely by the BD normalization. The dependence on the specific field content enters through the invariant amplitude $\mathcal{M}$ and through the treatment of discrete quantum numbers. For unpolarized initial states, one averages over the initial spins, colors, and polarizations:

$$|\mathcal{M}|^2 \longrightarrow \overline{|\mathcal{M}|^2} = \frac{1}{g_A g_B} \sum_{\text{spins, colors, pols}} |\mathcal{M}|^2, \qquad g_i = (2s_i+1) \times (\text{color mult.}) \times \cdots \,. \tag{3.57}$$

The BD spin sums for external fermions are

$$\sum_s u_s(p)\bar{u}_s(p) = \frac{\not{p}+m}{2m}, \qquad \sum_s v_s(p)\bar{v}_s(p) = \frac{\not{p}-m}{2m}, \tag{3.58}$$

where each $1/(2m)$ originates from the BD normalization $\bar{u}u = 1$. These factors, together with the $\prod(2m_i)$ in eq. (3.55), cancel in the final cross section (see the box above). For massive vectors the polarization sum was given above. For massless gauge bosons, appropriate physical polarization prescriptions or gauge-fixing conditions must be employed.

Identical particles and symmetry factors. When the final state contains identical particles, the phase-space integration overcounts physically indistinguishable configurations. If among the n final particles there are groups of identical particles with multiplicities $n_1, n_2, \ldots$, the cross section must include the symmetry factor

$$d\sigma \;\to\; \frac{1}{n_1!\,n_2!\cdots}\, d\sigma\,. \tag{3.59}$$

The complete formula. Collecting all factors, the differential cross section for an unpolarized scattering process with possible identical particles in the final state is

$$\boxed{d\sigma = \frac{1}{\mathcal{F}} \cdot \frac{1}{n_1!\,n_2!\cdots} \cdot \left(\prod_{i\in\text{fermions}} 2m_i \right) \cdot \overline{|\mathcal{M}_{\text{BD}}|^2} \cdot d\Phi_n} \tag{3.60}$$

This expression serves as the starting point for all cross-section calculations in this course.

4 Feynman Rules

The Feynman rules provide a systematic algorithm for computing scattering amplitudes in quantum field theory. Rather than evaluating complicated operator products and time-ordered integrals directly, we translate each term in the perturbative expansion into a pictorial *Feynman diagram*, and then associate mathematical factors with each element of the diagram. This chapter derives the Feynman rules from first principles, applying the Dyson series [8] (developed in Chapter 3) together with Wick's theorem. For complementary and particularly clear presentations, see also Peskin and Schroeder [6], Srednicki [11], and Schwartz [12].

4.1 The perturbative expansion of the S-matrix

As derived in the previous chapter, the S-matrix in the interaction picture is given by the Dyson series:

$$S = T\exp\left(-i\int d^4x\,\mathcal{H}_I(x)\right) = \sum_{n=0}^{\infty} S^{(n)}, \tag{4.1}$$

where the n-th order contribution is:

$$\boxed{S^{(n)} = \frac{(-i)^n}{n!}\int d^4x_1\cdots d^4x_n\, T\{\mathcal{H}_I(x_1)\cdots\mathcal{H}_I(x_n)\}}. \tag{4.2}$$

For QED, the interaction Hamiltonian density is:

$$\mathcal{H}_I(x) = -\mathcal{L}_I(x) = -e\,:\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x):, \tag{4.3}$$

where the normal ordering $:\cdots:$ is implicit in the definition of $\mathcal{H}_I$. The coupling constant $e>0$ is the positron charge (in natural units, $e=\sqrt{4\pi\alpha}\approx 0.303$).

To compute $\langle f|S|i\rangle$ for a scattering process, we must evaluate vacuum expectation values of time-ordered products of field operators. This can be seen as follows. The initial and final states $|i\rangle$, $|f\rangle$ are multi-particle states created by acting with creation operators on the vacuum: for instance, for a real scalar field, $|i\rangle = a^\dagger(\vec{p}_1)a^\dagger(\vec{p}_2)|0\rangle$, where $a^\dagger(\vec{p})$ is the creation operator appearing in the mode expansion (1.13) (and analogously for spinor and vector fields). The $S^{(n)}$ term in the Dyson series contains a time-ordered product of n interaction Hamiltonians, each of which is built from

field operators. Since the fields themselves are sums of creation and annihilation operators, the matrix element $\langle f|S^{(n)}|i\rangle$ reduces to a vacuum expectation value: the creation operators in $|i\rangle$ and $\langle f|$ can be traded for additional field operators using the relation $a^\dagger(\vec{p})|0\rangle = (2\pi)^{3/2}\sqrt{2E_p}\,\phi^{(-)}(\vec{p})|0\rangle$, so that ultimately we need to evaluate expressions of the form $\langle 0|T\{\text{fields}\}|0\rangle$. This is accomplished by **Wick's theorem**.

4.2 Wick's theorem

As argued above, the computation of scattering amplitudes $\langle f|S|i\rangle$ ultimately requires the evaluation of vacuum expectation values of time-ordered products of field operators. Wick's theorem is the fundamental tool that makes this computation feasible: it reduces complex operator products to sums of c-numbers (propagators) and normal-ordered terms, whose vacuum expectation value vanishes. The mode decompositions below are written in the BD convention; see Section 1.1 for how the expansion factors, (anti)commutators, and state normalizations fit together.

Normal ordering

Definition. For any product of field operators, the *normal-ordered* form $:\!O\!:$ is obtained by moving all creation operators to the left of all annihilation operators, *ignoring* any (anti)commutators that would arise. For fermionic fields, each transposition contributes a factor (-1).

Decomposition of fields. Any free field can be written as $\phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x)$, where:

- $\phi^{(+)}(x)$ contains annihilation operators (positive frequency: $e^{-ip\cdot x}$)
- $\phi^{(-)}(x)$ contains creation operators (negative frequency: $e^{+ip\cdot x}$)

For a real scalar field:

$$\phi^{(+)}(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2E_p}}\, a(\vec{p})\, e^{-ip\cdot x} \tag{4.4}$$

$$\phi^{(-)}(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2E_p}}\, a^\dagger(\vec{p})\, e^{+ip\cdot x}\,. \tag{4.5}$$

For a Dirac field:

$$\psi^{(+)}(x) = \int \frac{d^3p}{(2\pi)^{3/2}}\sqrt{\frac{m}{E_p}}\sum_s b_s(\vec{p})\, u_s(p)\, e^{-ip\cdot x}\,, \tag{4.6}$$

$$\psi^{(-)}(x) = \int \frac{d^3p}{(2\pi)^{3/2}}\sqrt{\frac{m}{E_p}}\sum_s d_s^\dagger(\vec{p})\, v_s(p)\, e^{+ip\cdot x}\,. \tag{4.7}$$

Examples of normal ordering.

$$\begin{aligned}
:\phi(x)\phi(y): &=: (\phi^{(+)}(x)+\phi^{(-)}(x))(\phi^{(+)}(y)+\phi^{(-)}(y)):\\
&=: \phi^{(-)}(x)\phi^{(-)}(y)+\phi^{(-)}(x)\phi^{(+)}(y)\\
&\quad +\phi^{(+)}(x)\phi^{(-)}(y)+\phi^{(+)}(x)\phi^{(+)}(y):\\
&= \phi^{(-)}(x)\phi^{(-)}(y)+\phi^{(-)}(x)\phi^{(+)}(y)\\
&\quad +\phi^{(-)}(y)\phi^{(+)}(x)+\phi^{(+)}(x)\phi^{(+)}(y)\,. \qquad (4.8)
\end{aligned}$$

In the second step we have simply expanded the product, but the normal-ordering symbol $:\cdots:$ still applies to the whole expression. In the third step, we *execute* the normal ordering: the first, second, and fourth terms are already normal-ordered (creation operators $\phi^{(-)}$ appear to the left of annihilation operators $\phi^{(+)}$), while the third term $\phi^{(+)}(x)\phi^{(-)}(y)$ is not. Normal ordering means we rewrite it as $\phi^{(-)}(y)\phi^{(+)}(x)$ by moving the creation operator $\phi^{(-)}(y)$ to the left, *without* introducing any commutator (this is what "ignoring commutators" means in the definition of normal ordering).

For fermions, with $\psi = \psi^{(+)} + \psi^{(-)}$ and $\bar\psi = \bar\psi^{(+)} + \bar\psi^{(-)}$:

$$:\bar\psi\psi: = \bar\psi^{(-)}\psi^{(-)} + \bar\psi^{(-)}\psi^{(+)} - \psi^{(-)}\bar\psi^{(+)} + \bar\psi^{(+)}\psi^{(+)}\,. \qquad (4.9)$$

The minus sign arises from anticommuting $\bar\psi^{(+)}$ past $\psi^{(-)}$.

Key property.

$$\boxed{\langle 0|\,:\mathcal{O}:\,|0\rangle = 0}\,. \qquad (4.10)$$

This holds because all annihilation operators act on $|0\rangle$ giving zero, and all creation operators act on $\langle 0|$ giving zero.

Contraction

Definition. The *contraction* of two fields is defined as:

$$\overbracket{\phi(x)\phi(y)} \equiv T\{\phi(x)\phi(y)\} - :\phi(x)\phi(y):\,. \qquad (4.11)$$

Explicit calculation for scalar fields. The key identity relating the T-product and normal ordering emerges from a straightforward calculation. We present it for both time orderings to show that the result is universal.

Case 1: $x^0 > y^0$. In this case $T\{\phi(x)\phi(y)\} = \phi(x)\phi(y)$. Expanding:

$$\begin{aligned}
\phi(x)\phi(y) &= \left(\phi^{(+)}(x)+\phi^{(-)}(x)\right)\left(\phi^{(+)}(y)+\phi^{(-)}(y)\right)\\
&= \phi^{(+)}(x)\phi^{(+)}(y)+\phi^{(-)}(x)\phi^{(+)}(y)\\
&\quad +\phi^{(+)}(x)\phi^{(-)}(y)+\phi^{(-)}(x)\phi^{(-)}(y)\,. \qquad (4.12)
\end{aligned}$$

The first, second, and fourth terms are already normal ordered. The third term is not: it has an annihilation operator ($\phi^{(+)}$) to the left of a creation operator ($\phi^{(-)}$). We reorder it using the commutator:

$$\phi^{(+)}(x)\phi^{(-)}(y) = \phi^{(-)}(y)\phi^{(+)}(x) + [\phi^{(+)}(x), \phi^{(-)}(y)]\,. \tag{4.13}$$

Since $[\phi^{(+)}(x), \phi^{(-)}(y)]$ is a c-number (as shown in the box below), we find:

Box 4.1 The commutator $[\phi^{(+)}(x), \phi^{(-)}(y)]$ is a c-number

Using the mode expansions (4.4) and (4.5):

$$\begin{aligned}[\phi^{(+)}(x), \phi^{(-)}(y)] &= \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2E_p}} \int \frac{d^3p'}{(2\pi)^{3/2}\sqrt{2E_{p'}}} \\ &\quad \times \underbrace{[a(\vec{p}), a^\dagger(\vec{p}\,')]}_{=\,\delta^{(3)}(\vec{p}-\vec{p}\,')} e^{-ip\cdot x}\, e^{+ip'\cdot y} \\ &= \int \frac{d^3p}{(2\pi)^3\, 2E_p}\, e^{-ip\cdot(x-y)}\,. \end{aligned} \tag{4.14}$$

This is a pure number (a c-number): it commutes with all operators, depends only on the spacetime separation $x-y$, and contains no creation or annihilation operators. It is, in fact, the positive-frequency Wightman function $D^{(+)}(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle$ (only the $\phi^{(+)}\phi^{(-)}$ piece survives in the VEV). The key point is that the commutator $[a(\vec{p}), a^\dagger(\vec{p}\,')]$ produces a delta function, which eliminates one of the two momentum integrals and leaves a purely numerical function of the coordinates.

$$T\{\phi(x)\phi(y)\} =: \phi(x)\phi(y) : + [\phi^{(+)}(x), \phi^{(-)}(y)] \qquad (x^0 > y^0)\,. \tag{4.15}$$

Case 2: $y^0 > x^0$. Now $T\{\phi(x)\phi(y)\} = \phi(y)\phi(x)$ (for bosons). Repeating the same steps with $x \leftrightarrow y$:

$$\begin{aligned} T\{\phi(x)\phi(y)\} &=: \phi(y)\phi(x) : + [\phi^{(+)}(y), \phi^{(-)}(x)] \\ &=: \phi(x)\phi(y) : + [\phi^{(+)}(y), \phi^{(-)}(x)]\,, \end{aligned} \tag{4.16}$$

where in the last step we used the fact that bosonic fields can be freely reordered inside the normal ordering.

Combining both cases. Equations (4.15) and (4.16) can be written as a single formula:

$$\begin{aligned} T\{\phi(x)\phi(y)\} &=: \phi(x)\phi(y) : \\ &\quad + [\phi^{(+)}(x), \phi^{(-)}(y)]\,\theta(x^0 - y^0) + [\phi^{(+)}(y), \phi^{(-)}(x)]\,\theta(y^0 - x^0)\,. \end{aligned} \tag{4.17}$$

We have established that

$$T\{\phi(x)\phi(y)\} =: \phi(x)\phi(y) : + \underbrace{c\text{-number function of } (x-y)}_{\text{contraction } \overbrace{\phi(x)\phi(y)}} . \tag{4.18}$$

We now show that the c-number remainder equals the vacuum expectation value of the time-ordered product. The argument is simple: take the vacuum expectation value of both sides of (4.18). On the left-hand side we get $\langle 0|T\{\phi(x)\phi(y)\}|0\rangle$. On the right-hand side, the normal-ordered term vanishes because in a normal-ordered product all annihilation operators stand to the right and destroy $|0\rangle$:

$$\langle 0| : \phi(x)\phi(y) : |0\rangle = 0 , \tag{4.19}$$

while the c-number passes through the VEV unchanged: $\langle 0|c|0\rangle = c \cdot \langle 0|0\rangle = c$. Therefore:

$$\boxed{\overbrace{\phi(x)\phi(y)} = T\{\phi(x)\phi(y)\} - : \phi(x)\phi(y) := \langle 0|T\{\phi(x)\phi(y)\}|0\rangle} \tag{4.20}$$

This is a key result: the contraction of two fields—defined algebraically as what remains after subtracting the normal-ordered product from the time-ordered product—is equal to the **Feynman propagator**, which has a direct physical interpretation as the amplitude for a particle to propagate from y to x (or from x to y).

The contraction equals the Feynman propagator.

$$\boxed{\overbrace{\phi(x)\phi(y)} = \langle 0|T\{\phi(x)\phi(y)\}|0\rangle = i\Delta_F(x-y)} . \tag{4.21}$$

Explicit calculation for Dirac fields. For fermions, the calculation parallels the scalar case but with anticommutators replacing commutators. Consider $x^0 > y^0$, so $T\{\psi_\alpha(x)\bar{\psi}_\beta(y)\} = \psi_\alpha(x)\bar{\psi}_\beta(y)$. Expanding:

$$\begin{aligned} \psi_\alpha(x)\bar{\psi}_\beta(y) &= \left(\psi_\alpha^{(+)}(x) + \psi_\alpha^{(-)}(x)\right)\left(\bar{\psi}_\beta^{(+)}(y) + \bar{\psi}_\beta^{(-)}(y)\right) \\ &= \psi_\alpha^{(+)}\bar{\psi}_\beta^{(+)} + \psi_\alpha^{(-)}\bar{\psi}_\beta^{(+)} + \psi_\alpha^{(+)}\bar{\psi}_\beta^{(-)} + \psi_\alpha^{(-)}\bar{\psi}_\beta^{(-)} . \end{aligned} \tag{4.22}$$

The problematic term is $\psi_\alpha^{(+)}\bar{\psi}_\beta^{(-)}$. To normal-order it, we use the anticommutator (note the sign):

$$\begin{aligned} \psi_\alpha^{(+)}(x)\bar{\psi}_\beta^{(-)}(y) &= -\bar{\psi}_\beta^{(-)}(y)\psi_\alpha^{(+)}(x) \\ &\quad + \{\psi_\alpha^{(+)}(x), \bar{\psi}_\beta^{(-)}(y)\} . \end{aligned} \tag{4.23}$$

The anticommutator is again a c-number. Collecting all terms:

$$T\{\psi_\alpha(x)\bar{\psi}_\beta(y)\} =: \psi_\alpha(x)\bar{\psi}_\beta(y) : + \{\psi_\alpha^{(+)}(x), \bar{\psi}_\beta^{(-)}(y)\} \qquad (x^0 > y^0)\,. \quad (4.24)$$

As in the scalar case, combining both time orderings and taking the vacuum expectation value gives:

$$\overbracket{\psi_\alpha(x)\bar{\psi}_\beta(y)} = \langle 0|T\{\psi_\alpha(x)\bar{\psi}_\beta(y)\}|0\rangle = iS_{F,\alpha\beta}(x-y)\,. \quad (4.25)$$

An important sign arises for the reversed contraction: $\overbracket{\bar{\psi}_\beta(y)\psi_\alpha(x)} = -\overbracket{\psi_\alpha(x)\bar{\psi}_\beta(y)}$, because fermionic fields must be anticommuted to bring them into the standard order.

Contractions for QED fields. The non-vanishing contractions are:

Contraction	Propagator
$\overbracket{\phi(x)\phi(y)}$	$i\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}e^{-ip\cdot(x-y)}$
$\overbracket{\psi_\alpha(x)\bar{\psi}_\beta(y)}$	$iS_{F,\alpha\beta}(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{i(\not{p}+m)_{\alpha\beta}}{p^2-m^2+i\epsilon}e^{-ip\cdot(x-y)}$
$\overbracket{A_\mu(x)A_\nu(y)}$	$iD_F^{\mu\nu}(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{-ig^{\mu\nu}}{p^2+i\epsilon}e^{-ip\cdot(x-y)}$

Important: $\overbracket{\bar{\psi}(x)\psi(y)} = -\overbracket{\psi(y)\bar{\psi}(x)} = -iS_F(y-x)$ due to fermionic anticommutation.

Vanishing contractions. The following contractions are zero:

$$\overbracket{\psi\psi} = 0, \quad \overbracket{\bar{\psi}\bar{\psi}} = 0, \quad \overbracket{\psi A} = 0, \quad \overbracket{\bar{\psi}A} = 0\,. \quad (4.26)$$

These vanish because the corresponding vacuum two-point functions are zero: there is no non-vanishing propagator connecting those field species and charge assignments.

Statement of Wick's theorem

For n field operators $\phi_1, \phi_2, \ldots, \phi_n$ (each may be ϕ, ψ, $\bar{\psi}$, A_μ, etc.):

$$
\begin{aligned}
T\{\phi_1 \cdots \phi_n\} =& :\phi_1 \cdots \phi_n: \\
&+ \sum_{i<j} (-1)^{P_{ij}} \overbracket{\phi_i \phi_j} \; :\prod_{\substack{k=1 \\ k\neq i,j}}^{n} \phi_k : \\
&+ \sum_{\substack{(i_1<j_1),(i_2<j_2) \\ \text{all distinct}}} (-1)^{P} \overbracket{\phi_{i_1} \phi_{j_1}} \, \overbracket{\phi_{i_2} \phi_{j_2}} \; : \prod_{k\notin\{i_1,j_1,i_2,j_2\}} \phi_k : \\
&+ \cdots
\end{aligned}
\tag{4.27}
$$

where $(-1)^P$ is the sign of the permutation of *fermionic* fields needed to bring each contracted pair together from the original ordering. For purely bosonic fields, all signs are $+1$.

Interpretation:

- The first line is the fully normal-ordered product (no contractions).
- The second line sums over all $\binom{n}{2}$ ways of choosing one pair: the pair is replaced by its contraction (c-number = propagator), while the remaining $n-2$ fields are normal-ordered. The sign $(-1)^{P_{ij}}$ counts the number of fermionic transpositions required to move ϕ_i and ϕ_j next to each other.
- The third line sums over all ways of choosing two disjoint pairs, and so on.
- Continue until all fields are contracted (n even) or one field remains (n odd).

Box 4.2 Proof of Wick's theorem

We prove by induction on the number of fields n.

Base case ($n = 2$). By definition of contraction:

$$
T\{\phi_1 \phi_2\} =: \phi_1 \phi_2 : + \overbracket{\phi_1 \phi_2} \, . \tag{4.28}
$$

This is exactly Wick's theorem for two fields. ✓

Inductive hypothesis. Assume Wick's theorem holds for $n-1$ fields.

Inductive step. We must prove it for n fields. Without loss of generality, assume t_n is the *earliest* time (if not, relabel). With the convention that time ordering puts the later field to the left, the earliest field is rightmost, so we can pull ϕ_n out on the right:

$$
T\{\phi_1 \cdots \phi_n\} = T\{\phi_1 \cdots \phi_{n-1}\} \cdot \phi_n \, , \tag{4.29}
$$

where ϕ_n now sits to the right of the time-ordered product of the remaining $n-1$ fields.

Decompose the field into creation and annihilation parts: $\phi_n = \phi_n^{(+)} + \phi_n^{(-)}$ where $\phi_n^{(+)}$ contains annihilation operators and $\phi_n^{(-)}$ contains creation operators. By the inductive hypothesis:

$$T\{\phi_1 \cdots \phi_{n-1}\} =: \phi_1 \cdots \phi_{n-1} : + (\text{terms with contractions}). \tag{4.30}$$

Consider the normal-ordered term $:\phi_1 \cdots \phi_{n-1}:$ multiplied by ϕ_n:

$$:\phi_1 \cdots \phi_{n-1} : \cdot \phi_n =: \phi_1 \cdots \phi_{n-1} : \cdot (\phi_n^{(+)} + \phi_n^{(-)}) \tag{4.31}$$

The $\phi_n^{(+)}$ part: Annihilation operators can be moved inside the normal ordering:

$$:\phi_1 \cdots \phi_{n-1} : \cdot \phi_n^{(+)} =: \phi_1 \cdots \phi_{n-1} \phi_n^{(+)} : . \tag{4.32}$$

The $\phi_n^{(-)}$ part: Creation operators must be moved to the left. For each field ϕ_j in the normal-ordered product:

$$\phi_j \phi_n^{(-)} = \phi_n^{(-)} \phi_j \pm [\phi_j, \phi_n^{(-)}]_{\pm}, \tag{4.33}$$

where $[\cdot,\cdot]_+$ is the anticommutator for fermions and $[\cdot,\cdot]_-$ is the commutator for bosons.
The (anti)commutator $[\phi_j, \phi_n^{(-)}]_{\pm}$ is a c-number equal to:

$$[\phi_j^{(+)}, \phi_n^{(-)}]_{\pm} = \langle 0 | \phi_j \phi_n | 0 \rangle = \overbrace{\phi_j \phi_n} \quad (\text{since } t_j > t_n, \text{ so } T\{\phi_j \phi_n\} = \phi_j \phi_n). \tag{4.34}$$

Therefore, moving $\phi_n^{(-)}$ through the normal-ordered product generates:

$$:\phi_1 \cdots \phi_{n-1} : \cdot \phi_n^{(-)} =: \phi_n^{(-)} \phi_1 \cdots \phi_{n-1} : + \sum_{j=1}^{n-1} (\pm)^j \overbrace{\phi_j \phi_n} \, :\phi_1 \cdots \hat{\phi}_j \cdots \phi_{n-1} :, \tag{4.35}$$

where $\hat{\phi}_j$ means ϕ_j is omitted and $(\pm)^j$ accounts for fermionic signs.
Combining both parts:

$$:\phi_1 \cdots \phi_{n-1} : \cdot \phi_n =: \phi_1 \cdots \phi_n : + \sum_{j=1}^{n-1} (\pm)^j \overbrace{\phi_j \phi_n} \, :\phi_1 \cdots \hat{\phi}_j \cdots \phi_{n-1} : . \tag{4.36}$$

The same procedure applied to terms with contractions in the inductive hypothesis generates all remaining terms in Wick's theorem. By induction, the theorem is proved. □

Examples of Wick's theorem

Example 1: Two scalar fields.

$$T\{\phi_1\phi_2\} =: \phi_1\phi_2 : + \overbracket{\phi_1\phi_2}\,. \tag{4.37}$$

This is the definition of contraction, showing that $T\{\cdot\} =: \cdot :$ +contractions.

Example 2: Four scalar fields. Using the notation $\langle ij\rangle \equiv \overbracket{\phi_i\phi_j} = i\Delta_F(x_i - x_j)$:

$$\begin{aligned} T\{\phi_1\phi_2\phi_3\phi_4\} =: \phi_1\phi_2\phi_3\phi_4 : \\ + \langle 12\rangle\ :\phi_3\phi_4: + \langle 13\rangle\ :\phi_2\phi_4: + \langle 14\rangle\ :\phi_2\phi_3: \\ + \langle 23\rangle\ :\phi_1\phi_4: + \langle 24\rangle\ :\phi_1\phi_3: + \langle 34\rangle\ :\phi_1\phi_2: \\ + \langle 12\rangle\langle 34\rangle + \langle 13\rangle\langle 24\rangle + \langle 14\rangle\langle 23\rangle\,. \end{aligned} \tag{4.38}$$

There are $\binom{4}{2} = 6$ single contractions. The number of *double* (i.e. complete) contractions is $\frac{1}{2!}\binom{4}{2}\binom{2}{2} = 3$: we first choose 2 fields out of 4 for the first pair ($\binom{4}{2}$ ways), then the remaining 2 form the second pair ($\binom{2}{2} = 1$ way), and we divide by 2! because the ordering of the two pairs is irrelevant. These are precisely the three terms $\langle 12\rangle\langle 34\rangle$, $\langle 13\rangle\langle 24\rangle$, and $\langle 14\rangle\langle 23\rangle$.

Example 3: Two Dirac fields (tracking signs).

$$T\{\psi_\alpha(x)\bar\psi_\beta(y)\} =: \psi_\alpha(x)\bar\psi_\beta(y) : + \overbracket{\psi_\alpha(x)\bar\psi_\beta(y)}\,, \tag{4.39}$$

where $\overbracket{\psi_\alpha(x)\bar\psi_\beta(y)} = iS_{F,\alpha\beta}(x-y)$.

For the reversed order:

$$T\{\bar\psi_\beta(y)\psi_\alpha(x)\} =: \bar\psi_\beta(y)\psi_\alpha(x) : + \overbracket{\bar\psi_\beta(y)\psi_\alpha(x)}\,. \tag{4.40}$$

Note: $\overbracket{\bar\psi_\beta(y)\psi_\alpha(x)} = -\overbracket{\psi_\alpha(x)\bar\psi_\beta(y)} = -iS_{F,\alpha\beta}(x-y)$ due to anticommutation.

Example 4: QED vertex at second order. At second order in the Dyson series for QED, we encounter the time-ordered product of two normal-ordered interaction vertices:

$$T\Big\{ :\bar\psi(x)\gamma^\mu\psi(x)A_\mu(x): \ :\bar\psi(y)\gamma^\nu\psi(y)A_\nu(y): \Big\}\,. \tag{4.41}$$

This is *not* the same as the time-ordered product of six individual field operators. Wick's theorem for the T-product of normal-ordered products gives:

$$T\{:O_1::O_2:\} =: O_1O_2 : + \text{all contractions between } \textit{different} \text{ groups}\,, \tag{4.42}$$

i.e. one sums over all possible contractions, but **only between fields belonging to different normal-ordered factors**—contractions of two fields *within the same*

$:\cdots:$ are absent. The reason is the following: a contraction is defined as $\overbracket{\phi_i\phi_j} = T\{\phi_i\phi_j\} - :\phi_i\phi_j:$, i.e. it measures the difference between time-ordering and normal-ordering. At coincident points this difference is *not* zero in general; indeed it is precisely what would generate singular self-contractions such as $i\Delta_F(0)$. What matters here is a different statement: Wick's theorem for the time-ordered product of *normal-ordered factors* tells us that only contractions between fields belonging to different factors appear explicitly. Thus

$$T\{:O_1::O_2:\} =:O_1O_2: + \text{all contractions between different factors}, \tag{4.43}$$

while contractions internal to the same normal-ordered vertex are absent by construction. Physically, normal ordering the interaction Hamiltonian $\mathcal{H}_I =:\cdots:$ is precisely the prescription that removes these self-contractions, which would otherwise generate tadpole-like contact terms at a single vertex. Signs from fermionic anticommutations apply as before.

The resulting contraction patterns are:

(i) Photon contraction only:

$$:\bar\psi(x)\gamma^\mu\psi(x)::\bar\psi(y)\gamma^\nu\psi(y):\cdot\overbracket{A_\mu(x)A_\nu(y)}\,. \tag{4.44}$$

This gives scattering processes like $e^+e^- \to \mu^+\mu^-$ (four external fermions, internal photon); see Section 5.2.

(ii) One fermion contraction:

$$:\bar\psi(x)\gamma^\mu A_\mu(x)::\gamma^\nu\psi(y)A_\nu(y):\cdot\overbracket{\psi(x)\bar\psi}(y)\,. \tag{4.45}$$

This gives Compton scattering (two external fermions, two external photons, internal fermion); see Section 5.4.

(iii) Both fermion contractions (closed loop):

$$\overbracket{\psi(x)\bar\psi}(y)\cdot\overbracket{\psi(y)\bar\psi}(x)\cdot:A_\mu(x)A_\nu(y):\cdot\gamma^\mu\gamma^\nu\,. \tag{4.46}$$

This gives vacuum polarization (two external photons, fermion loop); see Section 6.2. The trace arises from the closed spinor index chain.

Note that there is no term with, say, $\overbracket{\psi(x)\bar\psi}(x)$: this would be a contraction within the same normal-ordered factor $:\bar\psi(x)\gamma^\mu\psi(x)A_\mu(x):$, and is excluded by the rule above.

4.3 Scalar field theory: $\lambda\phi^4$

Before tackling the complexities of QED, we develop the Feynman rules for the simplest interacting quantum field theory: a real scalar field with quartic self-interaction. This theory, while lacking direct phenomenological application, serves as the ideal pedagogical laboratory for understanding perturbation theory, renormalization, and the renormalization group.

The Lagrangian

The Lagrangian density for $\lambda\phi^4$ theory is:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4, \tag{4.47}$$

where $\phi(x)$ is a real scalar field, m is the mass, and $\lambda > 0$ is a dimensionless coupling constant. The factor $1/4!$ is conventional and simplifies combinatorics.

The interaction Hamiltonian density is:

$$\mathcal{H}_I = -\mathcal{L}_I = \frac{\lambda}{4!}\phi^4. \tag{4.48}$$

Dimensional analysis. In $d = 4$ spacetime dimensions:

- $[\phi] = 1$ (mass dimension)
- $[\partial_\mu] = 1$
- $[m] = 1$
- $[\lambda] = 0$ (dimensionless)

The dimensionlessness of λ makes ϕ^4 theory *renormalizable* in $d = 4$; the precise meaning of this statement and the general classification of interactions based on power counting are discussed in Section 6.7.

Why ϕ^4 and not ϕ^3? One might wonder why the simplest scalar interaction is taken to be quartic rather than cubic. A ϕ^3 coupling would actually be *super-renormalizable* in $d = 4$ (the coupling has mass dimension $[\lambda_3] = 1$), so renormalizability is not the issue. The real problem is **stability**: the potential $V(\phi) = \frac{\lambda_3}{3!}\phi^3$ is unbounded from below—for $\phi \to -\infty$ (if $\lambda_3 > 0$) the energy decreases without limit, so the vacuum is unstable and the theory has no ground state. By contrast, $V(\phi) = \frac{\lambda}{4!}\phi^4$ with $\lambda > 0$ is bounded from below and possesses a stable vacuum. Moreover, ϕ^4 theory enjoys a discrete $\mathbb{Z}_2$ symmetry $\phi \to -\phi$, which forbids odd-power interactions such as ϕ^3 and ϕ and ensures that the vacuum at $\phi = 0$ is a true minimum.

Derivation of the Feynman rules for $\lambda\phi^4$

We now derive the Feynman rules systematically from the Dyson series and Wick's theorem.

The S-matrix expansion. The S-matrix is given by the Dyson series (4.1):

$$S = T\exp\left(-i\int d^4x\,\mathcal{H}_I(x)\right) = \mathbb{1} + S^{(1)} + S^{(2)} + \cdots, \tag{4.49}$$

where for ϕ^4 theory:

$$S^{(n)} = \frac{(-i)^n}{n!}\left(\frac{\lambda}{4!}\right)^n \int d^4x_1\cdots d^4x_n\, T\{\phi^4(x_1)\cdots\phi^4(x_n)\}\,. \tag{4.50}$$

First order: complete classification of $S^{(1)}$. At first order,

$$S^{(1)} = -i\frac{\lambda}{4!}\int d^4x\, T\{\phi^4(x)\}\,. \tag{4.51}$$

Because all four fields sit at the *same* spacetime point, a first-order Wick contraction can occur in two different ways:

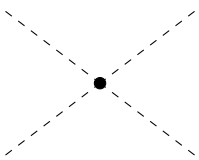
- a field contracts with an external particle state;
- two fields at the same vertex contract with each other, producing a factor $i\Delta_F(0)$.

If $r = 0, 1, 2$ self-contractions occur at the vertex, then $2r$ fields are used internally and the number of external legs is

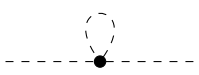
$$E = 4 - 2r\,. \tag{4.52}$$

Therefore the *full* first-order operator $S^{(1)}$ contains exactly three topologies:

- $r = 0$ ($E = 4$)**:** the elementary quartic contact vertex,

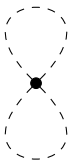


  which underlies the matrix elements $0 \to 4$, $1 \to 3$, $2 \to 2$, $3 \to 1$, $4 \to 0$.
- $r = 1$ ($E = 2$)**:** the tadpole topology,



which contributes to two-leg amplitudes (for example the $1 \to 1$ matrix element, or equivalently the two-point function).

- $r = 2$ ($E = 0$)**:** the vacuum bubble ("figure-eight"),



which contributes to the vacuum-to-vacuum amplitude $\langle 0|S^{(1)}|0\rangle$.

Among these three first-order topologies, the elementary interaction vertex is the four-leg contribution with no self-contractions. We now isolate this term and derive its factor explicitly. If one normal-orders the interaction Hamiltonian, the tadpole and vacuum-bubble topologies are absent from the outset.

Derivation of the vertex factor. Consider the amplitude with four incoming particles:

$$\langle 0|S^{(1)}|p_1,p_2,p_3,p_4\rangle = \frac{-i\lambda}{4!}\int d^4x \times \langle 0|T\{\phi^4(x)\}|p_1,p_2,p_3,p_4\rangle\,. \tag{4.53}$$

The initial state is $|p_1,p_2,p_3,p_4\rangle = a^\dagger(\vec{p}_1)a^\dagger(\vec{p}_2)a^\dagger(\vec{p}_3)a^\dagger(\vec{p}_4)|0\rangle$. We must contract all four field operators $\phi(x)$ with the four creation operators.

Using the mode expansion $\phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x)$ where $\phi^{(+)}$ contains the annihilation operators:

$$\phi^{(+)}(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2E_k}}\, a(\vec{k})\, e^{-ik\cdot x}, \tag{4.54}$$

the **contraction** of $\phi^{(+)}(x)$ with a single one-particle state $|p_i\rangle = a^\dagger(\vec{p}_i)|0\rangle$ is:

$$\langle 0|\phi^{(+)}(x)|p_i\rangle = \frac{e^{-ip_i\cdot x}}{(2\pi)^{3/2}\sqrt{2E_{p_i}}}\,. \tag{4.55}$$

This is the basic building block: each field $\phi^{(+)}(x)$, when contracted with a specific particle of momentum p_i, contributes a plane wave $e^{-ip_i\cdot x}$ (times the normalization factor).

For $\phi^4(x)$ to annihilate all four particles, we need all four $\phi^{(+)}$ components, each contracted with a different particle. There are 4! ways to assign the four fields to the four particles; each assignment gives the same result:

$$\langle 0|\phi^{(+)}(x)\phi^{(+)}(x)\phi^{(+)}(x)\phi^{(+)}(x)|p_1,p_2,p_3,p_4\rangle = 4!\prod_{i=1}^{4}\frac{1}{(2\pi)^{3/2}\sqrt{2E_{p_i}}}\, e^{-i(p_1+p_2+p_3+p_4)\cdot x}\,. \tag{4.56}$$

Therefore, denoting the external-leg normalization by $\mathcal{N} \equiv \prod_{i=1}^{4}[(2\pi)^{3/2}\sqrt{2E_{p_i}}]^{-1}$:

$$\begin{aligned}\langle 0|S^{(1)}|p_1,p_2,p_3,p_4\rangle &= \frac{-i\lambda}{4!}\cdot 4!\,\mathcal{N}\int d^4x\, e^{-i(p_1+p_2+p_3+p_4)\cdot x} \\ &= -i\lambda\,\mathcal{N}\,(2\pi)^4\delta^{(4)}(p_1+p_2+p_3+p_4)\,. \end{aligned} \tag{4.57}$$

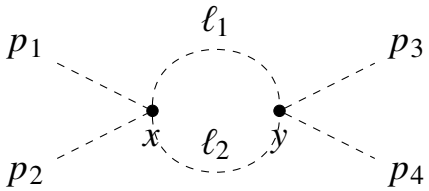
The product $\prod_i[(2\pi)^{3/2}\sqrt{2E_{p_i}}]^{-1}$ is the standard external-leg normalization factor that accompanies every S-matrix element in the Bjorken–Drell convention; it will reappear systematically in the cross-section formula (Chapter 5) and cancel against the flux and phase-space factors.

> **Result:** The vertex factor is $\boxed{-i\lambda}$ and momentum is conserved at the vertex. The 4! from the Lagrangian is exactly cancelled by the 4! ways of contracting four external lines with four fields.

Derivation of the propagator. Consider the second-order contribution to $2 \to 2$ scattering with two vertices at x and y. To generate the connected bubble topology, we must perform (for example) two cross-vertex contractions, i.e. two fields at x contracted with two fields at y:

$$\begin{aligned}&\langle 0|T\{\phi_a(x)\phi_b(x)\phi_c(y)\phi_d(y)\}|0\rangle \\ &\qquad\supset \underbrace{\langle 0|T\{\phi_a(x)\phi_c(y)\}|0\rangle}_{i\Delta_F(x-y)}\;\underbrace{\langle 0|T\{\phi_b(x)\phi_d(y)\}|0\rangle}_{i\Delta_F(x-y)} = \big[i\Delta_F(x-y)\big]^2 .\end{aligned} \tag{4.58}$$

The alternative pairing $c \leftrightarrow d$ gives the same integrand; this exchange is precisely the origin of the $S = 2$ symmetry factor for the bubble.



Each contraction contributes one propagator. In momentum space, the pair of contractions gives:

$$\big[i\Delta_F(x-y)\big]^2 = \prod_{j=1}^{2}\int \frac{d^4\ell_j}{(2\pi)^4}\,\frac{i}{\ell_j^2 - m^2 + i\epsilon}\,e^{-i\ell_j\cdot(x-y)}\,. \tag{4.59}$$

When we integrate over the vertex positions x and y, the exponentials combine with those from external lines to produce momentum-conserving delta functions.

Let us see this explicitly for the s-channel diagram, where the two incoming particles are annihilated at x and the two outgoing ones are created at y, with two internal propagators connecting the vertices.

Suppressing normalization and combinatorial prefactors, the spacetime integrals are:

$$\int d^4x\, d^4y\; \underbrace{e^{-ip_1\cdot x}\, e^{-ip_2\cdot x}}_{\text{incoming at } x}\; \underbrace{e^{+ip_3\cdot y}\, e^{+ip_4\cdot y}}_{\text{outgoing at } y} \prod_{j=1}^{2} \int \frac{d^4\ell_j}{(2\pi)^4} \frac{i}{\ell_j^2 - m^2 + i\epsilon}\, e^{-i\ell_j\cdot(x-y)}\,. \tag{4.60}$$

Each propagator $e^{-i\ell_j\cdot(x-y)}$ contributes a phase $e^{-i\ell_j\cdot x}$ at vertex x and $e^{+i\ell_j\cdot y}$ at vertex y. Collecting all x-dependent and y-dependent phases:

$$\text{integrate over } x: \quad \int d^4x\, e^{-i(p_1+p_2+\ell_1+\ell_2)\cdot x} = (2\pi)^4\, \delta^{(4)}(p_1+p_2+\ell_1+\ell_2)\,, \tag{4.61}$$

$$\text{integrate over } y: \quad \int d^4y\, e^{+i(p_3+p_4+\ell_1+\ell_2)\cdot y} = (2\pi)^4\, \delta^{(4)}(p_3+p_4+\ell_1+\ell_2)\,. \tag{4.62}$$

Each delta function enforces **momentum conservation at its vertex**: the total four-momentum flowing into the vertex equals the total momentum flowing out.

Using (4.61) to eliminate $\ell_2 = -(p_1+p_2) - \ell_1$ and substituting into (4.62):

$$(2\pi)^4\, \delta^{(4)}\big(p_3 + p_4 - (p_1+p_2)\big)\,, \tag{4.63}$$

which is **overall** four-momentum conservation. One loop momentum ($\ell \equiv \ell_1$) remains a free integration variable—this is the loop integral $\int d^4\ell/(2\pi)^4$ in the Feynman rules.

In general, V vertex integrations produce V delta functions, of which one enforces overall momentum conservation and the remaining $V-1$ fix internal momenta, leaving $L = I - V + 1$ independent loop momenta to be integrated over (see eq. (4.64) below).

After all vertex integrations are performed, what remains for each internal line is simply the momentum-space propagator:

Result: The propagator factor is $\boxed{\dfrac{i}{p^2 - m^2 + i\epsilon}}$ where p is determined by momentum conservation.

Loop integrals and momentum conservation. For a diagram with V vertices and I internal lines:

- Each vertex contributes an integral $\int d^4x_i$ and a factor $(-i\lambda/4!)$
- Each internal line contributes a propagator with momentum p_j
- The integrals over vertex positions produce V momentum-conserving delta functions

The number of independent loop momenta is:

$$L = I - V + 1\,. \tag{4.64}$$

(This is Euler's formula for connected graphs.) Each loop momentum requires an integration $\int d^4\ell/(2\pi)^4$.

Symmetry factors. Let us understand where overcounting arises. The Dyson series at order n contains a factor $1/n!$, and each ϕ^4 vertex carries $1/4!$. When we enumerate all Wick contractions we generate every possible Feynman diagram—but some contractions that *look* different at the algebraic level produce *the same* diagram. This happens whenever the diagram possesses an internal symmetry: a permutation of internal lines or vertices that maps the diagram onto itself, leaving all external legs fixed. Each such permutation corresponds to a distinct Wick contraction that gives an identical integral, so the sum over contractions overcounts by a factor equal to the number of these permutations.

More concretely, consider the nth-order term in the Dyson series:

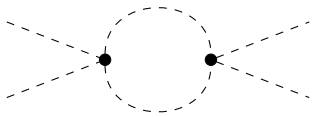
$$S^{(n)} = \frac{1}{n!}\left(\frac{-i\lambda}{4!}\right)^n \int d^4x_1 \cdots d^4x_n\, T\{\phi^4(x_1)\cdots\phi^4(x_n)\}\,. \tag{4.65}$$

The factor $1/n!$ compensates for the $n!$ permutations of the vertex labels $x_1,\ldots,x_n$ (the time-ordered product is symmetric). Similarly, the $(4!)^n$ in the denominator compensates for the $4!$ permutations of the four fields at each vertex. However, if a diagram has an additional symmetry—for instance, two internal lines connecting the same pair of vertices can be swapped, or a loop can be traversed in either direction—then some of those permutations produce identical integrands. The Wick sum counts them as separate contributions, so the result is too large by a factor S.

The **symmetry factor** S is the order of the group of internal automorphisms of the diagram (permutations of internal lines and vertices that leave the diagram and all external legs unchanged). The correct amplitude is:

$$\text{Amplitude} = \frac{1}{S} \times (\text{naive Feynman rules})\,. \tag{4.66}$$

Example 1: For the s-channel bubble diagram in $2 \to 2$ scattering:



The two internal lines can be exchanged $\Rightarrow S = 2$.

Example 2: The "figure-eight" vacuum diagram.

This diagram arises from the vacuum-to-vacuum amplitude at first order:

$$\langle 0|S^{(1)}|0\rangle = \frac{-i\lambda}{4!}\int d^4x\,\langle 0|T\{\phi(x)\phi(x)\phi(x)\phi(x)\}|0\rangle\,. \tag{4.67}$$

Since there are no external particles, all four fields at the vertex must be contracted among themselves. By Wick's theorem (Example 2 in (4.38) with $x_1 = x_2 = x_3 = x_4 = x$), there are 3 complete contractions, each consisting of two propagators that start and end at the same point:

$$\langle 0|T\{\phi^4(x)\}|0\rangle = 3\left[i\Delta_F(0)\right]^2. \tag{4.68}$$

Each factor $i\Delta_F(0) = \overbrace{\phi(x)\phi(x)}$ represents a closed loop—a propagator from x back to x—giving the characteristic figure-eight shape with two loops meeting at a single vertex:

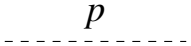
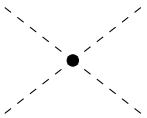
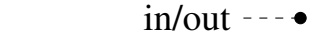
The symmetry factor is $S = 8$: each of the two loops can be traversed in two directions ($2\times 2 = 4$), and the two loops can be exchanged ($\times 2$). Equivalently, $S = 4!/3 = 8$, since the $4!$ from the Lagrangian overcounts by a factor of 3 (the number of Wick contractions) relative to the single diagram.

The full first-order vacuum amplitude is therefore

$$\langle 0|S^{(1)}|0\rangle = \frac{-i\lambda}{4!}\cdot 3\cdot\left[i\Delta_F(0)\right]^2\int d^4x = \frac{-i\lambda}{8}\left[i\Delta_F(0)\right]^2\cdot VT\,, \tag{4.69}$$

where $VT = \int d^4x$ is the spacetime volume. Note that $\Delta_F(0)$ is UV-divergent—this is a first example of a divergence that must be handled by renormalization.

Summary: Feynman rules for $\lambda\phi^4$

Element	Diagram	Factor
Propagator	p 	$\dfrac{i}{p^2 - m^2 + i\epsilon}$
Vertex		$-i\lambda$
External line	in/out 	1

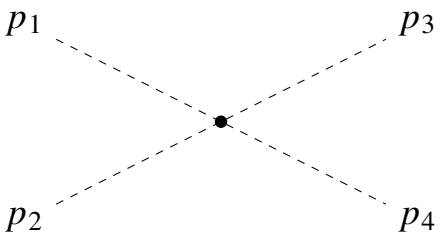
Procedural rules:

1. Draw all topologically distinct connected diagrams
2. Label external momenta p_i (incoming positive) and internal momenta q_j
3. Write $-i\lambda$ for each vertex
4. Write $\dfrac{i}{q_j^2 - m^2 + i\epsilon}$ for each internal line
5. Impose momentum conservation at each vertex: include $(2\pi)^4\delta^{(4)}(\sum p)$
6. Integrate $\displaystyle\int \frac{d^4\ell}{(2\pi)^4}$ over each independent loop momentum
7. Divide by the symmetry factor S
8. External lines contribute factor 1 (for scalars)

Remark on the (2π) *factors.* Each internal line brings $\int d^4q_j/(2\pi)^4$, while each vertex brings $(2\pi)^4\delta^{(4)}(\sum p)$. Of the V delta functions, $V-1$ are used to fix $V-1$ internal momenta, cancelling the corresponding integrals and $(2\pi)^4$ factors. The remaining delta function enforces overall momentum conservation, $(2\pi)^4\delta^{(4)}(p_{\text{in}} - p_{\text{out}})$, and is factored out in the definition of $i\mathcal{M}$. After this reduction, only $L = I - V + 1$ independent loop integrals $\int d^4\ell/(2\pi)^4$ survive, with all (2π) factors exactly accounted for.

Tree-level scattering: $\phi\phi \to \phi\phi$

The simplest scattering process is $2 \to 2$ scattering. At tree level (order λ), there is a single diagram:



The amplitude is simply:[1]

$$\boxed{i\mathcal{M}^{(1)} = -i\lambda}\,. \tag{4.70}$$

This is remarkably simple: the tree-level amplitude is just the coupling constant (times $-i$).

The Mandelstam variables for this process are:

$$s = (p_1 + p_2)^2, \qquad t = (p_1 - p_3)^2, \qquad u = (p_1 - p_4)^2, \tag{4.71}$$

with the constraint $s + t + u = 4m^2$. At tree level, $\mathcal{M}$ is independent of these kinematic variables.

One-loop corrections: systematic calculation

At order λ^2, we have contributions from the second-order term in the Dyson series:

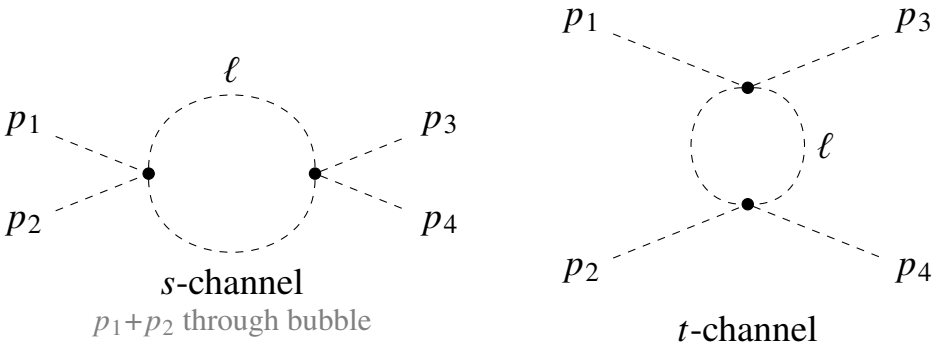
$$S^{(2)} = \frac{(-i)^2}{2!}\left(\frac{\lambda}{4!}\right)^2 \int d^4x\, d^4y\, T\{\phi^4(x)\phi^4(y)\}\,. \tag{4.72}$$

Applying Wick's theorem to $T\{\phi^4(x)\phi^4(y)\}$ generates all possible contractions. For $2 \to 2$ scattering ($p_1, p_2 \to p_3, p_4$), the connected diagrams at one loop are three **bubble diagrams**, distinguished by *which pair of external particles meets at each vertex*. The three possibilities correspond to the three Mandelstam channels:

- **s-channel**: Both incoming particles (p_1, p_2) enter the *same* vertex; both outgoing (p_3, p_4) leave from the other. The total momentum flowing through the internal lines is $p_1 + p_2$, so the bubble depends on $s = (p_1 + p_2)^2$.
- **t-channel**: One incoming and one outgoing particle meet at each vertex—specifically, p_1 and p_3 at one vertex, p_2 and p_4 at the other. The momentum transfer through the bubble is $p_1 - p_3$, so it depends on $t = (p_1 - p_3)^2$.
- **u-channel**: Same topology as the t-channel, but with the outgoing particles swapped: p_1 and p_4 meet at one vertex, p_2 and p_3 at the other. The momentum transfer is $p_1 - p_4$, and the bubble depends on $u = (p_1 - p_4)^2$.

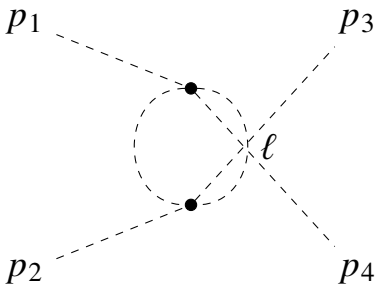
[1] By convention, the connected part of the S-matrix element is written as $\langle f|S|i\rangle = i(2\pi)^4\delta^{(4)}(p_f - p_i)\,\mathcal{M}$, so the Feynman rules directly compute $i\mathcal{M}$. Therefore here $\mathcal{M}^{(1)} = -\lambda$, while the diagram gives $i\mathcal{M}^{(1)} = -i\lambda$.

The diagrams are:



s-channel
p_1+p_2 through bubble

t-channel
p_1-p_3 through bubble



u-channel
p_1-p_4 through bubble

Note the key visual difference: in the *s*-channel, the bubble is *horizontal* (connecting the incoming side to the outgoing side), while in the *t*- and *u*-channels it is *vertical* (connecting top to bottom). The *t*- and *u*-channel diagrams have the same topology—they differ only in which outgoing particle (p_3 or p_4) is paired with p_1. Physically, the *s*-channel probes the *total energy* of the collision, while the *t*- and *u*-channels probe the *momentum transfer* between specific pairs of particles.

The amplitudes are:

$$i\mathcal{M}_s = \frac{(-i\lambda)^2}{2}\int \frac{d^4\ell}{(2\pi)^4}\frac{i}{\ell^2-m^2+i\epsilon}\cdot\frac{i}{(\ell-p_1-p_2)^2-m^2+i\epsilon}, \tag{4.73}$$

$$i\mathcal{M}_t = \frac{(-i\lambda)^2}{2}\int \frac{d^4\ell}{(2\pi)^4}\frac{i}{\ell^2-m^2+i\epsilon}\cdot\frac{i}{(\ell-p_1+p_3)^2-m^2+i\epsilon}, \tag{4.74}$$

$$i\mathcal{M}_u = \frac{(-i\lambda)^2}{2}\int \frac{d^4\ell}{(2\pi)^4}\frac{i}{\ell^2-m^2+i\epsilon}\cdot\frac{i}{(\ell-p_1+p_4)^2-m^2+i\epsilon}. \tag{4.75}$$

In each case, the factor $1/2$ is the **symmetry factor**: the two internal propagators can be exchanged without changing the diagram. Each amplitude is a function of one Mandelstam variable: $\mathcal{M}_s = \mathcal{M}_s(s)$, $\mathcal{M}_t = \mathcal{M}_t(t)$, $\mathcal{M}_u = \mathcal{M}_u(u)$, reflecting which kinematic invariant flows through the bubble.

The total one-loop correction to $2 \to 2$ scattering is:

$$i\mathcal{M}^{(2)}_{2\to2} = i\mathcal{M}_s + i\mathcal{M}_t + i\mathcal{M}_u\,. \tag{4.76}$$

Evaluation of the loop integral

Consider the generic bubble integral:

$$I(p^2) = \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{(\ell^2 - m^2 + i\epsilon)[(\ell - p)^2 - m^2 + i\epsilon]}. \tag{4.77}$$

This integral is **logarithmically divergent**. To see this, note that for large $|\ell|$:

$$I(p^2) \sim \int^{\Lambda} \frac{d^4\ell}{\ell^4} \sim \int^{\Lambda} \frac{\ell^3 d\ell}{\ell^4} = \int^{\Lambda} \frac{d\ell}{\ell} \sim \ln\Lambda. \tag{4.78}$$

Dimensional regularization. The standard modern approach is to analytically continue to $d = 4 - \varepsilon$ dimensions:

$$\int \frac{d^4\ell}{(2\pi)^4} \to \mu^{\varepsilon} \int \frac{d^d\ell}{(2\pi)^d}, \tag{4.79}$$

where μ is an arbitrary mass scale introduced to keep λ dimensionless.

Using Feynman parameters and standard techniques (see Appendix), one finds:

$$I(p^2) = \frac{i}{16\pi^2}\left[\frac{2}{\varepsilon} - \gamma_E + \ln(4\pi) - \int_0^1 dx\, \ln\frac{m^2 - x(1-x)p^2}{\mu^2}\right], \tag{4.80}$$

where $\gamma_E \approx 0.5772$ is the Euler-Mascheroni constant.

The total one-loop amplitude. Each channel contributes a copy of this integral with p^2 replaced by the corresponding Mandelstam variable: $\mathcal{M}_s$ depends on $I(s)$, $\mathcal{M}_t$ on $I(t)$, and $\mathcal{M}_u$ on $I(u)$. Summing all three channels, the total one-loop amplitude (4.76) becomes

$$i\mathcal{M}^{(1\text{-loop})} = -\frac{i\lambda^2}{32\pi^2}\left[\frac{6}{\varepsilon} + \text{finite}(s,t,u)\right], \tag{4.81}$$

where the factor of 6 arises because each of the three channels contributes $2/\varepsilon$, and the finite part is a calculable function of s, t, u (and m^2/μ^2) from the logarithmic integrals in (4.80).

The $6/\varepsilon$ pole means the one-loop amplitude is UV divergent: it blows up as $\varepsilon \to 0$ ($d \to 4$). This divergence must be cancelled by a **coupling counterterm** δ_λ of order λ^2—this is the subject of Chapter 6 (Renormalization). The finite remainder is physical and contains, among other things, the logarithmic dependence on the energy scale that drives the *running* of the coupling constant.

Box 4.3 Summary: $\lambda\phi^4$ Feynman rules

Propagator: $\dfrac{i}{p^2 - m^2 + i\epsilon}$

Vertex: $-i\lambda$

Rules:

- Momentum conservation at each vertex
- $\displaystyle\int \frac{d^4\ell}{(2\pi)^4}$ for each loop
- Divide by symmetry factor S
- No fermion signs (bosonic theory)

Key results:

- Tree-level $2 \to 2$: $\mathcal{M} = -\lambda$
- One-loop integrals contain UV divergences (see Chapter on Renormalization)

4.4 Feynman rules for QED

We now turn to the physically relevant case of quantum electrodynamics. With the sign conventions adopted in these notes (Convention A, see the detailed discussion in Chapter 5), the QED interaction Lagrangian density for the electron field is

$$\mathscr{L}_{\text{int}} = +e\,\bar{\psi}\gamma^{\mu}\psi\,A_{\mu}\,, \tag{4.82}$$

where $e > 0$ is the positron charge (the electron has charge $q_e = -e$) and the coupling constant $\alpha = e^2/(4\pi) \simeq 1/137$ is dimensionless, making QED a renormalizable theory (see Section 6.7). The S-matrix is then

$$S = T\exp\!\left(i\int d^4x\,\mathscr{L}_{\text{int}}(x)\right) = T\exp\!\left(ie\int d^4x\;{:}\bar{\psi}\gamma^{\mu}\psi\,A_{\mu}{:}\right). \tag{4.83}$$

We derive the Feynman rules by applying Wick's theorem order by order, starting from the first-order term—which, as we shall see, gives no physical contribution—and then analysing the second-order term, from which all leading QED processes originate.

First-order contributions: why they vanish

The first-order term in the expansion (4.83) is:

$$S^{(1)} = ie\int d^4x\;{:}\bar{\psi}(x)\gamma^{\mu}\psi(x)\,A_{\mu}(x){:}\;. \tag{4.84}$$

Let us systematically examine all possible matrix elements $\langle f|S^{(1)}|i\rangle$.

The QED vertex structure. The interaction ${:}\bar{\psi}\gamma^{\mu}\psi A_{\mu}{:}$ contains:

- One $\bar{\psi}$ field (can create e^- or annihilate e^+)
- One ψ field (can annihilate e^- or create e^+)
- One A_{μ} field (can create or annihilate γ)

At a single vertex, the interaction ${:}\bar{\psi}\gamma^{\mu}\psi A_{\mu}{:}$ involves exactly three field operators, each of which can create or annihilate a particle. This gives $2^3 = 8$ possible first-order processes.

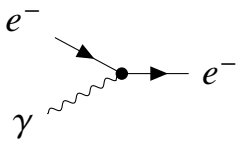
Diagram conventions.

Time direction. In these notes, **time flows from left to right**: initial-state particles enter from the left, and final-state particles exit to the right. This is the most common convention in particle physics textbooks (e.g. Peskin–Schroeder [6], Schwartz [12]). Other conventions exist in the literature: Srednicki [11] draws time flowing upward, while some condensed-matter and older texts use time flowing downward or to the left. The physics is of course independent of the choice.

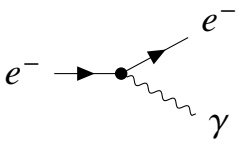
Vertex dots. Each interaction vertex is marked with a **dot** (•) to distinguish it from a mere crossing of lines. The dot represents the spacetime point x over which we integrate in the Dyson series.

Fermion arrows. The arrow on a fermion line follows the flow of *fermion number* (equivalently, the flow of negative charge), **not** the direction of the particle's momentum. For electrons, the arrow points along the direction of propagation (left to right for initial-state electrons, right for final-state); for positrons, the arrow points *opposite* to the direction of propagation. This convention arises naturally from the structure of the interaction: $\bar{\psi}$ creates an e^- or annihilates an e^+ at the arrow's tail, while ψ annihilates an e^- or creates an e^+ at the arrow's head. Thus, an incoming positron is represented by an outgoing arrow, and an outgoing positron by an incoming arrow.

Electron processes:

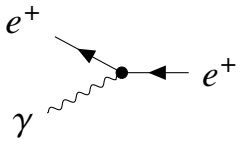


(a) $e^-\gamma \to e^-$

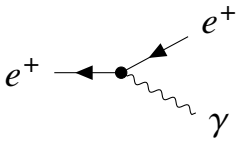


(b) $e^- \to e^-\gamma$

Positron processes:

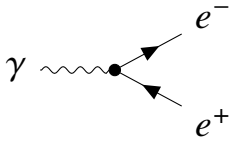


(c) $e^+\gamma \to e^+$

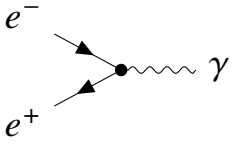


(d) $e^+ \to e^+\gamma$

Pair creation/annihilation:

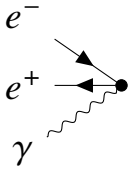


(e) $\gamma \to e^+e^-$

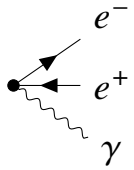


(f) $e^+e^- \to \gamma$

All particles in/out (vacuum transitions):



(g) $e^+e^-\gamma \to 0$



(h) $0 \to e^+e^-\gamma$

Why all first-order processes are forbidden. Each vertex conserves four-momentum. Let us show this explicitly for diagram (a), $e^-(p) + \gamma(k) \to e^-(p')$.

The amplitude is obtained by sandwiching $S^{(1)}$ between the initial and final states. The mode expansions (4.4)–(4.5) of the fields ψ, $\bar{\psi}$, and A_μ assign a plane-wave factor $e^{\pm iq\cdot x}$ to each external particle at the vertex located at x:

- the incoming electron (ψ annihilates it): factor $e^{-ip\cdot x}$,
- the incoming photon (A_μ annihilates it): factor $e^{-ik\cdot x}$,
- the outgoing electron ($\bar{\psi}$ creates it): factor $e^{+ip'\cdot x}$.

The amplitude therefore contains the spacetime integral:

$$\int d^4x\, e^{-ip\cdot x}\, e^{-ik\cdot x}\, e^{+ip'\cdot x} = \int d^4x\, e^{i(p'-p-k)\cdot x}\,. \tag{4.85}$$

We now use the integral representation of the Dirac delta:

$$\int d^4x\, e^{iq\cdot x} = (2\pi)^4\,\delta^{(4)}(q)\,, \tag{4.86}$$

which holds because each of the four components gives $\int dx^\mu\, e^{iq_\mu x^\mu} = 2\pi\,\delta(q_\mu)$. Applying this with $q = p' - p - k$:

$$\int d^4x\, e^{i(p'-p-k)\cdot x} = (2\pi)^4\,\delta^{(4)}(p'-p-k)\,, \tag{4.87}$$

which enforces $p + k = p'$, i.e. four-momentum conservation at the vertex. The same mechanism operates at every vertex and for every process: the integral over each vertex position produces a momentum-conserving delta function.

Consider diagram (a): $e^-(p) + \gamma(k) \to e^-(p')$. Momentum conservation requires:

$$p + k = p'\,. \tag{4.88}$$

But for on-shell particles: $p^2 = p'^2 = m_e^2$ and $k^2 = 0$. Then:

$$p'^2 = (p+k)^2 = p^2 + 2p\cdot k + k^2 = m_e^2 + 2p\cdot k\,. \tag{4.89}$$

This equals m_e^2 only if $p\cdot k = 0$. But for physical photons, $p\cdot k = E_e E_\gamma - |\vec{p}||\vec{k}|\cos\theta = E_e E_\gamma(1 - \beta_e\cos\theta) > 0$ (where $\beta_e = |\vec{p}|/E_e$) unless $E_\gamma = 0$.

Conclusion: All first-order QED processes violate energy-momentum conservation for on-shell external particles. Therefore:

$$\langle f|S^{(1)}|i\rangle = 0 \quad \text{for all physical processes}\,. \tag{4.90}$$

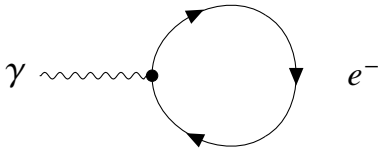
The leading non-trivial contributions come from $S^{(2)}$.

Vacuum diagrams at first order. One might consider the vacuum-to-vacuum amplitude $\langle 0|S^{(1)}|0\rangle$. However:

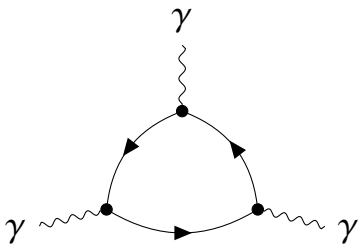
$$\langle 0| :\bar{\psi}\gamma^\mu\psi A_\mu: |0\rangle = 0\,, \tag{4.91}$$

because normal-ordered products have zero vacuum expectation value. There are no first-order vacuum diagrams in QED.

Tadpole diagrams and Furry's theorem. At higher orders, one might consider *tadpole diagrams*: a closed fermion loop attached to a single external photon line.



Tadpole



3-photon vertex (also zero)

This diagram corresponds to a one-point function $\langle 0|j^\mu(x)|0\rangle$ where $j^\mu = \bar{\psi}\gamma^\mu\psi$ is the electromagnetic current. Such diagrams **vanish identically in QED** due to *Furry's theorem*:

> **Furry's theorem:** Any Feynman diagram with a closed fermion loop and an *odd* number of photon vertices vanishes.

Proof. QED is invariant under **charge conjugation** C, the discrete symmetry that exchanges particles with antiparticles. On the fields, it acts as:

$$\psi \xrightarrow{C} C\bar{\psi}^\top\,, \qquad \bar{\psi} \xrightarrow{C} -\psi^\top C^{-1}\,, \qquad A_\mu \xrightarrow{C} -A_\mu\,, \tag{4.92}$$

where $C = i\gamma^0\gamma^2$ is the **charge conjugation matrix**, satisfying the defining property $C^{-1}\gamma^\mu C = -(\gamma^\mu)^\top$ (see eq. (9.153)). The photon field changes sign because the electromagnetic current j^μ changes sign when all charges are reversed.

One can verify that the QED interaction $\mathscr{L}_{\text{int}} = e\bar{\psi}\gamma^\mu\psi A_\mu$ is invariant: the current picks up a minus sign (shown below) and so does A_μ, giving an overall $(-1)^2 = +1$.

Under C, the electromagnetic current transforms as:

$$j^\mu = \bar{\psi}\gamma^\mu\psi \xrightarrow{C} -\bar{\psi}\gamma^\mu\psi = -j^\mu\,. \tag{4.93}$$

Since the vacuum is C-invariant: $C|0\rangle = |0\rangle$, we have:

$$\langle 0|j^\mu|0\rangle = \langle 0|C^\dagger j^\mu C|0\rangle = -\langle 0|j^\mu|0\rangle = 0\,. \tag{4.94}$$

More generally, a fermion loop with n photon insertions contributes a factor $(-1)^n$ under charge conjugation. For odd n, this gives a sign flip, forcing the amplitude to vanish.

Consequence: Tadpole diagrams, photon three-point vertices, and all diagrams with an odd number of photons attached to a single fermion loop are zero in QED. This significantly simplifies higher-order calculations.

4.5 Second-order contributions: complete derivation

The second-order term is:

$$S^{(2)} = \frac{(ie)^2}{2!} \int d^4x\, d^4y\, T\{:\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x)::\bar{\psi}(y)\gamma^\nu\psi(y)A_\nu(y):\}\,. \quad (4.95)$$

We must evaluate $\langle f|S^{(2)}|i\rangle$ for various initial and final states using Wick's theorem.

Field content at second order. The operator $:\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x):$ $:\bar{\psi}(y)\gamma^\nu\psi(y)A_\nu(y):$ contains:

- Two $\bar{\psi}$ fields: $\bar{\psi}(x)$ and $\bar{\psi}(y)$
- Two ψ fields: $\psi(x)$ and $\psi(y)$
- Two A_μ fields: $A_\mu(x)$ and $A_\nu(y)$

Complete Wick expansion. Applying Wick's theorem to the time-ordered product, we obtain all possible contractions. Using the shorthand notation:

$$\bar{\psi}_x \equiv \bar{\psi}(x), \quad \psi_x \equiv \psi(x), \quad A_x \equiv A_\mu(x), \quad \text{etc.}, \quad (4.96)$$

The complete expansion is (omitting γ^μ, γ^ν for clarity):

$$T\{:\bar\psi_x\psi_x A_x::\bar\psi_y\psi_y A_y:\} =$$

(0) No contractions:

$$:\bar\psi_x\psi_x A_x\bar\psi_y\psi_y A_y:$$

(1) Single contractions:

$$\overbracket{A_x A_y}\cdot :\bar\psi_x\psi_x\bar\psi_y\psi_y:$$
$$\overbracket{\psi_x\bar\psi_y}\cdot :\bar\psi_x A_x\psi_y A_y:$$
$$\overbracket{\psi_y\bar\psi_x}\cdot :\psi_x A_x\bar\psi_y A_y:$$

(2) Double contractions:

$$\overbracket{A_x A_y}\cdot\overbracket{\psi_x\bar\psi_y}\cdot :\bar\psi_x\psi_y:$$
$$\overbracket{A_x A_y}\cdot\overbracket{\psi_y\bar\psi_x}\cdot :\psi_x\bar\psi_y:$$
$$\overbracket{\psi_x\bar\psi_y}\cdot\overbracket{\psi_y\bar\psi_x}\cdot :A_x A_y:$$

(3) Triple contractions:

$$\overbracket{A_x A_y}\cdot\overbracket{\psi_x\bar\psi_y}\cdot\overbracket{\psi_y\bar\psi_x}\quad\text{(fully contracted)}. \tag{4.97}$$

Physical interpretation of each term:

- **(0) No contractions**: No fields are contracted between the two vertices x and y. Each vertex then acts independently, producing what amounts to two *first-order* processes happening separately. The result is a **disconnected diagram**—the product of two independent first-order diagrams. Since each first-order process is forbidden by energy-momentum conservation (Section 4.4), this contribution vanishes. More generally, disconnected diagrams factorise and are absorbed into the vacuum-to-vacuum amplitude; they do not contribute to scattering amplitudes (this is formalised by the linked-cluster theorem)
- **(1a) Photon contraction only**: $\overbracket{A_x A_y}$ → internal photon, 4 external fermions (e.g., $e^+e^-\to\mu^+\mu^-$, Bhabha, Møller)
- **(1b,c) Single fermion contraction**: $\overbracket{\psi\bar\psi}$ → internal fermion, 2 external photons + 2 external fermions (Compton)
- **(2a,b) Photon + fermion**: → internal photon + fermion, 2 external fermions (self-energy correction)
- **(2c) Two fermion contractions**: $\overbracket{\psi_x\bar\psi_y}\cdot\overbracket{\psi_y\bar\psi_x}$ → fermion loop, 2 external photons (vacuum polarization)
- **(3) Fully contracted**: → vacuum bubble (no external lines, contributes to $\langle 0|S|0\rangle$)

Wick's theorem tells us to sum over all contractions. Different contraction patterns give different physical processes.

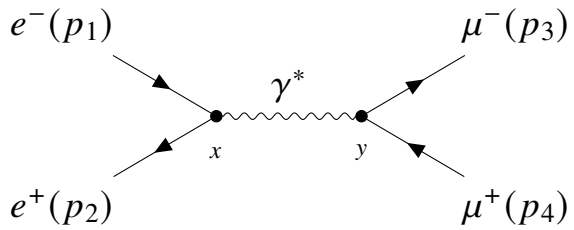
Classification of second-order diagrams

Case 1: Contract both photon fields (internal photon line).

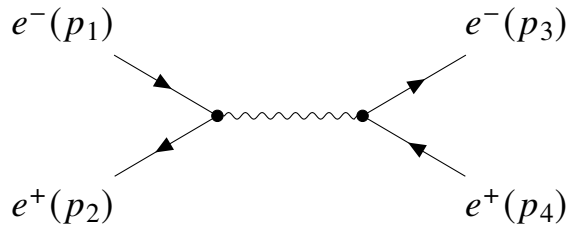
$$\overbrace{A_\mu(x)A_\nu(y)} = iD_F^{\mu\nu}(x-y)\,. \tag{4.98}$$

The remaining fermion fields $\bar\psi(x),\psi(x),\bar\psi(y),\psi(y)$ connect to external states.

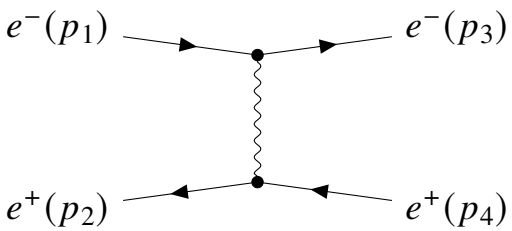
This gives processes with **two fermion lines connected by a photon**:



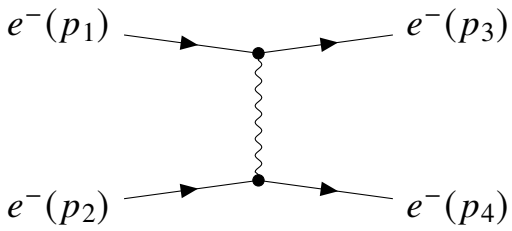
$e^+e^- \to \mu^+\mu^-$ (s-channel)



Bhabha s-channel



Bhabha t-channel

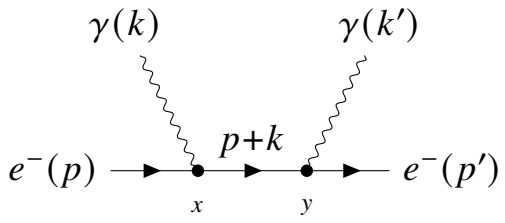


Møller scattering

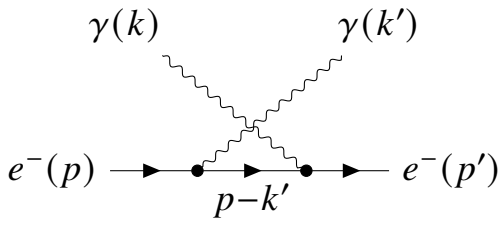
Case 2: Contract one fermion pair (internal fermion line).

$$\overbrace{\psi(x)\bar\psi(y)} = iS_F(x-y) \quad \text{or} \quad \overbrace{\psi(y)\bar\psi(x)} = iS_F(y-x)\,. \tag{4.99}$$

The remaining fields connect to external states, including two external photons.



Compton (s-channel)

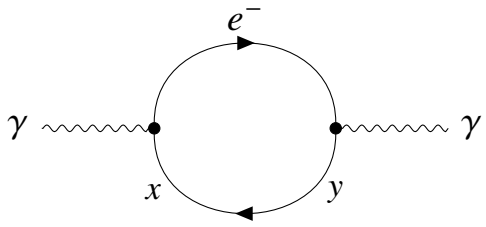


Compton (u-channel)

Case 3: Contract both fermion pairs (vacuum polarization).

$$\overbrace{\psi(x)\bar{\psi}}(y)\cdot\overbrace{\psi(y)\bar{\psi}}(x) = iS_F(x-y)\cdot iS_F(y-x)\,. \tag{4.100}$$

This creates a closed fermion loop with two external photons:



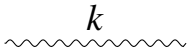
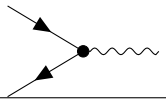
Vacuum polarization (photon self-energy)

Case 4: No contractions between different vertices. If we don't contract fields between x and y, we get disconnected diagrams or vacuum fluctuations. These contribute to the overall phase but not to scattering amplitudes.

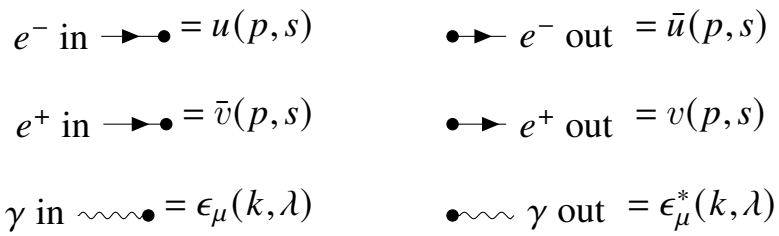
4.6 Feynman rules in momentum space

The calculation above illustrates the general pattern. We now summarize the Feynman rules.

Graphical elements

Element	Diagram	Factor
Fermion propagator		$\dfrac{i(\not{p}+m)}{p^2-m^2+i\epsilon}$
Photon propagator		$\dfrac{-ig_{\mu\nu}}{k^2+i\epsilon}$
QED vertex		$+ie\gamma^\mu$

External line factors



Derivation of external line factors

These factors arise from contracting field operators with external states.

Incoming electron. The state $|e^-(p,s)\rangle = b_s^\dagger(\vec{p})|0\rangle$. The field $\psi(x)$ acting on this state, using the BD field expansion (eq. (**??**)) and the anticommutator $\{b_{s'}(\vec{k}), b_s^\dagger(\vec{p})\} = \delta^{(3)}(\vec{k}-\vec{p})\delta_{s's}$:

$$\begin{aligned}\psi(x)|e^-(p,s)\rangle &= \int \frac{d^3k}{(2\pi)^{3/2}}\sqrt{\frac{m}{E_k}}\sum_{s'} b_{s'}(\vec{k})u_{s'}(k)e^{-ik\cdot x}\cdot b_s^\dagger(\vec{p})|0\rangle \\ &= \underbrace{\frac{1}{(2\pi)^{3/2}}\sqrt{\frac{m}{E_p}}}_{\text{BD wavefunction norm.}} u_s(p)\,e^{-ip\cdot x}|0\rangle\,. \end{aligned} \tag{4.101}$$

Stripping the BD normalisation. The prefactor $(2\pi)^{-3/2}\sqrt{m/E_p}$ is the Bjorken–Drell wavefunction normalisation. By convention, when writing the Feynman rule for the external line this factor is *stripped off*: it is absorbed into the conversion from the box-normalised matrix element $\mathcal{M}_{\text{box}}$ to the stripped BD amplitude $\mathcal{M}_{\text{BD}}$ that appears in the cross-section master formula (cf. eq. (3.38) and the discussion in Chapter 3). The Feynman rule for an incoming electron is then simply $u(p,s)$. The position integral $\int d^4x\, e^{-ip\cdot x}$ at the vertex produces the momentum-conserving

delta function. What remains as the external-line factor is therefore $u(p,s)$, with the understanding that the BD wavefunction normalisation has been moved into the box-to-stripped conversion. The same convention is applied to all four external-fermion factors below.

Outgoing electron. The state $\langle e^-(p',s')| = \langle 0|b_{s'}(\vec{p}')$. Acting with $\bar{\psi}(x)$:

$$\langle e^-(p',s')|\bar{\psi}(x) = \langle 0|\bar{u}_{s'}(p')e^{+ip'\cdot x}\,. \tag{4.102}$$

Factor: $\bar{u}(p',s')$.

Incoming positron. $|e^+(p,s)\rangle = d_s^\dagger(\vec{p})|0\rangle$. The field $\bar{\psi}(x)$ annihilates positrons when acting on kets:

$$\bar{\psi}(x)|e^+(p,s)\rangle = \bar{v}_s(p)e^{+ip\cdot x}|0\rangle\,. \tag{4.103}$$

Factor: $\bar{v}(p,s)$.

Outgoing positron. $\langle e^+(p',s')| = \langle 0|d_{s'}(\vec{p}')$:

$$\langle e^+(p',s')|\psi(x) = \langle 0|v_{s'}(p')e^{-ip'\cdot x}\,. \tag{4.104}$$

Factor: $v(p',s')$.

Additional rules

1. **Momentum conservation at each vertex.**
   Origin: Each vertex carries a spacetime integral $\int d^4x$. When fields are replaced by their Fourier representations, the position integral produces:

   $$\int d^4x\, e^{i(p_1+p_2+p_3)\cdot x} = (2\pi)^4\delta^{(4)}(p_1+p_2+p_3)\,, \tag{4.105}$$

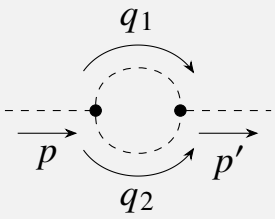
   where all momenta are defined as *incoming*. This delta function enforces momentum conservation at every vertex.
   Rule: Assign a factor $(2\pi)^4\delta^{(4)}(\sum_i p_i^{\text{in}})$ to each vertex, with the sign convention that momenta flowing into the vertex are positive.
2. **Loop integration.**
   Origin: After imposing momentum conservation at all vertices, there remain undetermined momenta circulating in closed loops. These arise because the diagram has more internal lines than independent momentum constraints.
   For a diagram with I internal lines and V vertices, the number of independent loop momenta is:

   $$L = I - V + 1\,. \tag{4.106}$$

Box 4.4 Why $L = I - V + 1$

Idea. Each internal line carries a momentum we must determine. Each vertex fixes one relation among these momenta (conservation). So: L = (unknowns) – (independent equations).

Concrete example. Consider the s-channel bubble in $2 \to 2$ scattering ($V = 2$, $I = 2$):



There are $I = 2$ unknown internal momenta: q_1 and q_2. Vertex 1 gives $p = q_1 + q_2$, vertex 2 gives $q_1 + q_2 = p'$. But these two equations are not independent: the second is just "$p = p'$" once you substitute the first. So there is only $V - 1 = 1$ independent equation, leaving $L = 2 - 1 = 1$ free momentum. We can choose $q_1 = \ell$ (the loop momentum), and then $q_2 = p - \ell$ is fixed.

General argument.

(a) *Unknowns*: I internal momenta.
(b) *Equations*: V vertices, each imposing 4-momentum conservation.
(c) *Redundancy*: one equation is always redundant, because if we *add* the conservation laws of all V vertices, every internal momentum cancels (it enters one vertex and exits another with opposite sign). What remains is just $\sum p_{\text{ext,in}} = \sum p_{\text{ext,out}}$—overall conservation of external momenta, which tells us nothing about the internal ones.
(d) *Result*: $L = I - (V - 1) = I - V + 1$.

Examples.

- *Tree diagrams*: $I = V - 1 \Rightarrow L = 0$. All internal momenta are fixed—no integrals.
- *Figure-eight*: $V = 1, I = 2 \Rightarrow L = 2$. Two independent loop momenta.

Rule: For each independent loop momentum ℓ, include an integration:

$$\int \frac{d^4\ell}{(2\pi)^4}\,. \tag{4.107}$$

The factor $(2\pi)^{-4}$ matches the $(2\pi)^4$ from the momentum-conserving delta functions.

3. **Fermion line ordering.**

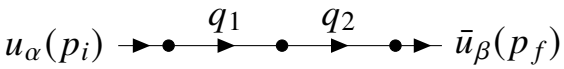
Origin: Matrix multiplication of spinor indices. Consider a generic S-matrix element for a fermion scattering through three vertices. From the Dyson series we have schematically:

$$\langle p_f|S|p_i\rangle \sim \langle 0|\, b_{p_f}\, \bar\psi(x_3)\,\gamma\, \underbrace{\psi(x_3)\bar\psi(x_2)}_{\text{contract}\to S_F(q_2)}\, \gamma\, \underbrace{\psi(x_2)\bar\psi(x_1)}_{\text{contract}\to S_F(q_1)}\, \gamma\psi(x_1)\, b^\dagger_{p_i}|0\rangle\,. \tag{4.108}$$

Each object carries spinor indices ($\alpha,\beta,\cdots = 1,\ldots,4$). Making them explicit:

- $b^\dagger_{p_i}|0\rangle$ produces $u_\alpha(p_i)$ from $\psi_\alpha(x_1)$;
- the vertex at x_1 contributes $(\gamma^{\mu_1})_{\zeta\alpha}$, with α contracted with u_α;
- the Wick contraction $\overbrace{\psi_\epsilon(x_2)\bar\psi_\zeta}(x_1) = S_{F,\epsilon\zeta}(q_1)$, with ζ contracted with the vertex $\gamma_{\zeta\alpha}$;
- the vertex at x_2 contributes $(\gamma^{\mu_2})_{\delta\epsilon}$, with ϵ contracted with $S_{F,\epsilon\zeta}$;
- the contraction $\overbrace{\psi_\gamma(x_3)\bar\psi_\delta}(x_2) = S_{F,\gamma\delta}(q_2)$;
- the vertex at x_3 contributes $(\gamma^{\mu_3})_{\beta\gamma}$;
- $\langle 0|b_{p_f}$ produces $\bar u_\beta(p_f)$ from $\bar\psi_\beta(x_3)$.

Assembling all factors and summing over repeated indices, we obtain for the fermion line:



$$\bar u_\beta(p_f)\cdot(\gamma)_{\beta\gamma}\cdot S_{\gamma\delta}(q_2)\cdot(\gamma)_{\delta\epsilon}\cdot S_{\epsilon\zeta}(q_1)\cdot(\gamma)_{\zeta\alpha}\cdot u_\alpha(p_i)\,. \tag{4.109}$$

This is simply matrix multiplication: each repeated index is summed, and the result is a scalar in spinor space.
Rule: Read spinor factors along the fermion line *opposite* to the arrow direction:

$$\bar u(p_f)\,\gamma^{\mu_3}\,S_F(q_2)\,\gamma^{\mu_2}\,S_F(q_1)\,\gamma^{\mu_1}\,u(p_i)\,. \tag{4.110}$$

4. **Closed fermion loop.**
Origin: A closed loop involves a contraction pattern like:

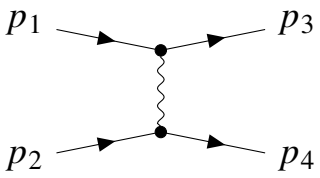
$$\overbrace{\psi_\alpha(x_1)\bar\psi_\beta}(x_2)\,\overbrace{\psi_\beta(x_2)\bar\psi_\gamma}(x_3)\cdots\overbrace{\psi_\omega(x_n)\bar\psi_\alpha}(x_1)\,. \tag{4.111}$$

The spinor indices form a closed chain: $\alpha\to\beta\to\gamma\to\cdots\to\omega\to\alpha$.
This closed chain of matrix multiplications yields a *trace* over spinor indices:

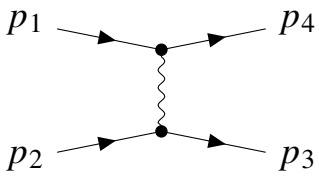
$$\mathrm{Tr}\big[\gamma^{\mu_1}S_F(x_1-x_2)\gamma^{\mu_2}S_F(x_2-x_3)\cdots\gamma^{\mu_n}S_F(x_n-x_1)\big]\,. \tag{4.112}$$

The factor of (-1) arises from anticommuting fermionic fields to form the closed contraction pattern (see Section 4.7).
Rule: For each closed fermion loop, include a factor (-1) and take the trace over Dirac indices.

5. **Relative signs between diagrams.**
Origin: When computing $\langle f|S|i\rangle$, the external fermion operators must be contracted with the field operators. Different diagrams may require different orderings of these contractions.
Consider $e^-e^- \to e^-e^-$ (Møller scattering). The initial state $|i\rangle = b^\dagger(p_1)\, b^\dagger(p_2)|0\rangle$ involves two identical fermions, and we contract with $b(p_3)\, b(p_4)$ in the final state. Two diagrams exist:



Direct: $1 \to 3,\ 2 \to 4$



Exchange: $1 \to 4,\ 2 \to 3$

The exchange diagram requires permuting the final-state labels: $(p_3, p_4) \to (p_4, p_3)$. Since $b(p_3)b(p_4) = -b(p_4)b(p_3)$, the two diagrams have opposite signs.
Rule: Diagrams related by exchange of two identical external fermions differ by a factor (-1).

6. **Symmetry factor.**
Origin: The factor $1/n!$ in $S^{(n)}$ is cancelled by the $n!$ ways of assigning the interaction vertices to the spacetime integration points $x_1, \ldots, x_n$. To see how this works explicitly, consider Compton scattering at second order ($n = 2$). The Dyson series gives:

$$S^{(2)} = \frac{(-i)^2}{2!} \int d^4x_1\, d^4x_2\, T\{H_I(x_1)\, H_I(x_2)\}\,. \tag{4.113}$$

Applying Wick's theorem to the time-ordered product, each Feynman diagram (say, the s-channel) arises from *two* distinct contractions:

- *Contraction A:* the incoming photon is absorbed at x_1, the outgoing photon is emitted at x_2;
- *Contraction B:* the incoming photon is absorbed at x_2, the outgoing photon is emitted at x_1.

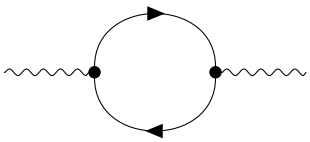
Since x_1 and x_2 are dummy integration variables, both contractions yield the *same* integral. The factor of 2 from these two equivalent contractions cancels the $1/2!$ from the Dyson series:

$$\frac{1}{2!} \times 2 = 1\,. \tag{4.114}$$

The same cancellation occurs for the u-channel diagram. This is a general feature: at order n, the $n!$ permutations of the vertices $x_1,\ldots,x_n$ produce $n!$ identical Wick contractions for each diagram topology, exactly cancelling the $1/n!$ prefactor. This is *why the Feynman rules do not include the* $1/n!$ *factor* from the Dyson series.

However, if the diagram has internal symmetries (automorphisms), some of these $n!$ assignments produce identical contributions even *before* relabelling, so the cancellation is only partial.

Example: Vacuum polarization has a symmetry under exchanging the two fermion propagators:



The two internal fermion lines can be exchanged without changing the diagram $\Rightarrow S = 1$ (the exchange is equivalent to $x \leftrightarrow y$, already accounted for).

General rule: S equals the number of permutations of internal lines and vertices that leave the diagram unchanged (its automorphism group order). For most tree-level QED diagrams, $S = 1$.

4.7 The fermion minus sign: detailed proof

Origin. The minus signs arise from the anticommutation of fermionic field operators in Wick's theorem.

Closed loop. Consider a fermion loop connecting n vertices at $x_1, x_2, \ldots, x_n$. The corresponding Feynman-diagram expression is:

$$(-1)\,\mathrm{Tr}\left[iS_F(x_1-x_2)\gamma^{\mu_2}\cdot iS_F(x_2-x_3)\gamma^{\mu_3}\cdots iS_F(x_n-x_1)\gamma^{\mu_1}\right], \tag{4.115}$$

where the overall factor of (-1) is the fermion loop sign we wish to derive. The trace arises because the spinor indices form a closed chain.

Box 4.5 Proof of the (-1) sign for a closed fermion loop

Warm-up: $n = 2$ (vacuum polarization). The relevant piece of the Wick expansion is (suppressing γ matrices):

$$\bar{\psi}_1\psi_1\bar{\psi}_2\psi_2\,,$$

where the subscripts label vertices. To form the loop, we need the contractions $\overbracket{\psi_1\bar{\psi}_2}$ and $\overbracket{\psi_2\bar{\psi}_1}$. Starting from the natural (normal-ordering) sequence

$\bar\psi_1\psi_1\bar\psi_2\psi_2$, we must bring ψ_1 next to $\bar\psi_2$, and ψ_2 next to $\bar\psi_1$. In detail:

$$\begin{aligned}&\bar\psi_1\psi_1\bar\psi_2\psi_2\\&= \bar\psi_1\underbrace{\psi_1\bar\psi_2}_{\text{swap}}\psi_2 = -\bar\psi_1\bar\psi_2\psi_1\psi_2 \qquad (1\text{ swap}\to -1)\end{aligned}$$

Now contract: $\overbrace{\psi_1\bar\psi_2}$ is already adjacent (reading $\bar\psi_2\psi_1$ from right to left gives $\psi_1\bar\psi_2$), and $\overbrace{\psi_2\bar\psi_1}$ closes the loop. The single swap gives one factor of (-1). Compare with an open fermion line, where we would contract $\overbrace{\psi_1\bar\psi_2}$ only, leaving $\bar\psi_1$ and ψ_2 as external legs—no rearrangement needed, no sign.

General case: n vertices. For an open fermion line through n vertices, the contractions are:

$$\overbrace{\psi_1\bar\psi_2}\,\overbrace{\psi_2\bar\psi_3}\cdots\overbrace{\psi_{n-1}\bar\psi_n},$$

with $\bar\psi_1$ and ψ_n connecting to external states. Call the sign required to bring the fields into this contracted order σ_{open}.

For a *closed* loop, we need the additional contraction $\overbrace{\psi_n\bar\psi_1}$: the field $\bar\psi_1$ must now also be contracted with ψ_n. Starting from the open-line arrangement, $\bar\psi_1$ sits at the far left. To bring it next to ψ_n, we must move it past the $n-1$ other fermionic fields ($\psi_1,\bar\psi_2,\psi_2,\ldots$ are already paired, but $\bar\psi_1$ must anticommute past each contracted pair). A careful counting shows that closing the loop always requires an *odd* number of additional transpositions relative to the open line, giving one extra factor of (-1):

$$\sigma_{\text{loop}} = (-1)\times\sigma_{\text{open}}\,.$$

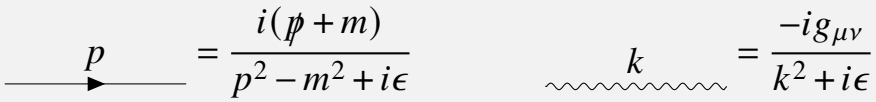
Since σ_{open} is already absorbed into the standard Feynman rules for open fermion lines, the only new ingredient for a closed loop is the additional factor of $\boxed{(-1)}$.

Exchange diagrams. For processes involving identical external fermions, diagrams that differ by exchanging two such external fermion lines carry a relative minus sign. This is the situation in Møller scattering discussed above. In Bhabha scattering, the relative sign between the s- and t-channel contributions is obtained by keeping a consistent ordering of the external fermionic operators in the Wick/LSZ reduction.

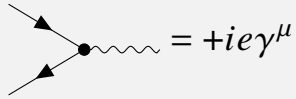
4.8 Summary: QED Feynman rules

Box 4.6 Complete QED Feynman rules (momentum space)

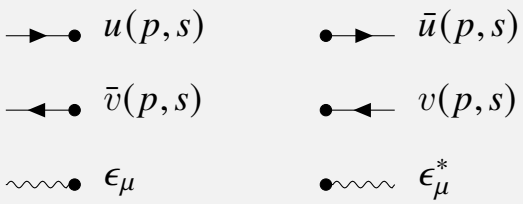
Propagators:



Vertex:



External lines:



Rules:

- Conserve momentum at each vertex
- Integrate $\int \frac{d^4\ell}{(2\pi)^4}$ over each loop
- Factor (-1) for each closed fermion loop
- Relative (-1) for fermion exchange
- Divide by symmetry factor S

Part II

Quantum Electrodynamics

5 QED Scattering Processes

In this chapter, we apply the Feynman rules of quantum electrodynamics to compute cross sections for fundamental scattering processes. These calculations illustrate the power and elegance of the diagrammatic approach developed by Feynman, and provide quantitative predictions that have been tested to extraordinary precision.

5.1 Overview of QED processes

The QED interaction vertex

The fundamental interaction in QED is the coupling of a photon to a charged fermion.

Charge conventions. We adopt the following conventions throughout these notes:

- The symbol e denotes the **positron charge**, a positive quantity: $e > 0$. Numerically, $e \simeq 0.303$ in natural units, or equivalently $e = \sqrt{4\pi\alpha}$ with $\alpha \simeq 1/137$.
- The **electron** carries charge $-e$ and the **positron** carries charge $+e$.
- Similarly, μ^- has charge $-e$ and μ^+ has charge $+e$.

The QED Lagrangian. For a Dirac fermion field ψ with electric charge q, the QED Lagrangian is:

$$\mathscr{L}_{\text{QED}} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{5.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength tensor and D_μ is the gauge-covariant derivative.

Box 5.1 Sign conventions in QED: a systematic treatment

There are **two independent convention choices** that are often confused:

1. Whether the symbol e denotes the **positive magnitude** ($e > 0$) or the **signed** electron charge ($e < 0$).
2. Whether the covariant derivative is written with $+iq$ or $-iq$ in front of the gauge field.

The fundamental starting point: interaction energy. For a static point charge q in an external electrostatic potential $\phi(\vec{x})$, the potential energy is:

$$V = q\,\phi(\vec{x})\,. \tag{5.2}$$

Since $L = T - V$, the interaction Lagrangian is $L_{\text{int}} = -V = -q\phi$. In field theory with the Dirac current $j^\mu = \bar{\psi}\gamma^\mu\psi$ (the probability/number current, *without* the charge factor), the covariant generalization is:

$$\boxed{\mathscr{L}_{\text{int}} = -q\,j^\mu A_\mu = -q\,\bar{\psi}\gamma^\mu\psi\,A_\mu}\,. \tag{5.3}$$

Mnemonic: *Potential energy is $+q\phi$, so Lagrangian interaction is $-q\phi$.* This pins down $\mathscr{L}_{\text{int}} = -q\,j^\mu A_\mu$.

Covariant derivative conventions. Two common conventions exist for implementing gauge invariance:

Convention A (used in these notes, consistent with energy argument):

$$\boxed{D_\mu = \partial_\mu + iq\,A_\mu}\,, \tag{5.4}$$

with gauge transformations $\psi \to e^{-iq\alpha(x)}\psi$ and $A_\mu \to A_\mu + \partial_\mu\alpha$.
Expanding the Dirac Lagrangian:

$$\begin{aligned}\bar{\psi}(i\gamma^\mu D_\mu - m)\psi &= \bar{\psi}(i\partial\!\!\!/ - m)\psi + \bar{\psi}(i\gamma^\mu \cdot iqA_\mu)\psi \\ &= \bar{\psi}(i\partial\!\!\!/ - m)\psi - q\,\bar{\psi}\gamma^\mu\psi\,A_\mu\,.\end{aligned} \tag{5.5}$$

Hence:

$$\boxed{\mathscr{L}_{\text{int}} = -q\,\bar{\psi}\gamma^\mu\psi\,A_\mu \qquad \text{(Convention A)}}\,. \tag{5.6}$$

This matches exactly the energy-based derivation eq. (5.3).

Convention B (used in some texts):

$$D_\mu = \partial_\mu - iq\,A_\mu\,. \tag{5.7}$$

This gives $\mathscr{L}_{\text{int}} = +q\,\bar{\psi}\gamma^\mu\psi\,A_\mu$, with opposite sign.
Key point: The sign is a convention; physics is unchanged. If you switch from A to B, you can compensate by $q \to -q$ or $A_\mu \to -A_\mu$.
Our convention: electron charge $q_e = -e$ with $e > 0$. We define:

- $e > 0$ is the **magnitude** of the electron charge (the positron charge)
- The **electron** has charge $q_e = -e < 0$

- The **positron** has charge $q_{e^+} = +e > 0$

With Convention A and $q = -e$ for the electron:

$$D_\mu = \partial_\mu + i(-e)A_\mu = \partial_\mu - ie\,A_\mu\,, \tag{5.8}$$

$$\boxed{\mathscr{L}_{\text{int}} = +e\bar{\psi}\gamma^\mu\psi\,A_\mu}\,, \qquad e > 0, \quad q_e = -e\,. \tag{5.9}$$

From Lagrangian to Feynman rule. The S-matrix expansion is:

$$S = T\exp\left(i\int d^4x\,\mathscr{L}_{\text{int}}(x)\right). \tag{5.10}$$

The vertex factor is obtained by: *take i times the coefficient of $A_\mu\bar{\psi}\psi$ in $\mathscr{L}_{int}$, attach γ^μ.*
From eq. (5.6) with charge q:

$$\boxed{\text{vertex} = i\times(-q)\times\gamma^\mu = -iq\,\gamma^\mu}\,. \tag{5.11}$$

For the **electron** ($q = -e$):

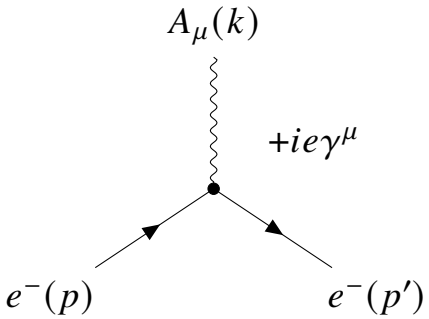
$$\boxed{\text{vertex} = -i(-e)\gamma^\mu = +ie\,\gamma^\mu \qquad \text{(our convention)}}\,. \tag{5.12}$$

Memory rule: Vertex $= i\times$ (coefficient of $A_\mu\bar{\psi}\psi$ in $\mathscr{L}_{\text{int}}$) $\times\gamma^\mu$.
Comparison with other textbooks. Different books package the same physics differently. The key is: *what is $\mathscr{L}_{int}$ and hence the vertex?*

Text	D_μ	$\mathscr{L}_{\text{int}}$	Vertex
These notes (with $q_e = -e$)	$\partial_\mu + iqA_\mu$ $= \partial_\mu - ieA_\mu$	$-q\bar{\psi}\gamma^\mu\psi A_\mu$ $= +e\bar{\psi}\gamma^\mu\psi A_\mu$	$-iq\gamma^\mu$ $= +ie\gamma^\mu$
Peskin–Schroeder [6]	$\partial_\mu + ieA_\mu$	$-e\bar{\psi}\gamma^\mu\psi A_\mu$	$-ie\gamma^\mu$
Greiner et al. [13]	$\partial_\mu + ieA_\mu$	$-e\bar{\psi}\gamma^\mu\psi A_\mu$	$-ie\gamma^\mu$
Bjorken–Drell [10]	$\partial_\mu + ieA_\mu$	$-e\bar{\psi}\gamma^\mu\psi A_\mu$	$-ie\gamma^\mu$
Mandl–Shaw [1]	$\partial_\mu - ieA_\mu$	$+e\bar{\psi}\gamma^\mu\psi A_\mu$	$+ie\gamma^\mu$

Note: Peskin–Schroeder [6] write $D_\mu = \partial_\mu + ieA_\mu$ and $\mathscr{L}_{\text{int}} = -e\bar{\psi}\gamma^\mu\psi A_\mu$, with $e > 0$ denoting the magnitude of the electron charge. This is a different sign convention from the one adopted in these notes, but it describes the same physics once used consistently. Our convention (and Mandl–Shaw's [1]) is directly consistent with the energy argument.

Feynman rule for the vertex. From eq. (5.9), the Feynman rule for the electron-photon vertex is:



The Feynman rule is $\boxed{+ie\gamma^\mu}$ for electrons and muons (charge $q = -e$), with momentum conservation $p + k = p'$ at the vertex. For a general fermion of charge q, the vertex is $-iq\gamma^\mu$.

Classification of tree-level processes

QED processes can be classified according to the number and type of external particles. At tree level (lowest order in $\alpha = e^2/4\pi$), the fundamental $2 \to 2$ scattering processes are:

Process	Name	Channel
$e^-e^- \to e^-e^-$	Møller scattering	t, u
$e^+e^- \to e^+e^-$	Bhabha scattering	s, t
$e^+e^- \to \mu^+\mu^-$	Muon pair production	s
$e^+e^- \to \gamma\gamma$	Pair annihilation	t, u
$e^-\gamma \to e^-\gamma$	Compton scattering	s, u
$e^-p \to e^-p$	Electron-proton scattering	t

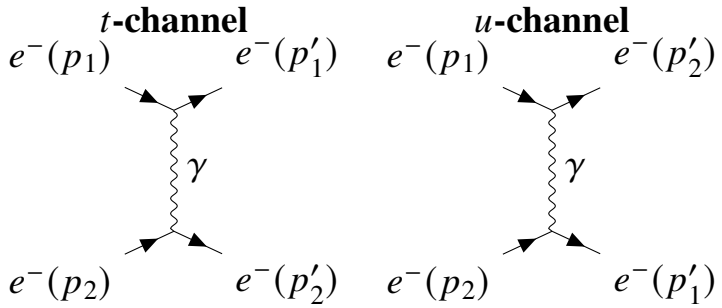
The "channel" indicates which Mandelstam variable (s, t, or u) appears in the propagator denominator, corresponding to different physical mechanisms:

- **s-channel**: particle-antiparticle annihilation into a virtual particle
- **t-channel**: exchange of a virtual particle between the two scattering particles
- **u-channel**: like t-channel but with crossed final-state particles

Feynman diagrams for classic QED processes

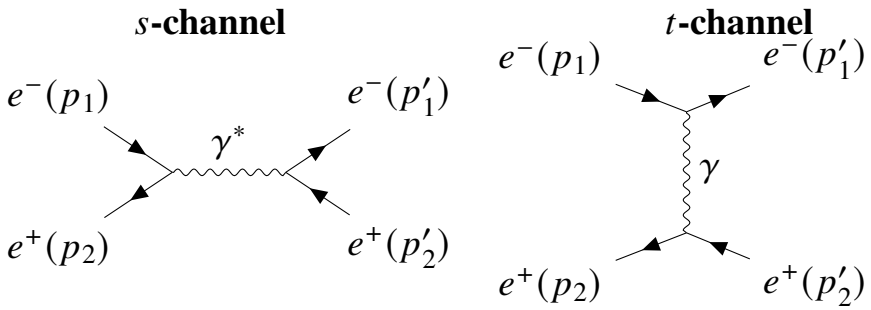
Møller scattering ($e^-e^- \to e^-e^-$)

Two electrons scatter via photon exchange. There are two diagrams at tree level (the u-channel diagram accounts for the indistinguishability of the final-state electrons):



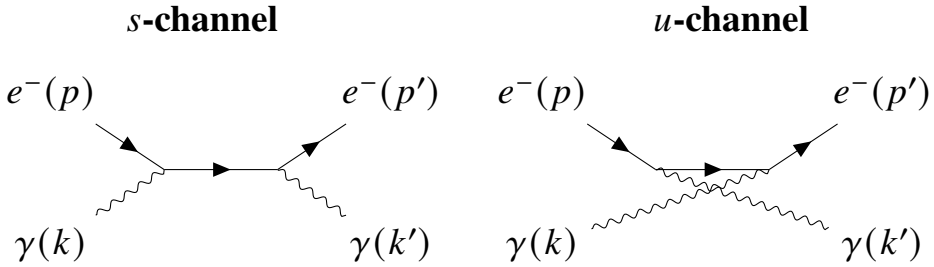
Bhabha scattering ($e^+e^- \to e^+e^-$)

Electron-positron elastic scattering proceeds via both annihilation (s-channel) and exchange (t-channel):



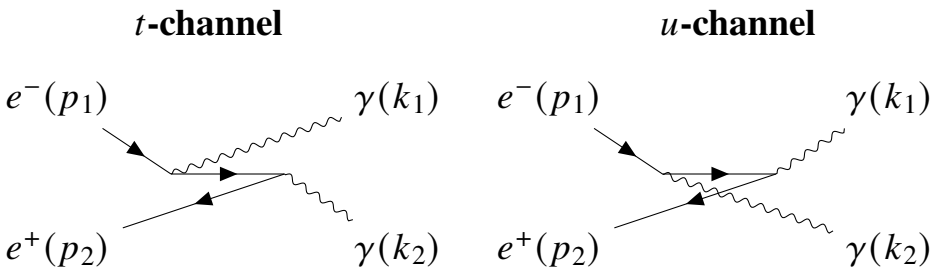
Compton scattering ($e^-\gamma \to e^-\gamma$)

The scattering of a photon by an electron involves two diagrams:



Pair annihilation ($e^+e^- \to \gamma\gamma$)

An electron-positron pair annihilates into two photons:



Remark. [Crossing symmetry] These processes are related by *crossing symmetry*: replacing an incoming particle with an outgoing antiparticle (or vice versa) transforms one process into another. For example, $e^+e^- \to \gamma\gamma$ is the crossed version of Compton scattering $e^-\gamma \to e^-\gamma$.

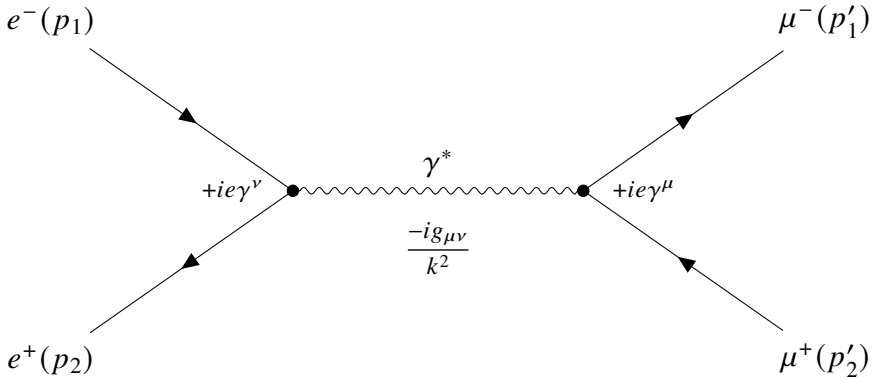
5.2 Muon pair production: $e^+e^- \to \mu^+\mu^-$

We now present a detailed calculation of the cross section for the process $e^+e^- \to \mu^+\mu^-$. This is one of the cleanest QED processes to analyze because: (i) only the s-channel contributes at tree level, since QED contains no vertex that converts an electron into a muon; (ii) it provides a "standard candle" for measuring luminosity at e^+e^- colliders; (iii) the theoretical prediction agrees with experiment to remarkable precision.

Kinematics and Feynman diagram. Consider the process:

$$e^-(p_1) + e^+(p_2) \to \mu^-(p_1') + \mu^+(p_2') \,. \tag{5.13}$$

At tree level, the only contributing diagram is the s-channel annihilation through a virtual photon:



Scattering amplitude. Applying the Feynman rules, the scattering amplitude is:

$$\begin{aligned} i\mathcal{M} = {} & \bar{u}(p_1', s_1')(+ie\gamma^\mu)v(p_2', s_2') \cdot \frac{-ig_{\mu\nu}}{k^2} \\ & \cdot \bar{v}(p_2, s_2)(+ie\gamma^\nu)u(p_1, s_1) \,. \end{aligned} \tag{5.14}$$

Simplifying (note that $(+ie)^2 = -e^2$):

$$\boxed{\mathcal{M} = \frac{e^2}{k^2}\left[\bar{u}(p_1')\gamma_\nu v(p_2')\right]\left[\bar{v}(p_2)\gamma^\nu u(p_1)\right]} \tag{5.15}$$

where $k = p_1 + p_2 = p_1' + p_2'$ is the four-momentum of the virtual photon, and $k^2 = s$ is the center-of-mass energy squared.

BD normalization factors. In the BD convention each external fermion field contributes a factor $\sqrt{m/E}$ from the mode expansion (3.25). For this process there are four external fermions, so the BD amplitude carries an overall factor

$$\mathcal{N}_{\text{BD}} = \sqrt{\frac{m_e^2 m_\mu^2}{E_1 E_2 E_1' E_2'}}\,. \tag{5.16}$$

As explained in Section 1.1, this factor combines with the $\prod(2m_i) = (2m_e)^2(2m_\mu)^2$ in the BD cross section formula (3.55) and the $1/(2m)^2$ from the BD spin sums (3.58), and all mass factors cancel in the final cross section. In the spin sums below we work with the BD completeness relations; the $1/(2m)^2$ prefactors are carried explicitly through the leptonic tensors until the cancellation is shown at the level of the cross section.

Spin-averaged amplitude squared. For unpolarized beams, we average over initial spins (factor $\frac{1}{4}$) and sum over final spins:

$$\overline{|\mathcal{M}|^2} \equiv \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2\,. \tag{5.17}$$

From eq. (5.15), we have:

$$\mathcal{M} = \frac{e^2}{s}\left[\bar{u}_\mu(p_1')\gamma_\nu v_\mu(p_2')\right]\left[\bar{v}_e(p_2)\gamma^\nu u_e(p_1)\right], \tag{5.18}$$

where we suppress spin labels for clarity and used $k^2 = s$. The complex conjugate is:

$$\mathcal{M}^* = \frac{e^2}{s}\left[\bar{v}_\mu(p_2')\gamma_\rho u_\mu(p_1')\right]\left[\bar{u}_e(p_1)\gamma^\rho v_e(p_2)\right], \tag{5.19}$$

where we used $(\bar{\psi}_1\gamma_\nu\psi_2)^* = \bar{\psi}_2\gamma_\nu\psi_1$.

The amplitude squared is therefore:

$$\begin{aligned}|\mathcal{M}|^2 &= \mathcal{M}\mathcal{M}^* \\ &= \frac{e^4}{s^2}\left[\bar{u}_\mu(p_1')\gamma_\nu v_\mu(p_2')\right]\left[\bar{v}_e(p_2)\gamma^\nu u_e(p_1)\right] \\ &\quad\times\left[\bar{v}_\mu(p_2')\gamma_\rho u_\mu(p_1')\right]\left[\bar{u}_e(p_1)\gamma^\rho v_e(p_2)\right].\end{aligned} \tag{5.20}$$

From spinor products to traces. To perform the spin sums, we use the **complete-**

ness relations appropriate to the BD normalization:

$$\boxed{\sum_s u_\sigma(p,s)\bar{u}_\tau(p,s) = \frac{(\not{p}+m)_{\sigma\tau}}{2m}} \tag{5.21}$$

$$\boxed{\sum_s v_\sigma(p,s)\bar{v}_\tau(p,s) = \frac{(\not{p}-m)_{\sigma\tau}}{2m}}, \tag{5.22}$$

where σ,τ are spinor indices. These $1/(2m)$ factors can be carried explicitly or absorbed into external-state normalizations; below we keep them explicit.

We now show in detail how the spin sum converts the product of spinor bilinears into traces. Consider first the muon sector. We can write the spinor bilinears with explicit spinor indices:

$$\left[\bar{u}_\mu(p_1')\gamma_\nu v_\mu(p_2')\right] = \bar{u}_{\mu,\sigma}(p_1')(\gamma_\nu)_\sigma{}^\tau\, v_{\mu,\tau}(p_2') \tag{5.23}$$

$$\left[\bar{v}_\mu(p_2')\gamma_\rho u_\mu(p_1')\right] = \bar{v}_{\mu,\xi}(p_2')(\gamma_\rho)_\xi{}^\zeta\, u_{\mu,\zeta}(p_1')\,. \tag{5.24}$$

Their product, summed over muon spins s_1' and s_2', is:

$$\begin{aligned}
&\sum_{s_1',s_2'}\left[\bar{u}_\mu(p_1')\gamma_\nu v_\mu(p_2')\right]\left[\bar{v}_\mu(p_2')\gamma_\rho u_\mu(p_1')\right] \\
&= \sum_{s_1',s_2'} \bar{u}_{\mu,\sigma}(p_1')(\gamma_\nu)_\sigma{}^\tau\, v_{\mu,\tau}(p_2')\,\bar{v}_{\mu,\xi}(p_2')(\gamma_\rho)_\xi{}^\zeta\, u_{\mu,\zeta}(p_1') \\
&= (\gamma_\nu)_\sigma{}^\tau\,(\gamma_\rho)_\xi{}^\zeta \sum_{s_2'} v_{\mu,\tau}(p_2')\bar{v}_{\mu,\xi}(p_2') \sum_{s_1'} u_{\mu,\zeta}(p_1')\bar{u}_{\mu,\sigma}(p_1')\,.
\end{aligned} \tag{5.25}$$

Using the completeness relations:

$$\sum_{s_2'} v_{\mu,\tau}(p_2')\bar{v}_{\mu,\xi}(p_2') = \frac{(\not{p}_2' - m_\mu)_{\tau\xi}}{2m_\mu} \tag{5.26}$$

$$\sum_{s_1'} u_{\mu,\zeta}(p_1')\bar{u}_{\mu,\sigma}(p_1') = \frac{(\not{p}_1' + m_\mu)_{\zeta\sigma}}{2m_\mu}\,. \tag{5.27}$$

Substituting:

$$\begin{aligned}
&\sum_{s_1',s_2'}\left[\bar{u}_\mu(p_1')\gamma_\nu v_\mu(p_2')\right]\left[\bar{v}_\mu(p_2')\gamma_\rho u_\mu(p_1')\right] \\
&= \frac{1}{(2m_\mu)^2}(\gamma_\nu)_\sigma{}^\tau\,(\not{p}_2' - m_\mu)_{\tau\xi}\,(\gamma_\rho)_\xi{}^\zeta\,(\not{p}_1' + m_\mu)_{\zeta\sigma} \\
&= \frac{1}{(2m_\mu)^2}\left[(\not{p}_1' + m_\mu)\gamma_\nu(\not{p}_2' - m_\mu)\gamma_\rho\right]_{\sigma\sigma} \\
&= \frac{1}{(2m_\mu)^2}\mathrm{Tr}\left[(\not{p}_1' + m_\mu)\gamma_\nu(\not{p}_2' - m_\mu)\gamma_\rho\right],
\end{aligned} \tag{5.28}$$

where in the last step we recognized the sum over the repeated index σ as a trace.

By exactly the same procedure for the electron sector:

$$\sum_{s_1,s_2} \left[\bar{v}_e(p_2)\gamma^\nu u_e(p_1)\right]\left[\bar{u}_e(p_1)\gamma^\rho v_e(p_2)\right]$$
$$= \frac{1}{(2m_e)^2}\mathrm{Tr}\left[(\not{p}_2 - m_e)\gamma^\nu(\not{p}_1 + m_e)\gamma^\rho\right]. \tag{5.29}$$

Therefore, the spin-averaged amplitude squared is:

$$\boxed{\overline{|\mathcal{M}|^2} = \frac{e^4}{4s^2} L^{(\mu)}_{\nu\rho} \cdot L^{(e)\nu\rho}}, \tag{5.30}$$

where we define the **leptonic tensors** (retaining the BD normalization factors):

$$L^{(\mu)}_{\nu\rho} \equiv \frac{1}{(2m_\mu)^2}\mathrm{Tr}\left[(\not{p}'_1 + m_\mu)\gamma_\nu(\not{p}'_2 - m_\mu)\gamma_\rho\right], \tag{5.31}$$
$$L^{(e)\nu\rho} \equiv \frac{1}{(2m_e)^2}\mathrm{Tr}\left[(\not{p}_2 - m_e)\gamma^\nu(\not{p}_1 + m_e)\gamma^\rho\right]. \tag{5.32}$$

The $1/(2m)^2$ prefactors originate from the BD completeness relations (5.21)–(5.22). They will cancel exactly against the external-state normalization factors when computing the cross section (see below, eq. (5.71)).

Evaluation of the muon trace. Expanding the product $(\not{p}'_1 + m_\mu)\gamma_\nu(\not{p}'_2 - m_\mu)\gamma_\rho$ inside the trace in eq. (5.31):

$$L^{(\mu)}_{\nu\rho} = \frac{1}{(2m_\mu)^2}\Big[\mathrm{Tr}(\not{p}'_1\gamma_\nu\not{p}'_2\gamma_\rho) - m_\mu\mathrm{Tr}(\not{p}'_1\gamma_\nu\gamma_\rho)$$
$$+ m_\mu\mathrm{Tr}(\gamma_\nu\not{p}'_2\gamma_\rho) - m_\mu^2\mathrm{Tr}(\gamma_\nu\gamma_\rho)\Big]. \tag{5.33}$$

The second and third terms vanish because they contain an odd number of gamma matrices (three each). Thus:

$$L^{(\mu)}_{\nu\rho} = \frac{1}{(2m_\mu)^2}\Big[\mathrm{Tr}(\not{p}'_1\gamma_\nu\not{p}'_2\gamma_\rho) - m_\mu^2\mathrm{Tr}(\gamma_\nu\gamma_\rho)\Big]. \tag{5.34}$$

For the first term, we write $\not{p}'_1 = p'_{1\alpha}\gamma^\alpha$ and $\not{p}'_2 = p'_{2\beta}\gamma^\beta$:

$$\mathrm{Tr}(\not{p}'_1\gamma_\nu\not{p}'_2\gamma_\rho) = p'_{1\alpha}p'_{2\beta}\mathrm{Tr}(\gamma^\alpha\gamma_\nu\gamma^\beta\gamma_\rho). \tag{5.35}$$

Using the trace formula:

$$\mathrm{Tr}(\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta) = 4(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\delta}g^{\beta\gamma}), \tag{5.36}$$

we have:

$$\begin{aligned}\mathrm{Tr}(\gamma^\alpha\gamma_\nu\gamma^\beta\gamma_\rho) &= 4(g^\alpha{}_\nu g^\beta{}_\rho - g^{\alpha\beta}g_{\nu\rho} + g^\alpha{}_\rho g^\beta{}_\nu)\\ &= 4(\delta^\alpha_\nu\delta^\beta_\rho - g^{\alpha\beta}g_{\nu\rho} + \delta^\alpha_\rho\delta^\beta_\nu)\,. \end{aligned} \tag{5.37}$$

Contracting with $p'_{1\alpha}p'_{2\beta}$:

$$p'_{1\alpha}p'_{2\beta}\mathrm{Tr}(\gamma^\alpha\gamma_\nu\gamma^\beta\gamma_\rho) = 4\big(p'_{1\nu}p'_{2\rho} - (p'_1\cdot p'_2)g_{\nu\rho} + p'_{1\rho}p'_{2\nu}\big)\,. \tag{5.38}$$

For the second term: $\mathrm{Tr}(\gamma_\nu\gamma_\rho) = 4g_{\nu\rho}$.
Combining:

$$\boxed{L^{(\mu)}_{\nu\rho} = \frac{1}{m_\mu^2}\left[p'_{1\nu}p'_{2\rho} + p'_{1\rho}p'_{2\nu} - (p'_1\cdot p'_2 + m_\mu^2)g_{\nu\rho}\right]}\,. \tag{5.39}$$

Evaluation of the electron trace. By the same calculation for the electron tensor $L^{(e)\nu\rho}$:

$$\boxed{L^{(e)\nu\rho} = \frac{1}{m_e^2}\left[p_2^\nu p_1^\rho + p_2^\rho p_1^\nu - (p_1\cdot p_2 + m_e^2)g^{\nu\rho}\right]}\,. \tag{5.40}$$

Contraction of the leptonic tensors. The spin-averaged amplitude squared is:

$$\overline{|\mathcal{M}|^2} = \frac{e^4}{4s^2}L^{(\mu)}_{\nu\rho}L^{(e)\nu\rho} = \frac{e^4}{4s^2}\cdot\frac{1}{(2m_\mu)^2(2m_e)^2}\,\mathrm{Tr}[\mu]\,\mathrm{Tr}[e]\,. \tag{5.41}$$

The contraction of the two traces (without the $1/(2m)^2$ prefactors) proceeds as follows. Writing:

$$\mathrm{Tr}[\mu]_{\nu\rho} = 4\left[p'_{1\nu}p'_{2\rho} + p'_{1\rho}p'_{2\nu} - A\,g_{\nu\rho}\right] \quad \text{where } A = p'_1\cdot p'_2 + m_\mu^2 \tag{5.42}$$

$$\mathrm{Tr}[e]^{\nu\rho} = 4\left[p_2^\nu p_1^\rho + p_2^\rho p_1^\nu - B\,g^{\nu\rho}\right] \quad \text{where } B = p_1\cdot p_2 + m_e^2\,. \tag{5.43}$$

The contraction gives:

$$\begin{aligned}L^{(\mu)}_{\nu\rho}L^{(e)\nu\rho} = 16\Big[&(p'_{1\nu}p'_{2\rho})(p_2^\nu p_1^\rho) + (p'_{1\nu}p'_{2\rho})(p_2^\rho p_1^\nu)\\ &+ (p'_{1\rho}p'_{2\nu})(p_2^\nu p_1^\rho) + (p'_{1\rho}p'_{2\nu})(p_2^\rho p_1^\nu)\\ &- A(p_2^\nu p_1^\rho g_{\nu\rho}) - A(p_2^\rho p_1^\nu g_{\nu\rho})\\ &- B(p'_{1\nu}p'_{2\rho}g^{\nu\rho}) - B(p'_{1\rho}p'_{2\nu}g^{\nu\rho}) + 4AB\Big]\,. \end{aligned} \tag{5.44}$$

Evaluating each term using $p'_{1\nu}p_2^\nu = p'_1 \cdot p_2$, etc.:

$$(p'_{1\nu}p'_{2\rho})(p_2^\nu p_1^\rho) = (p'_1 \cdot p_2)(p'_2 \cdot p_1) \tag{5.45}$$
$$(p'_{1\nu}p'_{2\rho})(p_2^\rho p_1^\nu) = (p'_1 \cdot p_1)(p'_2 \cdot p_2) \tag{5.46}$$
$$(p'_{1\rho}p'_{2\nu})(p_2^\nu p_1^\rho) = (p'_2 \cdot p_2)(p'_1 \cdot p_1) \tag{5.47}$$
$$(p'_{1\rho}p'_{2\nu})(p_2^\rho p_1^\nu) = (p'_1 \cdot p_2)(p'_2 \cdot p_1) \tag{5.48}$$
$$p_2^\nu p_1^\rho g_{\nu\rho} = p_1 \cdot p_2 \tag{5.49}$$
$$p'_{1\nu}p'_{2\rho}g^{\nu\rho} = p'_1 \cdot p'_2 . \tag{5.50}$$

Collecting terms:

$$L_{\nu\rho}^{(\mu)} L^{(e)\nu\rho} = 16\Big[2(p'_1 \cdot p_2)(p'_2 \cdot p_1) + 2(p'_1 \cdot p_1)(p'_2 \cdot p_2) - 2A(p_1 \cdot p_2) - 2B(p'_1 \cdot p'_2) + 4AB\Big] . \tag{5.51}$$

Substituting $A = p'_1 \cdot p'_2 + m_\mu^2$ and $B = p_1 \cdot p_2 + m_e^2$, and simplifying:

$$\mathrm{Tr}[\mu]_{\nu\rho}\,\mathrm{Tr}[e]^{\nu\rho} = 32\Big[(p_1 \cdot p'_1)(p_2 \cdot p'_2) + (p_1 \cdot p'_2)(p_2 \cdot p'_1) + m_\mu^2(p_1 \cdot p_2) + m_e^2(p'_1 \cdot p'_2) + 2m_e^2 m_\mu^2\Big] . \tag{5.52}$$

Including the BD normalization prefactors:

$$\boxed{\overline{|\mathcal{M}|^2} = \frac{e^4}{4s^2} \cdot \frac{1}{(2m_\mu)^2(2m_e)^2}\,\mathrm{Tr}[\mu]_{\nu\rho}\,\mathrm{Tr}[e]^{\nu\rho}} . \tag{5.53}$$

Center-of-mass kinematics. In the center-of-mass (CM) frame, the momenta are:

$$p_1 = (E, \vec{p}), \qquad p'_1 = (E, \vec{p}\,') \tag{5.54}$$
$$p_2 = (E, -\vec{p}), \qquad p'_2 = (E, -\vec{p}\,') , \tag{5.55}$$

where energy conservation requires all particles to have the same energy E. The relevant scalar products are:

$$p_1 \cdot p_2 = E^2 + |\vec{p}|^2 \tag{5.56}$$
$$p'_1 \cdot p'_2 = E^2 + |\vec{p}\,'|^2 \tag{5.57}$$
$$p_1 \cdot p'_1 = E^2 - |\vec{p}||\vec{p}\,'|\cos\theta \tag{5.58}$$
$$p_1 \cdot p'_2 = E^2 + |\vec{p}||\vec{p}\,'|\cos\theta , \tag{5.59}$$

where θ is the scattering angle between $\vec{p}$ and $\vec{p}\,'$, $|\vec{p}| = \sqrt{E^2 - m_e^2}$, and $|\vec{p}\,'| = \sqrt{E^2 - m_\mu^2}$.

Mandelstam variables. For a $2 \to 2$ scattering process, the Mandelstam variables are:

$$s \equiv (p_1 + p_2)^2, \qquad t \equiv (p_1 - p_1')^2, \qquad u \equiv (p_1 - p_2')^2 . \tag{5.60}$$

These satisfy $s + t + u = 2m_e^2 + 2m_\mu^2$. In the CM frame: $s = 4E^2$.

Ultrarelativistic limit. In the limit $E \gg m_e, m_\mu$, we have $|\vec{p}| \approx |\vec{p}\,'| \approx E$, and the Mandelstam variables simplify to:

$$s = 4E^2, \qquad t = -\frac{s}{2}(1 - \cos\theta), \qquad u = -\frac{s}{2}(1 + \cos\theta) . \tag{5.61}$$

The scalar products become:

$$p_1 \cdot p_2 = p_1' \cdot p_2' = \frac{s}{2}, \qquad p_1 \cdot p_1' = -\frac{t}{2}, \qquad p_1 \cdot p_2' = -\frac{u}{2} . \tag{5.62}$$

Substituting into eq. (5.52) with $m_e = m_\mu = 0$:

$$\mathrm{Tr}[\mu]_{\nu\rho}\,\mathrm{Tr}[e]^{\nu\rho} = 32\left[\frac{t^2}{4} + \frac{u^2}{4}\right] = 8(t^2 + u^2) . \tag{5.63}$$

Expressing in terms of the scattering angle:

$$t^2 + u^2 = \frac{s^2}{4}\left[(1 - \cos\theta)^2 + (1 + \cos\theta)^2\right] = \frac{s^2}{2}(1 + \cos^2\theta) . \tag{5.64}$$

Therefore:

$$\overline{|\mathcal{M}|^2} = \frac{e^4}{4s^2} \cdot \frac{8(t^2 + u^2)}{(2m_\mu)^2 (2m_e)^2} = \frac{2e^4 (t^2 + u^2)}{(2m_\mu)^2 (2m_e)^2 s^2} . \tag{5.65}$$

Note that $\overline{|\mathcal{M}|^2}$ still carries the BD normalization factors $1/(2m)^2$; these will cancel in the cross section.

Differential cross section and cancellation of BD normalization factors.
We now derive $d\sigma/d\Omega$ in the CM frame from the BD master formula (3.55). The two-body Lorentz-invariant phase space (3.45) reads

$$d\Phi_2 = (2\pi)^4 \delta^{(4)}(p_A + p_B - p_1' - p_2') \frac{d^3 p_1'}{(2\pi)^3 2E_1'} \frac{d^3 p_2'}{(2\pi)^3 2E_2'} . \tag{5.66}$$

In the CM frame $\vec{p}_A + \vec{p}_B = 0$, so the three-momentum delta function enforces $\vec{p}_2' = -\vec{p}_1'$. Writing $|\vec{p}_1'| = |\vec{p}_2'| \equiv |\vec{p}\,'|$ and $d^3 p_1' = |\vec{p}\,'|^2 \, d|\vec{p}\,'| \, d\Omega$, integration over $\vec{p}_2'$ yields

$$d\Phi_2 = \frac{1}{(2\pi)^2} \frac{|\vec{p}\,'|^2 \, d|\vec{p}\,'| \, d\Omega}{4E_1' E_2'} \delta(\sqrt{s} - E_1' - E_2') . \tag{5.67}$$

To perform the $|\vec{p}\,'|$ integration we use $\delta(f(x)) = \delta(x-x_0)/|f'(x_0)|$, which requires the Jacobian $d(E_1'+E_2')/d|\vec{p}\,'|$. Since both final-state energies depend on the *same* variable $|\vec{p}\,'|$ through $E_i' = \sqrt{|\vec{p}\,'|^2 + m_i'^2}$, differentiation gives

$$\frac{dE_i'}{d|\vec{p}\,'|} = \frac{|\vec{p}\,'|}{E_i'}, \tag{5.68}$$

so that

$$\frac{d(E_1'+E_2')}{d|\vec{p}\,'|} = \frac{|\vec{p}\,'|}{E_1'} + \frac{|\vec{p}\,'|}{E_2'} = |\vec{p}\,'|\,\frac{E_2'+E_1'}{E_1'E_2'} = \frac{|\vec{p}\,'|\sqrt{s}}{E_1'E_2'}, \tag{5.69}$$

where in the last step we used $E_1'+E_2' = \sqrt{s}$ on the support of the delta function. The delta function identity then gives

$$d\Phi_2\Big|_{\text{CM}} = \frac{1}{(2\pi)^2}\,\frac{|\vec{p}\,'|^2}{4E_1'E_2'}\,\frac{E_1'E_2'}{|\vec{p}\,'|\sqrt{s}}\,d\Omega = \frac{|\vec{p}\,'|}{16\pi^2\sqrt{s}}\,d\Omega\,. \tag{5.70}$$

Substituting into the master formula (3.55) with the CM-frame flux factor $\mathcal{F} = 4|\vec{p}\,|\sqrt{s}$ (cf. eq. (3.53)):

$$\begin{aligned}\left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} &= \frac{1}{4|\vec{p}\,|\sqrt{s}}\left(\prod_{i\in\text{fermions}} 2m_i\right)\overline{|\mathcal{M}_{\text{BD}}|^2}\,\frac{|\vec{p}\,'|}{16\pi^2\sqrt{s}} \\ &= \frac{1}{64\pi^2 s}\,\frac{|\vec{p}\,'|}{|\vec{p}|}\left(\prod_{i\in\text{fermions}} 2m_i\right)\overline{|\mathcal{M}_{\text{BD}}|^2}\,.\end{aligned} \tag{5.71}$$

For $e^+e^- \to \mu^+\mu^-$ with four external fermions: $\prod(2m_i) = (2m_e)^2(2m_\mu)^2$.

Using eq. (5.53), the factor $(2m_e)^2(2m_\mu)^2$ from the cross-section formula *exactly cancels* the $1/(2m_e)^2(2m_\mu)^2$ from the BD completeness relations in the leptonic tensors:

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \frac{1}{64\pi^2 s}\,\frac{|\vec{p}\,'|}{|\vec{p}|}\cdot\cancel{(2m_e)^2(2m_\mu)^2}\cdot\frac{e^4}{4s^2}\cdot\frac{\text{Tr}[\mu]\,\text{Tr}[e]}{\cancel{(2m_\mu)^2(2m_e)^2}} \\ &= \frac{1}{64\pi^2 s}\,\frac{|\vec{p}\,'|}{|\vec{p}|}\cdot\frac{e^4}{4s^2}\cdot\text{Tr}[\mu]\,\text{Tr}[e]\,.\end{aligned} \tag{5.72}$$

All mass factors from the BD normalisation drop out, as they must: the cross section is a physical observable independent of the spinor normalisation convention.

In the ultrarelativistic limit ($|\vec{p}\,'| = |\vec{p}\,|$, $m_e = m_\mu = 0$), using eq. (5.63) and $\alpha = e^2/4\pi$:

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}(1+\cos^2\theta)}\,. \tag{5.73}$$

The factor $(1+\cos^2\theta)$ reflects the spin-1 nature of the intermediate photon. At $\theta = 0$ (forward) and $\theta = \pi$ (backward), the cross section is maximized, while at $\theta = \pi/2$ it has a minimum.

Total cross section. Integrating over the solid angle:

$$\begin{aligned}\sigma_{\text{tot}} &= \int \frac{d\sigma}{d\Omega}\, d\Omega = 2\pi \cdot \frac{\alpha^2}{4s} \int_{-1}^{+1} (1+x^2)\, dx \\ &= \frac{\pi\alpha^2}{2s} \cdot \frac{8}{3}.\end{aligned} \tag{5.74}$$

Therefore:

$$\boxed{\sigma_{\text{tot}}(e^+e^- \to \mu^+\mu^-) = \frac{4\pi\alpha^2}{3s}}. \tag{5.75}$$

Numerical estimate. Using $\hbar c \simeq 0.197$ GeV·fm and $\alpha \simeq 1/137$, at $\sqrt{s} = 1$ GeV:

$$\sigma \simeq 87 \text{ nb}. \tag{5.76}$$

A useful practical formula is:

$$\boxed{\sigma(e^+e^- \to \mu^+\mu^-) \simeq \frac{87 \text{ nb}}{s/\text{GeV}^2}}. \tag{5.77}$$

Remark. [The "point cross section"] The result $\sigma \propto 1/s$ is characteristic of point-like electromagnetic scattering. Any deviation from this behavior signals the presence of structure (form factors) or new physics. This cross section serves as the normalization standard for the R-ratio:

$$R \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}, \tag{5.78}$$

which provides crucial information about quark charges and the number of colors.

Comparison with experiment. The QED prediction $\sigma = 4\pi\alpha^2/3s$ has been tested at numerous e^+e^- colliders. Figure 5.1 shows a comparison between the theoretical cross section and experimental measurements from various facilities.

The agreement between theory and experiment is remarkable, typically at the percent level or better. This precision test of QED provides strong evidence that the electron and muon are indeed point-like particles with no internal structure down to scales of $\sim 10^{-18}$ m.

Summary of results. We collect the main results of this calculation in the following table:

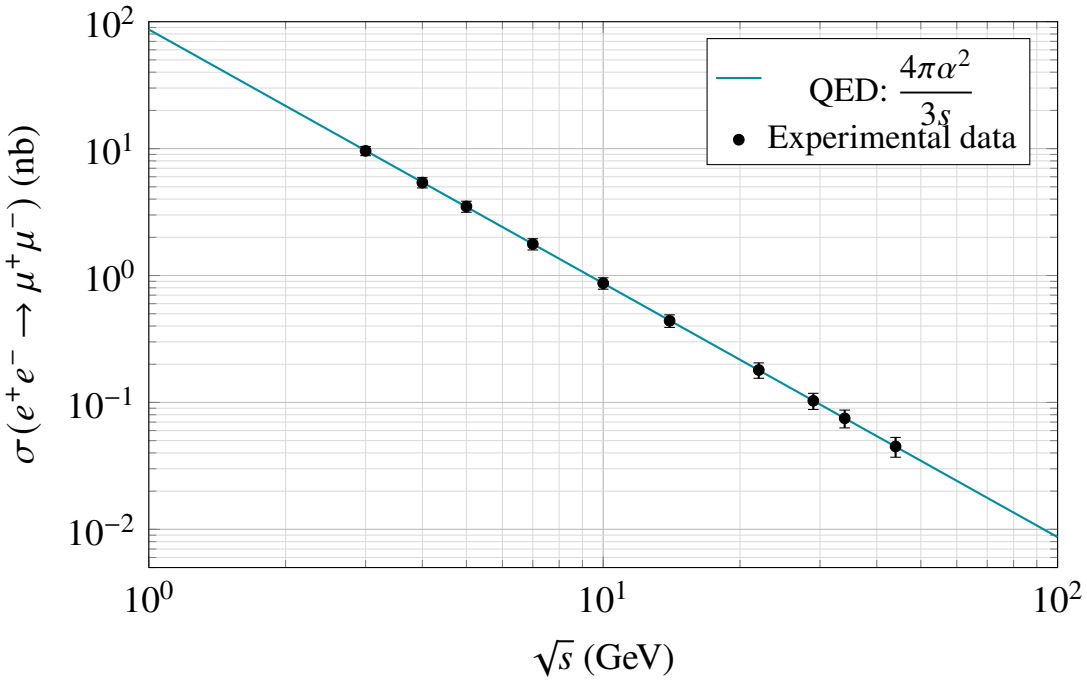


Figure 5.1. Total cross section for $e^+e^- \to \mu^+\mu^-$ as a function of center-of-mass energy $\sqrt{s}$. The solid line shows the QED prediction $\sigma = 4\pi\alpha^2/3s \simeq 86.8\ \text{nb}/(\sqrt{s}/\text{GeV})^2$. The data points represent measurements from various e^+e^- colliders (SPEAR, DORIS, PETRA, PEP, TRISTAN). The excellent agreement over more than two orders of magnitude in energy demonstrates the validity of QED as a description of electromagnetic interactions.

Quantity	**Expression (ultrarelativistic limit)**
Amplitude squared	$\overline{\lvert\mathcal{M}\rvert^2} = \dfrac{2e^4(t^2+u^2)}{s^2} = e^4(1+\cos^2\theta)$ (*)
Differential cross section	$\dfrac{d\sigma}{d\Omega} = \dfrac{\alpha^2}{4s}(1+\cos^2\theta)$
Total cross section	$\sigma_{\text{tot}} = \dfrac{4\pi\alpha^2}{3s}$

(*) Here $\overline{\lvert\mathcal{M}\rvert^2}$ denotes the covariant (physical) amplitude squared, i.e. $\prod(2m_i)\times\overline{\lvert\mathcal{M}_{\text{BD}}\rvert^2}$, in which the BD normalization factors have already cancelled (cf. eq. (5.72)).

Helicity considerations

The angular distribution $(1+\cos^2\theta)$ derived above has a deep physical interpretation in terms of helicity conservation. In the ultrarelativistic limit, helicity becomes a powerful organizing principle for understanding scattering amplitudes.

Helicity and chirality. Recall that the **helicity** of a particle is the component of its spin along its direction of motion, $h = \pm\frac{1}{2}$ for spin-$\frac{1}{2}$ particles. We denote right-handed ($h = +1/2$) and left-handed ($h = -1/2$) states as R and L respectively.

The **chirality** operators $P_{L,R} = \frac{1}{2}(1 \mp \gamma^5)$ project onto definite chirality states. For a massive fermion, helicity and chirality eigenstates do not coincide. However, in the ultrarelativistic limit $E \gg m$:

$$\text{helicity} \approx \text{chirality} \quad (m \to 0)\,. \tag{5.79}$$

More precisely, the overlap between a helicity eigenstate and the "wrong" chirality state is suppressed by $O(m/E)$.

Chirality conservation at the QED vertex. The electromagnetic interaction $\bar\psi\gamma^\mu\psi A_\mu$ can be decomposed as:

$$\bar\psi\gamma^\mu\psi = \bar\psi_L\gamma^\mu\psi_L + \bar\psi_R\gamma^\mu\psi_R\,, \tag{5.80}$$

where $\psi_{L,R} = P_{L,R}\psi$. There are *no* cross-terms $\bar\psi_L\gamma^\mu\psi_R$ because:

$$\bar\psi_L\gamma^\mu\psi_R = \bar\psi P_R\gamma^\mu P_R\psi = \bar\psi\gamma^\mu P_L P_R\psi = 0\,, \tag{5.81}$$

using $\gamma^\mu\gamma^5 = -\gamma^5\gamma^\mu$ and $P_L P_R = 0$.

Box 5.2 Helicity conservation in QED

In the massless limit, the QED vertex **conserves chirality**, and hence helicity:

- A left-handed fermion remains left-handed after emitting/absorbing a photon
- A right-handed fermion remains right-handed

This leads to strict **helicity selection rules** for scattering amplitudes.

Helicity vs chirality for antiparticles. For particles in the massless limit, helicity and chirality coincide: a right-handed electron ($h = +1/2$) is right-chiral ($P_R u_R = u_R$), and a left-handed electron ($h = -1/2$) is left-chiral ($P_L u_L = u_L$). This follows directly from the massless Dirac equation: for any solution of $\not{p}\psi = 0$ with $p^\mu = E(1, \hat{p})$, one can show that $\hat{p} \cdot \vec{\Sigma}\, \psi = \gamma^5 \psi$ (see the box below for details).

For antiparticles, the relationship is **inverted**: a right-handed positron ($h = +1/2$) is *left*-chiral. The origin of this inversion is explained in the box below; the key point is that the antiparticle creation operator $b^{s\dagger}$ produces a state with physical helicity $-s$ (not $+s$), while the spinor v^s has the same chirality-helicity relation as u^s. Denoting by the subscripts L, R the *physical helicity*:

$$\text{Particles:} \quad P_R u_R = u_R, \quad P_L u_L = u_L \qquad \text{(chirality = helicity)}, \tag{5.82}$$

$$\text{Antiparticles:} \quad P_L v_R = v_R, \quad P_R v_L = v_L \qquad \text{(chirality = } -\text{helicity)}. \tag{5.83}$$

Here $u_{L,R}$ denote the particle spinors with helicity $\mp\frac{1}{2}$, while $v_{L,R}$ denote the v spinors that enter the Feynman amplitude for a physical antiparticle with helicity $\mp\frac{1}{2}$.

Explicit verification. We verify eqs. (5.82)–(5.83) with the ultrarelativistic spinors along $\hat{z}$ in the Dirac representation. From eqs. (1.176)–(1.177), in the limit $m \to 0$ ($p_z/(E+m) \to 1$), dropping the common prefactor:

$$u(\uparrow) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad u(\downarrow) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \quad v(\uparrow) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad v(\downarrow) = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}. \tag{5.84}$$

The helicity operator for motion along z is $\hat{h} = \frac{1}{2}\Sigma^3 = \frac{1}{2}\,\mathrm{diag}(1, -1, 1, -1)$. Direct multiplication shows that $u(\uparrow)$ and $v(\uparrow)$ have spinor-helicity $+\frac{1}{2}$, while $u(\downarrow)$ and $v(\downarrow)$ have spinor-helicity $-\frac{1}{2}$.

For particles, spinor-helicity = physical helicity, so $u_R = u(\uparrow)$, $u_L = u(\downarrow)$. For antiparticles, the physical helicity is $-s$ (Fact 2 in the box below), so $v_R = v(\downarrow)$, $v_L = v(\uparrow)$.

Using $\gamma^5_{\rm D} = \left(\begin{smallmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{smallmatrix}\right)$:

$$P_R = \tfrac{1}{2}\begin{pmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & \mathbb{1} \end{pmatrix}, \qquad P_L = \tfrac{1}{2}\begin{pmatrix} \mathbb{1} & -\mathbb{1} \\ -\mathbb{1} & \mathbb{1} \end{pmatrix}. \tag{5.85}$$

Particles (chirality = helicity):

$$P_R\, u_R = \tfrac{1}{2}\begin{pmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & \mathbb{1} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = u_R\,, \qquad P_L\, u_L = \tfrac{1}{2}\begin{pmatrix} \mathbb{1} & -\mathbb{1} \\ -\mathbb{1} & \mathbb{1} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} = u_L\,. \checkmark \tag{5.86}$$

Antiparticles (chirality = −helicity):

$$P_L\, v_R = \tfrac{1}{2}\begin{pmatrix} \mathbb{1} & -\mathbb{1} \\ -\mathbb{1} & \mathbb{1} \end{pmatrix}\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} = v_R\,, \qquad P_R\, v_L = \tfrac{1}{2}\begin{pmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & \mathbb{1} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = v_L\,. \checkmark \tag{5.87}$$

The inversion for antiparticles is evident: $v_R = v(\downarrow)$ has its upper and lower two-components with *opposite* signs (unlike u_R where they have the *same* sign), which causes P_L rather than P_R to act as the identity on it.

Box 5.3 Why v_R is left-chiral

The inversion arises from the interplay of two properties of the Dirac field quantization.

Fact 1: Chirality equals spinor helicity (for both u and v). In the massless limit, both u^s and v^s satisfy the same equation $\not{p}\,\psi = 0$. For any such solution with $p^\mu = E(1, \hat{p})$, the Dirac equation implies

$$\hat{p}\cdot\vec{\Sigma}\,\psi = \gamma^5\,\psi\,, \tag{5.88}$$

so the helicity eigenvalue of the *spinor* equals its γ^5 eigenvalue (chirality). This holds for both u^s and v^s.

Fact 2: Physical antiparticle helicity $= -s$. In the Dirac field expansion $\psi \sim a^s u^s e^{-ipx} + b^{s\dagger} v^s e^{+ipx}$, the particle state $a^{s\dagger}|0\rangle$ has physical helicity $+s$, but $b^{s\dagger}|0\rangle$ has physical helicity $-s$.
To see why, consider how ψ transforms under a rotation by θ about the z-axis (at $\vec{p} = 0$ for simplicity):

$$U(R)\,\psi\,U(R)^{-1} = S(R)\,\psi\,, \qquad S(R) = e^{-i\theta\Sigma^3/2}\,. \tag{5.89}$$

The spinor matrix $S(R)$ acts on u^s and v^s identically: $S(R)\,u^s = e^{-is\theta}\,u^s$, $S(R)\,v^s = e^{-is\theta}\,v^s$. Matching both sides of the field expansion term by term:

$$U(R)\,a^s\,U(R)^{-1} = e^{-is\theta}\,a^s\,, \qquad U(R)\,b^{s\dagger}\,U(R)^{-1} = e^{-is\theta}\,b^{s\dagger}\,. \tag{5.90}$$

Both a^s and $b^{s\dagger}$ pick up the *same* phase $e^{-is\theta}$, because both multiply a spinor inside ψ. Taking the Hermitian conjugate of the first relation:

$$U(R)\,a^{s\dagger}\,U(R)^{-1} = e^{+is\theta}\,a^{s\dagger}\,. \tag{5.91}$$

For infinitesimal θ, these become:

$$[J_z,\, a^{s\dagger}] = +s\,a^{s\dagger}\,, \qquad [J_z,\, b^{s\dagger}] = -s\,b^{s\dagger}\,. \tag{5.92}$$

The sign difference is now clear: $a^{s\dagger}$ is the *conjugate* of a^s (which appears in ψ), picking up the opposite phase $e^{+is\theta}$; while $b^{s\dagger}$ appears *directly* in ψ, retaining the phase $e^{-is\theta}$.

Result. A right-handed positron ($h_{\text{phys}} = +\frac{1}{2}$) is created by $b^{s\dagger}$ with $s = -\frac{1}{2}$. By Fact 1, the spinor $v^{s=-1/2}$ satisfies $\gamma^5 v = -v$, i.e. it is left-chiral: $P_L v_R = v_R$.

Intuitively, the inversion can also be understood from the Dirac hole picture: a right-handed positron is a "hole" in the left-handed negative-energy electron sea; the missing electron was left-chiral (chirality = helicity for particles), and the hole inherits this chirality.

Taking the Dirac adjoint of $P_L v_R = v_R$ and using $P_L^\dagger = P_L$ together with $\gamma^0 P_R = P_L \gamma^0$ (which follows from $\{\gamma^0, \gamma^5\} = 0$), one obtains

$$\bar{v}_R\,P_R = \bar{v}_R\,. \tag{5.93}$$

This is the identity used in the helicity selection rules below.

Helicity selection rules for $e^+e^- \to \mu^+\mu^-$. At the electron vertex, the current is $j_e^\mu = \bar{v}(p_2)\gamma^\mu u(p_1)$, where $u(p_1)$ is the electron spinor and $v(p_2)$ is the positron spinor. Let us check which helicity combinations give non-zero currents.

Case 1: $e_L^- + e_R^+$ (left-handed electron, right-handed positron)

Using $P_L u_L = u_L$ and $P_L v_R = v_R$:

$$\bar{v}_R\gamma^\mu u_L = \bar{v}_R P_R \gamma^\mu P_L u_L = \bar{v}_R \gamma^\mu P_L u_L \neq 0\,, \tag{5.94}$$

where we used $P_R\gamma^\mu = \gamma^\mu P_L$. This is non-zero because $P_L u_L = u_L$.

Case 2: $e_R^- + e_L^+$ (right-handed electron, left-handed positron)

Using $P_R u_R = u_R$ and $P_R v_L = v_L$:

$$\bar{v}_L \gamma^\mu u_R = \bar{v}_L P_L \gamma^\mu P_R u_R = \bar{v}_L \gamma^\mu P_R u_R \neq 0\,. \tag{5.95}$$

This is also non-zero.

Case 3: $e_L^- + e_L^+$ (both left-handed)

Using $P_L u_L = u_L$ and $P_R v_L = v_L$:

$$\bar{v}_L \gamma^\mu u_L = \bar{v}_L P_L \gamma^\mu P_L u_L = \bar{v}_L \gamma^\mu \underbrace{P_R P_L}_{=0} u_L = 0\,. \tag{5.96}$$

This vanishes because $P_R P_L = 0$.

Case 4: $e_R^- + e_R^+$ (both right-handed)

Similarly:

$$\bar{v}_R \gamma^\mu u_R = \bar{v}_R P_R \gamma^\mu P_R u_R = \bar{v}_R \gamma^\mu \underbrace{P_L P_R}_{=0} u_R = 0\,. \tag{5.97}$$

Summary: The virtual photon can only be produced from:

- $e_L^- + e_R^+ \to \gamma^*$ (left-handed electron, right-handed positron)
- $e_R^- + e_L^+ \to \gamma^*$ (right-handed electron, left-handed positron)

The key insight is that chirality conservation, combined with the inverted helicity-chirality relation for antiparticles, requires the electron and positron to have *opposite* helicities. Similarly, at the muon vertex, the virtual photon can only produce:

- $\gamma^* \to \mu_L^- + \mu_R^+$
- $\gamma^* \to \mu_R^- + \mu_L^+$

Therefore, only **four** helicity amplitudes are non-vanishing (out of the $2^4 = 16$ possible combinations):

$$\begin{aligned} &\mathcal{M}(e_L^- e_R^+ \to \mu_L^- \mu_R^+), && \mathcal{M}(e_L^- e_R^+ \to \mu_R^- \mu_L^+), \\ &\mathcal{M}(e_R^- e_L^+ \to \mu_L^- \mu_R^+), && \mathcal{M}(e_R^- e_L^+ \to \mu_R^- \mu_L^+)\,. \end{aligned} \tag{5.98}$$

Box 5.4 Wigner d-matrices: group theory and scattering

Group-theoretic origin. The Wigner d-matrices arise as matrix elements of the **irreducible representations of SU(2)**, the universal covering group of the rotation group SO(3). For each spin $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$, there exists a $(2j+1)$-dimensional irreducible representation.

Euler angles. Any rotation in three dimensions can be decomposed into three

successive rotations. In the z-y-z **convention** (standard in quantum mechanics), the Euler angles (α, β, γ) parametrize a rotation as:

1. Rotate by angle γ about the original z-axis
2. Rotate by angle β about the (new) y'-axis
3. Rotate by angle α about the (new) z''-axis

The ranges are $\alpha \in [0, 2\pi)$, $\beta \in [0, \pi]$, $\gamma \in [0, 2\pi)$.

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The figure shows the original axes $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ in **blue** and the rotated axes $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in **red**. The **line of nodes** $\mathbf{N}$ (green) is the intersection of the xy-plane (blue ellipse) and the XY-plane (red ellipse). The three Euler angles are:

- α: angle from $\mathbf{x}$ to $\mathbf{N}$, measured in the xy-plane
- β: angle from $\mathbf{z}$ to $\mathbf{Z}$ (the nutation angle)
- γ: angle from $\mathbf{N}$ to $\mathbf{X}$, measured in the XY-plane

Equivalently, using rotations about the *fixed* (original) axes, the same result is achieved by applying rotations in reverse order: first α about z, then β about y, then γ about z:

$$R(\alpha, \beta, \gamma) = R_z(\alpha)\, R_y(\beta)\, R_z(\gamma)\,. \tag{5.99}$$

This is the form used in the quantum mechanical representation below.

Representation of rotations. A general rotation $R(\alpha, \beta, \gamma)$ is represented in the spin-j Hilbert space by:

$$D^j_{m,m'}(\alpha, \beta, \gamma) = \langle j, m | e^{-i\alpha J_z} e^{-i\beta J_y} e^{-i\gamma J_z} | j, m' \rangle = e^{-im\alpha} d^j_{m,m'}(\beta) e^{-im'\gamma}\,. \tag{5.100}$$

The factorization into phases $e^{-im\alpha}$ and $e^{-im'\gamma}$ follows because the basis states $|j, m\rangle$ are **eigenstates of** J_z:

$$J_z |j, m\rangle = m |j, m\rangle \quad \Rightarrow \quad e^{-i\gamma J_z} |j, m'\rangle = e^{-i\gamma m'} |j, m'\rangle\,. \tag{5.101}$$

Thus rotations about the z-axis are diagonal in this basis and contribute only phase factors. In contrast, J_y does not commute with J_z, so $e^{-i\beta J_y}$ mixes different m values—this is what makes $d^j_{m,m'}(\beta)$ a non-trivial matrix.

The **small d-matrix** $d^j_{m,m'}(\theta)$ corresponds to a pure rotation about the y-axis:

$$d^j_{m,m'}(\theta) = \langle j,m|e^{-i\theta J_y}|j,m'\rangle = D^j_{m,m'}(0,\theta,0)\,, \tag{5.102}$$

where the angular momentum generators satisfy the Lie algebra $[J_i,J_j] = i\epsilon_{ijk}J_k$.

Explicit formulas for $j = 1$. The d-matrix for spin-1 is a 3×3 matrix:

$$d^1(\theta) = \begin{pmatrix} \frac{1+\cos\theta}{2} & -\frac{\sin\theta}{\sqrt{2}} & \frac{1-\cos\theta}{2} \\ \frac{\sin\theta}{\sqrt{2}} & \cos\theta & -\frac{\sin\theta}{\sqrt{2}} \\ \frac{1-\cos\theta}{2} & \frac{\sin\theta}{\sqrt{2}} & \frac{1+\cos\theta}{2} \end{pmatrix} \tag{5.103}$$

These are related to spherical harmonics: $d^j_{m,0}(\theta) = \sqrt{\frac{4\pi}{2j+1}}\,Y_{jm}(\theta,0)^*$.

Key properties (following from group representation theory):

- $d^j_{m,m'}(0) = \delta_{m,m'}$ (identity element)
- $d^j_{m,m'}(\theta_1+\theta_2) = \sum_{m''} d^j_{m,m''}(\theta_1)d^j_{m'',m'}(\theta_2)$ (group multiplication)
- $d^j_{m,m'}(\theta)^* = d^j_{m',m}(\theta) = d^j_{m,m'}(-\theta)$ (unitarity)
- $d^j_{-m,-m'}(\theta) = (-1)^{m-m'}d^j_{m,m'}(\theta)$ (reflection symmetry)

Physical meaning in scattering: the sign in $\lambda_B - \lambda_C$. Consider a decay $A \to B + C$ where particle A (spin j, helicity λ_A) decays into B and C moving back-to-back. Let the decay axis make angle θ with the original z-axis.
The helicity of each daughter is defined as the spin projection along *its own direction of motion*:

- Particle B moves along $+\hat{z}'$ with helicity $\lambda_B \Rightarrow$ spin projection along $+\hat{z}'$ is λ_B
- Particle C moves along $-\hat{z}'$ with helicity $\lambda_C \Rightarrow$ spin projection along $-\hat{z}'$ is λ_C

To find C's contribution to the *total* $J_{z'}$ (along $+\hat{z}'$), we must flip the sign: the spin projection of C along $+\hat{z}'$ is $-\lambda_C$. Therefore:

$$J^{\text{final}}_{z'} = \lambda_B + (-\lambda_C) = \lambda_B - \lambda_C\,. \tag{5.104}$$

The amplitude is then proportional to $d^j_{\lambda_A,\lambda_B-\lambda_C}(\theta)$, connecting the initial helicity λ_A to the final net helicity $\lambda_B - \lambda_C$.

Angular dependence from helicity amplitudes. The angular dependence of each helicity amplitude is determined by angular momentum conservation. The initial e^+e^- state with definite helicities has a well-defined z-component of angular momentum $J_z = h_{e^-} - h_{e^+}$ (taking the e^- direction as the z-axis).

For $e^-_L e^+_R$: $J_z = -\frac{1}{2} - (+\frac{1}{2}) = -1$

For $e^-_R e^+_L$: $J_z = +\frac{1}{2} - (-\frac{1}{2}) = +1$

The virtual photon (spin-1) thus carries $J_z = \pm 1$. When it decays to $\mu^+\mu^-$ at angle θ, the amplitude picks up a factor from the rotation matrix $d^1_{m,m'}(\theta)$:

$$\mathcal{M}(e^-_L e^+_R \to \mu^-_L \mu^+_R) \propto d^1_{-1,-1}(\theta) = \frac{1+\cos\theta}{2} \tag{5.105}$$

$$\mathcal{M}(e^-_L e^+_R \to \mu^-_R \mu^+_L) \propto d^1_{-1,+1}(\theta) = \frac{1-\cos\theta}{2} \tag{5.106}$$

$$\mathcal{M}(e^-_R e^+_L \to \mu^-_L \mu^+_R) \propto d^1_{+1,-1}(\theta) = \frac{1-\cos\theta}{2} \tag{5.107}$$

$$\mathcal{M}(e^-_R e^+_L \to \mu^-_R \mu^+_L) \propto d^1_{+1,+1}(\theta) = \frac{1+\cos\theta}{2}\,. \tag{5.108}$$

Squaring and summing (for unpolarized beams, each initial helicity combination has equal weight):

$$\begin{aligned}\sum_{\text{helicities}} |\mathcal{M}|^2 &\propto 2\left(\frac{1+\cos\theta}{2}\right)^2 + 2\left(\frac{1-\cos\theta}{2}\right)^2 \\ &= \frac{1}{2}\left[(1+\cos\theta)^2 + (1-\cos\theta)^2\right] \\ &= 1+\cos^2\theta\,.\end{aligned} \tag{5.109}$$

This provides an elegant derivation of the angular distribution from helicity considerations alone.

Physical interpretation. The $(1+\cos^2\theta)$ distribution can be understood as follows:

- The $\cos^2\theta$ term arises from configurations where the muon helicity matches the electron helicity ($LL \to LL$ or $RR \to RR$ in shorthand notation). These amplitudes are maximal for forward/backward scattering ($\theta = 0, \pi$).
- The constant term (unity) comes from helicity-flip configurations ($LL \to RR$ or $RR \to LL$). These are maximal at $\theta = \pi/2$.

The absence of a $\cos\theta$ term (forward-backward asymmetry) reflects the parity conservation of QED: the process is symmetric under $\theta \to \pi - \theta$.

5.3 Bhabha scattering: $e^+e^- \to e^+e^-$

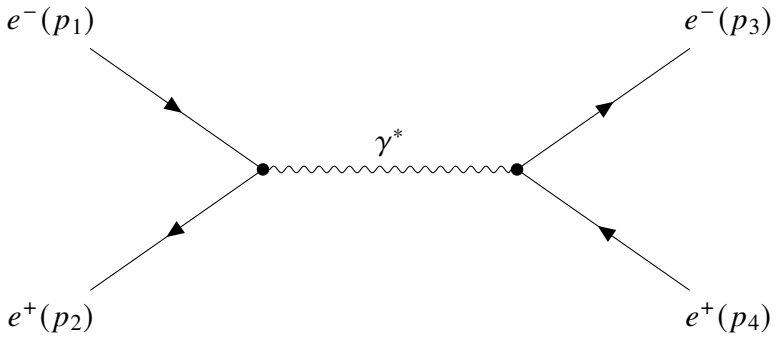
Bhabha scattering is the elastic scattering of electrons and positrons: $e^+e^- \to e^+e^-$. Unlike muon pair production, the same external state can be reached through

two distinct tree-level channels (s and t). Their interference, combined with the fermionic sign conventions, produces a rich angular distribution.

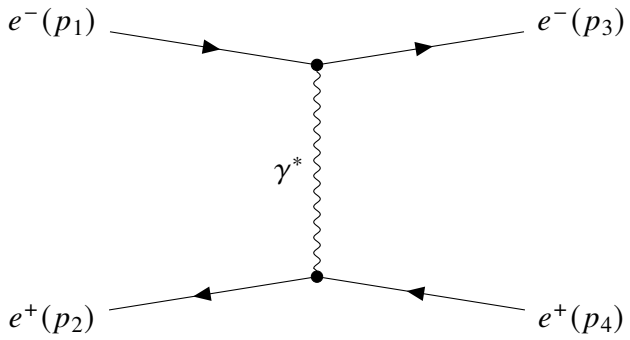
Kinematics and Feynman diagrams. Consider the process:

$$e^-(p_1) + e^+(p_2) \to e^-(p_3) + e^+(p_4)\,. \tag{5.110}$$

At tree level, two diagrams contribute:



(a) s-channel (annihilation)



(b) t-channel (exchange)

In the s-channel, the electron and positron annihilate into a virtual photon with momentum $k = p_1 + p_2$, which then creates a new e^+e^- pair. In the t-channel, a photon is exchanged between the electron and positron lines, with momentum transfer $q = p_1 - p_3$.

Relative sign from Fermi statistics. The two diagrams differ by the assignment of which external particles are handled at each vertex. Recall that at a QED vertex $\bar{\psi}\gamma^\mu\psi$:

- ψ can annihilate an e^- (contributing u) or create an e^+ (contributing v),
- $\bar{\psi}$ can create an e^- (contributing $\bar{u}$) or annihilate an e^+ (contributing $\bar{v}$).

s-channel (annihilation): both incoming particles are absorbed at vertex x, both outgoing particles are emitted at vertex y.

- Vertex x: $\psi(x)$ annihilates $e^-(p_1) \to u(p_1)$; $\bar{\psi}(x)$ annihilates $e^+(p_2) \to \bar{v}(p_2)$.
  Current: $\bar{v}(p_2)\gamma^\mu u(p_1)$.
- Vertex y: $\bar{\psi}(y)$ creates $e^-(p_3) \to \bar{u}(p_3)$; $\psi(y)$ creates $e^+(p_4) \to v(p_4)$.
  Current: $\bar{u}(p_3)\gamma^\nu v(p_4)$.

t-channel (exchange): the electron scatters at vertex x, the positron at vertex y.

- Vertex x: $\psi(x)$ annihilates $e^-(p_1) \to u(p_1)$; $\bar{\psi}(x)$ creates $e^-(p_3) \to \bar{u}(p_3)$.
  Current: $\bar{u}(p_3)\gamma^\mu u(p_1)$.
- Vertex y: $\bar{\psi}(y)$ annihilates $e^+(p_2) \to \bar{v}(p_2)$; $\psi(y)$ creates $e^+(p_4) \to v(p_4)$.
  Current: $\bar{v}(p_2)\gamma^\nu v(p_4)$.

The two channels differ by the exchange of the final-state fermion lines: $v(p_4)$ and $u(p_1)$ swap their vertex assignments. To determine the relative sign, note that the s-channel operator structure (suppressing gamma matrices and spacetime arguments) is

$$\bar{u}_3\, v_4\, \bar{v}_2\, u_1\,, \tag{5.111}$$

while the t-channel has

$$\bar{u}_3\, u_1\, \bar{v}_2\, v_4\,. \tag{5.112}$$

To bring the t-channel ordering into the s-channel ordering, we must move u_1 past $\bar{v}_2$ (one transposition of fermionic operators), and then v_4 past $\bar{v}_2 u_1$ (two more transpositions):

$$\bar{u}_3\, u_1\, \bar{v}_2\, v_4 \xrightarrow{u_1 \leftrightarrow \bar{v}_2} -\bar{u}_3\, \bar{v}_2\, u_1\, v_4 \xrightarrow{v_4 \text{ to front}} +\bar{u}_3\, \bar{v}_2\, v_4\, u_1 \to -\bar{u}_3\, v_4\, \bar{v}_2\, u_1\,. \tag{5.113}$$

Three transpositions give a factor $(-1)^3 = -1$, so the total amplitude is:

$$\boxed{\mathcal{M} = \mathcal{M}_s - \mathcal{M}_t}\,. \tag{5.114}$$

Individual amplitudes. The s-channel amplitude is (with $k^2 = (p_1 + p_2)^2 = s$):

$$\boxed{\mathcal{M}_s = \frac{e^2}{s}\big[\bar{u}(p_3)\gamma^\mu v(p_4)\big]\big[\bar{v}(p_2)\gamma_\mu u(p_1)\big]}\,. \tag{5.115}$$

The t-channel amplitude is (with $q^2 = (p_1 - p_3)^2 = t$):

$$\boxed{\mathcal{M}_t = \frac{e^2}{t}\big[\bar{u}(p_3)\gamma^\nu u(p_1)\big]\big[\bar{v}(p_2)\gamma_\nu v(p_4)\big]}\,. \tag{5.116}$$

Spin-averaged amplitude squared. For unpolarized beams, the spin-averaged amplitude squared is:

$$\begin{aligned}\overline{|\mathcal{M}|^2} &= \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_s - \mathcal{M}_t|^2 \\ &= \frac{1}{4}\sum_{\text{spins}} \left(|\mathcal{M}_s|^2 + |\mathcal{M}_t|^2 - \mathcal{M}_s\mathcal{M}_t^* - \mathcal{M}_s^*\mathcal{M}_t\right) \\ &= X_{ss} + X_{tt} + X_{st}\,, \end{aligned} \tag{5.117}$$

where we define:

$$X_{ss} = \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_s|^2 \tag{5.118}$$

$$X_{tt} = \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_t|^2 \tag{5.119}$$

$$X_{st} = -\frac{1}{4}\sum_{\text{spins}} \left(\mathcal{M}_s\mathcal{M}_t^* + \mathcal{M}_s^*\mathcal{M}_t\right) = -\frac{1}{2}\sum_{\text{spins}} \text{Re}(\mathcal{M}_s\mathcal{M}_t^*)\,. \tag{5.120}$$

Calculation of X_{ss}. This is structurally identical to the $e^+e^- \to \mu^+\mu^-$ calculation (Section 5.2), with the replacement $m_\mu \to m_e$. We have:

$$|\mathcal{M}_s|^2 = \frac{e^4}{s^2}\left[\bar{u}(p_3)\gamma^\mu v(p_4)\right]\left[\bar{v}(p_2)\gamma_\mu u(p_1)\right]\left[\bar{v}(p_4)\gamma^\rho u(p_3)\right]\left[\bar{u}(p_1)\gamma_\rho v(p_2)\right]. \tag{5.121}$$

We now show in detail how to convert this product of spinor bilinears into traces. Writing the spinor indices explicitly:

$$\begin{aligned}|\mathcal{M}_s|^2 = \frac{e^4}{s^2}&\left[\bar{u}_\alpha(p_3)(\gamma^\mu)_{\alpha\beta}v_\beta(p_4)\right]\left[\bar{v}_\gamma(p_2)(\gamma_\mu)_{\gamma\delta}u_\delta(p_1)\right] \\ &\times\left[\bar{v}_\epsilon(p_4)(\gamma^\rho)_{\epsilon\zeta}u_\zeta(p_3)\right]\left[\bar{u}_\eta(p_1)(\gamma_\rho)_{\eta\theta}v_\theta(p_2)\right]. \end{aligned} \tag{5.122}$$

When we sum over spins, we use the completeness relations (BD normalization):

$$\sum_s u_\alpha(p,s)\bar{u}_\beta(p,s) = \frac{(\not{p}+m)_{\alpha\beta}}{2m} \tag{5.123}$$

$$\sum_s v_\alpha(p,s)\bar{v}_\beta(p,s) = \frac{(\not{p}-m)_{\alpha\beta}}{2m}\,. \tag{5.124}$$

For the p_3 electron: $\sum_{s_3} u_\zeta(p_3)\bar{u}_\alpha(p_3) = (\not{p}_3 + m)_{\zeta\alpha}/(2m)$
For the p_4 positron: $\sum_{s_4} v_\beta(p_4)\bar{v}_\epsilon(p_4) = (\not{p}_4 - m)_{\beta\epsilon}/(2m)$

For the p_1 electron: $\sum_{s_1} u_\delta(p_1)\bar{u}_\eta(p_1) = (\not{p}_1 + m)_{\delta\eta}/(2m)$
For the p_2 positron: $\sum_{s_2} v_\theta(p_2)\bar{v}_\gamma(p_2) = (\not{p}_2 - m)_{\theta\gamma}/(2m)$
Substituting these into the spin sum and grouping indices that contract:

$$\sum_{\text{spins}} |\mathcal{M}_s|^2 = \frac{e^4}{s^2}\frac{1}{(2m)^4}(\not{p}_3 + m)_{\zeta\alpha}(\gamma^\mu)_{\alpha\beta}(\not{p}_4 - m)_{\beta\epsilon}(\gamma^\rho)_{\epsilon\zeta} \times (\not{p}_2 - m)_{\theta\gamma}(\gamma_\mu)_{\gamma\delta}(\not{p}_1 + m)_{\delta\eta}(\gamma_\rho)_{\eta\theta}\,. \tag{5.125}$$

In the first line, the indices form a closed loop: $\zeta \to \alpha \to \beta \to \epsilon \to \zeta$. This is precisely the definition of a trace:

$$(\not{p}_3 + m)_{\zeta\alpha}(\gamma^\mu)_{\alpha\beta}(\not{p}_4 - m)_{\beta\epsilon}(\gamma^\rho)_{\epsilon\zeta} = \mathrm{Tr}\big[(\not{p}_3 + m)\gamma^\mu(\not{p}_4 - m)\gamma^\rho\big]\,. \tag{5.126}$$

Similarly, the second line gives:

$$(\not{p}_2 - m)_{\theta\gamma}(\gamma_\mu)_{\gamma\delta}(\not{p}_1 + m)_{\delta\eta}(\gamma_\rho)_{\eta\theta} = \mathrm{Tr}\big[(\not{p}_2 - m)\gamma_\mu(\not{p}_1 + m)\gamma_\rho\big]\,. \tag{5.127}$$

Therefore:

$$\boxed{\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_s|^2 = \frac{e^4}{4s^2}\frac{1}{(2m)^4}\mathrm{Tr}\big[(\not{p}_3 + m)\gamma^\mu(\not{p}_4 - m)\gamma^\rho\big]\,\mathrm{Tr}\big[(\not{p}_2 - m)\gamma_\mu(\not{p}_1 + m)\gamma_\rho\big]}\,. \tag{5.128}$$

In the ultra-relativistic limit ($m \to 0$), the BD normalization factors $(2m)^{-4}$ cancel against $\prod(2m_i)$ in the cross section formula (cf. eq. (5.72)), so from now on we drop them and use the standard completeness relations $\sum u\bar{u} = \not{p}$, $\sum v\bar{v} = \not{p}$ in the massless limit. Using the trace identity:

$$\mathrm{Tr}(\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta) = 4(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\delta}g^{\beta\gamma})\,, \tag{5.129}$$

we obtain:

$$\mathrm{Tr}(\not{p}_3\gamma^\mu\not{p}_4\gamma^\rho) = 4(p_3^\mu p_4^\rho - (p_3\cdot p_4)g^{\mu\rho} + p_3^\rho p_4^\mu) \tag{5.130}$$

$$\mathrm{Tr}(\not{p}_2\gamma_\mu\not{p}_1\gamma_\rho) = 4(p_{2\mu}p_{1\rho} - (p_2\cdot p_1)g_{\mu\rho} + p_{2\rho}p_{1\mu})\,. \tag{5.131}$$

Contracting these two traces:

$$\begin{aligned}
&\mathrm{Tr}(\not{p}_3\gamma^\mu\not{p}_4\gamma^\rho)\,\mathrm{Tr}(\not{p}_2\gamma_\mu\not{p}_1\gamma_\rho)\\
&= 16\big[(p_3\cdot p_2)(p_4\cdot p_1) - (p_3\cdot p_4)(p_1\cdot p_2) + (p_3\cdot p_1)(p_4\cdot p_2)\\
&\qquad - (p_3\cdot p_4)(p_1\cdot p_2) + 4(p_3\cdot p_4)(p_1\cdot p_2) - (p_3\cdot p_4)(p_1\cdot p_2)\\
&\qquad + (p_4\cdot p_2)(p_3\cdot p_1) - (p_3\cdot p_4)(p_1\cdot p_2) + (p_4\cdot p_1)(p_3\cdot p_2)\big]\\
&= 32\big[(p_1\cdot p_3)(p_2\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3)\big]\,.
\end{aligned} \tag{5.132}$$

Therefore:

$$\boxed{X_{ss} = \frac{8e^4}{s^2}\left[(p_1\cdot p_3)(p_2\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3)\right]}\,. \tag{5.133}$$

Calculation of X_{tt}. For the t-channel:

$$|\mathcal{M}_t|^2 = \frac{e^4}{t^2}\left[\bar{u}(p_3)\gamma^\nu u(p_1)\right]\left[\bar{v}(p_2)\gamma_\nu v(p_4)\right]\left[\bar{u}(p_1)\gamma^\sigma u(p_3)\right]\left[\bar{v}(p_4)\gamma_\sigma v(p_2)\right]. \tag{5.134}$$

Converting to traces (in the $m \to 0$ limit):

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}_t|^2 = \frac{e^4}{4t^2}\,\mathrm{Tr}(\not{p}_3\gamma^\nu\not{p}_1\gamma^\sigma)\,\mathrm{Tr}(\not{p}_2\gamma_\nu\not{p}_4\gamma_\sigma) \tag{5.135}$$

Following the same procedure as for X_{ss}:

$$\mathrm{Tr}(\not{p}_3\gamma^\nu\not{p}_1\gamma^\sigma)\,\mathrm{Tr}(\not{p}_2\gamma_\nu\not{p}_4\gamma_\sigma) = 32\left[(p_1\cdot p_2)(p_3\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3)\right]. \tag{5.136}$$

Therefore:

$$\boxed{X_{tt} = \frac{8e^4}{t^2}\left[(p_1\cdot p_2)(p_3\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3)\right]}\,. \tag{5.137}$$

Calculation of X_{st} (interference term). The interference term requires more care. We have:

$$\begin{aligned}-\mathcal{M}_s^*\mathcal{M}_t = -\frac{e^4}{st}&\left[\bar{u}(p_1)\gamma^\mu v(p_2)\right]\left[\bar{v}(p_4)\gamma_\mu u(p_3)\right]\\ &\times\left[\bar{u}(p_3)\gamma^\nu u(p_1)\right]\left[\bar{v}(p_2)\gamma_\nu v(p_4)\right].\end{aligned} \tag{5.138}$$

To convert this to a trace, we write the product of all four spinor bilinears with explicit indices and use the completeness relations. The key observation is that all spinor indices must be contracted in a single closed loop. Tracing the index chain $\bar{u}(p_1) \to v(p_2) \to \bar{v}(p_2) \to v(p_4) \to \bar{v}(p_4) \to u(p_3) \to \bar{u}(p_3) \to u(p_1)$ and summing over spins (in the $m \to 0$ limit):

$$\frac{1}{4}\sum_{\text{spins}}\left(-\mathcal{M}_s^*\mathcal{M}_t\right) = -\frac{e^4}{4st}\,\mathrm{Tr}(\not{p}_1\gamma^\mu\not{p}_2\gamma_\nu\not{p}_4\gamma_\mu\not{p}_3\gamma^\nu)\,. \tag{5.139}$$

This is a trace of eight gamma matrices. We contract the μ indices first using $\gamma^\mu\not{a}\not{b}\not{c}\gamma_\mu = -2\not{c}\not{b}\not{a}$:

$$\mathrm{Tr}(\not{p}_1\gamma^\mu\not{p}_2\gamma_\nu\not{p}_4\gamma_\mu\not{p}_3\gamma^\nu) = -2\,\mathrm{Tr}(\not{p}_1\not{p}_4\gamma_\nu\not{p}_2\not{p}_3\gamma^\nu)\,. \tag{5.140}$$

Next, we use $\gamma_\nu \not{a} \not{b} \gamma^\nu = 4(a \cdot b)$:

$$\begin{aligned} -2\,\mathrm{Tr}(\not{p}_1 \not{p}_4 \gamma_\nu \not{p}_2 \not{p}_3 \gamma^\nu) &= -2 \cdot 4 (p_2 \cdot p_3)\,\mathrm{Tr}(\not{p}_1 \not{p}_4) \\ &= -32\,(p_2 \cdot p_3)(p_1 \cdot p_4)\,. \end{aligned} \tag{5.141}$$

Using $p_2 \cdot p_3 = -u/2$ and $p_1 \cdot p_4 = -u/2$ in the massless limit:

$$-32 \cdot \frac{u^2}{4} = -8u^2\,. \tag{5.142}$$

Therefore, in the massless limit this contribution simplifies to:

$$\frac{1}{4} \sum_{\text{spins}} \left(-\mathcal{M}_s^* \mathcal{M}_t\right) = -\frac{e^4}{4st} \cdot (-8u^2) = \frac{2e^4 u^2}{st}\,. \tag{5.143}$$

To obtain the full interference term, recall from eq. (5.117) that:

$$X_{st} = -\frac{1}{4} \sum_{\text{spins}} \left(\mathcal{M}_s \mathcal{M}_t^* + \mathcal{M}_s^* \mathcal{M}_t\right). \tag{5.144}$$

For any complex number z, we have $z + z^* = 2\,\mathrm{Re}(z)$. Applying this with $z = \mathcal{M}_s^* \mathcal{M}_t$:

$$\mathcal{M}_s \mathcal{M}_t^* + \mathcal{M}_s^* \mathcal{M}_t = (\mathcal{M}_s^* \mathcal{M}_t)^* + \mathcal{M}_s^* \mathcal{M}_t = 2\,\mathrm{Re}(\mathcal{M}_s^* \mathcal{M}_t)\,. \tag{5.145}$$

Therefore:

$$X_{st} = -\frac{1}{4} \sum_{\text{spins}} 2\,\mathrm{Re}(\mathcal{M}_s^* \mathcal{M}_t) = -\frac{1}{2} \sum_{\text{spins}} \mathrm{Re}(\mathcal{M}_s^* \mathcal{M}_t) = -\frac{1}{2}\,\mathrm{Re}\left[\sum_{\text{spins}} \mathcal{M}_s^* \mathcal{M}_t\right], \tag{5.146}$$

where in the last step we used the linearity of the real part, $\mathrm{Re}(\sum_i z_i) = \sum_i \mathrm{Re}(z_i)$.

Since the trace calculation gives a real result (all scalar products $p_i \cdot p_j$ are real), we can express the interference term directly in Mandelstam variables. In the massless limit this becomes:

$$\boxed{X_{st} = \frac{4e^4 u^2}{st}}\,. \tag{5.147}$$

Result in Mandelstam variables. In the center-of-mass frame with $m_e \to 0$, we have:

$$p_1 = (E, \vec{p}), \qquad p_2 = (E, -\vec{p}) \tag{5.148}$$

$$p_3 = (E, \vec{p}\,'), \qquad p_4 = (E, -\vec{p}\,')\,, \tag{5.149}$$

with $|\vec{p}| = |\vec{p}\,'| = E$ in the massless limit. The Mandelstam variables are:

$$s = (p_1 + p_2)^2 = 4E^2 \tag{5.150}$$

$$t = (p_1 - p_3)^2 = -2E^2(1 - \cos\theta) \tag{5.151}$$

$$u = (p_1 - p_4)^2 = -2E^2(1 + \cos\theta)\,, \tag{5.152}$$

where θ is the scattering angle. They satisfy $s + t + u = 0$ (for massless particles).

The scalar products become:

$$p_1 \cdot p_2 = \frac{s}{2}, \qquad p_3 \cdot p_4 = \frac{s}{2} \tag{5.153}$$

$$p_1 \cdot p_3 = -\frac{t}{2}, \qquad p_2 \cdot p_4 = -\frac{t}{2} \tag{5.154}$$

$$p_1 \cdot p_4 = -\frac{u}{2}, \qquad p_2 \cdot p_3 = -\frac{u}{2}\,. \tag{5.155}$$

Substituting into eqs. (5.133), (5.137), and (5.147):

$$X_{ss} = \frac{8e^4}{s^2}\left[\frac{t^2}{4} + \frac{u^2}{4}\right] = \frac{2e^4}{s^2}(t^2 + u^2) \tag{5.156}$$

$$X_{tt} = \frac{8e^4}{t^2}\left[\frac{s^2}{4} + \frac{u^2}{4}\right] = \frac{2e^4}{t^2}(s^2 + u^2) \tag{5.157}$$

$$X_{st} = \frac{4e^4u^2}{st}\,. \tag{5.158}$$

The total spin-averaged amplitude squared is:

$$\boxed{\overline{|\mathcal{M}|^2} = 2e^4\left[\frac{t^2+u^2}{s^2} + \frac{s^2+u^2}{t^2} + \frac{2u^2}{st}\right]}\,. \tag{5.159}$$

Differential cross section. We now derive the differential cross section from $\overline{|\mathcal{M}|^2}$, showing every step explicitly. The starting point is the BD master formula (3.55):

$$d\sigma = \frac{1}{\mathcal{F}}\left(\prod_{i\in\text{fermions}} 2m_i\right)\overline{|\mathcal{M}_{\text{BD}}|^2}\, d\Phi_2\,. \tag{5.160}$$

As shown for $e^+e^- \to \mu^+\mu^-$ (eq. (5.72)), the factor $\prod(2m_i)$ cancels the $1/(2m)^2$ from the BD spin sums, so the net result is equivalent to using:

$$d\sigma = \frac{1}{\mathcal{F}}\,\overline{|\mathcal{M}|^2}\, d\Phi_2\,, \tag{5.161}$$

where $\overline{|\mathcal{M}|^2}$ denotes the trace expression *without* the $1/(2m)^2$ prefactors.

We now evaluate the flux factor $\mathcal{F}$ and the two-body phase space $d\Phi_2$ in the CM frame.

Step 1: The flux factor. By definition (eq. (3.53)), $\mathcal{F} = 4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}$. In the CM frame, $\vec{p}_1 + \vec{p}_2 = 0$, so $p_1 = (E_1, \vec{p})$ and $p_2 = (E_2, -\vec{p})$. Then:

$$p_1 \cdot p_2 = E_1 E_2 + |\vec{p}|^2 . \tag{5.162}$$

Also $s = (p_1 + p_2)^2 = (E_1 + E_2)^2$, so $E_1 + E_2 = \sqrt{s}$. Computing:

$$\begin{aligned}(p_1 \cdot p_2)^2 - m_1^2 m_2^2 &= (E_1 E_2 + |\vec{p}|^2)^2 - (E_1^2 - |\vec{p}|^2)(E_2^2 - |\vec{p}|^2) \\ &= E_1^2 E_2^2 + 2E_1 E_2 |\vec{p}|^2 + |\vec{p}|^4 - E_1^2 E_2^2 + (E_1^2 + E_2^2)|\vec{p}|^2 - |\vec{p}|^4 \\ &= (2E_1 E_2 + E_1^2 + E_2^2)\,|\vec{p}|^2 = (E_1 + E_2)^2\,|\vec{p}|^2 = s\,|\vec{p}|^2 .\end{aligned} \tag{5.163}$$

Therefore:

$$\mathcal{F} = 4\sqrt{s}\,|\vec{p}\,| . \tag{5.164}$$

Step 2: The two-body phase space $d\Phi_2$. The Lorentz-invariant n-body phase space (eq. (3.45)) for $n = 2$ reads:

$$d\Phi_2 = (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4)\, \frac{d^3 p_3}{(2\pi)^3\, 2E_3}\, \frac{d^3 p_4}{(2\pi)^3\, 2E_4} . \tag{5.165}$$

We evaluate this in the CM frame where $\vec{p}_1 + \vec{p}_2 = 0$, so the delta function enforces $\vec{p}_3 + \vec{p}_4 = 0$ and $E_3 + E_4 = \sqrt{s}$.

Integrate over $\vec{p}_4$: the three-momentum delta function $\delta^{(3)}(\vec{p}_3 + \vec{p}_4)$ sets $\vec{p}_4 = -\vec{p}_3$, removing $d^3 p_4$ and one factor of $(2\pi)^3$:

$$d\Phi_2 = \frac{(2\pi)^4}{(2\pi)^6}\, \delta(E_3 + E_4 - \sqrt{s})\, \frac{d^3 p_3}{2E_3 \cdot 2E_4}\bigg|_{\vec{p}_4 = -\vec{p}_3} = \frac{1}{4\pi^2}\, \delta(E_3 + E_4 - \sqrt{s})\, \frac{d^3 p_3}{4E_3 E_4} . \tag{5.166}$$

Switch to spherical coordinates: write $d^3 p_3 = |\vec{p}\,'|^2\, d|\vec{p}\,'|\, d\Omega$ where $|\vec{p}\,'| \equiv |\vec{p}_3|$ and $d\Omega = \sin\theta\, d\theta\, d\varphi$:

$$d\Phi_2 = \frac{1}{16\pi^2}\, \delta(E_3 + E_4 - \sqrt{s})\, \frac{|\vec{p}\,'|^2\, d|\vec{p}\,'|}{E_3 E_4}\, d\Omega . \tag{5.167}$$

Integrate over $|\vec{p}\,'|$: the remaining delta function fixes $|\vec{p}\,'|$ via $E_3(|\vec{p}\,'|) + E_4(|\vec{p}\,'|) = \sqrt{s}$, where $E_i = \sqrt{|\vec{p}\,'|^2 + m_i^2}$. To evaluate this, we need:

$$\frac{d(E_3 + E_4)}{d|\vec{p}\,'|} = \frac{|\vec{p}\,'|}{E_3} + \frac{|\vec{p}\,'|}{E_4} = |\vec{p}\,'|\, \frac{E_3 + E_4}{E_3 E_4} = |\vec{p}\,'|\, \frac{\sqrt{s}}{E_3 E_4} , \tag{5.168}$$

where in the last step we used $E_3 + E_4 = \sqrt{s}$ (imposed by the delta function). Therefore:

$$\delta(E_3 + E_4 - \sqrt{s}) = \frac{E_3 E_4}{|\vec{p}\,'|\sqrt{s}}\,\delta(|\vec{p}\,'| - |\vec{p}\,'|_0)\,, \tag{5.169}$$

where $|\vec{p}\,'|_0$ is the value satisfying the energy constraint. Substituting into (5.167) and integrating over $|\vec{p}\,'|$:

$$d\Phi_2 = \frac{1}{16\pi^2}\frac{|\vec{p}\,'|^2}{E_3 E_4}\cdot\frac{E_3 E_4}{|\vec{p}\,'|\sqrt{s}}\,d\Omega = \frac{|\vec{p}\,'|}{16\pi^2\sqrt{s}}\,d\Omega\,. \tag{5.170}$$

Step 3: Combining flux and phase space. Substituting (5.164) and (5.170) into (5.161):

$$\frac{d\sigma}{d\Omega} = \frac{1}{4\sqrt{s}\,|\vec{p}\,|}\cdot\frac{|\vec{p}\,'|}{16\pi^2\sqrt{s}}\,\overline{|\mathcal{M}|^2} = \frac{|\vec{p}\,'|}{64\pi^2 s\,|\vec{p}\,|}\,\overline{|\mathcal{M}|^2}\,. \tag{5.171}$$

This is the **general** $2 \to 2$ **CM-frame cross section**, valid for arbitrary masses.

Step 4: Ultrarelativistic limit. For $m_e \to 0$ (Bhabha scattering at $\sqrt{s} \gg m_e$), both initial and final particles are effectively massless: $|\vec{p}\,| = |\vec{p}\,'| = E = \sqrt{s}/2$. Therefore $|\vec{p}\,'|/|\vec{p}\,| = 1$ and:

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\,\overline{|\mathcal{M}|^2}}\,. \tag{5.172}$$

Substituting eq. (5.159) and using $e^4 = (4\pi\alpha)^2 = 16\pi^2\alpha^2$:

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \frac{1}{64\pi^2 s}\cdot 2e^4\left[\frac{t^2+u^2}{s^2} + \frac{s^2+u^2}{t^2} + \frac{2u^2}{st}\right] \\ &= \frac{2\cdot 16\pi^2\alpha^2}{64\pi^2 s}\left[\frac{t^2+u^2}{s^2} + \frac{s^2+u^2}{t^2} + \frac{2u^2}{st}\right].\end{aligned} \tag{5.173}$$

Therefore:

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2s}\left[\frac{t^2+u^2}{s^2} + \frac{s^2+u^2}{t^2} + \frac{2u^2}{st}\right]}\,. \tag{5.174}$$

This is the Bhabha cross section in Lorentz-invariant form. Note the three contributions: s-channel ($\propto 1/s^2$), t-channel ($\propto 1/t^2$), and interference ($\propto 1/(st)$).

Angular distribution. To express (5.174) in terms of the scattering angle θ, we substitute the CM-frame Mandelstam variables $t = -\frac{s}{2}(1-\cos\theta)$ and $u =$

$-\frac{s}{2}(1+\cos\theta)$. Each ratio becomes:

$$\frac{t^2+u^2}{s^2} = \frac{s^2(1-\cos\theta)^2/4 + s^2(1+\cos\theta)^2/4}{s^2} = \frac{1}{2}(1+\cos^2\theta)\,, \tag{5.175}$$

$$\frac{s^2+u^2}{t^2} = \frac{s^2+s^2(1+\cos\theta)^2/4}{s^2(1-\cos\theta)^2/4} = \frac{4+(1+\cos\theta)^2}{(1-\cos\theta)^2}\,, \tag{5.176}$$

$$\frac{2u^2}{st} = \frac{2\cdot s^2(1+\cos\theta)^2/4}{s\cdot[-s(1-\cos\theta)/2]} = -\frac{(1+\cos\theta)^2}{1-\cos\theta}\,. \tag{5.177}$$

Substituting into (5.174):

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2s}\left[\frac{1}{2}(1+\cos^2\theta) + \frac{4+(1+\cos\theta)^2}{(1-\cos\theta)^2} - \frac{(1+\cos\theta)^2}{1-\cos\theta}\right]}\,. \tag{5.178}$$

The three terms have a clear physical origin: the first is the familiar s-channel annihilation contribution (identical to $e^+e^- \to \mu^+\mu^-$), the second is the t-channel exchange (which diverges as $\theta \to 0$), and the third is the s-t interference.

Physical features. Several notable features distinguish Bhabha scattering from muon pair production:

- **Forward divergence**: The $1/t^2$ term from the t-channel causes the cross section to diverge as $\theta \to 0$ (forward scattering). This is characteristic of Coulomb scattering—the long-range nature of the electromagnetic interaction leads to a preference for small-angle scattering.
- **Interference**: The s-t interference term $2u^2/(st)$ is negative for backward scattering ($\cos\theta < 0$, where $u > 0$, $t < 0$, $s > 0$), reducing the cross section relative to the sum of s- and t-channel contributions alone.
- **Annihilation channel**: At $\theta = 90°$ ($t = u = -s/2$), all three terms contribute comparably. At high energies and large angles, the annihilation diagram becomes relatively more important.
- **Luminosity monitoring**: Despite the forward divergence, Bhabha scattering at specific angles provides a precisely calculable process used for luminosity measurements at e^+e^- colliders.

Box 5.5 Comparison with $e^+e^- \to \mu^+\mu^-$

For muon pair production, only the s-channel contributes, giving:

$$\overline{|\mathcal{M}|^2}_{e^+e^-\to\mu^+\mu^-} = \frac{2e^4(t^2+u^2)}{s^2}\,. \tag{5.179}$$

This corresponds to setting $X_{tt} = X_{st} = 0$ in the Bhabha formula. The angular distribution $(1+\cos^2\theta)$ is characteristic of spin-1 intermediate states coupling

to spin-$\frac{1}{2}$ fermions. In Bhabha scattering, the additional t-channel and interference terms modify this pattern, creating the characteristic forward peak.

The forward divergence: physics and regularization

The $1/t^2$ divergence as $\theta \to 0$ deserves careful discussion. This is *not* an artifact of the calculation but reflects the physics of long-range Coulomb interactions.

Origin of the divergence. In classical mechanics, the Rutherford cross section for Coulomb scattering also diverges as $\theta \to 0$:

$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{Rutherford}} = \frac{\alpha^2}{4E^2 \sin^4(\theta/2)} \xrightarrow{\theta\to 0} \infty\,. \tag{5.180}$$

The divergence arises because the Coulomb potential $V(r) \sim 1/r$ has infinite range. Particles with arbitrarily large impact parameters still experience a (small) deflection, leading to an infinite total cross section. This is fundamentally different from scattering off a finite-range potential, where the cross section is finite.

Why the total cross section diverges. Integrating the differential cross section:

$$\begin{aligned}\sigma_{\text{tot}} &= \int \frac{d\sigma}{d\Omega}\, d\Omega \sim \int_0^\pi \frac{\sin\theta\, d\theta}{(1-\cos\theta)^2} \\ &\sim \int_0 \frac{d\theta}{\theta^3} \to \infty\,.\end{aligned} \tag{5.181}$$

The integral diverges at the lower limit. Physically, this means that *every* electron-positron pair in a beam "scatters" to some degree—most by unmeasurably small angles.

Regularization in practice. Several mechanisms regularize this divergence:

1. **Angular cutoff**: In any experiment, detectors have finite angular acceptance. Defining a minimum scattering angle $\theta_{\min}$ renders the cross section finite:

$$\sigma(\theta > \theta_{\min}) = \int_{\theta_{\min}}^\pi \frac{d\sigma}{d\Omega}\, d\Omega < \infty\,. \tag{5.182}$$

2. **Screening**: In matter, the Coulomb potential is screened at large distances by other charges. This effectively gives the photon a small mass $m_\gamma \sim 1/\lambda_D$ (where λ_D is the Debye length), cutting off the divergence.

3. **Vacuum polarization**: At the quantum level, virtual e^+e^- pairs (and other charged particles) momentarily appear from the vacuum and modify the photon propagator. The bare propagator $1/t$ is replaced by:

$$\frac{1}{t} \to \frac{1}{t\big[1-\Pi(t)\big]}, \tag{5.183}$$

where $\Pi(t)$ is the vacuum polarization function. In renormalized QED, gauge invariance implies $\Pi(0) = 0$, so vacuum polarization does *not* remove the Coulombic $1/t$ pole in the forward direction: the photon remains massless, and the small-angle enhancement persists. The practical regularization of the forward divergence therefore comes from angular cuts, screening in matter, finite beam size, and finite detector resolution.
4. **Finite beam size**: Real beams have finite spatial extent, which limits the maximum impact parameter and thus the minimum scattering angle.

Detailed comparison: Bhabha vs. muon pair production

The structural differences between $e^+e^- \to e^+e^-$ and $e^+e^- \to \mu^+\mu^-$ provide insight into the role of channel interference and crossing structure in QED.

Channel structure.

	Bhabha ($e^+e^- \to e^+e^-$)	**Muon production** ($e^+e^- \to \mu^+\mu^-$)
s-channel	✓ (annihilation)	✓ (annihilation)
t-channel	✓ (exchange)	× (forbidden)
Interference	✓ (s-t)	×

The t-channel is forbidden in muon production because it would require an $e \to \mu$ transition, which violates lepton flavor conservation at tree level.

Angular distributions. Writing the differential cross sections explicitly:

$$\left(\frac{d\sigma}{d\Omega}\right)_{e^+e^-\to\mu^+\mu^-} = \frac{\alpha^2}{4s}(1+\cos^2\theta)\,, \tag{5.184}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{e^+e^-\to e^+e^-} = \frac{\alpha^2}{2s}\Bigg[\underbrace{\frac{1+\cos^2\theta}{2}}_{\text{from } s\text{-channel}} + \underbrace{\frac{4+(1+\cos\theta)^2}{(1-\cos\theta)^2}}_{\text{from } t\text{-channel}} - \underbrace{\frac{(1+\cos\theta)^2}{1-\cos\theta}}_{\text{interference}}\Bigg]. \tag{5.185}$$

Key differences:

- **Forward region** ($\theta \to 0$): Muon production $\to \frac{\alpha^2}{2s}$ (finite); Bhabha $\to \infty$ (divergent)
- **Backward region** ($\theta \to \pi$): Both finite, but Bhabha is suppressed by destructive s-t interference
- **At** $\theta = 90°$: Muon production gives $\frac{\alpha^2}{4s}$; Bhabha gives $\frac{3\alpha^2}{2s}$ (enhanced by t-channel)

Ratio of cross sections. At large angles, the t-channel contribution becomes subdominant. To see why, recall that the t-channel amplitude scales as $1/t$ where $t = -2E^2(1-\cos\theta)$. At backward angles ($\theta \to \pi$):

$$|t| = 2E^2(1-\cos\theta) \xrightarrow{\theta\to\pi} 4E^2 = s\,. \tag{5.186}$$

Thus $X_{tt} \sim 1/t^2 \sim 1/s^2$, which is the same order as $X_{ss} \sim 1/s^2$. At backward angles, the interference term $X_{st} \sim 1/(st)$ is *destructive*, so the total Bhabha cross section remains of the same parametric order as the annihilation contribution rather than being forward-enhanced.

Defining the ratio $R(\theta) = (d\sigma/d\Omega)_{\text{Bhabha}}/(d\sigma/d\Omega)_{\mu\mu}$:

$$R(\theta) \xrightarrow{\theta\to\pi} 2 + O(1+\cos\theta)\,. \tag{5.187}$$

At small angles, the ratio diverges as $R(\theta) \sim 1/\theta^4$ due to the $1/t^2$ enhancement.

Helicity amplitude analysis

The helicity structure of Bhabha scattering extends the analysis presented for $e^+e^- \to \mu^+\mu^-$ in Section 5.2. We now have two channels whose interference creates a richer helicity structure.

Review: helicity selection rules. As discussed in Section 5.2, for massless fermions helicity is conserved at QED vertices. The virtual photon couples only to $f\bar{f}$ pairs with opposite helicities. The initial state $e^-_L e^+_R$ gives $J_z = h_{e^-} - h_{e^+} = -\frac{1}{2} - \frac{1}{2} = -1$, while $e^-_R e^+_L$ gives $J_z = +1$. Configurations like $e^-_L e^+_L$ have $J_z = 0$ and do not couple to a transverse photon.

***s*-channel amplitudes (same as $\mu^+\mu^-$).** The s-channel in Bhabha scattering has exactly the same helicity structure as muon pair production. Using the Wigner d-matrices introduced in the box of Section 5.2:

$$\mathcal{M}_s(LR \to LR) \propto \frac{e^2}{s} d^1_{-1,-1}(\theta) = \frac{e^2}{s}\frac{1+\cos\theta}{2} \tag{5.188}$$

$$\mathcal{M}_s(LR \to RL) \propto \frac{e^2}{s} d^1_{-1,+1}(\theta) = \frac{e^2}{s}\frac{1-\cos\theta}{2}, \tag{5.189}$$

with analogous expressions for $RL \to LR$ and $RL \to RL$ by the substitution $\theta \to \pi - \theta$ in the d-matrices.

t-channel amplitudes. The t-channel has a different helicity structure. Here, the electron and positron lines are separate—each scatters independently off the exchanged photon. Helicity is conserved along each fermion line:

- $e^-_L \to e^-_L$ (electron line preserves helicity)
- $e^+_R \to e^+_R$ (positron line preserves helicity)

Thus the t-channel only contributes to helicity-conserving configurations:

$$\begin{aligned}\mathcal{M}_t(LR \to LR) &= \frac{e^2}{t}\,[\bar{u}_L(p_3)\gamma^\mu u_L(p_1)] \\ &\quad \times [\bar{v}_R(p_2)\gamma_\mu v_R(p_4)] \\ &\propto \frac{e^2}{t}\cdot s \cdot \frac{1+\cos\theta}{2}\,. \end{aligned} \tag{5.190}$$

The factor $s(1+\cos\theta)/2$ arises from the spinor products. Crucially, $\mathcal{M}_t(LR \to RL) = 0$ because helicity flip is forbidden in the t-channel.

Total amplitudes and interference. For the helicity-conserving configuration $LR \to LR$, both channels contribute:

$$\mathcal{M}(LR \to LR) = \mathcal{M}_s - \mathcal{M}_t = e^2\left[\frac{1}{s} - \frac{1}{t}\right]\frac{1+\cos\theta}{2}\cdot(\text{spinor factor})\,. \tag{5.191}$$

The relative minus sign from Fermi statistics is crucial. Since $s > 0$ and $t < 0$, both terms have the *same sign*, leading to constructive interference at forward angles and destructive interference at backward angles.

For the helicity-flip configuration $LR \to RL$, only the s-channel contributes:

$$\mathcal{M}(LR \to RL) = \mathcal{M}_s = \frac{e^2}{s}\,\frac{1-\cos\theta}{2}\cdot(\text{spinor factor})\,. \tag{5.192}$$

Angular distribution from helicity sum. Squaring and summing over helicities (factor of 2 for $LR \leftrightarrow RL$ symmetry):

$$\sum_{\text{hel}}|\mathcal{M}|^2 = 2\left|\frac{e^2(1+\cos\theta)}{2}\left(\frac{1}{s}-\frac{1}{t}\right)\right|^2 + 2\left|\frac{e^2(1-\cos\theta)}{2s}\right|^2 + (L \leftrightarrow R)\,. \tag{5.193}$$

The first term (helicity-conserving) contains the t-channel pole and s-t interference; the second term (helicity-flip) is pure s-channel. This reproduces the trace calculation and explains the physical origin of each contribution in eq. (5.159).

Experimental applications and precision tests

Bhabha scattering plays a central role in the physics program of e^+e^- colliders, serving both as a luminosity monitor and as a probe of QED at high precision.

Luminosity determination. The integrated luminosity $\mathcal{L}$ relates the number of observed events N to the cross section σ:

$$N = \mathcal{L} \cdot \sigma \,. \tag{5.194}$$

To measure $\mathcal{L}$, one needs a process with: (i) large cross section (high statistics), (ii) clean experimental signature, and (iii) precisely calculable theoretical cross section. Small-angle Bhabha scattering satisfies all three criteria:

- The $1/t^2$ enhancement gives a large rate at small angles
- The e^+e^- final state has a distinctive back-to-back topology
- Being a pure QED process, radiative corrections are calculable to high precision

Precision at LEP. At LEP, small-angle Bhabha scattering (SABS) was measured with remarkable precision. The OPAL, ALEPH, DELPHI, and L3 experiments achieved:

- **Experimental precision**: $\sim 0.05\%$–0.08% using silicon-tungsten luminometers
- **Theoretical precision**: The Monte Carlo generator BHLUMI 4.04 achieved 0.061% at LEP1 (Z pole) and 0.122% at LEP2 (higher energies)
- **Combined uncertainty**: $\lesssim 0.1\%$ on the luminosity measurement

This precision was essential for the electroweak precision tests at LEP, where cross sections and asymmetries were measured with per-mille accuracy.

Running of α. Large-angle Bhabha scattering at LEP was used to measure the running of the QED coupling $\alpha(q^2)$. The OPAL collaboration measured $\alpha(t)$ for space-like momentum transfer, confirming the predicted logarithmic running from vacuum polarization effects.

Radiative corrections. For precision applications, higher-order corrections must be included:

1. **Virtual corrections**: One-loop vertex and box diagrams modify the amplitude at $O(\alpha)$
2. **Real emission**: Bremsstrahlung $e^+e^- \to e^+e^-\gamma$ contributes when soft/-collinear photons are undetected

3. **Vacuum polarization**: Modifies the photon propagator, introducing hadronic uncertainties
4. **Weak corrections**: Z boson exchange becomes relevant near $\sqrt{s} \sim M_Z$

The complete $O(\alpha)$ correction to the Bhabha cross section is typically ~ 10–30%, depending on the experimental cuts. Two-loop corrections at $O(\alpha^2)$ are at the per-mille level and have been computed for precision applications.

Future colliders. At proposed future e^+e^- colliders (FCC-ee, CEPC, ILC), the target luminosity precision is 10^{-4} or better. This requires:

- Complete two-loop QED calculations for SABS
- Precise treatment of initial-state radiation and beamstrahlung
- Control of hadronic vacuum polarization uncertainties
- Understanding of potential new physics contributions to the luminosity process

Numerical estimates and comparison with data

Order of magnitude of cross sections. The characteristic scale for QED cross sections is set by dimensional analysis:

$$\sigma \sim \frac{\alpha^2}{s}. \tag{5.195}$$

Using $\alpha \approx 1/137$ and the conversion factor $\hbar c \approx 197\ \text{MeV} \cdot \text{fm}$:

$$\sigma \sim \frac{\pi\alpha^2}{s} = \frac{\pi}{137^2} \cdot \frac{(197\ \text{MeV} \cdot \text{fm})^2}{s} \approx \frac{87\ \text{nb}}{(E_{\text{cm}}/\text{GeV})^2}. \tag{5.196}$$

Total cross section for $e^+e^- \to \mu^+\mu^-$. Integrating the differential cross section over the full solid angle:

$$\sigma_{e^+e^-\to\mu^+\mu^-} = \frac{4\pi\alpha^2}{3s} \approx \frac{87\ \text{nb}}{(E_{\text{cm}}/\text{GeV})^2}. \tag{5.197}$$

At $\sqrt{s} = 10$ GeV (typical B-factory energy): $\sigma \approx 0.87$ nb.
At $\sqrt{s} = 91$ GeV (Z pole): $\sigma \approx 10$ pb (enhanced by Z resonance).
At $\sqrt{s} = 200$ GeV (LEP2): $\sigma \approx 2.2$ pb.

Bhabha scattering cross section. The Bhabha total cross section diverges due to the forward singularity. With an angular cut $\theta > \theta_{\text{min}}$:

$$\sigma_{\text{Bhabha}}(\theta > \theta_{\text{min}}) \approx \frac{2\pi\alpha^2}{s}\left[\frac{1}{\sin^2(\theta_{\text{min}}/2)} + \frac{1}{3}(1+\cos^2\theta_{\text{min}}) + \cdots\right]. \tag{5.198}$$

For small-angle Bhabha scattering with $\theta_{\min} = 30$ mrad at $\sqrt{s} = 91$ GeV:

$$\sigma_{\text{SABS}} \sim \frac{2\pi\alpha^2}{s \cdot \theta^2_{\min}} \approx 100 \text{ nb}. \tag{5.199}$$

This large cross section (compared to ~ 10 pb for $\mu^+\mu^-$) makes SABS ideal for luminosity monitoring.

Angular distribution: theory vs. experiment. Figure 5.2 shows the differential cross section as a function of scattering angle, comparing Bhabha scattering with muon pair production.

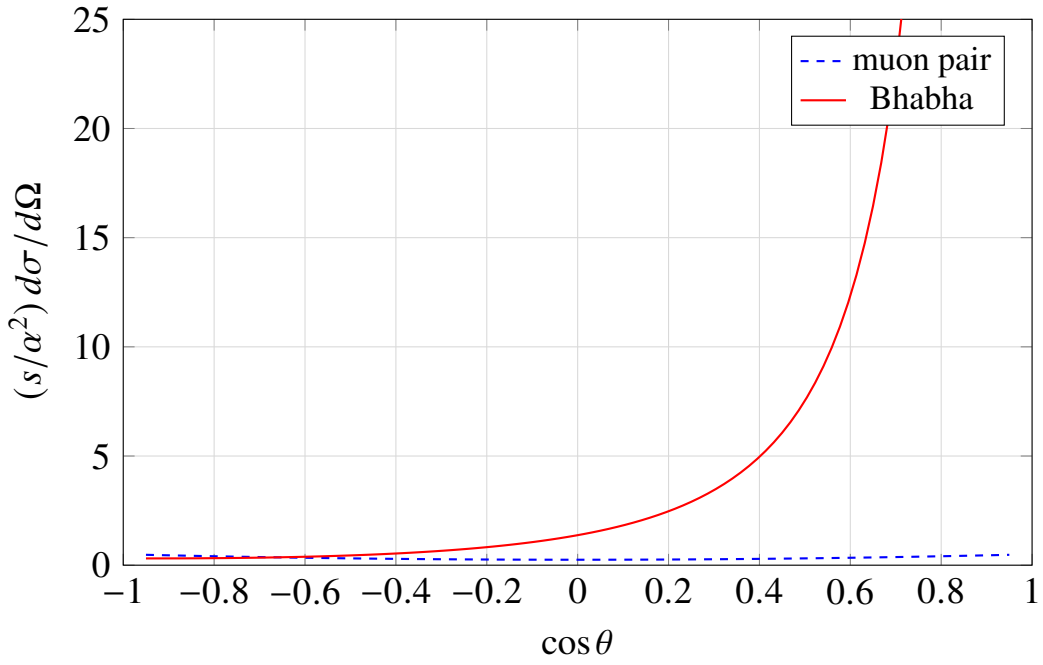


Figure 5.2. Comparison of the differential cross sections (in units of α^2/s) for Bhabha scattering ($e^+e^- \to e^+e^-$, solid red) and muon pair production ($e^+e^- \to \mu^+\mu^-$, dashed blue). The Bhabha cross section diverges as $\theta \to 0$ ($\cos\theta \to 1$) due to the t-channel pole, while muon production remains finite. At backward angles ($\cos\theta \to -1$), the two processes have similar magnitudes due to destructive s-t interference in Bhabha scattering.

Experimental verification. The Bhabha cross section has been measured with high precision at numerous e^+e^- colliders. Table 5.1 summarizes key measurements.

At LEP, the measured Bhabha cross sections agreed with QED predictions (including radiative corrections) to better than 0.1%, providing one of the most stringent tests of QED at high energies. No deviations from the Standard Model prediction were observed, setting limits on new physics such as contact interactions or extra dimensions.

Experiment	$\sqrt{s}$ (GeV)	Angular range	Precision
PETRA (1980s)	14–47	Large angle	$\sim 2\%$
LEP1 (1989–95)	91	25–58 mrad	0.05%
LEP2 (1996–2000)	130–209	25–58 mrad	0.1%
BaBar (1999–2008)	10.6	Large angle	0.5%
Belle (1999–2010)	10.6	Large angle	0.5%

Table 5.1. Summary of Bhabha scattering measurements at e^+e^- colliders. The LEP experiments achieved sub-per-mille precision, essential for electroweak tests.

5.4 Compton scattering: $e^-\gamma \to e^-\gamma$

Compton scattering is the elastic scattering of a photon off a free electron: $e^-(p) + \gamma(k) \to e^-(p') + \gamma(k')$. This process, first studied experimentally by Arthur Compton in 1923, provided crucial evidence for the particle nature of light. The quantum field theoretic treatment not only reproduces the classical Thomson limit at low energies but also predicts the full energy and angular dependence through the Klein–Nishina formula.

Kinematics and amplitude

Momentum assignments. Consider the scattering process:

$$e^-(p) + \gamma(k) \to e^-(p') + \gamma(k')\,. \tag{5.200}$$

The four-momenta satisfy:

- Incoming electron: $p^2 = m^2$
- Incoming photon: $k^2 = 0$
- Outgoing electron: $p'^2 = m^2$
- Outgoing photon: $k'^2 = 0$

with energy-momentum conservation $p + k = p' + k'$.

Mandelstam variables. The Mandelstam variables for Compton scattering are:

$$s = (p+k)^2 = m^2 + 2p\cdot k \tag{5.201}$$

$$t = (k-k')^2 = -2k\cdot k' \tag{5.202}$$

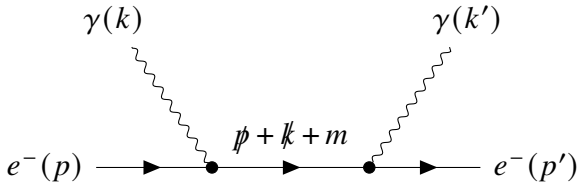
$$u = (p-k')^2 = m^2 - 2p\cdot k'\,, \tag{5.203}$$

satisfying the constraint $s + t + u = 2m^2$.

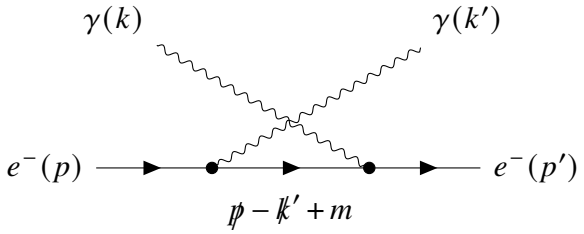
From energy-momentum conservation, we have the useful relations:

$$\boxed{p\cdot k = p'\cdot k', \qquad p\cdot k' = p'\cdot k, \qquad p\cdot p' = m^2 + k\cdot k'}\,. \tag{5.204}$$

Tree-level Feynman diagrams. At leading order, two diagrams contribute to Compton scattering:



(a) s-channel



(b) u-channel

In the s-channel diagram, the incoming photon is absorbed first, creating an intermediate electron with momentum $p+k$. In the u-channel diagram, the outgoing photon is emitted first, and the intermediate electron has momentum $p-k'$.

Note on channels. Unlike Bhabha scattering, there is no t-channel diagram for Compton scattering because a photon cannot couple directly to another photon (at tree level). The t-channel would require a fermion exchange between the two photon lines, which is not possible.

Individual amplitudes. Applying the QED Feynman rules, the s-channel amplitude is:

$$i\mathcal{M}_s = \bar{u}(p')(ie\gamma^\beta)\,\varepsilon^*_\beta(k',\lambda')\,\frac{i(\not{p}+\not{k}+m)}{(p+k)^2-m^2}\,(ie\gamma^\alpha)\,\varepsilon_\alpha(k,\lambda)\,u(p)\,. \tag{5.205}$$

Simplifying with $(p+k)^2-m^2 = 2p\cdot k$:

$$\boxed{\mathcal{M}_s = -\frac{e^2}{2p\cdot k}\,\bar{u}(p')\gamma^\beta\varepsilon^*_\beta(k',\lambda')(\not{p}+\not{k}+m)\gamma^\alpha\varepsilon_\alpha(k,\lambda)\,u(p)}\,. \tag{5.206}$$

The u-channel amplitude is:

$$\boxed{\mathcal{M}_u = -\frac{e^2}{-2p\cdot k'}\,\bar{u}(p')\gamma^\alpha\varepsilon_\alpha(k,\lambda)(\not{p}-\not{k}'+m)\gamma^\beta\varepsilon^*_\beta(k',\lambda')\,u(p)}\,. \tag{5.207}$$

Note the sign change in the denominator: $(p-k')^2-m^2 = -2p\cdot k'$.

Total amplitude. The total amplitude is the sum of both diagrams:

$$\mathcal{M} = \mathcal{M}_s + \mathcal{M}_u \,. \tag{5.208}$$

Unlike Bhabha scattering, there is **no relative minus sign** between the two diagrams because both describe the same fermion line—there is no exchange of identical fermions.

Compact notation. It is convenient to factor out the polarization vectors and write:

$$\mathcal{M} = \varepsilon^*_\beta(k',\lambda')\,\varepsilon_\alpha(k,\lambda)\,\mathcal{M}^{\alpha\beta}\,, \tag{5.209}$$

where the tensor amplitude is:

$$\begin{aligned}\mathcal{M}^{\alpha\beta} &= -e^2\,\bar{u}(p')\left[\frac{\gamma^\beta(\not{p}+\not{k}+m)\gamma^\alpha}{2p\cdot k}+\frac{\gamma^\alpha(\not{p}-\not{k}'+m)\gamma^\beta}{-2p\cdot k'}\right]u(p)\\ &= -e^2\,\bar{u}(p')\left[\frac{\gamma^\beta(\not{p}+\not{k}+m)\gamma^\alpha}{2p\cdot k}-\frac{\gamma^\alpha(\not{p}-\not{k}'+m)\gamma^\beta}{2p\cdot k'}\right]u(p)\,.\end{aligned} \tag{5.210}$$

Ward identity: general argument from gauge invariance. For any QED process involving external photons, the Feynman amplitude can be written in the form

$$\mathcal{M} = \varepsilon^\alpha(k_1,\lambda_1)\,\varepsilon^\beta(k_2,\lambda_2)\cdots\mathcal{M}_{\alpha\beta\ldots}(k_1,k_2,\ldots)\,, \tag{5.211}$$

with one polarization vector $\varepsilon(k_i,\lambda_i)$ for each external photon, and the **tensor amplitude** $\mathcal{M}_{\alpha\beta\ldots}$ independent of these polarization vectors. (This follows from the QED Feynman rules, since each external photon line contributes exactly one factor of ε.)

Now, the polarization vectors are gauge dependent. For a free photon described in a Lorentz gauge by $A^\mu(x) = \text{const.}\;\varepsilon^\mu(k,\lambda)\,e^{\pm ikx}$, the gauge transformation $A^\mu \to A^\mu + \partial^\mu f$ with $f(x) = \tilde{f}(k)\,e^{\pm ikx}$ implies

$$\varepsilon^\mu(k,\lambda) \;\longrightarrow\; \varepsilon^\mu(k,\lambda) \pm i k^\mu \tilde{f}(k)\,. \tag{5.212}$$

Since only the *sum* of all Feynman diagrams at a given order—i.e. the full matrix element $\mathcal{M}$—is a physical observable, it must be gauge invariant. Invariance of eq. (5.211) under the replacement $\varepsilon^\alpha(k_i,\lambda_i) \to \varepsilon^\alpha(k_i,\lambda_i) + c\,k_i^\alpha$ (with c an arbitrary constant) for *any* external photon requires:

$$\boxed{k_1^\alpha\,\mathcal{M}_{\alpha\beta\ldots}(k_1,k_2,\ldots) = k_2^\beta\,\mathcal{M}_{\alpha\beta\ldots}(k_1,k_2,\ldots) = \cdots = 0}\,. \tag{5.213}$$

This is the **Ward identity**: contracting the tensor amplitude with the four-momentum of any external photon gives zero. It holds for *any* QED process, to all orders in perturbation theory, provided all diagrams of a given order are included.

Remark. The individual contributions from single Feynman diagrams are in general *not* gauge invariant. For Compton scattering, neither $\mathcal{M}_s$ nor $\mathcal{M}_u$ alone satisfies the Ward identity—only their sum $\mathcal{M} = \mathcal{M}_s + \mathcal{M}_u$ does. This is a concrete manifestation of the fact that gauge invariance constrains the full amplitude, not its individual diagrammatic contributions.

Ward identity for Compton scattering: explicit verification. For the specific case of Compton scattering, the Ward identity reads

$$k_\alpha \mathcal{M}^{\alpha\beta} = 0, \qquad k'_\beta \mathcal{M}^{\alpha\beta} = 0\,. \tag{5.214}$$

Let us verify $k_\alpha \mathcal{M}^{\alpha\beta} = 0$ explicitly by direct calculation.

Contracting with k_α (i.e., replacing $\gamma^\alpha \to \not{k}$):

$$k_\alpha \mathcal{M}^{\alpha\beta} = -e^2 \bar{u}(p') \left[\frac{\gamma^\beta (\not{p} + \not{k} + m)\not{k}}{2p\cdot k} - \frac{\not{k}(\not{p} - \not{k}' + m)\gamma^\beta}{2p\cdot k'} \right] u(p)\,. \tag{5.215}$$

First term: We compute $(\not{p} + \not{k} + m)\not{k}\, u(p)$.

Using $\not{p}\not{k} = 2p\cdot k - \not{k}\not{p}$ and the Dirac equation $(\not{p} - m)u(p) = 0$:

$$\begin{aligned}(\not{p} + \not{k} + m)\not{k}\, u(p) &= [\not{p}\not{k} + \not{k}^2 + m\not{k}]u(p) \\ &= [2p\cdot k - \not{k}\not{p} + m\not{k}]u(p) \quad (\text{since } k^2 = 0) \\ &= 2p\cdot k\, u(p) - \not{k}(\not{p} - m)u(p) \\ &= 2p\cdot k\, u(p)\,.\end{aligned} \tag{5.216}$$

Therefore: $\dfrac{(\not{p} + \not{k} + m)\not{k}}{2p\cdot k} u(p) = u(p)$.

Second term: We compute $\bar{u}(p')\not{k}(\not{p} - \not{k}' + m)$.

Using $\not{k}\not{a} = 2k\cdot a - \not{a}\not{k}$ and the kinematic relation $p - k' = p' - k$:

$$\bar{u}(p')\not{k}(\not{p} - \not{k}' + m) = \bar{u}(p')[2k\cdot(p - k') - (\not{p} - \not{k}')\not{k} + m\not{k}]\,. \tag{5.217}$$

Now, $(\not{p} - \not{k}') = (\not{p}' - \not{k})$, so using $\bar{u}(p')(\not{p}' - m) = 0$:

$$\begin{aligned}\bar{u}(p')(\not{p} - \not{k}')\not{k} &= \bar{u}(p')(\not{p}' - \not{k})\not{k} = [m\bar{u}(p') - \bar{u}(p')\not{k}]\not{k} \\ &= m\bar{u}(p')\not{k} \quad (\text{since } \not{k}^2 = 0)\,.\end{aligned} \tag{5.218}$$

Therefore:

$$\begin{aligned}\bar{u}(p')\not{k}(\not{p} - \not{k}' + m) &= 2(p\cdot k - k\cdot k')\bar{u}(p') - m\bar{u}(p')\not{k} + m\bar{u}(p')\not{k} \\ &= 2(p\cdot k - k\cdot k')\bar{u}(p')\,.\end{aligned} \tag{5.219}$$

Using the kinematic identity $p\cdot k - k\cdot k' = p'\cdot k = p\cdot k'$:

$$\frac{\bar{u}(p')\not{k}(\not{p} - \not{k}' + m)}{2p\cdot k'} = \bar{u}(p')\,. \tag{5.220}$$

Combining: The two terms give:

$$k_\alpha \mathcal{M}^{\alpha\beta} = -e^2[\bar{u}(p')\gamma^\beta u(p) - \bar{u}(p')\gamma^\beta u(p)] = 0 \quad \checkmark . \tag{5.221}$$

The proof of $k'_\beta \mathcal{M}^{\alpha\beta} = 0$ is analogous.

Squared amplitude. For unpolarized electrons and photons, we must average over initial spins and polarizations and sum over final ones:

$$\overline{|\mathcal{M}|^2} = \frac{1}{2} \sum_{\text{electron spins}} \frac{1}{2} \sum_{\text{photon polarizations}} |\mathcal{M}|^2 . \tag{5.222}$$

The factor of $\frac{1}{4}$ comes from averaging over the two electron spin states and two photon polarization states.

Photon polarization sum. For real (on-shell) photons, the polarization sum gives:

$$\sum_{\lambda=1,2} \varepsilon_\mu(k,\lambda)\, \varepsilon^*_\nu(k,\lambda) = -g_{\mu\nu} . \tag{5.223}$$

This replacement is valid for physical amplitudes that satisfy the Ward identity, which ensures gauge invariance: $k_\mu \mathcal{M}^{\mu\nu} = k'_\nu \mathcal{M}^{\mu\nu} = 0$.

Structure of the calculation. The squared amplitude can be written as:

$$\overline{|\mathcal{M}|^2} = \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}_s + \mathcal{M}_u|^2 = \frac{1}{4} \sum_{\text{spins}} \left(|\mathcal{M}_s|^2 + |\mathcal{M}_u|^2 + 2\text{Re}(\mathcal{M}_s \mathcal{M}_u^*) \right) . \tag{5.224}$$

We define:

$$X_{ss} = \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}_s|^2 \tag{5.225}$$

$$X_{uu} = \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}_u|^2 \tag{5.226}$$

$$X_{su} = \frac{1}{4} \sum_{\text{spins}} 2\text{Re}(\mathcal{M}_s \mathcal{M}_u^*) . \tag{5.227}$$

Spin sums and the Klein–Nishina formula

Calculation of X_{ss}. After performing the spin sum using the BD completeness relation $\sum_s u(p)\bar{u}(p) = (\not{p} + m)/(2m)$ and the polarization sum $\sum_\lambda \varepsilon_\mu(k,\lambda)\, \varepsilon^*_\nu(k,\lambda) = -g_{\mu\nu}$, we obtain:

$$X_{ss} = \frac{e^4}{4(p\cdot k)^2} \text{Tr}\left[\gamma^\alpha (\not{p} + \not{k} + m) \gamma^\beta \frac{\not{p}' + m}{2m} \gamma_\beta (\not{p} + \not{k} + m) \gamma_\alpha \frac{\not{p} + m}{2m} \right] . \tag{5.228}$$

Using the notation $\not{f}_1 = \not{p} + \not{k}$, this becomes:

$$X_{ss} = \frac{e^4}{16m^2(p\cdot k)^2}\,\mathrm{Tr}\big[\gamma^\alpha(\not{f}_1+m)\gamma^\beta(\not{p}'+m)\gamma_\beta(\not{f}_1+m)\gamma_\alpha(\not{p}+m)\big]\,. \quad (5.229)$$

Detailed trace calculation. We use the gamma matrix contraction identities:

$$\begin{aligned} \gamma_\mu\gamma^\mu &= 4, & \gamma_\mu\gamma^\alpha\gamma^\mu &= -2\gamma^\alpha \\ \gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\mu &= 4g^{\alpha\beta}, \\ \gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\mu &= -2\gamma^\gamma\gamma^\beta\gamma^\alpha\,. \end{aligned} \quad (5.230)$$

Step 1: Contract $\gamma^\beta\ldots\gamma_\beta$ first.

$$\gamma^\beta(\not{p}'+m)\gamma_\beta = \gamma^\beta\not{p}'\gamma_\beta + m\gamma^\beta\gamma_\beta = -2\not{p}' + 4m\,. \quad (5.231)$$

Then:

$$\begin{aligned} \gamma^\beta(\not{p}'+m)\gamma_\beta(\not{f}_1+m) &= (-2\not{p}'+4m)(\not{f}_1+m) \\ &= -2\not{p}'\not{f}_1 - 2m\not{p}' + 4m\not{f}_1 + 4m^2\,. \end{aligned} \quad (5.232)$$

Step 2: Multiply by $(\not{f}_1+m)$ on the left.

$$\begin{aligned} (\not{f}_1+m)(-2\not{p}'\not{f}_1-2m\not{p}'+4m\not{f}_1+4m^2) = {} & -2\not{f}_1\not{p}'\not{f}_1 - 2m\not{f}_1\not{p}' + 4m\not{f}_1^2 + 4m^2\not{f}_1 \\ & - 2m\not{p}'\not{f}_1 - 2m^2\not{p}' + 4m^2\not{f}_1 + 4m^3 \end{aligned} \quad (5.233)$$

Using $\not{f}_1^2 = f_1^2 = (p+k)^2 = m^2 + 2p\cdot k$ and $\not{a}\not{b} + \not{b}\not{a} = 2a\cdot b$:

$$= -2\not{f}_1\not{p}'\not{f}_1 - 4m(f_1\cdot p') - 2m^2\not{p}' + 8m^2\not{f}_1 + 8m^3 + 8m(p\cdot k)\,. \quad (5.234)$$

Step 3: Contract $\gamma^\alpha\ldots\gamma_\alpha$ and take the trace.

The trace $T = \mathrm{Tr}[\gamma^\alpha(\ldots)\gamma_\alpha(\not{p}+m)]$ has six contributions:

(A) $\mathrm{Tr}[\gamma^\alpha(-2\not{f}_1\not{p}'\not{f}_1)\gamma_\alpha(\not{p}+m)]$:

Using $\gamma^\alpha\not{a}\not{b}\not{c}\gamma_\alpha = -2\not{c}\not{b}\not{a}$:

$$= \mathrm{Tr}[4\not{f}_1\not{p}'\not{f}_1\not{p}] = 4\times4[(f_1\cdot p')(f_1\cdot p) - f_1^2(p'\cdot p) + (f_1\cdot p)(f_1\cdot p')] \quad (5.235)$$

$$= 32(f_1\cdot p)(f_1\cdot p') - 16(m^2+2p\cdot k)(p\cdot p') \quad (5.236)$$

(B) $\mathrm{Tr}[\gamma^\alpha(-4mf_1\cdot p')\gamma_\alpha(\not{p}+m)]$:

Using $\gamma^\alpha\gamma_\alpha = 4$ and $\mathrm{Tr}[\not{p}+m] = 4m$:

$$= -4m(f_1\cdot p')\times4\times4m = -64m^2(f_1\cdot p') \quad (5.237)$$

(C) $\mathrm{Tr}[\gamma^\alpha(-2m^2\not{p}')\gamma_\alpha(\not{p}+m)]$:
Using $\gamma^\alpha \not{a}\gamma_\alpha = -2\not{a}$ and $\mathrm{Tr}[\not{p}'\not{p}] = 4(p'\cdot p)$:

$$= -2m^2 \times (-2) \times \mathrm{Tr}[\not{p}'(\not{p}+m)] = 4m^2 \times 4(p'\cdot p) = 16m^2(p\cdot p') \tag{5.238}$$

(D) $\mathrm{Tr}[\gamma^\alpha(8m^2\not{f}_1)\gamma_\alpha(\not{p}+m)]$:

$$= 8m^2\times(-2)\times\mathrm{Tr}[\not{f}_1(\not{p}+m)] = -16m^2\times4(f_1\cdot p) = -64m^2(f_1\cdot p) \tag{5.239}$$

(E) $\mathrm{Tr}[\gamma^\alpha(8m^3)\gamma_\alpha(\not{p}+m)]$:

$$= 8m^3\times4\times4m = 128m^4 \tag{5.240}$$

(F) $\mathrm{Tr}[\gamma^\alpha(8m\,p\cdot k)\gamma_\alpha(\not{p}+m)]$:

$$= 8m(p\cdot k)\times4\times4m = 128m^2(p\cdot k) \tag{5.241}$$

Step 4: Sum all contributions.

$$\begin{aligned} T = {} & 32(f_1\cdot p)(f_1\cdot p') - 16(m^2+2p\cdot k)(p\cdot p') - 64m^2(f_1\cdot p') \\ & + 16m^2(p\cdot p') - 64m^2(f_1\cdot p) + 128m^4 + 128m^2(p\cdot k) \end{aligned} \tag{5.242}$$

Regrouping:

$$\begin{aligned} T = {} & 32(f_1\cdot p)(f_1\cdot p') - 16(m^2+2p\cdot k)(p\cdot p') - 32m^2(f_1\cdot p') \\ & + 16m^2(p\cdot p') - 32m^2(f_1\cdot p') - 64m^2(f_1\cdot p) + 64m^4 + 64m^2(m^2+2p\cdot k) \end{aligned} \tag{5.243}$$

Step 5: Simplify using kinematics.

$$\boxed{X_{ss} = 2e^4\left[\frac{m^4+(p\cdot k)m^2+(p\cdot k)(p\cdot k')}{(p\cdot k)^2}\right]}. \tag{5.244}$$

Calculation of X_{uu}. By the crossing symmetry $k \leftrightarrow -k'$ (which exchanges the two diagrams), we have:

$$\boxed{X_{uu} = 2e^4\left[\frac{m^4-(p\cdot k')m^2+(p\cdot k)(p\cdot k')}{(p\cdot k')^2}\right]}. \tag{5.245}$$

Calculation of X_{su} (interference term). The interference term is:

$$X_{su} = \frac{e^4}{2(p\cdot k)(p\cdot k')}\cdot\frac{1}{4m^2}\,\mathrm{Tr}\left[\gamma^\beta(\not{f}_1+m)\gamma^\alpha(\not{p}'+m)\gamma_\beta(\not{f}_2+m)\gamma_\alpha(\not{p}+m)\right],$$

(5.246)

where $\not{f}_2 = \not{p} - \not{k}'$.

After computing the trace using the identities:

$$\gamma^\beta \not{f}_1 \gamma^\alpha \not{p}' \gamma_\beta = -2\not{p}' \gamma^\alpha \not{f}_1 \tag{5.247}$$

$$\gamma^\beta \gamma^\alpha \not{p} \gamma_\beta = 4p^\alpha , \tag{5.248}$$

we obtain:

$$\boxed{X_{su} = -2e^4 \left[\frac{m^2(2m^2 + p\cdot k - p\cdot k')}{(p\cdot k)(p\cdot k')} \right]} . \tag{5.249}$$

Combining the results. Adding eqs. (5.244), (5.245), and (5.249):

$$\begin{aligned} \overline{|\mathcal{M}|^2} &= X_{ss} + X_{uu} + X_{su} \\ &= 2e^4 \left[\frac{p\cdot k'}{p\cdot k} + \frac{p\cdot k}{p\cdot k'} + 2m^2 \left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'} \right) + m^4 \left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'} \right)^2 \right] . \end{aligned} \tag{5.250}$$

Lab frame kinematics. In the laboratory frame where the electron is initially at rest ($\vec{p} = 0$, $p = (m, \vec{0})$):

$$p\cdot k = m\omega \tag{5.251}$$

$$p\cdot k' = m\omega' , \tag{5.252}$$

where ω and ω' are the initial and final photon energies.

Derivation of the Compton formula. Energy-momentum conservation $p + k = p' + k'$ gives $p' = p + k - k'$. Squaring both sides:

$$p'^2 = (p + k - k')^2 = p^2 + k^2 + k'^2 + 2p\cdot k - 2p\cdot k' - 2k\cdot k' . \tag{5.253}$$

Using $p^2 = p'^2 = m^2$ and $k^2 = k'^2 = 0$:

$$m^2 = m^2 + 2p\cdot k - 2p\cdot k' - 2k\cdot k' . \tag{5.254}$$

In the lab frame with $p = (m, \vec{0})$, $k = (\omega, \vec{k})$, $k' = (\omega', \vec{k}')$:

$$\begin{aligned} &p\cdot k = m\omega, \qquad p\cdot k' = m\omega' \\ &k\cdot k' = \omega\omega' - \vec{k}\cdot\vec{k}' = \omega\omega'(1 - \cos\theta) . \end{aligned} \tag{5.255}$$

Substituting:

$$0 = 2m\omega - 2m\omega' - 2\omega\omega'(1 - \cos\theta) . \tag{5.256}$$

Dividing by $2m\omega\omega'$:

$$\boxed{\frac{1}{\omega'} - \frac{1}{\omega} = \frac{1}{m}(1-\cos\theta)}\,. \tag{5.257}$$

This is the **Compton formula**. Equivalently:

$$\omega' = \frac{\omega}{1+\dfrac{\omega}{m}(1-\cos\theta)}\,. \tag{5.258}$$

This can be rewritten in terms of the **Compton wavelength** $\lambda_C = 1/m$:

$$\boxed{\lambda' - \lambda = \lambda_C(1-\cos\theta)}\,. \tag{5.259}$$

The maximum wavelength shift occurs at $\theta = \pi$ (backscattering): $\Delta\lambda_{\max} = 2\lambda_C$.

Differential cross section. The differential cross section in the lab frame is:

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\frac{1}{m^2}\left(\frac{\omega'}{\omega}\right)^2 \overline{|\mathcal{M}|^2}\,. \tag{5.260}$$

Substituting the squared amplitude (5.250) and using the lab frame relations:

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta\right]}\,. \tag{5.261}$$

This is the **Klein–Nishina formula**, derived by Oskar Klein and Yoshio Nishina in 1929.

Using the Compton formula to eliminate ω'/ω:

$$\frac{\omega'}{\omega} = \frac{1}{1+\dfrac{\omega}{m}(1-\cos\theta)} \equiv \frac{1}{1+x(1-\cos\theta)}\,, \tag{5.262}$$

where $x = \omega/m$ is the photon energy in units of the electron mass.

Box 5.6 The Klein–Nishina formula: properties and limits

Alternative forms. Introducing $y = \omega'/\omega$ and using $\sin^2\theta = 1-\cos^2\theta$, the Klein–Nishina formula can be written as:

$$\frac{d\sigma}{d\Omega} = \frac{r_e^2}{2}y^2\left(y + \frac{1}{y} - \sin^2\theta\right)\,, \tag{5.263}$$

where $r_e = \alpha/m \approx 2.82$ fm is the classical electron radius.

Low-energy limit: Thomson scattering. When $\omega \ll m$ (i.e., $x \to 0$), we have

$\omega' \approx \omega$ and:

$$\frac{d\sigma}{d\Omega} \xrightarrow{\omega \ll m} \frac{r_e^2}{2}(1+\cos^2\theta)\,. \tag{5.264}$$

This is the **Thomson formula** for classical electromagnetic scattering. The $(1+\cos^2\theta)$ factor reflects the **dipole radiation pattern** of a classical oscillating charge, which can be understood as follows.

In the low-energy limit, the incoming photon's electric field forces the electron to oscillate as an electric dipole. The radiation emitted by an oscillating dipole has an angular distribution $\propto \sin^2\alpha$, where α is the angle between the observation direction $\hat{n}$ and the dipole axis (i.e. the polarization direction $\vec{\varepsilon}$ of the incoming photon). Since $\sin^2\alpha = 1-(\hat{n}\cdot\vec{\varepsilon})^2$, the scattered intensity for a *given* linear polarization is $\propto 1-(\hat{n}\cdot\vec{\varepsilon})^2$.

For *unpolarized* light, we average over two orthogonal transverse polarizations $\vec{\varepsilon}_1, \vec{\varepsilon}_2$ perpendicular to the beam axis $\hat{z}$. Choosing $\hat{n}$ in the xz-plane at angle θ from $\hat{z}$, and $\vec{\varepsilon}_1 = \hat{x}$, $\vec{\varepsilon}_2 = \hat{y}$:

$$\frac{1}{2}\sum_{\lambda=1,2}\left[1-(\hat{n}\cdot\vec{\varepsilon}_\lambda)^2\right] = \frac{1}{2}\left[(1-\sin^2\theta)+1\right] = \frac{1}{2}(1+\cos^2\theta)\,. \tag{5.265}$$

The first term $(1-\sin^2\theta = \cos^2\theta)$ comes from the polarization in the scattering plane, which radiates less at $\theta = \pi/2$; the second term (1) comes from the polarization perpendicular to the scattering plane, which radiates isotropically in θ. Their average gives the characteristic $(1+\cos^2\theta)$ distribution (up to the overall factor $1/2$ absorbed in the cross section).

The total Thomson cross section is:

$$\sigma_T = \frac{8\pi r_e^2}{3} = \frac{8\pi\alpha^2}{3m^2} \approx 0.665 \text{ barn}\,. \tag{5.266}$$

High-energy limit. When $\omega \gg m$:

- For forward scattering ($\theta \approx 0$): $\omega' \approx \omega$, and the cross section remains approximately Thomson.
- For large angles ($\theta \sim \pi$): $\omega' \approx m/2$, independent of ω. The cross section becomes:

$$\frac{d\sigma}{d\Omega}(\theta\sim\pi) \sim \frac{\alpha^2}{2\omega m} \ll \sigma_T\,. \tag{5.267}$$

The total cross section decreases logarithmically at high energies:

$$\sigma \xrightarrow{\omega \gg m} \frac{\pi\alpha^2}{m\omega}\left(\ln\frac{2\omega}{m}+\frac{1}{2}\right). \tag{5.268}$$

Recoil effects. The factor $(\omega'/\omega)^2$ in the Klein–Nishina formula accounts for:

1. Phase space reduction: fewer final states available when $\omega' < \omega$
2. Energy conservation: the recoiling electron carries away energy

At high energies, this recoil factor strongly suppresses large-angle scattering.

Physical discussion and applications

Polarization dependence. For polarized photons, the differential cross section depends on the relative orientation of the initial and final polarization vectors. Choosing the scattering plane as the xz-plane, we can decompose the polarization into components parallel ($\parallel$) and perpendicular ($\perp$) to this plane.

The polarized cross sections follow from the polarized Klein–Nishina formula (5.337) by substituting the values of $(\varepsilon\cdot\varepsilon')^2$ for each polarization channel (see eq. (5.343) and the discussion in Section 5.4): $(\varepsilon\cdot\varepsilon')^2 = \cos^2\theta$ for $\parallel\to\parallel$, $(\varepsilon\cdot\varepsilon')^2 = 1$ for $\perp\to\perp$, and $\varepsilon\cdot\varepsilon' = 0$ for $\parallel\leftrightarrow\perp$. Using $r_e = \alpha/m$:

$$\frac{d\sigma}{d\Omega}(\parallel\to\parallel) = \frac{r_e^2}{4}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega'}{\omega}+\frac{\omega}{\omega'}-2+4\cos^2\theta\right] \tag{5.269}$$

$$\frac{d\sigma}{d\Omega}(\perp\to\perp) = \frac{r_e^2}{4}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega'}{\omega}+\frac{\omega}{\omega'}+2\right] \tag{5.270}$$

$$\frac{d\sigma}{d\Omega}(\parallel\to\perp) = \frac{d\sigma}{d\Omega}(\perp\to\parallel) = 0\,. \tag{5.271}$$

In the linear polarization basis adapted to the scattering plane, the mixed channels $\parallel\to\perp$ and $\perp\to\parallel$ vanish at tree level. This is the basis-dependent selection rule encoded in the formulas above.

Asymmetry. For linearly polarized photons, the azimuthal distribution of scattered photons is:

$$\frac{d\sigma}{d\Omega} \propto 1 - \frac{\sin^2\theta}{1+\cos^2\theta+(\omega'/\omega+\omega/\omega'-2)}\cos(2\phi)\,, \tag{5.272}$$

where ϕ is the azimuthal angle relative to the initial polarization direction. This asymmetry is used in polarimetry to measure photon polarization.

Comparison with other QED processes. Table 5.2 compares the key features of the fundamental QED processes studied in this chapter.

Distinctive features of Compton scattering.

Process	Channels	Propagator	σ scaling	Angular behavior
$e^+e^- \to \mu^+\mu^-$	s only	γ^* (timelike)	α^2/s	$1+\cos^2\theta$
Bhabha	s, t	γ^* (both)	divergent (forward)	$1/t^2$ pole
Møller	t, u	γ^* (spacelike)	divergent	$1/t^2 + 1/u^2$
Compton	s, u	e^* (fermion)	α^2/m^2	Thomson + recoil

Table 5.2. Comparison of tree-level QED processes. The "channels" column indicates which Mandelstam variables appear in the propagator denominators.

1. **Fermion propagator**: Unlike the other processes, Compton scattering involves an internal fermion line rather than a photon propagator. This leads to a qualitatively different energy dependence.
2. **No forward divergence**: The cross section remains finite at all angles. This is because the intermediate electron is massive, preventing any pole at $t = 0$.
3. **Energy scale**: The characteristic cross section is $\sigma \sim \alpha^2/m^2 \sim r_e^2$, independent of the photon energy at low energies. This contrasts with e^+e^- processes where $\sigma \sim \alpha^2/s$.
4. **Recoil effects**: The electron recoil causes a frequency shift that is absent in the Thomson (classical) limit. This was the key quantum signature observed by Compton.
5. **Channel structure**: The s- and u-channel amplitudes add without a relative sign (unlike Bhabha), because there is no fermion exchange—the same electron line appears in both diagrams.

Crossing relations. Compton scattering is related by crossing to other QED processes:

- $e^-\gamma \to e^-\gamma$ (Compton): direct process
- $e^+e^- \to \gamma\gamma$ (pair annihilation): $s \leftrightarrow u$ crossing
- $\gamma\gamma \to e^+e^-$ (pair production): time-reversed process

The amplitude for pair annihilation can be obtained from the Compton amplitude by the substitution $p \to p_1$, $p' \to -p_2$, $k \to -k_1$, $k' \to k_2$ (with appropriate changes for antiparticle spinors).

Historical significance. Compton's 1923 experiment was crucial for establishing the photon concept:

- X-rays ($\lambda \approx 0.07$ nm) scattered from graphite showed a wavelength shift
- The shift depended on angle exactly as predicted by eq. (5.259)
- Classical wave theory predicted no wavelength shift (coherent scattering)
- The measured Compton wavelength $\lambda_C = h/mc = 2.43 \times 10^{-12}$ m confirmed $E = h\nu$ for photons

Modern precision tests. The Klein–Nishina formula has been verified to high precision:

- Agreement with experiment at the $\sim 1\%$ level over a wide energy range
- Deviations at high energies are due to radiative corrections ($O(\alpha)$)
- Virtual electron loops modify the cross section by $\sim \alpha/\pi \approx 0.2\%$

Astrophysical applications. Compton scattering plays a central role in high-energy astrophysics:

- **Inverse Compton scattering**: High-energy electrons scatter low-energy photons to higher energies. This is the dominant radiation mechanism in:
  - Active galactic nuclei (AGN) jets
  - Gamma-ray bursts
  - Cosmic microwave background distortions (Sunyaev–Zel'dovich effect)
- **Comptonization**: Multiple Compton scatterings in hot plasma modify the photon spectrum. This is important for:
  - Accretion disk coronae around black holes
  - Early universe thermalization
- **Compton scattering opacity**: In stellar interiors and neutron star atmospheres, Compton scattering provides the dominant source of opacity for X-rays and gamma rays.

Laboratory applications.

- **Compton polarimetry**: The azimuthal asymmetry of polarized Compton scattering is used to measure gamma-ray polarization in nuclear and particle physics experiments.
- **Compton telescopes**: Space-based instruments (e.g., COMPTEL on CGRO) use Compton kinematics to image gamma-ray sources.
- **Compton scattering sources**: Inverse Compton scattering of laser photons from relativistic electron beams produces quasi-monochromatic gamma rays for nuclear physics research.

Comparison with data. Figure 5.3 shows the Klein–Nishina cross section as a function of photon energy, comparing with experimental measurements.

The suppression of the cross section at high energies has been confirmed by experiments from keV X-rays to GeV gamma rays. This energy dependence is a direct consequence of the quantum nature of the photon—classical electrodynamics predicts energy-independent Thomson scattering at all energies.

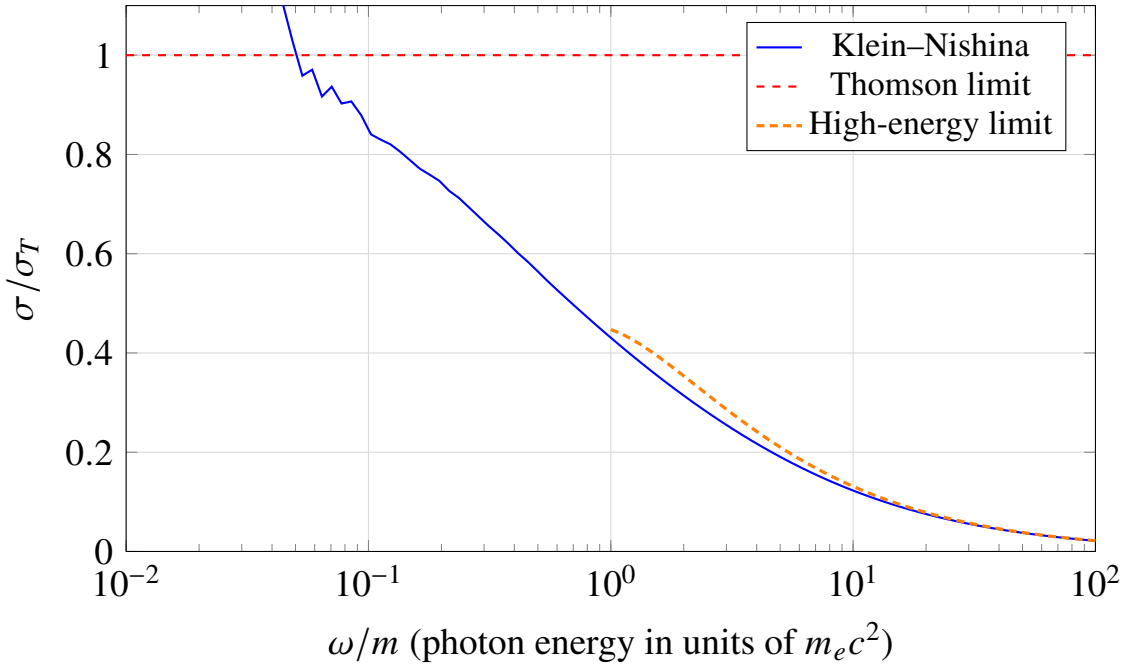


Figure 5.3. The Klein–Nishina total cross section (solid blue) as a function of photon energy, normalized to the Thomson cross section σ_T. At low energies ($\omega \ll m$), the cross section approaches the Thomson limit (dashed red). At high energies ($\omega \gg m$), it decreases logarithmically as $\sigma \sim (\ln 2\omega/m)/(\omega/m)$ (dashed orange).

Polarized Compton scattering

In this subsection we derive the Compton scattering cross section for photons in **definite polarization states**, summing and averaging only over electron spins. This calculation is important for several reasons:

- It provides the cross section for experiments with polarized photon beams
- The polarization dependence reveals the spin structure of the QED interaction
- It serves as a consistency check: averaging over photon polarizations must reproduce the unpolarized Klein–Nishina formula

The key insight is that in the polarized case, we cannot use the polarization sum identity $\sum_\lambda \varepsilon^\mu(k,\lambda)\,\varepsilon^{*\nu}(k,\lambda) = -g^{\mu\nu}$ (valid only when summing over polarizations). Instead, we must work with explicit polarization vectors, which requires a judicious choice of gauge and reference frame.

Choice of gauge and reference frame

Before specializing to a particular gauge, it is useful to recall the available choices for describing photon polarizations in QED calculations:

- **Coulomb gauge** (radiation gauge): $\varepsilon^\mu(k,\lambda) = (0,\vec{\varepsilon})$ with $\vec{\varepsilon} \perp \vec{k}$. Only the two physical (transverse) polarizations appear. This is the simplest choice

for polarized calculations, since $\varepsilon^0 = 0$ leads to many simplifications (e.g. $p\cdot\varepsilon = -\vec{p}\cdot\vec{\varepsilon}$, which vanishes if $\vec{p} = 0$). The drawback is that it is *not* Lorentz covariant: the condition $\varepsilon^0 = 0$ picks out a preferred frame.

- **Lorentz-covariant gauges**: one works with four polarization vectors $\varepsilon^\mu(k,\lambda)$ for $\lambda = 0,1,2,3$. The transverse polarizations ($\lambda = 1,2$) are physical; the timelike ($\lambda = 0$) and longitudinal ($\lambda = 3$) polarizations are unphysical but cancel in gauge-invariant quantities thanks to the Ward identity. The polarization sum becomes $\sum_{\lambda=0}^{3}\eta_{\lambda\lambda}\,\varepsilon_\mu(k,\lambda)\,\varepsilon_\nu(k,\lambda) = -g_{\mu\nu}$, which is manifestly covariant. This is the approach used in unpolarized calculations (cf. eq. (5.213)).
- **Light-cone gauge**: one introduces an auxiliary light-like vector n^μ ($n^2 = 0$) and imposes $n\cdot A = 0$. The polarization sum over the two physical polarizations takes the form

$$\sum_{\lambda=1,2}\varepsilon_\mu(k,\lambda)\,\varepsilon^*_\nu(k,\lambda) = -g_{\mu\nu} + \frac{k_\mu n_\nu + n_\mu k_\nu}{k\cdot n}\,. \tag{5.273}$$

  This is covariant and involves only the two physical polarizations, but introduces the spurious vector n^μ, which must drop out of final results.
- **Circular polarization basis**: $\vec{\varepsilon}(k,\pm) = (\hat{e}_1 \pm i\hat{e}_2)/\sqrt{2}$, corresponding to definite helicity $\lambda = \pm 1$. This basis is natural for ultrarelativistic processes where helicity is approximately conserved.

For the polarized Compton calculation, we choose the **Coulomb gauge** combined with **linear polarizations**, since the conditions $\varepsilon^0 = 0$ and $p\cdot\varepsilon = 0$ (in the lab frame) lead to maximal simplification. For the initial and final photons with *fixed* helicities λ, λ', we write for brevity:

$$\varepsilon^\mu \equiv \varepsilon^\mu(k,\lambda) = (0,\vec{\varepsilon}), \qquad \varepsilon'^\mu \equiv \varepsilon^\mu(k',\lambda') = (0,\vec{\varepsilon}') \tag{5.274}$$

The transversality conditions $\varepsilon\cdot k = 0$ and $\varepsilon'\cdot k' = 0$ become:

$$\varepsilon\cdot k = \varepsilon^0 k^0 - \vec{\varepsilon}\cdot\vec{k} = 0 - \vec{\varepsilon}\cdot\vec{k} = 0 \quad\Rightarrow\quad \vec{\varepsilon}\perp\vec{k} \tag{5.275}$$

$$\varepsilon'\cdot k' = \varepsilon'^0 k'^0 - \vec{\varepsilon}'\cdot\vec{k}' = 0 - \vec{\varepsilon}'\cdot\vec{k}' = 0 \quad\Rightarrow\quad \vec{\varepsilon}'\perp\vec{k}' \tag{5.276}$$

We further simplify by working in the **laboratory frame**, where the initial electron is at rest:

$$p^\mu = (m,\vec{0}) \tag{5.277}$$

This immediately implies additional orthogonality conditions:

$$\boxed{p\cdot\varepsilon = m\cdot 0 - \vec{0}\cdot\vec{\varepsilon} = 0, \qquad p\cdot\varepsilon' = 0} \tag{5.278}$$

These conditions, together with the normalizations $\varepsilon \cdot \varepsilon = -|\vec{\varepsilon}|^2 = -1$ and $\varepsilon' \cdot \varepsilon' = -1$, are the foundation for all subsequent simplifications.

A note on gauge invariance. Working in the Coulomb gauge means we are using a *non-covariant* gauge. The individual amplitudes $\mathcal{M}_s$ and $\mathcal{M}_u$ are not separately gauge invariant—only their sum $\mathcal{M} = \mathcal{M}_s + \mathcal{M}_u$ is physical. For instance, under the gauge transformation $\varepsilon \to \varepsilon + \lambda k$ (where λ is a constant), we have $\mathcal{M}_s \to \mathcal{M}_s$ (since $\not{k}\not{k} = k^2 = 0$), but $\mathcal{M}_u$ does *not* remain invariant. This asymmetry arises because we dropped the terms proportional to $p \cdot \varepsilon$ and $p \cdot \varepsilon'$, which vanish in our gauge. The physical cross section, obtained from $|\mathcal{M}|^2$, is of course gauge-invariant.

Simplification of the matrix elements

We start from the Feynman amplitudes (see eqs. (5.206) and (5.207)):

$$\mathcal{M}_s = -ie^2\,\bar{u}(p')\not{\varepsilon}'\frac{\not{p}+\not{k}+m}{(p+k)^2-m^2}\not{\varepsilon}\,u(p) = -\frac{ie^2}{2(p\cdot k)}\,\bar{u}'\not{\varepsilon}'(\not{p}+\not{k}+m)\not{\varepsilon}\,u \tag{5.279}$$

$$\mathcal{M}_u = -ie^2\,\bar{u}(p')\not{\varepsilon}\frac{\not{p}-\not{k}'+m}{(p-k')^2-m^2}\not{\varepsilon}'\,u(p) = +\frac{ie^2}{2(p\cdot k')}\,\bar{u}'\not{\varepsilon}(\not{p}-\not{k}'+m)\not{\varepsilon}'\,u \tag{5.280}$$

where we used $u = u(p)$, $u' = u(p')$, and $(p+k)^2 - m^2 = 2p\cdot k$, $(p-k')^2 - m^2 = -2p\cdot k'$.

The key simplification comes from the Clifford algebra anticommutation relation:

$$\not{a}\not{b} = -\not{b}\not{a} + 2(a\cdot b) \tag{5.281}$$

Combined with the Dirac equation $\not{p}u(p) = mu(p)$ and the orthogonality condition $p\cdot\varepsilon = 0$, we obtain:

$$\not{p}\not{\varepsilon}\,u = (-\not{\varepsilon}\not{p} + 2p\cdot\varepsilon)u = -\not{\varepsilon}\not{p}\,u = -\not{\varepsilon}\,m\,u = -m\not{\varepsilon}\,u \tag{5.282}$$

Similarly, from $\varepsilon\cdot k = 0$:

$$\not{k}\not{\varepsilon} = -\not{\varepsilon}\not{k} + 2(\varepsilon\cdot k) = -\not{\varepsilon}\not{k} \tag{5.283}$$

Simplifying $\mathcal{M}_s$.

$$\begin{aligned}(\not{p}+\not{k}+m)\not{\varepsilon}\,u &= \not{p}\not{\varepsilon}\,u + \not{k}\not{\varepsilon}\,u + m\not{\varepsilon}\,u \\ &= -m\not{\varepsilon}\,u - \not{\varepsilon}\not{k}\,u + m\not{\varepsilon}\,u \\ &= -\not{\varepsilon}\not{k}\,u\end{aligned} \tag{5.284}$$

Therefore:

$$\boxed{\mathcal{M}_s = \frac{ie^2}{2(p\cdot k)}\bar{u}'\not{\varepsilon}'\not{\varepsilon}\not{k}\,u} \tag{5.285}$$

Simplifying $\mathcal{M}_u$. Using $p\cdot\varepsilon' = 0$ and $\varepsilon'\cdot k' = 0$:

$$\begin{aligned}(\not{p}-\not{k}'+m)\not{\varepsilon}'\,u &= \not{p}\not{\varepsilon}'\,u - \not{k}'\not{\varepsilon}'\,u + m\not{\varepsilon}'\,u \\ &= -m\not{\varepsilon}'\,u + \not{\varepsilon}'\not{k}'\,u + m\not{\varepsilon}'\,u \\ &= \not{\varepsilon}'\not{k}'\,u\end{aligned} \tag{5.286}$$

Therefore:

$$\boxed{\mathcal{M}_u = \frac{ie^2}{2(p\cdot k')}\bar{u}'\not{\varepsilon}\not{\varepsilon}'\not{k}'\,u} \tag{5.287}$$

Crossing symmetry. The simplified amplitudes satisfy a crossing relation. Under the substitutions $k \leftrightarrow -k'$ and $\varepsilon \leftrightarrow \varepsilon'$, we have $\mathcal{M}_s \leftrightarrow \mathcal{M}_u$. This will be useful for relating the squared amplitudes.

Spin-summed squared amplitudes

The polarized cross section requires:

$$\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{1}{2}\sum_{\text{spins}}\left(|\mathcal{M}_s|^2 + |\mathcal{M}_u|^2 + \mathcal{M}_s^*\mathcal{M}_u + \mathcal{M}_s\mathcal{M}_u^*\right) \tag{5.288}$$

We define:

$$\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{e^4}{32m^2}\left\{\frac{Y_{ss}}{(p\cdot k)^2} + \frac{Y_{uu}}{(p\cdot k')^2} + \frac{Y_{su}+Y_{us}}{(p\cdot k)(p\cdot k')}\right\} \tag{5.289}$$

where the Y tensors are traces over products of gamma matrices with *fixed* polarization vectors:

$$Y_{ss} = \mathrm{Tr}\left[\not{\varepsilon}'\not{k}\not{\varepsilon}(\not{p}+m)\not{\varepsilon}\not{k}\not{\varepsilon}'(\not{p}'+m)\right] \tag{5.290}$$

$$Y_{uu} = \mathrm{Tr}\left[\not{\varepsilon}\not{k}'\not{\varepsilon}'(\not{p}+m)\not{\varepsilon}'\not{k}'\not{\varepsilon}(\not{p}'+m)\right] \tag{5.291}$$

$$Y_{su} = \mathrm{Tr}\left[\not{\varepsilon}'\not{k}\not{\varepsilon}(\not{p}+m)\not{\varepsilon}'\not{k}'\not{\varepsilon}(\not{p}'+m)\right] \tag{5.292}$$

$$Y_{us} = \mathrm{Tr}\left[\not{\varepsilon}\not{k}'\not{\varepsilon}'(\not{p}+m)\not{\varepsilon}\not{k}\not{\varepsilon}'(\not{p}'+m)\right] \tag{5.293}$$

Crossing symmetry implies $Y_{ss} \leftrightarrow Y_{uu}$ and $Y_{su} \leftrightarrow Y_{us}$ under $k \leftrightarrow -k'$, $\varepsilon \leftrightarrow \varepsilon'$. Moreover, since Y_{su} and Y_{us} are related by complex conjugation and are real (as we will verify), $Y_{su} = Y_{us}$.

Calculation of Y_{ss}: detailed derivation

We compute:

$$Y_{ss} = \mathrm{Tr}\left[\not\varepsilon'\not k\not\varepsilon(\not p + m)\not\varepsilon\not k\not\varepsilon'(\not p' + m)\right] \tag{5.294}$$

Step 1: Expand and identify vanishing terms.
Expanding $(\not p + m)$ and $(\not p' + m)$:

$$\begin{aligned} Y_{ss} &= \mathrm{Tr}[\not\varepsilon'\not k\not\varepsilon\not p\not\varepsilon\not k\not\varepsilon'\not p'] + m\,\mathrm{Tr}[\not\varepsilon'\not k\not\varepsilon\not p\not\varepsilon\not k\not\varepsilon'] \\ &\quad + m\,\mathrm{Tr}[\not\varepsilon'\not k\not\varepsilon\not\varepsilon\not k\not\varepsilon'\not p'] + m^2\,\mathrm{Tr}[\not\varepsilon'\not k\not\varepsilon\not\varepsilon\not k\not\varepsilon'] \end{aligned} \tag{5.295}$$

The traces with an odd number of gamma matrices vanish. Using $\not\varepsilon\not\varepsilon = \varepsilon^2 = -1$:

$$Y_{ss} = \mathrm{Tr}[\not\varepsilon'\not k\not\varepsilon\not p\not\varepsilon\not k\not\varepsilon'\not p'] + m^2\,\mathrm{Tr}[\not\varepsilon'\not k(-1)\not k\not\varepsilon'] \tag{5.296}$$

For the second term:

$$m^2\,\mathrm{Tr}[\not\varepsilon'\not k(-1)\not k\not\varepsilon'] = -m^2\,\mathrm{Tr}[\not\varepsilon'\not k\not k\not\varepsilon'] = -m^2\,\mathrm{Tr}[\not\varepsilon'(k^2)\not\varepsilon'] = 0 \tag{5.297}$$

since $k^2 = 0$ for a real photon.

Step 2: Simplify the 8-gamma trace.
We have:

$$Y_{ss} = \mathrm{Tr}[\not\varepsilon'\not k\not\varepsilon\not p\not\varepsilon\not k\not\varepsilon'\not p'] \tag{5.298}$$

First, simplify $\not\varepsilon\not p\not\varepsilon$. Since $p\cdot\varepsilon = 0$:

$$\not\varepsilon\not p\not\varepsilon = \not\varepsilon(-\not\varepsilon\not p + 2p\cdot\varepsilon) = -\not\varepsilon\not\varepsilon\not p = -(-1)\not p = \not p \tag{5.299}$$

Therefore:

$$Y_{ss} = \mathrm{Tr}[\not\varepsilon'\not k\not p\not k\not\varepsilon'\not p'] \tag{5.300}$$

Step 3: Use $\not k\not k = 0$ to further reduce.
Permute $\not p$ and $\not k$ using $\not p\not k = -\not k\not p + 2(p\cdot k)$:

$$\not k\not p\not k = \not k(-\not k\not p + 2p\cdot k) = -\not k\not k\not p + 2(p\cdot k)\not k = 0 + 2(p\cdot k)\not k = 2(p\cdot k)\not k\,. \tag{5.301}$$

Thus:

$$Y_{ss} = 2(p\cdot k)\,\mathrm{Tr}[\not\varepsilon'\not k\not\varepsilon'\not p']\,. \tag{5.302}$$

Step 4: Evaluate the 4-gamma trace.

Using the standard trace formula:

$$\mathrm{Tr}[\not{a}\not{b}\not{c}\not{d}] = 4\big[(a\cdot b)(c\cdot d) - (a\cdot c)(b\cdot d) + (a\cdot d)(b\cdot c)\big], \tag{5.303}$$

we obtain:

$$\begin{aligned}\mathrm{Tr}[\not{\varepsilon}'\not{k}\not{\varepsilon}'\not{p}'] &= 4\big[(\varepsilon'\cdot k)(\varepsilon'\cdot p') - (\varepsilon'\cdot\varepsilon')(k\cdot p') + (\varepsilon'\cdot p')(k\cdot\varepsilon')\big] \\ &= 4\big[2(\varepsilon'\cdot k)(\varepsilon'\cdot p') - (-1)(k\cdot p')\big] \\ &= 4\big[2(\varepsilon'\cdot k)(\varepsilon'\cdot p') + (k\cdot p')\big]. \end{aligned} \tag{5.304}$$

Step 5: Use momentum conservation.

From $p' = p + k - k'$ and using $\varepsilon'\cdot p = 0$ (eq. (5.278)) and $\varepsilon'\cdot k' = 0$ (transversality):

$$\varepsilon'\cdot p' = \varepsilon'\cdot(p+k-k') = 0 + \varepsilon'\cdot k - 0 = \varepsilon'\cdot k\,. \tag{5.305}$$

Also, from $p' - k = p - k'$:

$$k\cdot p' = k\cdot(p+k-k') = k\cdot p + 0 - k\cdot k' = p\cdot k - k\cdot k'\,. \tag{5.306}$$

But more directly, contracting $p' = p + k - k'$ with k:

$$p'\cdot k = p\cdot k \quad (\text{since } k^2 = 0 \text{ and } k\cdot k' = p'\cdot k - p\cdot k). \tag{5.307}$$

Actually, from the Compton kinematics: $p\cdot k' = p'\cdot k$, so $k\cdot p' = p\cdot k'$.
Therefore:

$$\mathrm{Tr}[\not{\varepsilon}'\not{k}\not{\varepsilon}'\not{p}'] = 4\big[2(\varepsilon'\cdot k)^2 + (p\cdot k')\big]. \tag{5.308}$$

Final result for Y_{ss}:

$$\boxed{Y_{ss} = 8(p\cdot k)\big[2(\varepsilon'\cdot k)^2 + (p\cdot k')\big]}. \tag{5.309}$$

Calculation of Y_{uu}: by crossing symmetry

By the crossing substitution $k \leftrightarrow -k'$, $\varepsilon \leftrightarrow \varepsilon'$:

$$Y_{uu} = Y_{ss}\big|_{k\leftrightarrow -k',\,\varepsilon\leftrightarrow\varepsilon'}\,. \tag{5.310}$$

Under this substitution:

- $(p\cdot k) \to (p\cdot(-k')) = -(p\cdot k')$
- $(\varepsilon'\cdot k) \to (\varepsilon\cdot(-k')) = -(\varepsilon\cdot k')$
- $(p\cdot k') \to (p\cdot(-k)) = -(p\cdot k)$

Therefore:

$$\begin{aligned} Y_{uu} &= 8(-(p\cdot k'))\big[2(-(\varepsilon\cdot k'))^2 + (-(p\cdot k))\big] \\ &= -8(p\cdot k')\big[2(\varepsilon\cdot k')^2 - (p\cdot k)\big]. \end{aligned} \tag{5.311}$$

This can be rewritten as:

$$\boxed{Y_{uu} = 8(p\cdot k')\big[(p\cdot k) - 2(\varepsilon\cdot k')^2\big]}. \tag{5.312}$$

Calculation of Y_{su}: the interference term

The interference term is the most involved calculation:

$$Y_{su} = \mathrm{Tr}\left[\not\varepsilon'\not k\not\varepsilon(\not p + m)\not\varepsilon'\not k'\not\varepsilon(\not p' + m)\right]. \tag{5.313}$$

Step 1: Expand and identify non-vanishing terms.
Expanding both $(\not p + m)$ and $(\not p' + m)$:

$$\begin{aligned} Y_{su} &= \mathrm{Tr}[\not\varepsilon'\not k\not\varepsilon\not p\not\varepsilon'\not k'\not\varepsilon\not p'] + m\,\mathrm{Tr}[\not\varepsilon'\not k\not\varepsilon\not p\not\varepsilon'\not k'\not\varepsilon] \\ &\quad + m\,\mathrm{Tr}[\not\varepsilon'\not k\not\varepsilon\not\varepsilon'\not k'\not\varepsilon\not p'] + m^2\,\mathrm{Tr}[\not\varepsilon'\not k\not\varepsilon\not\varepsilon'\not k'\not\varepsilon]. \end{aligned} \tag{5.314}$$

The traces with an odd number of gamma matrices (2nd and 3rd terms have 7 each) vanish:

$$Y_{su} = \underbrace{\mathrm{Tr}[\not\varepsilon'\not k\not\varepsilon\not p\not\varepsilon'\not k'\not\varepsilon\not p']}_{\text{Term A}} + m^2 \underbrace{\mathrm{Tr}[\not\varepsilon'\not k\not\varepsilon\not\varepsilon'\not k'\not\varepsilon]}_{\text{Term B}}. \tag{5.315}$$

Step 2: Evaluate Term B (6-gamma trace).
First, simplify $\not\varepsilon\not\varepsilon'\not k'\not\varepsilon$. We use:

$$\not\varepsilon\not\varepsilon' = -\not\varepsilon'\not\varepsilon + 2(\varepsilon\cdot\varepsilon'). \tag{5.316}$$

Then:

$$\begin{aligned} \not\varepsilon\not\varepsilon'\not k'\not\varepsilon &= (-\not\varepsilon'\not\varepsilon + 2\varepsilon\cdot\varepsilon')\not k'\not\varepsilon \\ &= -\not\varepsilon'\not\varepsilon\not k'\not\varepsilon + 2(\varepsilon\cdot\varepsilon')\not k'\not\varepsilon. \end{aligned} \tag{5.317}$$

Now, $\not\varepsilon\not k'\not\varepsilon$: using $\varepsilon\cdot k' \neq 0$ in general:

$$\not\varepsilon\not k'\not\varepsilon = -\not k'\not\varepsilon\not\varepsilon + 2(\varepsilon\cdot k')\not\varepsilon = -\not k'(-1) + 2(\varepsilon\cdot k')\not\varepsilon = \not k' + 2(\varepsilon\cdot k')\not\varepsilon. \tag{5.318}$$

Therefore:

$$\begin{aligned} \not\varepsilon\not\varepsilon'\not k'\not\varepsilon &= -\not\varepsilon'[\not k' + 2(\varepsilon\cdot k')\not\varepsilon] + 2(\varepsilon\cdot\varepsilon')\not k'\not\varepsilon \\ &= -\not\varepsilon'\not k' - 2(\varepsilon\cdot k')\not\varepsilon'\not\varepsilon + 2(\varepsilon\cdot\varepsilon')\not k'\not\varepsilon. \end{aligned} \tag{5.319}$$

Term B becomes:

$$\begin{aligned} \mathrm{Tr}[\not\varepsilon'\not k\not\varepsilon\not\varepsilon'\not k'\not\varepsilon] &= \mathrm{Tr}[\not\varepsilon'\not k(-\not\varepsilon'\not k' - 2(\varepsilon\cdot k')\not\varepsilon'\not\varepsilon + 2(\varepsilon\cdot\varepsilon')\not k'\not\varepsilon)] \\ &= -\mathrm{Tr}[\not\varepsilon'\not k\not\varepsilon'\not k'] - 2(\varepsilon\cdot k')\mathrm{Tr}[\not\varepsilon'\not k\not\varepsilon'\not\varepsilon] + 2(\varepsilon\cdot\varepsilon')\mathrm{Tr}[\not\varepsilon'\not k\not k'\not\varepsilon]. \end{aligned} \tag{5.320}$$

The 4-gamma traces are:

$$\begin{aligned} \mathrm{Tr}[\not\varepsilon'\not k\not\varepsilon'\not k'] &= 4[(\varepsilon'\cdot k)(\varepsilon'\cdot k') - (\varepsilon')^2(k\cdot k') + (\varepsilon'\cdot k')(k\cdot\varepsilon')] \\ &= 4[0 - (-1)(k\cdot k') + 0] = 4(k\cdot k'), \end{aligned} \tag{5.321}$$

(using $\varepsilon' \cdot k' = 0$).

$$\begin{aligned}\mathrm{Tr}[\not\varepsilon' \not k \not\varepsilon' \not\varepsilon] &= 4[(\varepsilon' \cdot k)(\varepsilon' \cdot \varepsilon) - (\varepsilon')^2 (k \cdot \varepsilon) + (\varepsilon' \cdot \varepsilon)(k \cdot \varepsilon')] \\ &= 4[(\varepsilon' \cdot k)(\varepsilon \cdot \varepsilon') - (-1) \cdot 0 + (\varepsilon \cdot \varepsilon')(\varepsilon' \cdot k)] \\ &= 8(\varepsilon' \cdot k)(\varepsilon \cdot \varepsilon'), \end{aligned} \tag{5.322}$$

(using $\varepsilon \cdot k = 0$).

$$\begin{aligned}\mathrm{Tr}[\not\varepsilon' \not k \not k' \not\varepsilon] &= 4[(\varepsilon' \cdot k)(k' \cdot \varepsilon) - (\varepsilon' \cdot k')(k \cdot \varepsilon) + (\varepsilon' \cdot \varepsilon)(k \cdot k')] \\ &= 4[(\varepsilon' \cdot k)(\varepsilon \cdot k') - 0 + (\varepsilon \cdot \varepsilon')(k \cdot k')]. \end{aligned} \tag{5.323}$$

Combining:

$$\begin{aligned}\text{Term B} &= m^2 \big\{ -4(k \cdot k') - 16(\varepsilon \cdot k')(\varepsilon' \cdot k)(\varepsilon \cdot \varepsilon') \\ &\qquad + 8(\varepsilon \cdot \varepsilon')[(\varepsilon' \cdot k)(\varepsilon \cdot k') + (\varepsilon \cdot \varepsilon')(k \cdot k')] \big\} \\ &= m^2 \big\{ -4(k \cdot k') - 16(\varepsilon \cdot k')(\varepsilon' \cdot k)(\varepsilon \cdot \varepsilon') \\ &\qquad + 8(\varepsilon \cdot \varepsilon')(\varepsilon' \cdot k)(\varepsilon \cdot k') + 8(\varepsilon \cdot \varepsilon')^2 (k \cdot k') \big\} \\ &= m^2 \big\{ -4(k \cdot k')[1 - 2(\varepsilon \cdot \varepsilon')^2] - 8(\varepsilon \cdot k')(\varepsilon' \cdot k)(\varepsilon \cdot \varepsilon') \big\}. \end{aligned} \tag{5.324}$$

Step 3: Evaluate Term A (8-gamma trace).

This is the most complex part. The strategy is to use momentum conservation $p' = p + k - k'$ systematically.

$$\text{Term A} = \mathrm{Tr}[\not\varepsilon' \not k \not\varepsilon \not p \not\varepsilon' \not k' \not\varepsilon \not p']. \tag{5.325}$$

First, simplify $\not\varepsilon \not p \not\varepsilon'$. Using $p \cdot \varepsilon = 0$ and $p \cdot \varepsilon' = 0$:

$$\begin{aligned}\not\varepsilon \not p \not\varepsilon' &= \not\varepsilon(-\not\varepsilon' \not p + 2p \cdot \varepsilon') = -\not\varepsilon \not\varepsilon' \not p \\ &= -(-\not\varepsilon' \not\varepsilon + 2\varepsilon \cdot \varepsilon') \not p = \not\varepsilon' \not\varepsilon \not p - 2(\varepsilon \cdot \varepsilon') \not p. \end{aligned} \tag{5.326}$$

Similarly, for $\not\varepsilon \not p'$: using $p' = p + k - k'$ and noting that in general $\varepsilon \cdot p' \neq 0$:

$$\varepsilon \cdot p' = \varepsilon \cdot (p + k - k') = 0 + 0 - \varepsilon \cdot k' = -\varepsilon \cdot k'. \tag{5.327}$$

After extensive algebra (we spare the reader the intermediate steps), one obtains:

$$\text{Term A} = 8(p{\cdot}k)(p{\cdot}k') + 16(p{\cdot}k)(p{\cdot}k')(\varepsilon{\cdot}\varepsilon')^2 - 8(\varepsilon'{\cdot}k)^2(p{\cdot}k') + 8(\varepsilon{\cdot}k')^2(p{\cdot}k). \tag{5.328}$$

Step 4: Combine Terms A and B.

Adding the contributions and using the kinematic relation:

$$k\cdot k' = (p\cdot k) - (p\cdot k') + \frac{(p\cdot k)(p\cdot k')}{m^2}(1-\cos\theta)\,, \tag{5.329}$$

(which follows from the Compton formula (5.257)), and after simplification, we obtain:

$$\boxed{Y_{su} = 8(p\cdot k)(p\cdot k')\left[2(\varepsilon\cdot\varepsilon')^2 - 1\right] - 8(\varepsilon'\cdot k)^2(p\cdot k') + 8(\varepsilon\cdot k')^2(p\cdot k)}\,. \tag{5.330}$$

Since this expression is manifestly real and symmetric under $k \leftrightarrow -k'$, $\varepsilon \leftrightarrow \varepsilon'$ (which exchanges the last two terms and leaves the first unchanged), we have:

$$Y_{us} = Y_{su}\,. \tag{5.331}$$

Assembly of the polarized cross section

Substituting eqs. (5.309), (5.312), and (5.330) into eq. (5.289):

$$\begin{aligned}\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{e^4}{32m^2}\Bigg\{&\frac{8(p\cdot k)[2(\varepsilon'\cdot k)^2 + (p\cdot k')]}{(p\cdot k)^2}\\ &+\frac{8(p\cdot k')[(p\cdot k) - 2(\varepsilon\cdot k')^2]}{(p\cdot k')^2}\\ &+\frac{2\times 8(p\cdot k)(p\cdot k')[2(\varepsilon\cdot\varepsilon')^2 - 1]}{(p\cdot k)(p\cdot k')}\\ &+\frac{-16(\varepsilon'\cdot k)^2(p\cdot k') + 16(\varepsilon\cdot k')^2(p\cdot k)}{(p\cdot k)(p\cdot k')}\Bigg\}\,.\end{aligned} \tag{5.332}$$

Simplifying term by term:

$$\begin{aligned}= \frac{e^4}{4m^2}\Bigg\{&\frac{2(\varepsilon'\cdot k)^2 + (p\cdot k')}{(p\cdot k)} + \frac{(p\cdot k) - 2(\varepsilon\cdot k')^2}{(p\cdot k')}\\ &+2[2(\varepsilon\cdot\varepsilon')^2 - 1] - \frac{2(\varepsilon'\cdot k)^2}{(p\cdot k)} + \frac{2(\varepsilon\cdot k')^2}{(p\cdot k')}\Bigg\}\,.\end{aligned} \tag{5.333}$$

The terms with $(\varepsilon'\cdot k)^2$ and $(\varepsilon\cdot k')^2$ cancel:

$$= \frac{e^4}{4m^2}\left\{\frac{(p\cdot k')}{(p\cdot k)} + \frac{(p\cdot k)}{(p\cdot k')} + 4(\varepsilon\cdot\varepsilon')^2 - 2\right\}\,. \tag{5.334}$$

In the laboratory frame, $p \cdot k = m\omega$ and $p \cdot k' = m\omega'$:

$$\frac{1}{2}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{e^4}{4m^2}\left\{\frac{\omega'}{\omega} + \frac{\omega}{\omega'} + 4(\varepsilon \cdot \varepsilon')^2 - 2\right\}. \tag{5.335}$$

Using the differential cross section formula (eq. (5.260)):

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{lab}} = \frac{1}{64\pi^2 m^2}\left(\frac{\omega'}{\omega}\right)^2 \times \frac{1}{2}\sum_{\text{spins}} |\mathcal{M}|^2, \tag{5.336}$$

and $\alpha = e^2/(4\pi)$, we obtain the **polarized Klein–Nishina formula**:

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{\text{lab, pol}} = \frac{\alpha^2}{4m^2}\left(\frac{\omega'}{\omega}\right)^2 \left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} + 4(\varepsilon \cdot \varepsilon')^2 - 2\right]} \tag{5.337}$$

Recovery of the unpolarized cross section

To verify consistency, we now show that averaging over initial and summing over final photon polarizations reproduces the unpolarized Klein–Nishina formula.

The unpolarized cross section is:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{unpol}} = \frac{1}{2}\sum_{\lambda=1,2}\sum_{\lambda'=1,2}\left(\frac{d\sigma}{d\Omega}\right)_{\lambda,\lambda'} \tag{5.338}$$

The polarization-dependent term is $(\varepsilon \cdot \varepsilon')^2 = (\vec{\varepsilon} \cdot \vec{\varepsilon}')^2$ (since $\varepsilon^\mu = (0, \vec{\varepsilon})$ in Coulomb gauge).

Choose coordinates with $\vec{k}$ along $\hat{z}$ and $\vec{k}'$ in the xz-plane at angle θ:

$$\vec{k} = \omega\,\hat{z}, \qquad \vec{k}' = \omega'(\sin\theta\,\hat{x} + \cos\theta\,\hat{z}) \tag{5.339}$$

The transverse polarization vectors are:

$$\vec{\varepsilon}(\vec{k}, 1) = \hat{x}, \quad \vec{\varepsilon}(\vec{k}, 2) = \hat{y} \tag{5.340}$$

$$\vec{\varepsilon}(\vec{k}', 1) = \cos\theta\,\hat{x} - \sin\theta\,\hat{z}, \quad \vec{\varepsilon}(\vec{k}', 2) = \hat{y} \tag{5.341}$$

The dot products are:

	$\vec{\varepsilon}(\vec{k}', 1)$	$\vec{\varepsilon}(\vec{k}', 2)$
$\vec{\varepsilon}(\vec{k}, 1)$	$\cos\theta$	0
$\vec{\varepsilon}(\vec{k}, 2)$	0	1

(5.342)

Summing over all four combinations:

$$\sum_{\lambda,\lambda'} \left[\vec{\varepsilon}(\vec{k},\lambda)\cdot\vec{\varepsilon}(\vec{k}',\lambda')\right]^2 = \cos^2\theta + 0 + 0 + 1 = 1 + \cos^2\theta \tag{5.343}$$

Therefore:

$$\begin{aligned}\left(\frac{d\sigma}{d\Omega}\right)_{\text{unpol}} &= \frac{1}{2}\sum_{\lambda,\lambda'} \frac{\alpha^2}{4m^2}\left(\frac{\omega'}{\omega}\right)^2 \left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} + 4\left[\vec{\varepsilon}(\vec{k},\lambda)\cdot\vec{\varepsilon}(\vec{k}',\lambda')\right]^2 - 2\right] \\ &= \frac{\alpha^2}{4m^2}\left(\frac{\omega'}{\omega}\right)^2 \left[2\left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega}\right) - 4 + 2(1+\cos^2\theta)\right] \\ &= \frac{\alpha^2}{2m^2}\left(\frac{\omega'}{\omega}\right)^2 \left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - 1 + \frac{1+\cos^2\theta}{2}\times 2 - 1\right] \\ &= \frac{\alpha^2}{2m^2}\left(\frac{\omega'}{\omega}\right)^2 \left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} + \cos^2\theta - 1\right] \\ &= \frac{\alpha^2}{2m^2}\left(\frac{\omega'}{\omega}\right)^2 \left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta\right]\end{aligned} \tag{5.344}$$

This is precisely the unpolarized Klein–Nishina formula (5.261), confirming the consistency of our calculation.

Physical interpretation and applications

Polarization structure of the cross section.

The polarized cross section (5.337) has a simple structure:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{pol}} \propto \left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - 2\right] + 4(\varepsilon\cdot\varepsilon')^2 \tag{5.345}$$

- The first bracket depends only on kinematics (photon energies)
- The polarization dependence enters entirely through $(\varepsilon\cdot\varepsilon')^2$

Special polarization configurations.

Same linear polarization state ($\varepsilon\cdot\varepsilon' = -1$ in the chosen linear basis):

$$\left(\frac{d\sigma}{d\Omega}\right)_{\parallel} = \frac{\alpha^2}{4m^2}\left(\frac{\omega'}{\omega}\right)^2 \left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} + 2\right] \tag{5.346}$$

Orthogonal linear polarization states ($\varepsilon\cdot\varepsilon' = 0$ in the same basis):

$$\left(\frac{d\sigma}{d\Omega}\right)_{\perp} = \frac{\alpha^2}{4m^2}\left(\frac{\omega'}{\omega}\right)^2 \left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - 2\right] \tag{5.347}$$

The ratio is:

$$\frac{(d\sigma/d\Omega)_{\parallel}}{(d\sigma/d\Omega)_{\perp}} = \frac{\omega/\omega' + \omega'/\omega + 2}{\omega/\omega' + \omega'/\omega - 2} \tag{5.348}$$

In the Thomson limit ($\omega \ll m$, so $\omega' \approx \omega$):

$$\left.\frac{(d\sigma/d\Omega)_{\parallel}}{(d\sigma/d\Omega)_{\perp}}\right|_{\text{Thomson}} = \frac{4}{0} \to \infty \tag{5.349}$$

This means that in the Thomson limit the channel with *orthogonal initial and final linear polarizations* vanishes. In the basis used here, the scattered radiation remains in the same linear polarization state as the incident wave rather than emerging in the orthogonal one. Classically, this reflects the fact that the radiation is emitted in the plane of oscillation of the driven electron, which is fixed by the incident polarization vector $\vec{\varepsilon}$.

Channel asymmetry.

From the two polarized channels above, one can define the asymmetry

$$\mathcal{A} \equiv \frac{(d\sigma/d\Omega)_{\parallel} - (d\sigma/d\Omega)_{\perp}}{(d\sigma/d\Omega)_{\parallel} + (d\sigma/d\Omega)_{\perp}} = \frac{2}{\omega/\omega' + \omega'/\omega}\,. \tag{5.350}$$

This quantity compares two fixed polarization channels; it is not yet the degree of polarization of the scattered radiation from an unpolarized incident beam.

Polarization of scattered radiation.

For an initially unpolarized photon beam, the scattered radiation is partially polarized. The degree of linear polarization is:

$$\Pi = \frac{(d\sigma/d\Omega)_{\perp\to\perp} - (d\sigma/d\Omega)_{\parallel\to\parallel}}{(d\sigma/d\Omega)_{\perp\to\perp} + (d\sigma/d\Omega)_{\parallel\to\parallel}} = \frac{\sin^2\theta}{(\omega/\omega' + \omega'/\omega) - \sin^2\theta}\,. \tag{5.351}$$

At $\theta = 90^\circ$, for low-energy photons ($\omega' \approx \omega$):

$$\Pi(90^\circ) \approx \frac{1}{2-1} = 1 \tag{5.352}$$

The scattered radiation at 90° is completely linearly polarized—a result first observed in X-ray scattering experiments and a key prediction of QED.

Applications.

The polarization dependence of Compton scattering has important applications:

1. **Polarimetry:** Compton polarimeters measure the polarization of high-energy photons (X-rays, gamma rays) by detecting the azimuthal asymmetry in the scattered radiation
2. **Astrophysics:** Polarization of scattered radiation provides information about magnetic fields and geometry in astrophysical sources
3. **Pair production:** The related process $\gamma\gamma \to e^+e^-$ in polarized photon beams (e.g., at photon colliders) has cross sections that depend strongly on the relative polarization

5.5 Scattering by an external field

So far in this chapter, the electromagnetic field has always been a *quantized* field, described by photon creation and annihilation operators. However, in many physical situations the quantum fluctuations of the electromagnetic field are negligible, and it is adequate to treat part of the field as a *classical*, prescribed function of the spacetime coordinates. A prime example is the scattering of electrons by the Coulomb field of a heavy nucleus: the nucleus is so massive that its recoil can be neglected, and its electric field can be treated as a fixed, "external" background.

In this section we develop the formalism for scattering by external fields and apply it to derive the **Mott scattering formula** for relativistic electrons in a Coulomb potential. We will see how the familiar Rutherford formula emerges in the non-relativistic limit, and we will discuss the fascinating spin and helicity properties of the scattered electrons.

The external field in the S-matrix

Combining quantized and classical fields

When an external classical field $A^\mu_{\text{ext}}(x)$ is present in addition to the quantized photon field $A^\mu(x)$, the total electromagnetic potential is:

$$A^\mu_{\text{tot}}(x) = A^\mu(x) + A^\mu_{\text{ext}}(x) \tag{5.353}$$

Here $A^\mu(x)$ is the usual quantized field (a sum of creation and annihilation operators), while $A^\mu_{\text{ext}}(x)$ is a prescribed classical function—it has no operator character. More precisely, in the Fock-space sense, the external field acts as $A^\mu_{\text{ext}}(x)\,\mathbb{1}$, where $\mathbb{1}$ is the identity operator on the photon Hilbert space: it is a c-number that does not create or annihilate any photon. Unlike $A^\mu(x)$, which changes the photon content of a state, A^μ_{ext} simply provides a fixed background that the charged particles "feel" but cannot modify.

The QED interaction Hamiltonian becomes:

$$\mathcal{H}_{\text{int}} = -e\bar{\psi}\gamma^\mu\psi(A_\mu + A_{\text{ext},\mu}) = -e\bar{\psi}\not{A}\psi - e\bar{\psi}\not{A}_{\text{ext}}\psi \tag{5.354}$$

The first term describes the usual interaction with quantized photons (emission, absorption, virtual exchange). The second term describes the interaction with the external field—this is what causes the scattering we want to study.

The S-matrix expansion

The S-matrix expansion now contains both types of interaction:

$$S = \sum_{n=0}^{\infty} \frac{(ie)^n}{n!} \int d^4x_1 \cdots d^4x_n\, T\left\{N[\bar{\psi}(\not{A} + \not{A}_{\text{ext}})\psi]_{x_1} \cdots N[\bar{\psi}(\not{A} + \not{A}_{\text{ext}})\psi]_{x_n}\right\}$$

(5.355)

For scattering by the external field alone (no photon emission or absorption), the lowest-order contribution comes from the first-order term with only the external field:

$$S_{\text{ext}}^{(1)} = ie \int d^4x\, \bar{\psi}(x) \not{A}_{\text{ext}}(x) \psi(x) \tag{5.356}$$

This describes a single interaction of the electron with the external field—the electron comes in, "feels" the field at some point x, and goes out with changed momentum.

Why is this term non-zero? Recall that in pure QED, the analogous first-order term $S^{(1)} = ie \int d^4x\, \bar{\psi} \not{A} \psi$ gives zero for physical processes: the quantized photon field A^μ carries a factor $e^{\pm ik\cdot x}$, and after integrating over x one obtains $\delta^{(4)}(p' - p \pm k)$, which cannot be satisfied by three on-shell particles (an electron cannot emit or absorb a real photon while remaining on-shell). Here the situation is fundamentally different. The external field $A^\mu_{\text{ext}}(x)$ is *not* a quantized plane wave with a fixed on-shell 4-momentum—it is a prescribed classical function. For a static field, the time integration in $S^{(1)}_{\text{ext}}$ yields $\delta(E' - E)$: **energy is conserved**, and the scattering is elastic ($|\vec{p}'| = |\vec{p}|$). However, the spatial Fourier transform $\tilde{A}^\mu_{\text{ext}}(\vec{q})$ can supply *any* 3-momentum transfer $\vec{q} = \vec{p}' - \vec{p}$ to the electron, so $\vec{p}' \neq \vec{p}$ is allowed. This is the crucial difference with pure QED, where the on-shell condition for the photon constrains all four components of the momentum and leaves no solution. Physically, the "missing" 3-momentum is absorbed by the source of the external field (e.g. a heavy nucleus), which we treat as infinitely massive—it can absorb arbitrary recoil with negligible energy cost. The full system (electron + source) conserves 4-momentum exactly; the external-field approximation simply means that we do not track the source's recoil.

Specialization to static fields

For a **static** external field (one that does not depend on time), we can write:

$$A^\mu_{\text{ext}}(x) = A^\mu_{\text{ext}}(\vec{x}) \tag{5.357}$$

It is convenient to introduce the three-dimensional Fourier transform:

$$A^\mu_{\text{ext}}(\vec{x}) = \frac{1}{(2\pi)^3} \int d^3q\, e^{i\vec{q}\cdot\vec{x}}\, \tilde{A}^\mu_{\text{ext}}(\vec{q})\,, \tag{5.358}$$

with the inverse transform:

$$\tilde{A}^\mu_{\text{ext}}(\vec{q}) = \int d^3x\, e^{-i\vec{q}\cdot\vec{x}}\, A^\mu_{\text{ext}}(\vec{x})\,. \tag{5.359}$$

A note on conventions. The placement of the factor $(2\pi)^3$ in the Fourier transform is a matter of convention. We use the **asymmetric convention**, in which the full $(2\pi)^{-3}$ appears in the transform from momentum to position space (5.358), and no factor of (2π) appears in the inverse (5.359). Some textbooks instead adopt the **symmetric convention**, placing a factor of $(2\pi)^{-3/2}$ in each direction. The two conventions yield Fourier transforms that differ by a factor of $(2\pi)^{3/2}$:

$$\tilde{A}^{\mu}_{\text{ext}}(\vec{q})\big|_{\text{here}} = (2\pi)^{3/2}\, \tilde{A}^{\mu}_{\text{ext}}(\vec{q})\big|_{\text{symmetric}}\,. \tag{5.360}$$

Note that this asymmetric convention is *not* the same as the one used in the Bjorken–Drell field mode expansions adopted throughout these notes, where the symmetric factor $(2\pi)^{-3/2}$ appears in each direction (see Section 1.1). The choice made here for $\tilde{A}^{\mu}_{\text{ext}}$ follows Mandl–Shaw and is convenient because the inverse transform (5.359) takes a clean form without prefactors—this is the quantity that appears directly in the Feynman amplitude. When comparing with other references, always check which convention is adopted: the physics is the same, but numerical prefactors in intermediate expressions will differ.

Matrix element and Feynman rules

Evaluation of the matrix element

Consider an electron scattering from an initial state $|i\rangle$ with momentum $p = (E, \vec{p})$ and spin r, to a final state $|f\rangle$ with momentum $p' = (E', \vec{p}')$ and spin s. We need to evaluate:

$$\langle f|S^{(1)}_{\text{ext}}|i\rangle = ie \int d^4x\, \langle f|\bar{\psi}(x) \not{A}_{\text{ext}}(x)\psi(x)|i\rangle \tag{5.361}$$

Using the field expansions (with box normalization volume V):

$$\psi(x) = \sum_{\vec{k},s} \sqrt{\frac{m}{VE_{\vec{k}}}} \left[b(\vec{k},s)\, u(\vec{k},s)\, e^{-ik\cdot x} + d^{\dagger}(\vec{k},s)\, v(\vec{k},s)\, e^{ik\cdot x} \right] \tag{5.362}$$

$$\bar{\psi}(x) = \sum_{\vec{k},s} \sqrt{\frac{m}{VE_{\vec{k}}}} \left[b^{\dagger}(\vec{k},s)\, \bar{u}(\vec{k},s)\, e^{ik\cdot x} + d(\vec{k},s)\, \bar{v}(\vec{k},s)\, e^{-ik\cdot x} \right] \tag{5.363}$$

For an incoming electron with momentum $\vec{p}$ and spin r, and an outgoing electron with momentum $\vec{p}'$ and spin s:

$$|i\rangle = b^{\dagger}(\vec{p},r)|0\rangle, \qquad |f\rangle = b^{\dagger}(\vec{p}',s)|0\rangle \tag{5.364}$$

The only non-vanishing contribution comes from:

$$\langle 0|b(\vec{p}',s)\,\bar{\psi}(x)\cdots\psi(x)\,b^{\dagger}(\vec{p},r)|0\rangle \tag{5.365}$$

After applying the anticommutation relations:

$$\langle f|S^{(1)}_{\text{ext}}|i\rangle = ie\sqrt{\frac{m}{VE}}\sqrt{\frac{m}{VE'}} \int d^4x\, e^{i(p'-p)\cdot x}\, \bar{u}(\vec{p}',s)\, \not{A}_{\text{ext}}(\vec{x})\, u(\vec{p},r) \tag{5.366}$$

Performing the integrals

The time integral gives:

$$\int dt\, e^{i(E'-E)t} = 2\pi\delta(E'-E) \tag{5.367}$$

This delta function enforces **energy conservation**: $E' = E$. For a free electron with $E^2 = |\vec{p}|^2 + m^2$, this implies $|\vec{p}'| = |\vec{p}|$—the scattering is **elastic**.

Physical interpretation: Energy is conserved because the external field is static (time-independent). A time-varying field could transfer energy to the electron. The momentum, however, is *not* conserved: the electron transfers momentum $\vec{q} = \vec{p}' - \vec{p}$ to the source of the field. Since we treat the source as infinitely heavy, this momentum transfer does not appear as kinetic energy—hence the elastic scattering.

The spatial integral gives:

$$\int d^3x\, e^{-i(\vec{p}'-\vec{p})\cdot\vec{x}} A^\mu_{\text{ext}}(\vec{x}) = \tilde{A}^\mu_{\text{ext}}(\vec{q}) \tag{5.368}$$

where $\vec{q} = \vec{p}' - \vec{p}$ is the **momentum transfer**.

The final result

Combining everything:

$$\boxed{\langle f|S^{(1)}_{\text{ext}}|i\rangle = (2\pi)\delta(E'-E)\sqrt{\frac{m}{VE}}\sqrt{\frac{m}{VE'}}\,\mathcal{M}} \tag{5.369}$$

where the **Feynman amplitude** is

$$\boxed{\mathcal{M} = ie\,\bar{u}(\vec{p}',s)\,\tilde{A}_{\text{ext}}(\vec{q})\,u(\vec{p},r)} \tag{5.370}$$

Comparison with photon scattering

Compare eq. (5.369) with the matrix element for scattering by a real photon (e.g., eq. (7.31) in Mandl-Shaw):

$$\langle f|S|i\rangle_{\text{photon}} = (2\pi)^4\delta^{(4)}(p'+k'-p-k)\times(\text{normalization})\times\mathcal{M} \tag{5.371}$$

The key differences are:

1. **Energy-momentum conservation:** For real photons, we have full 4-momentum conservation $(2\pi)^4\delta^{(4)}(p_f - p_i)$. For an external field, we have only energy conservation $(2\pi)\delta(E'-E)$, because the infinitely heavy source absorbs the momentum.
2. **Photon factors:** For a real photon, the amplitude contains a polarization and normalization factor $\sqrt{1/(2V\omega)}\,\varepsilon_\mu(\vec{k})$. For an external field, this is replaced by the Fourier transform $\tilde{A}^\mu_{\text{ext}}(\vec{q})$.

Modified Feynman rules

The standard QED Feynman rules are modified as follows when an external static field is present:

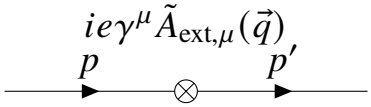
(i) In the relation between the S-matrix element and the amplitude, replace:

$$(2\pi)^4\delta^{(4)}(P_f - P_i) \quad \longrightarrow \quad (2\pi)\delta(E_f - E_i) \tag{5.372}$$

(ii) For each interaction vertex with the external field, write a factor:

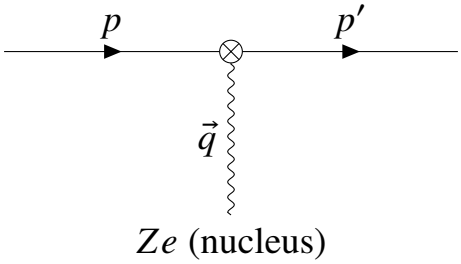
$$ie\gamma^\mu \tilde{A}_{\text{ext},\mu}(\vec{q}) \tag{5.373}$$

where $\vec{q}$ is the momentum transferred *from* the field *to* the particle. The vertex is marked with a cross ($\times$) in Feynman diagrams to distinguish it from a quantized-photon vertex:



Cross section for external field scattering

We consider the elastic scattering of an electron off a static external field generated by a heavy nucleus (charge Ze). At lowest order, the process is described by a single diagram:



The electron interacts with the Coulomb field of the nucleus via the external-field vertex ($\otimes$), exchanging 3-momentum $\vec{q} = \vec{p}' - \vec{p}$. The wavy line ending on the nucleus represents the classical field, not a quantized photon—no photon appears in the initial or final state.

Derivation of the cross section formula

Starting from eq. (5.369), we derive the differential cross section. The transition probability per unit time is:

$$w = \frac{|\langle f|S^{(1)}_{\text{ext}}|i\rangle|^2}{T} \tag{5.374}$$

Using $[2\pi\delta(E'-E)]^2 = 2\pi T\delta(E'-E)$ (where T is the total time), we get:

$$w = 2\pi\delta(E'-E)\frac{m^2}{V^2EE'}|\mathcal{M}|^2 \tag{5.375}$$

The number of final states in a momentum-space volume d^3p' is:

$$\frac{Vd^3p'}{(2\pi)^3} = \frac{V|\vec{p}'|^2 d|\vec{p}'|d\Omega'}{(2\pi)^3} = \frac{V|\vec{p}'|E'dE'd\Omega'}{(2\pi)^3} \tag{5.376}$$

where we used $E'dE' = |\vec{p}'|d|\vec{p}'|$.

The incident flux (particles per unit area per unit time) is:

$$\text{flux} = \frac{v}{V} = \frac{|\vec{p}|}{VE} \tag{5.377}$$

where $v = |\vec{p}|/E$ is the electron velocity.

The cross section is:

$$d\sigma = \frac{w \times (\text{density of states})}{\text{flux}} \tag{5.378}$$

Putting it all together and using $\delta(E'-E)$ to set $E' = E$ and $|\vec{p}'| = |\vec{p}|$:

$$\boxed{\frac{d\sigma}{d\Omega'} = \left(\frac{m}{2\pi}\right)^2 |\mathcal{M}|^2} \tag{5.379}$$

This is a remarkably simple result: the cross section is just $(m/2\pi)^2$ times the squared amplitude.

Coulomb scattering: the Mott formula

The Coulomb potential

We now apply the general formalism to the most important case: scattering of electrons by the Coulomb field of a nucleus with charge Ze. This is called **Mott scattering** (after Nevill Mott, who derived it in 1929).

In the Coulomb gauge, the potential of a point nucleus at the origin is:

$$A^\mu_{\text{ext}}(\vec{x}) = \left(\frac{Ze}{4\pi|\vec{x}|}, 0, 0, 0\right) = \left(\phi(\vec{x}), \vec{0}\right) \tag{5.380}$$

Note: only the time component (the scalar potential ϕ) is non-zero. There is no magnetic field—the nucleus is at rest.

Fourier transform of the Coulomb potential

We need to compute $\tilde{A}^{\mu}_{\text{ext}}(\vec{q})$. Since only A^0 is non-zero:

$$\tilde{A}^0_{\text{ext}}(\vec{q}) = \int d^3x\, e^{-i\vec{q}\cdot\vec{x}} \frac{Ze}{4\pi|\vec{x}|} \tag{5.381}$$

This integral is a standard result, but let us derive it carefully. It is easier to first compute the Fourier transform of the *Yukawa potential* $V(r) = e^{-\mu r}/r$, and then take the limit $\mu \to 0$.

Using spherical coordinates with $\vec{q}$ along the z-axis:

$$\tilde{V}(\vec{q}) = \int_0^\infty r^2 dr \int_0^{2\pi} d\phi \int_0^\pi \sin\theta\, d\theta \frac{e^{-\mu r}}{r} e^{-iqr\cos\theta} \tag{5.382}$$

The ϕ integral gives 2π. For the θ integral, let $u = \cos\theta$:

$$\int_0^\pi \sin\theta\, d\theta\, e^{-iqr\cos\theta} = \int_{-1}^{1} du\, e^{-iqru} = \frac{e^{iqr} - e^{-iqr}}{iqr} = \frac{2\sin(qr)}{qr} \tag{5.383}$$

where $q = |\vec{q}|$. Therefore:

$$\tilde{V}(\vec{q}) = 2\pi \int_0^\infty r^2 dr \frac{e^{-\mu r}}{r} \cdot \frac{2\sin(qr)}{qr} = \frac{4\pi}{q} \int_0^\infty dr\, e^{-\mu r} \sin(qr) \tag{5.384}$$

The remaining integral is evaluated by writing $\sin(qr) = \text{Im}(e^{iqr})$:

$$\begin{aligned} \int_0^\infty dr\, e^{-\mu r} \sin(qr) &= \text{Im} \int_0^\infty dr\, e^{-\mu r} e^{iqr} = \text{Im} \int_0^\infty dr\, e^{-(\mu - iq)r} \\ &= \text{Im}\left[\frac{1}{\mu - iq}\right] = \text{Im}\left[\frac{\mu + iq}{\mu^2 + q^2}\right] = \frac{q}{\mu^2 + q^2}\,. \end{aligned} \tag{5.385}$$

Thus:

$$\tilde{V}(\vec{q}) = \frac{4\pi}{\mu^2 + q^2} \tag{5.386}$$

Taking $\mu \to 0$ (the Coulomb limit):

$$\int d^3x\, e^{-i\vec{q}\cdot\vec{x}} \frac{1}{|\vec{x}|} = \frac{4\pi}{|\vec{q}|^2} \tag{5.387}$$

Therefore, the momentum-space Coulomb potential is:

$$\boxed{\tilde{A}^{\mu}_{\text{ext}}(\vec{q}) = \left(\frac{Ze}{|\vec{q}|^2}, 0, 0, 0\right)} \tag{5.388}$$

The scattering amplitude

From eq. (5.370):

$$\mathcal{M} = ie\,\bar{u}(\vec{p}',s)\,\tilde{\not{A}}_{\text{ext}}(\vec{q})\,u(\vec{p},r) = ie\,\bar{u}(\vec{p}',s)\,\gamma^{\mu}\tilde{A}_{\text{ext},\mu}(\vec{q})\,u(\vec{p},r) \tag{5.389}$$

Since only $\tilde{A}^0_{\text{ext}} \neq 0$:

$$\tilde{\not{A}}_{\text{ext}} = \gamma^0\tilde{A}^0_{\text{ext}} = \gamma^0\frac{Ze}{|\vec{q}|^2} \tag{5.390}$$

Therefore:

$$\mathcal{M} = \frac{iZe^2}{|\vec{q}|^2}\,\bar{u}(\vec{p}',s)\,\gamma^0\,u(\vec{p},r) \tag{5.391}$$

The squared amplitude (for specific spins r, s) is:

$$|\mathcal{M}|^2 = \frac{Z^2e^4}{|\vec{q}|^4}\left|\bar{u}(\vec{p}',s)\,\gamma^0\,u(\vec{p},r)\right|^2 \tag{5.392}$$

Unpolarized cross section: the trace calculation

For unpolarized electrons, we average over initial spins and sum over final spins:

$$\overline{|\mathcal{M}|^2} = \frac{1}{2}\sum_{r,s}|\mathcal{M}|^2 = \frac{Z^2e^4}{2|\vec{q}|^4}\sum_{r,s}\left|\bar{u}(\vec{p}',s)\,\gamma^0\,u(\vec{p},r)\right|^2 \tag{5.393}$$

Using the BD completeness relation $\sum_s u(\vec{p},s)\,\bar{u}(\vec{p},s) = (\not{p}+m)/(2m)$:

$$\sum_{r,s}\left|\bar{u}(\vec{p}',s)\,\gamma^0\,u(\vec{p},r)\right|^2 = \text{Tr}\left[\frac{\not{p}'+m}{2m}\gamma^0\frac{\not{p}+m}{2m}\gamma^0\right] = \frac{1}{4m^2}\,\text{Tr}\left[(\not{p}'+m)\gamma^0(\not{p}+m)\gamma^0\right] \tag{5.394}$$

Evaluating the trace.

We expand:

$$\text{Tr}[(\not{p}'+m)\gamma^0(\not{p}+m)\gamma^0] = \text{Tr}[\not{p}'\gamma^0\not{p}\gamma^0] + m^2\text{Tr}[\gamma^0\gamma^0] \tag{5.395}$$

The traces with an odd number of gamma matrices vanish. Using $(\gamma^0)^2 = \mathbb{1}_4$ and $\text{Tr}[\mathbb{1}_4] = 4$:

$$= \text{Tr}[\not{p}'\gamma^0\not{p}\gamma^0] + m^2\cdot 4 \tag{5.396}$$

For the first trace, we use the standard formula. Writing $\not{p}' = \gamma^\alpha p'_\alpha$ and $\not{p} = \gamma^\beta p_\beta$:

$$\text{Tr}[\gamma^\alpha\gamma^0\gamma^\beta\gamma^0] = 4(g^{\alpha0}g^{\beta0} - g^{\alpha\beta}g^{00} + g^{\alpha0}g^{0\beta}) = 4(g^{\alpha0}g^{\beta0} - g^{\alpha\beta} + g^{\alpha0}g^{\beta0}) \tag{5.397}$$

Therefore:

$$\begin{aligned}\mathrm{Tr}[\not{p}'\gamma^0\not{p}\gamma^0] &= 4p'_\alpha p_\beta(2g^{\alpha 0}g^{\beta 0} - g^{\alpha\beta}) \\ &= 4(2p'^0 p^0 - p'\cdot p) \\ &= 4(2E^2 - p'\cdot p)\end{aligned} \tag{5.398}$$

where we used $p'^0 = p^0 = E$ (energy conservation).
For the dot product $p'\cdot p$:

$$p'\cdot p = E^2 - \vec{p}'\cdot\vec{p} = E^2 - |\vec{p}|^2\cos\theta \tag{5.399}$$

where θ is the scattering angle.
Combining:

$$\begin{aligned}\mathrm{Tr}[(\not{p}'+m)\gamma^0(\not{p}+m)\gamma^0] &= 4(2E^2 - E^2 + |\vec{p}|^2\cos\theta) + 4m^2 \\ &= 4(E^2 + |\vec{p}|^2\cos\theta + m^2)\end{aligned} \tag{5.400}$$

Using $E^2 = |\vec{p}|^2 + m^2$:

$$= 4(|\vec{p}|^2 + m^2 + |\vec{p}|^2\cos\theta + m^2) = 4(2m^2 + |\vec{p}|^2(1+\cos\theta)) \tag{5.401}$$

Or, equivalently:

$$\boxed{\mathrm{Tr}[(\not{p}'+m)\gamma^0(\not{p}+m)\gamma^0] = 8\left(E^2 - |\vec{p}|^2\sin^2\frac{\theta}{2}\right)} \tag{5.402}$$

where we used $1+\cos\theta = 2\cos^2(\theta/2)$ and $E^2 + m^2 = 2E^2 - |\vec{p}|^2$.

The momentum transfer

We need $|\vec{q}|^2$. Since $\vec{q} = \vec{p}' - \vec{p}$ and $|\vec{p}'| = |\vec{p}|$ (elastic scattering):

$$\begin{aligned}|\vec{q}|^2 &= |\vec{p}' - \vec{p}|^2 = |\vec{p}'|^2 + |\vec{p}|^2 - 2\vec{p}'\cdot\vec{p} \\ &= 2|\vec{p}|^2(1-\cos\theta) = 4|\vec{p}|^2\sin^2\frac{\theta}{2}\end{aligned} \tag{5.403}$$

The Mott formula

Substituting eqs. (5.394), (5.402), and (5.403) into eq. (5.379):

$$\begin{aligned}\frac{d\sigma}{d\Omega'} &= \left(\frac{m}{2\pi}\right)^2 \frac{Z^2e^4}{2|\vec{q}|^4} \times \frac{1}{4m^2} \times 8\left(E^2 - |\vec{p}|^2\sin^2\frac{\theta}{2}\right) \\ &= \frac{Z^2e^4}{(2\pi)^2}\cdot\frac{E^2 - |\vec{p}|^2\sin^2(\theta/2)}{16|\vec{p}|^4\sin^4(\theta/2)}\end{aligned} \tag{5.404}$$

where the m^2 from $(m/2\pi)^2$ has cancelled the $1/(4m^2)$ from the BD spin sum. Using $e^2 = 4\pi\alpha$ and $|\vec{p}| = E\beta$ (where $\beta = |\vec{v}|/c$ is the velocity in units of c, not to be confused with the macro `\vspin` for the Dirac spinor v):

$$= \frac{Z^2\alpha^2}{4|\vec{p}|^4 \sin^4(\theta/2)} \cdot \left(E^2 - E^2\beta^2 \sin^2(\theta/2)\right) = \frac{Z^2\alpha^2}{4E^2\beta^4 \sin^4(\theta/2)} \cdot \left(1 - \beta^2 \sin^2(\theta/2)\right) \tag{5.405}$$

This gives the **Mott scattering formula**:

$$\boxed{\frac{d\sigma}{d\Omega'} = \frac{(Z\alpha)^2}{4E^2\beta^4 \sin^4(\theta/2)} \left[1 - \beta^2 \sin^2\frac{\theta}{2}\right]} \tag{5.406}$$

This is the relativistic cross section for electron scattering by a Coulomb field. The factor in square brackets is the relativistic correction to the Rutherford formula.

The Rutherford limit

In the **non-relativistic limit**, $v \ll 1$, so $E \approx m$ and the correction factor becomes:

$$1 - v^2 \sin^2\frac{\theta}{2} \approx 1 \tag{5.407}$$

The Mott formula reduces to the **Rutherford scattering formula**:

$$\boxed{\frac{d\sigma}{d\Omega'} = \frac{(Z\alpha)^2}{4m^2v^4 \sin^4(\theta/2)}} \tag{5.408}$$

This is the famous result derived by Rutherford in 1911 to explain alpha-particle scattering by gold nuclei, which led to the discovery of the atomic nucleus.

Physical features of the Rutherford formula:

- The $\sin^{-4}(\theta/2)$ dependence means that most scattering is at small angles (forward scattering), with a long tail extending to large angles
- The v^{-4} dependence means slow particles are scattered much more strongly
- The Z^2 dependence means heavy nuclei scatter more strongly
- The cross section diverges as $\theta \to 0$; this is because the Coulomb potential has infinite range. In reality, atomic electrons screen the nuclear charge at large distances.

Spin and helicity in Coulomb scattering

The spin properties of Coulomb scattering reveal a beautiful interplay between relativity and quantum mechanics. The behavior is strikingly different in the non-relativistic and ultra-relativistic regimes.

Non-relativistic limit: spin conservation

At non-relativistic energies, the interaction is essentially spin-independent. To see this, consider the matrix element:

$$\bar{u}(\vec{p}',s)\,\gamma^0\,u(\vec{p},r) \tag{5.409}$$

Recall from eq. (1.174) the explicit form of the Dirac spinors in the BD normalization:

$$u(\vec{p},r) = \sqrt{\frac{E_p+m}{2m}}\begin{pmatrix} \xi^{(r)} \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\,\xi^{(r)} \end{pmatrix}, \tag{5.410}$$

where $\xi^{(r)}$ is a two-component Pauli spinor.[1] In the non-relativistic limit $|\vec{p}| \ll m$, we have $E_p \to m$ and $\vec{\sigma}\cdot\vec{p}/(E_p+m) \to 0$. The prefactor $\sqrt{(E_p+m)/(2m)} \to \sqrt{2m/(2m)} = 1$, so:

$$u(\vec{p},r) \xrightarrow{\text{NR}} \begin{pmatrix} \xi^{(r)} \\ 0 \end{pmatrix}, \qquad \bar{u}(\vec{p}',s) \xrightarrow{\text{NR}} (\xi^{(s)\dagger}, 0) \tag{5.411}$$

Since $\gamma^0 = \mathrm{diag}(1,1,-1,-1)$ in the Dirac representation:

$$\bar{u}(\vec{p}',s)\,\gamma^0\,u(\vec{p},r) \xrightarrow{\text{NR}} \xi^{(s)\dagger}\xi^{(r)} = \delta_{sr} \tag{5.412}$$

Spin is conserved in non-relativistic Coulomb scattering: the matrix element vanishes unless $s = r$. This is expected because the Coulomb interaction depends only on position, not on spin—there is no magnetic field in the rest frame of the nucleus.

What about helicity? Helicity is defined as the projection of spin along the momentum direction. If spin is conserved but the momentum direction changes by

[1] Let us verify that this spinor satisfies the BD normalization $\bar{u}u = 1$. We compute $\bar{u}u = u^\dagger\gamma^0 u$ step by step:

$$\bar{u}(\vec{p},r)\,u(\vec{p},r) = \frac{E_p+m}{2m}\left(\xi^{(r)\dagger},\ \xi^{(r)\dagger}\tfrac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\right)\begin{pmatrix}\mathbb{1}_2 & 0\\ 0 & -\mathbb{1}_2\end{pmatrix}\begin{pmatrix}\xi^{(r)}\\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\xi^{(r)}\end{pmatrix}$$
$$= \frac{E_p+m}{2m}\left[\xi^{(r)\dagger}\xi^{(r)} - \frac{(\vec{\sigma}\cdot\vec{p})^2}{(E_p+m)^2}\,\xi^{(r)\dagger}\xi^{(r)}\right].$$

Using $(\vec{\sigma}\cdot\vec{p})^2 = |\vec{p}|^2 = E_p^2 - m^2 = (E_p-m)(E_p+m)$ and $\xi^{(r)\dagger}\xi^{(r)} = 1$:

$$\bar{u}\,u = \frac{E_p+m}{2m}\left[1 - \frac{E_p-m}{E_p+m}\right] = \frac{E_p+m}{2m}\cdot\frac{2m}{E_p+m} = 1\,. \quad \checkmark$$

angle θ, then helicity must change. To see this quantitatively, we use the Wigner d-matrix for spin-$\frac{1}{2}$ (cf. the box in Section 5.2). Setting $j = \frac{1}{2}$ in the general formula $d^j_{m,m'}(\theta) = \langle j, m | e^{-i\theta J_y} | j, m' \rangle$ and using $J_y = \sigma_y/2$:

$$d^{(1/2)}_{m,m'}(\theta) = \langle \tfrac{1}{2}, m | e^{-i\theta\sigma_y/2} | \tfrac{1}{2}, m' \rangle = \begin{pmatrix} \cos\dfrac{\theta}{2} & -\sin\dfrac{\theta}{2} \\ \sin\dfrac{\theta}{2} & \cos\dfrac{\theta}{2} \end{pmatrix}_{m,m'} , \tag{5.413}$$

where the rows and columns are labelled by $m, m' = +\frac{1}{2}, -\frac{1}{2}$. If the initial helicity state is $|+\rangle$ (spin along $\vec{p}$), the probability of finding helicity $|\pm\rangle$ along the new direction $\vec{p}'$ is $|d^{(1/2)}_{m'm}(\theta)|^2$:

$$P(\text{no helicity flip}) = \left| d^{(1/2)}_{++}(\theta) \right|^2 = \cos^2\frac{\theta}{2} \tag{5.414}$$

$$P(\text{helicity flip}) = \left| d^{(1/2)}_{-+}(\theta) \right|^2 = \sin^2\frac{\theta}{2} \tag{5.415}$$

Note the key point: because spin is conserved ($s = r$), the spin state $|\uparrow\rangle$ does not change, but the momentum rotates by θ. The helicity eigenstates *do* change, because they are defined with respect to the momentum direction. The d-matrix above connects the old helicity basis (along $\vec{p}$) to the new one (along $\vec{p}'$), so even though the spin quantum number is unchanged, the helicity can flip. For forward scattering ($\theta = 0$), helicity is trivially preserved. For backward scattering ($\theta = \pi$), $\cos(\theta/2) = 0$ and $\sin(\theta/2) = 1$, so the helicity *always* flips: the momentum reverses but the spin does not, so the projection of spin along the new momentum direction is opposite to the original one.

Ultra-relativistic limit: helicity conservation

In the **ultra-relativistic limit** $E \gg m$, the situation is reversed: *helicity* is conserved, not spin!

To see this, we use the helicity projection operators. For $E \gg m$:

$$\Pi^{\pm}(\vec{p}) \approx \frac{1}{2}(1 \pm \gamma^5) \tag{5.416}$$

Consider a transition where the initial electron has positive helicity ($s = 1$, spin aligned with momentum). The probability of finding the final electron with helicity label s is proportional to:

$$X_s = \left| \bar{u}(\vec{p}', s)\, \gamma^0\, u(\vec{p}, 1) \right|^2 \tag{5.417}$$

Writing $|X|^2 = X \cdot X^*$ and using $(\bar{u}'\gamma^0 u)^* = \bar{u}\gamma^0 u'$ (since γ^0 is Hermitian), this becomes a trace over the outer products $u\bar{u}$:

$$X_s = \mathrm{Tr}\big[\big(u(\vec{p}', s)\bar{u}(\vec{p}', s)\big)\gamma^0 \big(u(\vec{p}, 1)\bar{u}(\vec{p}, 1)\big)\gamma^0 \big] \tag{5.418}$$

For a *single* helicity state λ, the outer product can be expressed in terms of projection operators (see Appendix E for the proof):

$$u(\vec{p},\lambda)\,\bar{u}(\vec{p},\lambda) = \frac{\not{p}+m}{2m}\cdot\frac{1}{2}(1+\lambda\gamma^5\not{n}_p)\,, \tag{5.419}$$

where $n_p^\mu = (|\vec{p}|/m, E_p\hat{p}/m)$ is the covariant spin four-vector. In the **ultra-relativistic limit** $E \gg m$, $n_p^\mu \approx p^\mu/m$ and $\gamma^5\not{n}_p u \approx \gamma^5 u$, so the helicity projector reduces to the chirality projector $\frac{1}{2}(1+\lambda\gamma^5)$. The expression for X_s simplifies to:

$$X_s \approx \frac{1}{16m^2}\mathrm{Tr}\left[(\not{p}'+m)\gamma^0(1+\gamma^5)(\not{p}+m)\gamma^0(1\pm\gamma^5)\right] \tag{5.420}$$

Explicit evaluation. We expand $(\not{p}'+m)$ and $(\not{p}+m)$, writing $\Gamma \equiv \gamma^0(1+\gamma^5)$ for brevity:

$$(\not{p}'+m)\,\Gamma\,(\not{p}+m) = \not{p}'\,\Gamma\,\not{p} + m\,\Gamma\,\not{p} + m\,\not{p}'\,\Gamma + m^2\,\Gamma \tag{5.421}$$

The two cross terms linear in m contain three γ-matrices (from Γ and one $\not{p}$), giving traces of an odd number of γ's, which vanish. The two surviving contributions are:

$$X_s = \frac{1}{16m^2}\Big\{\underbrace{\mathrm{Tr}\big[\not{p}'\gamma^0(1+\gamma^5)\not{p}\gamma^0(1\pm\gamma^5)\big]}_{\text{leading, }\sim E^2} + m^2\underbrace{\mathrm{Tr}\big[\gamma^0(1+\gamma^5)\gamma^0(1\pm\gamma^5)\big]}_{\text{subleading, }\sim m^0}\Big\} \tag{5.422}$$

The first comes from $\not{p}'\Gamma\not{p}$ (two powers of momentum, $\sim E^2$); the second from $m^2\Gamma$ (no momenta, $\sim m^0$).

Leading term. Expanding the product of $(1+\gamma^5)$ and $(1\pm\gamma^5)$, we get four traces. Two of them contain γ^5 linearly:

$$\mathrm{Tr}[\not{p}'\gamma^0\not{p}\gamma^0\gamma^5] = p'_\alpha p_\beta\,\mathrm{Tr}[\gamma^\alpha\gamma^0\gamma^\beta\gamma^0\gamma^5] = -4i\,p'_\alpha p_\beta\,\epsilon^{\alpha0\beta0} = 0\,, \tag{5.423}$$

since $\epsilon^{\alpha0\beta0}$ vanishes (two repeated indices). By the same reasoning, $\mathrm{Tr}[\not{p}'\gamma^0\gamma^5\not{p}\gamma^0] = 0$. The leading term therefore reduces to:

$$\mathrm{Tr}[\not{p}'\gamma^0\not{p}\gamma^0] \pm \mathrm{Tr}[\not{p}'\gamma^0\gamma^5\not{p}\gamma^0\gamma^5] \equiv T_1 \pm T_2\,. \tag{5.424}$$

The key step is to show that $T_2 = T_1$. Anticommute γ^5 past $\not{p}$ using $\{\gamma^5,\gamma^\mu\}=0$, i.e. $\gamma^5\not{p} = -\not{p}\gamma^5$:

$$\begin{aligned} T_2 &= \mathrm{Tr}[\not{p}'\gamma^0\gamma^5\not{p}\gamma^0\gamma^5] \\ &= \mathrm{Tr}[\not{p}'\gamma^0(-\not{p}\gamma^5)\gamma^0\gamma^5] = -\mathrm{Tr}[\not{p}'\gamma^0\not{p}\,\gamma^5\gamma^0\,\gamma^5] \\ &= -\mathrm{Tr}[\not{p}'\gamma^0\not{p}\,(-\gamma^0\gamma^5)\,\gamma^5] = +\mathrm{Tr}[\not{p}'\gamma^0\not{p}\,\gamma^0\,(\gamma^5)^2] \\ &= \mathrm{Tr}[\not{p}'\gamma^0\not{p}\gamma^0] = T_1\,. \quad \checkmark \end{aligned} \tag{5.425}$$

(We used $\gamma^5\gamma^0 = -\gamma^0\gamma^5$ in the third line and $(\gamma^5)^2 = \mathbb{1}$ in the last.)

Therefore:

- **No helicity flip** ($s = 1$, taking $+$): leading $= T_1 + T_1 = 2T_1 = 8(2E^2 - p'\cdot p) \sim E^2$.
- **Helicity flip** ($s = 2$, taking $-$): leading $= T_1 - T_1 = 0$.

Subleading m^2 term. Using $(\gamma^0)^2 = \mathbb{1}$ and $\gamma^0\gamma^5\gamma^0 = -\gamma^5$:

$$\mathrm{Tr}[\gamma^0(1+\gamma^5)\gamma^0(1\pm\gamma^5)] = \mathrm{Tr}[(1-\gamma^5)(1\pm\gamma^5)]\,. \tag{5.426}$$

For $+$: $(1-\gamma^5)(1+\gamma^5) = 1-(\gamma^5)^2 = 0$. For $-$: $(1-\gamma^5)^2 = 2(1-\gamma^5)$, so Tr $= 2\,\mathrm{Tr}[\mathbb{1}_4] = 8$.
Collecting results:

$$X_1 \propto 2T_1 + 0 \sim E^2\,, \qquad X_2 \propto 0 + 8m^2 \sim m^2\,. \tag{5.427}$$

Therefore, in the ultra-relativistic limit:

$$\frac{X_2}{X_1} \sim \left(\frac{m}{E}\right)^2 \to 0 \tag{5.428}$$

Helicity is conserved in ultra-relativistic Coulomb scattering. The probability of helicity flip vanishes as $(m/E)^2$.

Physical interpretation

- **Non-relativistic regime:** The electron moves slowly, so its spin has time to "follow" the momentum direction changes caused by the electric field. The interaction is purely electric, with no magnetic component.
- **Ultra-relativistic regime:** In the electron's instantaneous rest frame, the purely electric field of the nucleus appears as both electric and magnetic fields (Lorentz transformation). The magnetic field couples to the electron's spin, and this spin-orbit coupling tends to preserve helicity. Mathematically, this is connected to chiral symmetry: for massless fermions, left- and right-handed components decouple completely.

The relativistic correction factor $[1 - v^2\sin^2(\theta/2)]$ in the Mott formula (5.406) is a direct consequence of this magnetic scattering. It suppresses large-angle scattering for relativistic electrons.

Summary: two limiting behaviors

	Non-relat. ($\lvert\vec{p}\rvert \ll m$)	Ultra-relat. ($E \gg m$)
Conserved quantity	Spin	Helicity
Helicity flip probability	$\sin^2(\theta/2)$	$\sim (m/E)^2 \to 0$
Physical origin	Electric interaction only	Magnetic spin-orbit coupling

This transition from spin conservation to helicity conservation as the energy increases is a beautiful illustration of how relativistic effects modify the quantum mechanical behavior of spin-1/2 particles.

5.6 Bremsstrahlung

When an electron is deflected by the Coulomb field of a heavy nucleus, it must emit electromagnetic radiation—this is a fundamental consequence of classical electrodynamics, which states that any accelerated charge radiates. The emitted radiation, and the process itself, are called **bremsstrahlung** (German for "braking radiation"). In this section, we develop the quantum field theoretic treatment of bremsstrahlung in the soft photon limit, deriving the cross section and exploring the famous **infrared divergence** that arises when the photon energy approaches zero.

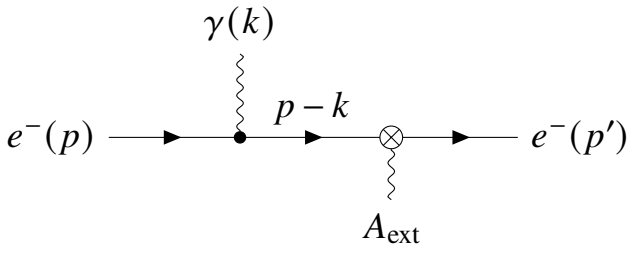
Bremsstrahlung is of considerable practical importance: it is the primary mechanism by which fast electrons lose energy as they pass through matter, and it is widely used in electron accelerators to produce intense photon beams for experiments.

The process and its Feynman diagrams

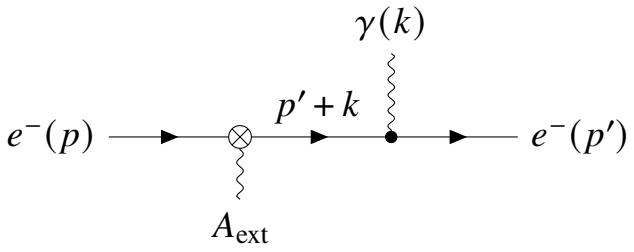
Consider an electron with initial momentum $p = (E, \vec{p})$ scattering in an external Coulomb field and emitting a photon with momentum $k = (\omega, \vec{k})$. The final electron has momentum $p' = (E', \vec{p}')$. Unlike pure Coulomb scattering, this process involves *both* the quantized electromagnetic field (to emit the photon) and the external field (to provide the momentum transfer to the source).

Why do we need both fields? If the electron only interacted with the external field, it could not emit a real photon—energy-momentum conservation would be violated. If it only interacted with the quantized field, there would be no source to absorb the recoil momentum. The external field of the heavy nucleus absorbs momentum without absorbing energy (because the nucleus is infinitely heavy), making the process kinematically allowed.

In lowest order, bremsstrahlung arises from the second-order term ($n = 2$) in the S-matrix expansion. There are two Feynman diagrams, shown below:



(a)



(b)

In diagram (a), the electron first emits the photon, then scatters from the external field. The intermediate electron has momentum $p - k$. In diagram (b), the electron first scatters, then emits the photon. The intermediate electron has momentum $p' + k$. Both diagrams must be included and their amplitudes added coherently.

The S-matrix element

Using the Feynman rules developed in the previous section, we can write down the S-matrix element directly. The structure is similar to Coulomb scattering, but now with an additional photon vertex and propagator.

Writing the amplitude

From the Feynman rules for external fields (Section 5.5) and for photon emission, the S-matrix element is:

$$\langle f|S|i\rangle = 2\pi\delta(E' + \omega - E)\sqrt{\frac{m}{VE}}\sqrt{\frac{m}{VE'}}\sqrt{\frac{1}{2V\omega}}\,\mathcal{M} \tag{5.429}$$

Note the energy conservation: The delta function enforces $E = E' + \omega$, meaning the initial electron energy equals the final electron energy plus the photon energy. This is different from pure Coulomb scattering where $E = E'$.

The Feynman amplitude $\mathcal{M}$ is the sum of the two diagrams:

$$\mathcal{M} = \mathcal{M}_a + \mathcal{M}_b \tag{5.430}$$

For diagram (a), reading from right to left along the fermion line:

- Final electron: $\bar{u}(\vec{p}')$
- External field vertex: $ie\gamma^\mu \tilde{A}_{\text{ext},\mu}(\vec{q})$
- Electron propagator: $iS_F(p-k) = \frac{i(\not{p}-\not{k}+m)}{(p-k)^2-m^2}$
- Photon emission vertex: $ie\gamma^\nu \varepsilon_\nu^*(k)$
- Initial electron: $u(\vec{p})$

For diagram (b):

- Final electron: $\bar{u}(\vec{p}')$
- Photon emission vertex: $ie\gamma^\nu \varepsilon_\nu^*(k)$
- Electron propagator: $iS_F(p'+k) = \frac{i(\not{p}'+\not{k}+m)}{(p'+k)^2-m^2}$
- External field vertex: $ie\gamma^\mu \tilde{A}_{\text{ext},\mu}(\vec{q})$
- Initial electron: $u(\vec{p})$

Writing $\not{\varepsilon} \equiv \gamma^\nu \varepsilon_\nu^*(k)$ and $\not{A}_{\text{ext}} \equiv \gamma^\mu \tilde{A}_{\text{ext},\mu}(\vec{q})$:

$$\mathcal{M} = -e^2 \bar{u}(\vec{p}') \left[\not{A}_{\text{ext}} iS_F(p-k) \not{\varepsilon} + \not{\varepsilon} iS_F(p'+k) \not{A}_{\text{ext}} \right] u(\vec{p}) \tag{5.431}$$

Simplifying the propagators

The electron propagators can be simplified. For the first propagator:

$$iS_F(p-k) = \frac{i(\not{p}-\not{k}+m)}{(p-k)^2-m^2} \tag{5.432}$$

$$= \frac{i(\not{p}-\not{k}+m)}{p^2-2p\cdot k+k^2-m^2} \tag{5.433}$$

Since the initial electron is on-shell ($p^2 = m^2$) and the photon is massless ($k^2 = 0$):

$$iS_F(p-k) = \frac{i(\not{p}-\not{k}+m)}{-2p\cdot k} \tag{5.434}$$

Similarly, for the second propagator:

$$iS_F(p'+k) = \frac{i(\not{p}'+\not{k}+m)}{(p'+k)^2-m^2} \tag{5.435}$$

$$= \frac{i(\not{p}'+\not{k}+m)}{p'^2+2p'\cdot k+k^2-m^2} = \frac{i(\not{p}'+\not{k}+m)}{2p'\cdot k} \tag{5.436}$$

Substituting back into eq. (5.431):

$$\boxed{\mathcal{M} = -ie^2 \bar{u}(\vec{p}') \left[\not{A}_{\text{ext}} \frac{\not{p}-\not{k}+m}{-2p\cdot k} \not{\varepsilon} + \not{\varepsilon} \frac{\not{p}'+\not{k}+m}{2p'\cdot k} \not{A}_{\text{ext}} \right] u(\vec{p})} \tag{5.437}$$

The momentum transfer to the external field is:

$$\vec{q} = \vec{p}' + \vec{k} - \vec{p} \tag{5.438}$$

Note that three-momentum is not conserved among the electron and photon alone—the difference is absorbed by the external field.

The cross section formula

The derivation of the cross section follows the same pattern as for Coulomb scattering, with the addition of a photon phase space factor.

Transition rate and phase space

The transition probability per unit time is:

$$w = \frac{|\langle f|S|i\rangle|^2}{T} \tag{5.439}$$

Using $[2\pi\delta(E'+\omega-E)]^2 = 2\pi T\delta(E'+\omega-E)$:

$$w = 2\pi\delta(E'+\omega-E)\frac{m^2}{V^2EE'}\frac{1}{2V\omega}|\mathcal{M}|^2 \tag{5.440}$$

The density of final states now includes both the scattered electron and the emitted photon:

$$\frac{Vd^3p'}{(2\pi)^3}\times\frac{Vd^3k}{(2\pi)^3} \tag{5.441}$$

Dividing by the incident flux $|\vec{p}|/(VE)$ and integrating over $|p'|$ using the energy delta function:

$$\boxed{d\sigma = \frac{m^2}{(2\pi)^5}\frac{1}{2\omega}\frac{|\vec{p}'|}{|\vec{p}|}|\mathcal{M}|^2\,d^3k\,d\Omega'} \tag{5.442}$$

where $d\Omega'$ is the solid angle for the scattered electron direction.

The Bethe-Heitler formula

Evaluating $|\mathcal{M}|^2$ from eq. (5.437), summing over final spins and averaging over initial spins, leads to the **Bethe-Heitler formula** for bremsstrahlung in a Coulomb field. This is a lengthy but straightforward calculation. The full formula is quite complex and we will not derive it here. Instead, we focus on the physically illuminating **soft photon limit**.

The soft photon approximation

When the emitted photon has very low energy ($\omega \to 0$), the calculation simplifies dramatically and reveals deep physical insights.

Simplification of the amplitude

In the soft photon limit $\omega \approx 0$, we have:

- $\vec{q} = \vec{p}' + \vec{k} - \vec{p} \approx \vec{p}' - \vec{p}$ (same as elastic scattering)
- $|\vec{p}'| \approx |\vec{p}|$ (elastic kinematics)
- We can neglect $\not{k}$ in the numerators of the propagators

With these approximations, the amplitude (5.437) becomes:

$$\mathcal{M} \approx -ie^2 \bar{u}(\vec{p}') \left[\not{A}_{\text{ext}} \frac{\not{p} + m}{-2p \cdot k} \not{\varepsilon} + \not{\varepsilon} \frac{\not{p}' + m}{2p' \cdot k} \not{A}_{\text{ext}} \right] u(\vec{p}) \tag{5.443}$$

Using the Dirac equation

We can simplify further using the Dirac equation and gamma matrix algebra. Recall that:

$$\not{p} u(\vec{p}) = m u(\vec{p}), \qquad \bar{u}(\vec{p}') \not{p}' = m \bar{u}(\vec{p}') \tag{5.444}$$

For the term $(\not{p} + m)\not{\varepsilon}$, we use the anticommutation relation $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$:

$$\not{p}\not{\varepsilon} = -\not{\varepsilon}\not{p} + 2(p \cdot \varepsilon) \tag{5.445}$$

$$\therefore \quad (\not{p} + m)\not{\varepsilon} = -\not{\varepsilon}\not{p} + 2(p \cdot \varepsilon) + m\not{\varepsilon} = -\not{\varepsilon}(\not{p} - m) + 2(p \cdot \varepsilon) \tag{5.446}$$

When acting on $u(\vec{p})$:

$$(\not{p} + m)\not{\varepsilon}\, u(\vec{p}) = -\not{\varepsilon}(\not{p} - m)u(\vec{p}) + 2(p \cdot \varepsilon)u(\vec{p}) = 2(p \cdot \varepsilon)u(\vec{p}) \tag{5.447}$$

since $(\not{p} - m)u(\vec{p}) = 0$.

Similarly, for $\not{\varepsilon}(\not{p}' + m)$:

$$\not{\varepsilon}\not{p}' = -\not{p}'\not{\varepsilon} + 2(\varepsilon \cdot p') \tag{5.448}$$

so

$$\not{\varepsilon}(\not{p}' + m) = -(\not{p}' - m)\not{\varepsilon} + 2(\varepsilon \cdot p') \tag{5.449}$$

When $\bar{u}(\vec{p}')$ acts from the left:

$$\bar{u}(\vec{p}')\not{\varepsilon}(\not{p}' + m) = 2(\varepsilon \cdot p')\bar{u}(\vec{p}') \tag{5.450}$$

since $\bar{u}(\vec{p}')(\not{p}' - m) = 0$.

The factorized amplitude

Substituting eqs. (5.447) and (5.450) into eq. (5.443):

$$\mathcal{M} \approx -ie^2\bar{u}(\vec{p}')\left[\not{A}_{\text{ext}}\frac{2(p\cdot\varepsilon)}{-2p\cdot k}+\frac{2(\varepsilon\cdot p')}{2p'\cdot k}\not{A}_{\text{ext}}\right]u(\vec{p}) \tag{5.451}$$

$$= -ie^2\bar{u}(\vec{p}')\not{A}_{\text{ext}}u(\vec{p})\left[-\frac{p\cdot\varepsilon}{p\cdot k}+\frac{p'\cdot\varepsilon}{p'\cdot k}\right] \tag{5.452}$$

Recognizing that $ie\bar{u}(\vec{p}')\not{A}_{\text{ext}}u(\vec{p}) = \mathcal{M}_0$ is the amplitude for elastic Coulomb scattering (eq. (5.370)), we obtain the beautiful result:

$$\boxed{\mathcal{M} = -e\mathcal{M}_0\left[\frac{p'\cdot\varepsilon}{p'\cdot k}-\frac{p\cdot\varepsilon}{p\cdot k}\right] \qquad (\omega\approx 0)} \tag{5.453}$$

Physical interpretation: The soft bremsstrahlung amplitude **factorizes** into the elastic scattering amplitude times a universal factor that depends only on the initial and final electron momenta and the photon polarization. This factorization is a deep result—it says that soft photon emission is completely determined by the external kinematics, independent of the details of the scattering process.

The soft bremsstrahlung cross section

Substituting eq. (5.453) into the cross section formula (5.442):

$$d\sigma = \frac{m^2}{(2\pi)^5}\frac{1}{2\omega}\frac{|\vec{p}'|}{|\vec{p}|}e^2|\mathcal{M}_0|^2\left|\frac{p'\cdot\varepsilon}{p'\cdot k}-\frac{p\cdot\varepsilon}{p\cdot k}\right|^2 d^3k\,d\Omega' \tag{5.454}$$

In the soft limit, $|\vec{p}'| = |\vec{p}|$, and comparing with the elastic cross section (5.379):

$$\left(\frac{d\sigma}{d\Omega'}\right)_0 = \left(\frac{m}{2\pi}\right)^2|\mathcal{M}_0|^2 \tag{5.455}$$

we obtain:

$$\boxed{\left(\frac{d\sigma}{d\Omega'}\right)_{\text{B}} = \left(\frac{d\sigma}{d\Omega'}\right)_0\frac{\alpha}{(2\pi)^2}\left[\frac{p'\cdot\varepsilon}{p'\cdot k}-\frac{p\cdot\varepsilon}{p\cdot k}\right]^2\frac{d^3k}{\omega} \qquad (\omega\approx 0)} \tag{5.456}$$

where $\alpha = e^2/(4\pi)$ is the fine structure constant.

Unpolarized cross section

If we do not observe the photon polarization, we must sum over the two transverse polarizations. Using the gauge-invariant polarization sum:

$$\sum_{\lambda=1,2} \varepsilon_\mu(k,\lambda)\,\varepsilon_\nu^*(k,\lambda) \to -g_{\mu\nu} \tag{5.457}$$

(the longitudinal terms vanish by current conservation), we obtain:

$$\sum_\lambda \left|\frac{p'\cdot\varepsilon}{p'\cdot k} - \frac{p\cdot\varepsilon}{p\cdot k}\right|^2 = -\left(\frac{p'^\mu}{p'\cdot k} - \frac{p^\mu}{p\cdot k}\right)\left(\frac{p'_\mu}{p'\cdot k} - \frac{p_\mu}{p\cdot k}\right) \tag{5.458}$$

Expanding:

$$= -\frac{p'^2}{(p'\cdot k)^2} + \frac{2p'\cdot p}{(p'\cdot k)(p\cdot k)} - \frac{p^2}{(p\cdot k)^2} \tag{5.459}$$

$$= -\frac{m^2}{(p'\cdot k)^2} + \frac{2p'\cdot p}{(p'\cdot k)(p\cdot k)} - \frac{m^2}{(p\cdot k)^2} \tag{5.460}$$

The unpolarized soft bremsstrahlung cross section is:

$$\boxed{\left(\frac{d\sigma}{d\Omega'}\right)_{\rm B} = \left(\frac{d\sigma}{d\Omega'}\right)_0 \frac{(-\alpha)}{(2\pi)^2}\left[\frac{p'}{p'\cdot k} - \frac{p}{p\cdot k}\right]^2 \frac{d^3k}{\omega} \qquad (\omega \approx 0)} \tag{5.461}$$

The overall minus sign ensures the cross section is positive (since $p'^2 = p^2 = m^2$ and the dominant term is $2p'\cdot p/(p'\cdot k)(p\cdot k)$ which is positive for small scattering angles).

The infrared divergence

Equation (5.461) reveals a remarkable and troubling feature: the cross section diverges as $\omega \to 0$. This is the famous **infrared catastrophe**.

Origin of the divergence

The divergence arises from the interplay of two factors with opposite ω-dependence.

(i) The soft factor diverges as $1/\omega$. In the soft limit $k \to 0$, the intermediate electron lines approach the mass shell: $(p-k)^2 - m^2 = -2p\cdot k \to 0$ and $(p'+k)^2 - m^2 = 2p'\cdot k \to 0$. Each term in the soft factor individually diverges as $1/\omega$, since $p\cdot k = \omega(E - |\vec{p}|\cos\theta_{pk})$ and $p'\cdot k = \omega(E - |\vec{p}|\cos\theta_{p'k})$, and one must verify that their difference does not cancel this divergence. Factoring out $1/\omega$:

$$\frac{p'\cdot\varepsilon}{p'\cdot k} - \frac{p\cdot\varepsilon}{p\cdot k} = \frac{1}{\omega}\left[\frac{p'\cdot\varepsilon}{E - |\vec{p}|\cos\theta_{p'k}} - \frac{p\cdot\varepsilon}{E - |\vec{p}|\cos\theta_{pk}}\right] \tag{5.462}$$

The expression in brackets is a finite, ω-independent angular factor. Crucially, it does *not* vanish: the two terms have different angular denominators ($\theta_{p'k} \neq \theta_{pk}$ in general, since $\vec{p}' \neq \vec{p}$ after scattering), so their difference is generically non-zero. The soft factor therefore behaves as $\sim 1/\omega$ and its **modulo squared** as $\sim 1/\omega^2$.

Note that the cancellation *would* occur in the forward limit $\vec{p}' = \vec{p}$ (no scattering), which is physically expected: a charge moving at constant velocity does not radiate.

(ii) The photon phase space contributes $\omega\, d\omega$. Writing $d^3k = |\vec{k}|^2 d|\vec{k}| d\Omega_k$ and noting that for massless photons $\omega = |\vec{k}|$:

$$\frac{d^3k}{\omega} = \frac{|\vec{k}|^2 d|\vec{k}|}{\omega} d\Omega_k = \omega\, d\omega\, d\Omega_k \tag{5.463}$$

(The extra $1/\omega$ in d^3k/ω comes from the Lorentz-invariant phase-space measure $d^3k/(2\omega)$.)

(iii) Combining propagators and phase space:

$$d\sigma_{\text{brem}} \sim \underbrace{\frac{1}{\omega^2}}_{\text{soft factor}^2} \times \underbrace{\omega\, d\omega\, d\Omega_k}_{\text{phase space}} = \frac{d\omega}{\omega} d\Omega_k \tag{5.464}$$

The angular integral $\int d\Omega_k$ is finite. But the integral over photon energy:

$$\int_0^{\Delta E} \frac{d\omega}{\omega} = \left[\ln \omega\right]_0^{\Delta E} \to +\infty \tag{5.465}$$

is **logarithmically divergent** at the lower limit. The phase-space suppression ($\sim \omega$) is not strong enough to overcome the propagator singularity ($\sim 1/\omega^2$); the net $1/\omega$ behaviour is at the borderline of integrability, and diverges.

Physical interpretation: A zero-energy photon cannot be distinguished from no photon at all, yet the cross section to emit such a photon is infinite—the **infrared catastrophe**.

Regularization with a photon mass

Following Feynman, we can regulate the divergence by giving the photon a small fictitious mass λ. At the end of the calculation, we take $\lambda \to 0$ to recover QED. With $k^2 = \lambda^2$:

$$\omega_\lambda = \sqrt{\lambda^2 + |\vec{k}|^2} \tag{5.466}$$

The propagator denominators become:

$$(p-k)^2 - m^2 = -2p\cdot k + \lambda^2 \tag{5.467}$$

$$(p'+k)^2 - m^2 = 2p'\cdot k + \lambda^2 \tag{5.468}$$

The modified amplitude is:

$$\mathcal{M} = -ie^2 \bar{u}(\vec{p}') \left[\not{A}_{\text{ext}} \frac{\not{p} - \not{k} + m}{-2p \cdot k + \lambda^2} \not{\varepsilon} + \not{\varepsilon} \frac{\not{p}' + \not{k} + m}{2p' \cdot k + \lambda^2} \not{A}_{\text{ext}} \right] u(\vec{p}) \tag{5.469}$$

For a massive photon, there are three polarization states. The polarization sum becomes:

$$\sum_{r=1}^{3} \varepsilon_{r\mu} \varepsilon_{r\nu}^* = -g_{\mu\nu} + \frac{k_\mu k_\nu}{\lambda^2} \tag{5.470}$$

The term $k_\mu k_\nu / \lambda^2$ vanishes by current conservation in the limit $\lambda \to 0$, so it does not affect the physical result.

The regularized bremsstrahlung cross section, integrated over photon energies $\lambda \leq \omega_\lambda \leq \Delta E$, is:

$$\left(\frac{d\sigma}{d\Omega'} \right)_{\text{B}} = \left(\frac{d\sigma}{d\Omega'} \right)_0 \alpha B(\lambda) \tag{5.471}$$

where

$$B(\lambda) = \frac{-1}{(2\pi)^2} \int_{\lambda \leq \omega_\lambda \leq \Delta E} \frac{d^3k}{\omega_\lambda} \left[\frac{2p'}{2p' \cdot k + \lambda^2} + \frac{2p}{-2p \cdot k + \lambda^2} \right]^2 \tag{5.472}$$

For $\lambda > 0$, the integral $B(\lambda)$ is finite. As $\lambda \to 0$, $B(\lambda) \to +\infty$ logarithmically.

The experimental cross section

In any real experiment, there is a finite energy resolution ΔE. A photon with energy $\omega < \Delta E$ cannot be distinguished from no photon at all. Therefore, what the detector reports as "elastic" is the sum

$$\left(\frac{d\sigma}{d\Omega'} \right)_{\text{Exp}}^{\text{elastic-like}} = \left(\frac{d\sigma}{d\Omega'} \right)_{\text{El}} + \left(\frac{d\sigma}{d\Omega'} \right)_{\text{B}}^{\text{soft}} \tag{5.473}$$

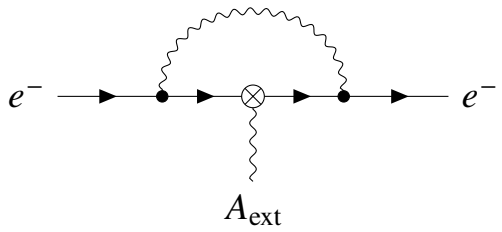
where:

- $(d\sigma/d\Omega')_{\text{El}}$ is the genuine elastic scattering cross section (no photon emitted) including its $O(\alpha)$ virtual radiative corrections (vertex, self-energy, etc.; see below).
- $(d\sigma/d\Omega')_{\text{B}}^{\text{soft}}$ is the *soft* bremsstrahlung contribution (5.471) integrated over $\lambda \leq \omega \leq \Delta E$ – the region where a photon *is* emitted but is too soft to be resolved by the detector.

Photons with $\omega > \Delta E$ (*hard* bremsstrahlung) *are* detected and constitute a separate, distinguishable inelastic channel, not included in the elastic-like signal.

Why the vertex correction enters at the same order

The soft bremsstrahlung cross section (5.471) is of order α relative to the lowest-order elastic cross section. For consistency, we must include the order-α **radiative corrections** to elastic scattering as well. These come from loop diagrams such as the **vertex correction**:



This correction is of the *same* order in α as bremsstrahlung, despite involving two extra vertices rather than one. To see why, we need to count powers of e carefully in the *cross section*, not in the amplitude.

Bremsstrahlung (one real photon emitted). The emission adds one QED vertex to the Born amplitude, so $\mathcal{M}_{\text{brem}} \sim e\,\mathcal{M}_0$. Since bremsstrahlung has a photon in the final state while elastic scattering does not, the two processes are *distinguishable*—there is no interference between $\mathcal{M}_0$ and $\mathcal{M}_{\text{brem}}$. The bremsstrahlung cross section is the **modulus squared** of the emission amplitude:

$$\sigma_{\text{brem}} \;\sim\; |\mathcal{M}_{\text{brem}}|^2 \;\sim\; e^2\,|\mathcal{M}_0|^2 \;\sim\; \alpha\,\sigma_0 \tag{5.474}$$

Vertex correction (one virtual photon exchanged in a loop). The loop adds two extra QED vertices (e^2) but no extra external particles. The corrected amplitude is $\mathcal{M}_0 + \delta\mathcal{M}$ with $\delta\mathcal{M} \sim e^2\,\mathcal{M}_0$. Since the final state is the *same* as the Born process (no photon), the correction contributes through **interference** with $\mathcal{M}_0$:

$$\sigma_{\text{el}} \;\sim\; |\mathcal{M}_0 + \delta\mathcal{M}|^2 = |\mathcal{M}_0|^2 + \underbrace{2\,\text{Re}(\mathcal{M}_0^*\,\delta\mathcal{M})}_{\sim e^2\,|\mathcal{M}_0|^2} + \underbrace{|\delta\mathcal{M}|^2}_{\sim e^4\,|\mathcal{M}_0|^2} \tag{5.475}$$

The interference term is *linear* in $\delta\mathcal{M}$, so despite $\delta\mathcal{M}$ carrying e^2, the correction to the cross section is:

$$\delta\sigma_{\text{el}} \;\sim\; e^2\,|\mathcal{M}_0|^2 \;\sim\; \alpha\,\sigma_0 \tag{5.476}$$

Summary. The key mechanism is:

	Real photon (bremsstrahlung)	**Virtual photon (vertex)**
Extra powers of e in $\mathcal{M}$	e^1	e^2
Enters σ via	modulus squared ($e^1 \to e^2$)	interference ($e^2 \to e^2$)
Correction to σ	both $\sim \alpha\,\sigma_0$	

One extra power of e in the amplitude, squared, gives e^2. Two extra powers of e, taken linearly (interference), also give e^2. The two mechanisms conspire to produce corrections of the same order—the necessary condition for their infrared divergences to cancel.

Including the vertex correction:

$$\left(\frac{d\sigma}{d\Omega'}\right)_{\text{El}} = \left(\frac{d\sigma}{d\Omega'}\right)_0 [1 + \alpha R(\lambda)] \tag{5.477}$$

where $R(\lambda)$ is a function of the photon mass regulator that also diverges as $\lambda \to 0$.

The Bloch-Nordsieck theorem

Combining eqs. (5.473), (5.471), and (5.477):

$$\boxed{\left(\frac{d\sigma}{d\Omega'}\right)_{\text{Exp}} = \left(\frac{d\sigma}{d\Omega'}\right)_0 \left\{1 + \alpha \left[B(\lambda) + R(\lambda)\right] + O(\alpha^2)\right\}} \tag{5.478}$$

The Bloch-Nordsieck theorem (1937) states: *The infrared divergences in $B(\lambda)$ and $R(\lambda)$ cancel exactly*, so that

$$\lim_{\lambda \to 0} [B(\lambda) + R(\lambda)] = \text{finite} \tag{5.479}$$

This remarkable result can be understood as follows:

- $B(\lambda) \to +\infty$ as $\lambda \to 0$ (too many soft photons emitted)
- $R(\lambda) \to -\infty$ as $\lambda \to 0$ (virtual photon loops diverge)
- The two divergences are intimately related and cancel exactly

Physical understanding

The infrared catastrophe arises from treating soft bremsstrahlung and elastic scattering as separate processes. But this separation is artificial: one *always* observes elastic scattering together with some soft bremsstrahlung (photons too soft to detect). The cross section for this combined process is finite.

The Bloch-Nordsieck theorem tells us that the infrared divergence is not a defect of QED, but rather a consequence of asking an unphysical question. The question "What is the probability of scattering with no photon emission?" has no well-defined answer in QED (the probability is zero!). The physically meaningful question is "What is the probability of scattering with photon emission below the detection threshold ΔE?" and this has a finite answer.

Exponentiation and the vanishing of pure elastic scattering

Including all orders of perturbation theory, the Bloch-Nordsieck theorem implies:

$$\left(\frac{d\sigma}{d\Omega'}\right)_{\text{Exp}} = \left(\frac{d\sigma}{d\Omega'}\right)_0 \exp\{\alpha\,[B(\lambda)+R(\lambda)]\} \tag{5.480}$$

The elastic cross section alone would be:

$$\left(\frac{d\sigma}{d\Omega'}\right)_{\text{El}} = \left(\frac{d\sigma}{d\Omega'}\right)_0 e^{\alpha R(\lambda)} \tag{5.481}$$

Since $R(\lambda) \to -\infty$ as $\lambda \to 0$:

$$\boxed{\lim_{\lambda\to 0}\left(\frac{d\sigma}{d\Omega'}\right)_{\text{El}} = 0} \tag{5.482}$$

Remarkable conclusion: The pure elastic cross section, without any photon emission, **vanishes exactly**. There is no truly elastic scattering—all observed scattering is accompanied by emission of radiation. This is the quantum field theoretic manifestation of the classical result that an accelerated charge must radiate.

Physical applications and summary

Radiative corrections in precision experiments

The experimental cross section (5.478) depends on the energy resolution ΔE:

$$\left(\frac{d\sigma}{d\Omega'}\right)_{\text{Exp}} = \left(\frac{d\sigma}{d\Omega'}\right)_0 [1+\alpha f(\Delta E)] \tag{5.483}$$

where $f(\Delta E)$ is the finite radiative correction. In high-energy experiments, these corrections can be as large as 50% in some kinematic regions. They must be carefully calculated and either:

- Applied to theoretical predictions before comparing with data
- Subtracted from experimental data to obtain "radiatively corrected" cross sections

Universality of soft photon emission

The factorization property of eq. (5.453) is universal: it applies to soft photon emission in *any* process involving charged particles, not just Coulomb scattering. The soft photon factor

$$-e\left[\frac{p'\cdot\varepsilon}{p'\cdot k} - \frac{p\cdot\varepsilon}{p\cdot k}\right] \tag{5.484}$$

depends only on the external charged particle momenta and is independent of the details of the hard scattering process.

Summary

Concept	Key result
Bremsstrahlung ampl.	Factorizes: $\mathcal{M} = -e\mathcal{M}_0 \times$ (soft factor)
Infrared divergence	Cross section $\sim \int d\omega/\omega$ diverges as $\omega \to 0$
Physical origin	Zero-mass photon; internal lines go on-shell
Bloch-Nordsieck theorem	Divergences cancel between real and virtual photons
Pure elastic scattering	Vanishes: all scattering is accompanied by radiation

The infrared problem in QED is completely understood and poses no obstacle to making precise predictions. It teaches us that the naive separation of "elastic" and "inelastic" scattering is unphysical when massless particles are involved, and that physically meaningful questions always have finite answers.

Part III

Renormalization

6 Renormalization in QED

Before tackling the renormalization of QED, we develop the key concepts in the simpler setting of $\lambda\phi^4$ theory. This scalar field theory exhibits all the essential features of renormalization—UV divergences, counterterms, running couplings—without the technical complications of spin and gauge invariance. Throughout this chapter, we employ dimensional regularization, the method introduced by 't Hooft and Veltman [14] that preserves gauge invariance. For a comprehensive treatment of renormalization theory, see Collins [15].

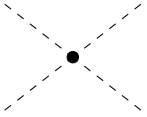
6.1 Warm-up: renormalization of $\lambda\phi^4$ theory

Recall the $\lambda\phi^4$ Lagrangian (see Chapter 4.3):

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 . \tag{6.1}$$

Feynman rules for $\lambda\phi^4$

Before computing loop corrections, let us collect the Feynman rules derived in Chapter 4.3. To each element of a diagram one assigns:

Element	Diagram	Factor
Propagator	p	$\dfrac{i}{p^2 - m^2 + i\epsilon}$
Vertex		$-i\lambda$
External line		1

In addition, one must:

1. draw all topologically distinct connected diagrams at the desired order in λ;

2. impose momentum conservation at each vertex;
3. integrate $\displaystyle\int \frac{d^4\ell}{(2\pi)^4}$ over each independent loop momentum;
4. divide by the symmetry factor S of the diagram.

With these rules in hand we can identify the one-loop diagrams that require renormalization. At order λ there is a single divergent contribution to the propagator (the **tadpole**), and at order λ^2 there is a divergent correction to the four-point vertex (the **bubble**).

Dimensional regularization: strategy

To handle the UV divergences that appear in loop integrals, we employ **dimensional regularization**: the spacetime dimension is analytically continued from 4 to $d = 4-\varepsilon$, and an arbitrary mass scale μ is introduced so that the coupling λ remains dimensionless:

$$\int \frac{d^4\ell}{(2\pi)^4} \longrightarrow \mu^{\varepsilon} \int \frac{d^d\ell}{(2\pi)^d}\,. \tag{6.2}$$

All d-dimensional momentum integrals that arise in the following reduce, after Feynman parametrization and a shift of the loop momentum, to the **master integral**

$$J_n(\Delta) \equiv \int \frac{d^d\ell}{(2\pi)^d}\,\frac{1}{(\ell^2-\Delta+i\epsilon)^n} = \frac{i(-1)^n}{(4\pi)^{d/2}}\,\frac{\Gamma\!\left(n-\frac{d}{2}\right)}{\Gamma(n)}\,\Delta^{\frac{d}{2}-n}, \tag{6.3}$$

which is derived by Wick rotation to Euclidean space followed by integration in d-dimensional spherical coordinates.

Here $\Gamma(z)$ is the **Euler gamma function**, the analytic continuation of the factorial: $\Gamma(n) = (n-1)!$ for positive integers n, and more generally

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt \qquad (\operatorname{Re} z > 0)\,, \tag{6.4}$$

extended to the whole complex plane by the recurrence $\Gamma(z+1) = z\,\Gamma(z)$. The function is **meromorphic**: it is analytic everywhere except for simple poles at the non-positive integers $z = 0, -1, -2, \ldots$, with residues

$$\operatorname*{Res}_{z=-k} \Gamma(z) = \frac{(-1)^k}{k!}, \qquad k = 0, 1, 2, \ldots \tag{6.5}$$

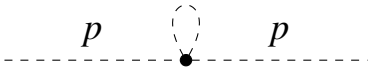
Near $z = 0$ one has the Laurent expansion $\Gamma(z) = 1/z - \gamma_E + O(z)$, with $\gamma_E \approx 0.5772$ the Euler–Mascheroni constant; the poles at negative integers follow via the recurrence.

The UV divergences of loop integrals manifest themselves precisely as these poles: in $d = 4-\varepsilon$ the argument $n - d/2 = n - 2 + \varepsilon/2$ hits a non-positive integer exactly when $n \leq 2$, producing a $1/\varepsilon$ singularity. For example, the tadpole ($n = 1$) needs $\Gamma(1 - d/2) = \Gamma(-1 + \varepsilon/2)$, which has a pole at $\varepsilon = 0$—the tadpole has superficial degree of divergence 2, and in dimensional regularization this manifests as a $1/\varepsilon$ pole multiplied by m^2 rather than as an explicit Λ^2. The bubble ($n = 2$) needs $\Gamma(2 - d/2) = \Gamma(\varepsilon/2)$, with a pole encoding the logarithmic divergence. For $n \geq 3$ the argument is strictly positive at $\varepsilon = 0$ and the integral is UV finite.

Since the manipulations involved (Feynman parametrization, momentum shift, reduction to J_n, expansion around $\varepsilon = 0$) are rather mechanical—but instructive on first encounter—we present them here only in outline, deferring the **fully detailed, step-by-step calculations** to Appendix G and the explicit QED computations in Chapter 7.1.

The tadpole: mass renormalization

The one-loop correction to the propagator comes from the tadpole diagram:



Reading the diagram with the Feynman rules: one vertex $(-i\lambda)$, one internal propagator $(i/(\ell^2 - m^2 + i\epsilon))$, and a symmetry factor $S = 2$. The origin of this factor is as follows: at the vertex, four ϕ-legs meet. Two of them are connected to the external lines, and the remaining two are contracted with each other to form the loop. Swapping these two internal legs yields exactly the same diagram—the loop propagator starts and ends at the same vertex regardless of which leg we call "first." This discrete exchange symmetry means the naïve Feynman rules overcount by a factor of 2, so we divide by $S = 2$. Equivalently, the two orientations of the loop momentum (clockwise or counterclockwise) correspond to the same single contraction.[1] This gives:

$$-i\Sigma_{\text{tadpole}} = \frac{-i\lambda}{2} \int \frac{d^4\ell}{(2\pi)^4} \frac{i}{\ell^2 - m^2 + i\epsilon}\,. \tag{6.6}$$

Power counting. For large $|\ell|$, the integrand behaves as $1/\ell^2$. Since $d^4\ell \sim \ell^3 d\ell$:

$$\int^{\Lambda} \frac{d^4\ell}{\ell^2} \sim \int^{\Lambda} \frac{\ell^3 d\ell}{\ell^2} \sim \Lambda^2\,. \tag{6.7}$$

[1] A quick combinatorial check: the vertex carries $(-i\lambda)/4!$, and there are $\binom{4}{2} = 6$ ways to choose which two legs form the loop, times $2! = 2$ ways to assign the remaining two legs to the two external lines, giving 12 distinct contractions. Since $4!/12 = 2 = S$, this confirms $S = 2$.

The integral is **quadratically divergent**.

Evaluation in dimensional regularization. In $d = 4-\varepsilon$ dimensions, the tadpole integral is a special case of the master integral (6.3) with $n = 1$ and $\Delta = m^2$. Expanding around $\varepsilon = 0$:[2]

$$\mu^{\varepsilon}\int \frac{d^d\ell}{(2\pi)^d}\frac{1}{\ell^2-m^2} = \frac{-im^2}{16\pi^2}\left[\frac{2}{\varepsilon}-\gamma_E+\ln(4\pi)-\ln\frac{m^2}{\mu^2}+1\right]+O(\varepsilon)\,, \tag{6.8}$$

where $\gamma_E \approx 0.5772$ is the Euler–Mascheroni constant. The $2/\varepsilon$ pole encodes the quadratic divergence that was $\sim \Lambda^2$ in the cutoff language.

The appearance of the logarithm should not be viewed as an artificial trick of the expansion. In dimensional regularization the loop integral is analytically continued away from $d = 4$, so dimensional analysis naturally produces factors such as μ^{ε} and $(m^2)^{-\varepsilon/2}$. Near $d = 4$ these combine into

$$\left(\frac{\mu^2}{m^2}\right)^{\varepsilon/2} = \exp\left[\frac{\varepsilon}{2}\ln\left(\frac{\mu^2}{m^2}\right)\right] = 1+\frac{\varepsilon}{2}\ln\left(\frac{\mu^2}{m^2}\right)+O(\varepsilon^2)\,,$$

so the logarithm is the derivative of a power with respect to its exponent, $\lim_{\alpha\to 0}\dfrac{x^{\alpha}-1}{\alpha} = \ln x$, rather than an arbitrary rewriting. What is scheme-dependent is only which *finite* combination is subtracted together with the pole—for example, in $\overline{\text{MS}}$ one absorbs the ubiquitous combination $2/\varepsilon-\gamma_E+\ln(4\pi)$ into the definition of the renormalized parameter. Combining now with the vertex factor and the symmetry factor:

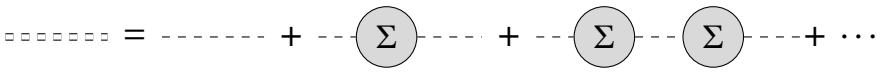
$$-i\Sigma_{\text{tadpole}} = \frac{-i\lambda m^2}{32\pi^2}\left[\frac{2}{\varepsilon}-\gamma_E+\ln(4\pi)-\ln\frac{m^2}{\mu^2}+1\right]. \tag{6.9}$$

The self-energy Σ computed above is a *one-particle irreducible* (1PI) insertion: it cannot be split into two disconnected pieces by cutting a single internal line. The full (dressed) propagator is obtained by **resumming** all possible chains of 1PI insertions on the bare propagator. Denoting the bare propagator by $G_0(p) = i/(p^2-m^2)$ and the 1PI self-energy by $-i\Sigma$, the dressed propagator is the geometric series

$$G(p) = G_0 + G_0\,(-i\Sigma)\,G_0 + G_0\,(-i\Sigma)\,G_0\,(-i\Sigma)\,G_0+\cdots = \frac{G_0}{1+i\Sigma\,G_0} = \frac{i}{p^2-m^2-\Sigma}\,, \tag{6.10}$$

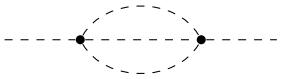
Diagrammatically, this resummation reads:

[2] The renormalization scale does not disappear: it combines with the factor $(m^2)^{1-\varepsilon/2}$ from (6.3). Indeed, $\mu^{\varepsilon}(m^2)^{1-\varepsilon/2} = m^2(\mu^2/m^2)^{\varepsilon/2} = m^2\big[1+\tfrac{\varepsilon}{2}\ln(\mu^2/m^2)+O(\varepsilon^2)\big]$, which is why the final answer contains the term $-\ln(m^2/\mu^2)$.

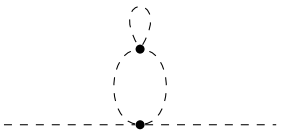


where the double dashed line denotes the full (dressed) propagator, single dashed lines are bare propagators G_0, and each shaded blob is a 1PI self-energy insertion $-i\Sigma$. This is just an algebraic identity: each 1PI blob is inserted between two bare propagators, and the infinite sum is a geometric series in $(-i\Sigma)\cdot G_0$.

A caveat on the meaning of Σ. The blob $-i\Sigma$ in the picture is *not* just the one-loop tadpole: by construction Σ denotes the sum of *all* 1PI self-energy diagrams, at every order in λ. One might worry that the chain above misses diagrams in which a self-energy correction sits inside another self-energy correction (a "bubble on top of a bubble," a tadpole dressing the internal line of another tadpole, a sunset, etc.). It does not: any such diagram is *either* itself 1PI—in which case it is a higher-order contribution to a single blob Σ—*or* it is 1P-reducible, and is then uniquely decomposed into a chain of strictly smaller 1PI blobs joined by single bare propagators, which is precisely what the geometric series sums. The partition of all propagator corrections into (chains of) 1PI pieces is unique; this is what makes the Dyson resummation an identity, not an approximation. In this sense, writing $\Sigma \approx \Sigma_{\text{tadpole}}$ is a *truncation* of the exact identity at order λ: at two loops Σ receives new 1PI contributions that are *not* generated by chaining one-loop tadpoles. Two characteristic examples in ϕ^4 theory are the **sunset** and the **nested tadpole**:



sunset (2 vertices, 3 internal lines)



nested tadpole

The **sunset** has two vertices joined by three internal propagators, with one external leg at each vertex (total: $2\times 4 = 8$ legs, 2 external plus $2\times 3 = 6$ accounted for by the three internal lines). Cutting any single line leaves the diagram connected, so it is genuinely 1PI and cannot be reduced to a chain of one-loop tadpoles. The **nested tadpole** has a one-loop tadpole attached to the main line at vertex w_1, but the internal propagator of that loop is itself dressed by another one-loop tadpole at w_2; again, no single cut disconnects it. Both diagrams are genuine $O(\lambda^2)$ contributions to Σ and must be included separately when one goes beyond leading order.

Using the one-loop result $\Sigma = \Sigma_{\text{tadpole}}$, the **full (dressed) propagator** becomes

$$\frac{i}{p^2 - m^2 - \Sigma_{\text{tadpole}}} = \frac{i}{p^2 - m^2_{\text{phys}} + O(\lambda^2)}\,, \tag{6.11}$$

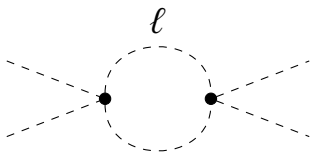
where we defined the **physical (renormalized) mass**

$$m^2_{\text{phys}} \equiv m^2 + \Sigma_{\text{tadpole}}\,. \tag{6.12}$$

The key point is that Σ_{tadpole} is UV-divergent, so the *bare* mass m^2 appearing in the Lagrangian (6.1) is *not* the physically measured mass. Instead, the bare mass must itself be divergent, chosen so that the combination $m^2 + \Sigma_{\text{tadpole}}$ yields a finite, measurable quantity m^2_{phys}. This is the essence of **mass renormalization**: the divergence is not in any observable—it is absorbed into the unobservable bare parameter of the Lagrangian.

The bubble: coupling renormalization

The one-loop correction to $2 \to 2$ scattering involves bubble diagrams in the s, t, and u channels:



Reading the s-channel diagram with the Feynman rules: two vertices ($(-i\mu^\varepsilon\lambda)^2$), two internal propagators, a loop integration, and a symmetry factor $S = 2$ (the two internal lines can be interchanged). Setting $P = p_1 + p_2$:

$$i\mathcal{M}_s = \frac{(-i\mu^\varepsilon\lambda)^2}{2}\int \frac{d^d\ell}{(2\pi)^d}\frac{i}{\ell^2 - m^2}\cdot\frac{i}{(\ell - P)^2 - m^2}\,. \tag{6.13}$$

Power counting. For large $|\ell|$, the integrand behaves as $1/\ell^4$:

$$\int^{\Lambda}\frac{d^4\ell}{\ell^4} \sim \int^{\Lambda}\frac{\ell^3 d\ell}{\ell^4} \sim \ln\Lambda\,. \tag{6.14}$$

The integral is **logarithmically divergent**.

Evaluation in dimensional regularization. One introduces a single Feynman parameter x to combine the two denominators, shifts the loop momentum $\ell \to \ell + xP$, and then applies the master integral (6.3) with $n = 2$ and $\Delta = m^2 - x(1-x)s$. The result is

$$I(s) = \mu^\varepsilon\int\frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2 - m^2)[(\ell - P)^2 - m^2]} = \frac{i}{16\pi^2}\left[\frac{2}{\varepsilon} - \gamma_E + \ln(4\pi) - F(s)\right], \tag{6.15}$$

where $s = P^2$ and $F(s)$ is the finite kinematic function (which depends on the renormalization scale μ)

$$F(s) = \int_0^1 dx\,\ln\frac{m^2 - x(1-x)s}{\mu^2}\,. \tag{6.16}$$

The full details of the Feynman-parameter technique and the evaluation of J_2 are given in Appendix G.

Carefully tracking the signs, $(-i\mu^\varepsilon\lambda)^2 = -\mu^{2\varepsilon}\lambda^2$ from the vertices and $(i)(i) = -1$ from the two propagators combine to give an overall $+\mu^{2\varepsilon}\lambda^2/2$, so that $i\mathcal{M}_s = (\mu^\varepsilon\lambda^2/2)\,I(s)$. Summing the three channels gives

$$i\mathcal{M}^{(1\text{-loop})} = +\frac{i\mu^\varepsilon\lambda^2}{32\pi^2}\left[\frac{6}{\varepsilon} + \text{finite}(s,t,u)\right], \tag{6.17}$$

so that, to order λ^2, the full $2 \to 2$ amplitude reads

$$i\mathcal{M}(s,t,u) = -i\mu^\varepsilon\lambda \;+\; \frac{i\mu^\varepsilon\lambda^2}{32\pi^2}\left[\frac{6}{\varepsilon} + \text{finite}(s,t,u)\right] + O(\lambda^3)\,. \tag{6.18}$$

Bare vs physical coupling. Exactly as for the mass, the parameter λ that appears in the Lagrangian (6.1) is the *bare* coupling, and it is *not* the quantity measured in a scattering experiment. The physical (renormalized) coupling λ_{phys} is *defined* as the value of the amplitude $-i\mathcal{M}$ at some reference kinematic point, say the symmetric point $s_0 = t_0 = u_0 = 4m_{\text{phys}}^2/3$:

$$\lambda_{\text{phys}} \;\equiv\; -\mathcal{M}(s_0,t_0,u_0)\,. \tag{6.19}$$

Reading this off from (6.18) and using $\lambda_{\text{phys}} = -\mathcal{M} = \mu^\varepsilon\lambda - (\mu^\varepsilon\lambda^2/32\pi^2)[6/\varepsilon + \text{finite}]$, one finds schematically

$$\lambda_{\text{phys}} = \mu^\varepsilon\lambda - \frac{\mu^\varepsilon\lambda^2}{32\pi^2}\left[\frac{6}{\varepsilon} + \text{finite}\right] + O(\lambda^3)\,, \tag{6.20}$$

Since at leading order $\mu^\varepsilon\lambda = \lambda_{\text{phys}} + O(\lambda_{\text{phys}}^2)$, this relation can be inverted iteratively. Writing $\text{finite}_0 \equiv \text{finite}(s_0,t_0,u_0)$ for the finite part evaluated at the reference kinematic point, one obtains

$$\mu^\varepsilon\lambda = \lambda_{\text{phys}} + \frac{\lambda_{\text{phys}}^2}{32\pi^2}\left[\frac{6}{\varepsilon} + \text{finite}_0\right] + O(\lambda_{\text{phys}}^3)\,. \tag{6.21}$$

In particular, the singular part of the bare coupling is

$$\mu^\varepsilon\lambda = \lambda_{\text{phys}} + \frac{3\lambda_{\text{phys}}^2}{16\pi^2}\,\frac{1}{\varepsilon} + \text{finite}\,,$$

so that λ_{phys} is *finite* precisely if the bare λ is divergent, with its divergence chosen to cancel the $6/\varepsilon$ pole of the bubble. This is the essence of **coupling renormalization**: just as for the mass, the divergence sits in the unobservable bare parameter, not in any measured quantity.

At this stage we have only established the existence of such a redefinition; the systematic machinery that implements it—counterterms, renormalization constants Z_λ, and the freedom in the finite part (the choice of *scheme*)—is developed in the next subsection.

Counterterms and the renormalized Lagrangian

We have seen that loop corrections produce UV-divergent shifts to the mass and the coupling constant. The strategy of **renormalization** is to recognize that the parameters m and λ appearing in the original Lagrangian (6.1) are not the physically measured quantities—they are **bare** (unobservable) parameters. The divergences are not a sickness of the theory; they are simply telling us that the relation between bare and physical parameters involves infinite corrections.

To make this systematic, we introduce **renormalization constants** Z_ϕ, Z_m, Z_λ and write the Lagrangian as

$$\mathcal{L} = \frac{1}{2} Z_\phi \, \partial_\mu \phi \, \partial^\mu \phi - \frac{1}{2} Z_m m^2 \phi^2 - \mu^\varepsilon \frac{\lambda Z_\lambda}{4!} \phi^4 , \tag{6.22}$$

where m and λ are now the *renormalized* (finite) parameters, whose precise values depend on the renormalization scheme and scale. The perturbative expansion is counted in powers of this renormalized coupling λ, not in powers of the poles in ε. Accordingly, each renormalization factor has the structure

$$Z = 1 + \sum_{n=1}^{\infty} \lambda^n \sum_{k=1}^{n} \frac{c_{nk}}{\varepsilon^k} ,$$

with numerical coefficients c_{nk} fixed order by order. Thus a term such as λ^2/ε is still an $O(\lambda^2)$ contribution: the pole is part of the regulator-dependent coefficient, not an extra power of the coupling. These divergent coefficients are chosen so that, at each order in λ, the poles from the Z-factors cancel the poles coming from loop integrals before the limit $\varepsilon \to 0$ is taken.

The idea is best understood by splitting the Lagrangian into two pieces: $\mathcal{L} = \mathcal{L}_{\text{ren}} + \mathcal{L}_{\text{ct}}$, where

$$\mathcal{L}_{\text{ren}} = \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \mu^\varepsilon \frac{\lambda}{4!} \phi^4 \tag{6.23}$$

has exactly the same form as the original Lagrangian (but now with *renormalized* parameters), and $\mathcal{L}_{\text{ct}}$ collects the **counterterms**—the leftover pieces. Let us derive $\mathcal{L}_{\text{ct}}$ explicitly.

Step 1: Splitting the Lagrangian. Start from the bare Lagrangian (6.22) and peel off the renormalized part (6.23). Writing $Z_\phi = 1 + \delta_\phi$ and $Z_m = 1 - \delta_m$, one finds

$$\frac{1}{2} Z_\phi \, (\partial\phi)^2 = \frac{1}{2} (\partial\phi)^2 + \frac{1}{2} \delta_\phi \, (\partial\phi)^2 , \tag{6.24}$$

$$-\frac{1}{2} Z_m m^2 \phi^2 = -\frac{1}{2} m^2 \phi^2 + \frac{1}{2} \delta_m \, m^2 \phi^2 , \tag{6.25}$$

where the second line uses $Z_m = 1 - \delta_m$, i.e. $-(1-\delta_m)m^2 = -m^2 + \delta_m m^2$. Similarly for the coupling: defining $\delta_\lambda = \lambda(Z_\lambda - 1)$, the quartic term splits as

$$-\mu^\varepsilon \frac{\lambda Z_\lambda}{4!}\phi^4 = -\mu^\varepsilon \frac{\lambda}{4!}\phi^4 - \mu^\varepsilon \frac{\delta_\lambda}{4!}\phi^4 .$$

Collecting all the leftover pieces:

$$\boxed{\mathcal{L}_{\rm ct} = \frac{1}{2}\delta_\phi\, \partial_\mu\phi\, \partial^\mu\phi + \frac{1}{2}\delta_m\, m^2\phi^2 - \mu^\varepsilon \frac{\delta_\lambda}{4!}\phi^4 ,} \tag{6.26}$$

with

$$\delta_\phi = Z_\phi - 1 , \qquad \delta_m = 1 - Z_m , \qquad \delta_\lambda = \lambda(Z_\lambda - 1) . \tag{6.27}$$

Notice the sign asymmetry: $\delta_\phi = Z_\phi - 1$ but $\delta_m = 1 - Z_m$. This traces back to the fact that the kinetic and mass terms have *opposite* signs in the Lagrangian ($+\frac{1}{2}(\partial\phi)^2$ vs $-\frac{1}{2}m^2\phi^2$), so peeling off $\mathcal{L}_{\rm ren}$ produces different signs for the two residuals. The payoff is that $\delta_m > 0$ (the counterterm adds a positive correction to the propagator), which will make the cancellation of divergences transparent.

Step 2: Deriving the counterterm Feynman rules. Since $\mathcal{L}_{\rm ct}$ has the same monomial structure as $\mathcal{L}_{\rm ren}$, it generates new vertices in the Feynman rules. The recipe is the same as for any interaction term in the Lagrangian:

1. Identify the relevant monomial in the interaction Lagrangian. Recall that in the Dyson series (Chapter 3), the nth-order term contains

   $$(-i)^n \int d^4x_1 \cdots d^4x_n\, \mathcal{H}_I(x_1)\cdots\mathcal{H}_I(x_n) ,$$

   and the interaction Hamiltonian density is $\mathcal{H}_I = -\mathcal{L}_{\rm int}$. The two minus signs ($-i$ and $-\mathcal{L}$) combine to give a factor of $+i$ for each vertex: a term $c\,\phi^n$ in $\mathcal{L}_{\rm int}$ produces a vertex factor starting with $i \times c$.
2. Go to momentum space, replacing each $\phi(x)$ by its Fourier transform $\tilde\phi(p)$ and each derivative ∂_μ by $-ip_\mu$. The integral $\int d^4x$ over the vertex position produces a momentum-conserving $(2\pi)^4\delta^{(4)}(\sum p_i)$.
3. Count the combinatorial factor: n identical fields can be contracted with n external legs in $n!$ ways; this cancels the $1/n!$ in front of the monomial (if present).

Let us work through each counterterm.

- **Mass counterterm:** The monomial is $+\frac{1}{2}\delta_m m^2\phi^2$. This is a two-point vertex (two ϕ-legs meeting at a single point). Going to momentum space: the field

$\phi(x)$ is replaced by $\tilde{\phi}(p) = \int d^4x\, \phi(x)\, e^{ip\cdot x}$, and a product $\phi(x)^2$ at the *same* spacetime point becomes, after integrating over x:

$$\int d^4x\ \phi(x)^2 = \int \frac{d^4p}{(2\pi)^4}\ \tilde{\phi}(p)\,\tilde{\phi}(-p)\,.$$

(The δ-function from $\int d^4x\, e^{i(p_1+p_2)\cdot x}$ forces $p_2 = -p_1$.) Therefore the contribution to the action is:

$$iS_{\rm ct} \supset i\cdot\frac{1}{2}\delta_m m^2 \int \frac{d^4p}{(2\pi)^4}\ \tilde{\phi}(p)\,\tilde{\phi}(-p)\,. \tag{6.28}$$

The vertex has two ϕ-fields (ϕ_a and ϕ_b, say) and two external legs carrying momenta p and $-p$. There are $2! = 2$ ways to assign which field contracts with which leg:

Assignment 1: Assignment 2:

Both assignments give the same integral, so the Wick contractions produce a factor of 2. But the Lagrangian monomial carries $\frac{1}{2}$, so: $\frac{1}{2}\times 2 = 1$.
What about the fields $\tilde{\phi}(p)\tilde{\phi}(-p)$ and the integral $\int d^4p/(2\pi)^4$? These are *not* part of the vertex factor—they represent the external legs of the diagram (the incoming and outgoing propagators). The Feynman rule for a vertex is the "kernel" that remains after stripping off all external propagators and field operators. What is left is simply

$$\text{mass counterterm vertex:} \qquad +\,i\,\delta_m\,m^2\,. \tag{6.29}$$

- **Field-strength counterterm:** The monomial is $+\frac{1}{2}\delta_\phi\,(\partial_\mu\phi)(\partial^\mu\phi)$. Going to momentum space, each derivative $\partial_\mu\phi(x)$ becomes $(-ip_\mu)\tilde{\phi}(p)$, so $(\partial_\mu\phi)(\partial^\mu\phi)$ produces a factor $(-ip_\mu)(ip^\mu) = p^2$, where p is the four-momentum flowing through the vertex. The same $\frac{1}{2}$ vs $2!$ cancellation as for the mass counterterm yields

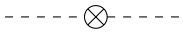
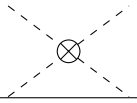
$$\text{field-strength counterterm vertex:} \qquad +\,i\,\delta_\phi\,p^2\,. \tag{6.30}$$

Note that p^2 here is *not* set equal to m^2: this counterterm is inserted on an internal propagator line where the particle is generically off-shell ($p^2 \neq m^2$). Its role is to correct the *momentum dependence* of the propagator (wave-function renormalization), unlike the mass counterterm which shifts only the constant part.

- **Coupling counterterm:** The monomial is $-\mu^{\varepsilon}\frac{\delta_\lambda}{4!}\phi^4$. This is a four-point vertex. Four external legs can connect to the four ϕ-fields in $4! = 24$ ways; the $\frac{1}{4!}$ cancels this, leaving

$$\text{coupling counterterm vertex:} \qquad -i\,\mu^{\varepsilon}\delta_\lambda\,. \tag{6.31}$$

These **counterterm vertices** are drawn with a cross ($\otimes$) to distinguish them from ordinary vertices:

Counterterm	**Diagram**	**Feynman rule**
Mass (δ_m)		$i\delta_m\, m^2$
Field strength (δ_ϕ)		$i\delta_\phi\, p^2$
Coupling (δ_λ)		$-i\mu^{\varepsilon}\delta_\lambda$

Step 3: Consistency check—does the propagator come out right? The mass and field-strength counterterms both insert on the propagator line, giving a combined two-point vertex factor

$$V_{\text{ct}}^{(2)} = i(\delta_\phi\, p^2 + \delta_m\, m^2)\,. \tag{6.32}$$

Inserting this into the Dyson series for the propagator:

$$G(p) = G_0 + G_0 \cdot V_{\text{ct}}^{(2)} \cdot G_0 + \cdots = \frac{i}{p^2 - m^2 + \delta_\phi\, p^2 + \delta_m\, m^2} = \frac{i}{Z_\phi\, p^2 - Z_m\, m^2}\,, \tag{6.33}$$

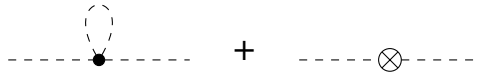
which is indeed the propagator from the bare Lagrangian (6.22). The two approaches—working directly with the bare Lagrangian or treating counterterms as extra Feynman rules—give the same result, as they must.

Step 4: How counterterms cancel divergences, order by order.

The key idea is simple: counterterm vertices are used *exactly like ordinary vertices*—insert them into diagrams, multiply by the appropriate factor, and integrate over internal momenta as usual. At each order in perturbation theory, we *choose* the counterterm coefficients (δ_m, δ_ϕ, δ_λ) so that the sum of all diagrams at that order is UV finite. Let us see how this works explicitly at order λ and order λ^2.

At order λ: mass renormalization. The only one-loop correction to the propagator is the tadpole. It contributes a self-energy insertion $-i\Sigma_{\text{tadpole}}$, which is UV divergent (it contains a $2/\varepsilon$ pole). At the *same order* in λ, the mass counterterm

δ_m (which is itself $O(\lambda)$) contributes a vertex $+i\delta_m m^2$ on the propagator line. The **renormalized** self-energy is their sum:

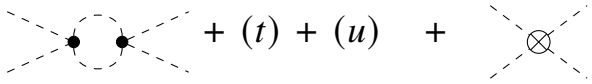
renormalized:  = finite. (6.34)

In formulas, using the tadpole result (6.9):

$$-i\Sigma_{\text{tadpole}} + i\delta_m m^2 = \frac{-i\lambda m^2}{32\pi^2}\left[\frac{2}{\varepsilon} + \text{finite}\right] + i\delta_m m^2 . \tag{6.35}$$

The $2/\varepsilon$ pole cancels if we choose $\delta_m = \frac{\lambda}{32\pi^2}\left[\frac{2}{\varepsilon} + (\text{scheme-dependent finite part})\right]$. What remains is a finite, calculable correction to the propagator—physically, a shift of the mass.

At order λ^2: coupling renormalization. The one-loop correction to $2 \to 2$ scattering comes from the three bubble diagrams (the s-, t-, and u-channel diagrams derived above). Each bubble is $O(\lambda^2)$ and logarithmically divergent. At the same order, the coupling counterterm δ_λ (which is $O(\lambda^2)$) contributes a single tree-level diagram with a $\otimes$ vertex:

renormalized:  = finite. (6.36)

Equivalently, at order λ^2 one can write explicitly

$$i\mathcal{M}_{\text{ren}}(s,t,u) = -i\mu^{\varepsilon}\lambda - i\mu^{\varepsilon}\delta_\lambda + i\mathcal{M}^{(\text{1-loop})}(s,t,u), \tag{6.37}$$

so the UV pole in $i\mathcal{M}^{(\text{1-loop})}$ is cancelled by choosing δ_λ with the same coefficient and opposite sign. The three bubbles together give a divergence proportional to $3\lambda^2/(16\pi^2\varepsilon)$ (the factor of 3 because there are three channels). The counterterm vertex contributes $-i\mu^{\varepsilon}\delta_\lambda$. The divergence cancels if $\delta_\lambda = \frac{3\lambda^2}{32\pi^2}\left[\frac{2}{\varepsilon} + \text{finite}\right]$. The remaining finite part is a measurable correction to the scattering amplitude—the one-loop renormalization of the coupling constant.

The pattern. At each order in λ, the counterterms contain just enough free parameters to absorb all the $1/\varepsilon$ poles. This works because $\lambda\phi^4$ is **renormalizable**: the divergences at any loop order have the same structure as terms already present in the Lagrangian (ϕ^2, $(\partial\phi)^2$, ϕ^4), so they can always be absorbed by adjusting δ_m, δ_ϕ, and δ_λ. No new types of counterterms are ever needed.

Using the one-loop results from the tadpole (6.9) and the bubble (6.15), we can read off the renormalization constants at one loop:

$$Z_\phi = 1 + O(\lambda^2) \qquad \text{(no wave function renormalization at one loop)}, \tag{6.38}$$

$$Z_m = 1 - \frac{\lambda}{32\pi^2}\left(\frac{2}{\varepsilon} + \text{finite}\right) + O(\lambda^2), \tag{6.39}$$

$$Z_\lambda = 1 + \frac{3\lambda}{32\pi^2}\left(\frac{2}{\varepsilon} + \text{finite}\right) + O(\lambda^2). \tag{6.40}$$

Since $Z_m < 1$, we have $\delta_m = 1 - Z_m > 0$: the counterterm correction is positive, precisely compensating the positive loop self-energy. The fact that $Z_\phi = 1$ at one loop reflects the absence of a one-loop correction to the propagator's momentum dependence in ϕ^4 theory (the tadpole is momentum-independent). The "finite" parts depend on the choice of renormalization scheme, as we discuss next.

Renormalization schemes

The counterterms must cancel the $2/\varepsilon$ divergences—but nothing prevents us from also subtracting some finite amount. This freedom is not a bug; it simply reflects a choice of *what we mean* by m and λ. Different conventions define different **renormalization schemes**.

To see concretely how this works, consider the mass counterterm. From (6.9), the divergent self-energy is

$$\Sigma_{\text{tadpole}} = \frac{\lambda m^2}{32\pi^2}\left[\frac{2}{\varepsilon} - \gamma_E + \ln(4\pi) - \ln\frac{m^2}{\mu^2} + 1\right]. \tag{6.41}$$

We must choose δ_m to cancel the divergence. The simplest option: subtract *only* the pole.

- **MS (minimal subtraction):** Set

$$\delta_m^{\text{MS}} = \frac{\lambda}{32\pi^2}\frac{2}{\varepsilon}. \tag{6.42}$$

  The counterterm removes the divergence and nothing else. The renormalized mass m then differs from the physical mass by a finite, calculable shift.
- **$\overline{\text{MS}}$ (modified minimal subtraction):** In practice, the combination $2/\varepsilon - \gamma_E + \ln(4\pi)$ appears in *every* dimensionally regularized integral. It is convenient to subtract all of it at once:

$$\delta_m^{\overline{\text{MS}}} = \frac{\lambda}{32\pi^2}\left[\frac{2}{\varepsilon} - \gamma_E + \ln(4\pi)\right]. \tag{6.43}$$

  This is by far the most widely used scheme in modern particle physics.
- **On-shell scheme:** Subtract *everything* so that the renormalized mass equals the physical mass (the pole of the propagator):

$$\delta_m^{\text{OS}} = \frac{\lambda}{32\pi^2}\left[\frac{2}{\varepsilon} - \gamma_E + \ln(4\pi) - \ln\frac{m^2}{\mu^2} + 1\right], \tag{6.44}$$

  i.e. $\delta_m^{\text{OS}} = \Sigma_{\text{tadpole}}/m^2$, so that the correction to the propagator vanishes identically at $p^2 = m^2$.

All three schemes remove the divergence; they differ only in how much finite "leftover" is absorbed into the definition of m. But *physical observables cannot depend on this bookkeeping choice*. Let us prove this explicitly.

Explicit proof: the physical mass is scheme-independent.

The physical (pole) mass m^2_{phys} is defined as the location of the pole of the full propagator:

$$m^2_{\text{phys}} = m^2 + \Sigma_{\text{tadpole}} - \delta_m\, m^2\,, \tag{6.45}$$

where m is the renormalized mass parameter and δ_m is the counterterm—both scheme-dependent. To connect this with the original bare theory, recall from (6.22) that the mass term in the Lagrangian is

$$-\frac{1}{2} Z_m m^2 \phi^2\,.$$

This is just the original bare mass term written in renormalized variables, so it is natural to *define*

$$m_0^2 \;\equiv\; Z_m m^2 \;=\; (1-\delta_m) m^2\,.$$

Thus m_0^2 is not a new physical quantity introduced out of nowhere: it is simply the **bare mass parameter** of the theory, rewritten in terms of the renormalized mass and the counterterm. The key point is that, for a fixed regularized theory, m_0^2 is held fixed while different renormalization schemes merely reshuffle how much of it is called "renormalized parameter" and how much is called "counterterm":

$$m_0^2 = (1-\delta_m)\, m^2 \quad\Longrightarrow\quad m^2_{\text{phys}} = m_0^2 + \Sigma_{\text{tadpole}}\,. \tag{6.46}$$

At the one-loop level, m_0 and Σ_{tadpole} (computed with the bare propagator) do not depend on the scheme, so neither does m_{phys}. (At higher orders, one must be more careful: the loop integrals themselves depend on which mass appears in the propagator, and the scheme independence of m_{phys} follows from the all-orders structure of the counterterms.) Let us verify scheme independence explicitly at one loop by direct calculation in two different schemes.

In the $\overline{MS}$ scheme: The renormalized mass is $m_{\overline{\text{MS}}}$ and

$$\delta_m^{\overline{\text{MS}}} = \frac{\lambda}{32\pi^2}\left[\frac{2}{\varepsilon} - \gamma_E + \ln(4\pi)\right].$$

The renormalized self-energy (tadpole minus counterterm) is:

$$\begin{aligned}
\Sigma_{\text{ren}}^{\overline{\text{MS}}} &= \Sigma_{\text{tadpole}} - \delta_m^{\overline{\text{MS}}}\, m^2_{\overline{\text{MS}}} \\
&= \frac{\lambda\, m^2_{\overline{\text{MS}}}}{32\pi^2}\left[\frac{2}{\varepsilon} - \gamma_E + \ln(4\pi) - \ln\frac{m^2_{\overline{\text{MS}}}}{\mu^2} + 1\right] - \frac{\lambda\, m^2_{\overline{\text{MS}}}}{32\pi^2}\left[\frac{2}{\varepsilon} - \gamma_E + \ln(4\pi)\right] \\
&= \frac{\lambda\, m^2_{\overline{\text{MS}}}}{32\pi^2}\left[1 - \ln\frac{m^2_{\overline{\text{MS}}}}{\mu^2}\right].
\end{aligned} \tag{6.47}$$

The $1/\varepsilon$ pole has cancelled, and we are left with a *finite* correction. The physical mass is:

$$\boxed{m^2_{\text{phys}} = m^2_{\overline{\text{MS}}} + \frac{\lambda m^2_{\overline{\text{MS}}}}{32\pi^2}\left[1 - \ln\frac{m^2_{\overline{\text{MS}}}}{\mu^2}\right] + O(\lambda^2)\,.} \tag{6.48}$$

In the on-shell scheme: By construction, $\delta_m^{\text{OS}} = \Sigma_{\text{tadpole}}/m^2_{\text{OS}}$, so:

$$\Sigma^{\text{OS}}_{\text{ren}} = \Sigma_{\text{tadpole}} - \delta_m^{\text{OS}}\, m^2_{\text{OS}} = \Sigma_{\text{tadpole}} - \Sigma_{\text{tadpole}} = 0\,.$$

The physical mass is:

$$\boxed{m^2_{\text{phys}} = m^2_{\text{OS}} + 0 = m^2_{\text{OS}}\,.} \tag{6.49}$$

Are these the same? Yes! From (6.48), the $\overline{\text{MS}}$ mass is related to the on-shell mass by

$$m^2_{\text{OS}} = m^2_{\overline{\text{MS}}} + \frac{\lambda m^2_{\overline{\text{MS}}}}{32\pi^2}\left[1 - \ln\frac{m^2_{\overline{\text{MS}}}}{\mu^2}\right] + O(\lambda^2)\,, \tag{6.50}$$

which is precisely (6.48). Both routes give the same m^2_{phys}—they must, since m_{phys} is what an experiment measures.

The physical picture. A renormalization scheme is nothing but a **bookkeeping convention**: it dictates how to split a fixed, scheme-independent total into "parameter" and "loop correction."

	Renormalized parameter	**Loop correction**
On-shell	$m^2_{\text{OS}} = m^2_{\text{phys}}$	$\Sigma^{\text{OS}}_{\text{ren}} = 0$
$\overline{\text{MS}}$	$m^2_{\overline{\text{MS}}} \neq m^2_{\text{phys}}$	$\Sigma^{\overline{\text{MS}}}_{\text{ren}} \neq 0$
Sum (physical)	$m^2_{\text{phys}} = m^2 + \Sigma_{\text{ren}}$	(same in both cases)

In the on-shell scheme, the parameter does all the work and the loop correction vanishes; in $\overline{\text{MS}}$, the parameter absorbs less and the loop correction carries the rest. The observable is always the *sum*, which is scheme-independent.

Box 6.1 Summary: Renormalization of $\lambda\phi^4$

Divergent diagrams:

- Tadpole (mass correction): quadratically divergent $\sim \Lambda^2$
- Bubble (vertex correction): logarithmically divergent $\sim \ln\Lambda$

Counterterms: Z_m cancels mass divergence, Z_λ cancels coupling divergence[a]

Key lesson: The same structure applies to QED, but with the complication of spin and gauge invariance. The Ward identity constrains the counterterms, leaving only three independent renormalization constants.

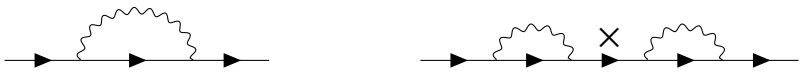
[a] Because the renormalization scale μ is arbitrary, the bare coupling λ_0 cannot depend on it. Requiring $\mu\, d\lambda_0/d\mu = 0$ implies that the renormalized coupling λ must *run* with μ; the rate of change is the **beta function** $\beta(\lambda) = \mu\, d\lambda/d\mu = 3\lambda^2/(16\pi^2) + O(\lambda^3)$. Since $\beta > 0$, the coupling grows with energy and eventually hits a *Landau pole*, signalling the breakdown of perturbation theory: ϕ^4 theory is not asymptotically free. We will return to the beta function in detail when discussing QED.

Introduction: why QED needs renormalization

When we compute higher-order corrections in quantum electrodynamics, we encounter loop integrals that diverge in the ultraviolet (UV). These divergences are not a sign that the theory is sick; rather, they signal that the "bare" parameters in the Lagrangian (e_0, m_0)—that is, the charge and mass appearing in the unrenormalized Lagrangian—are not the physical quantities we measure. Renormalization is the systematic procedure that absorbs these infinities into redefinitions of the parameters and fields, leaving finite predictions for observables.

One-particle irreducible (1PI) diagrams. Recall that a Feynman diagram is called **1PI** if it *cannot* be disconnected by cutting a single internal line (we already encountered this concept when resumming the self-energy in (6.10)). Diagrams that *can* be split by cutting one line are called *one-particle reducible* (1PR).

1PI (irreducible) **1PR (reducible)**



Cannot be split by cutting one line

Cutting the middle line disconnects the diagram

Why 1PI diagrams matter. The 1PI diagrams are the fundamental building blocks of perturbation theory for several reasons:

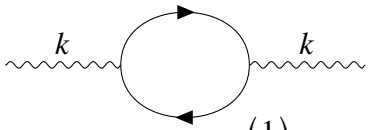
1. **No double counting:** Any Feynman diagram can be uniquely decomposed into 1PI subdiagrams connected by single propagators. Working with 1PI diagrams avoids overcounting contributions.
2. **Generating functionals:** The *effective action* $\Gamma[\phi]$ (the Legendre transform of the generating functional $W[J]$) generates precisely the 1PI diagrams. This is why Γ is also called the "1PI generating functional."[3]
3. **Self-energies and vertex functions:** The physically meaningful corrections to propagators (self-energies Σ, Π) and vertices (Γ^μ) are defined as sums of 1PI diagrams. The full propagator is then obtained by resumming all 1PI insertions via a geometric series (Dyson equation).

[3]The idea, briefly, is the following. One introduces an external source $J(x)$ coupled linearly to the field, and defines a generating functional $Z[J]$ whose functional derivatives yield all correlation functions. Taking $W[J] = -i \ln Z[J]$ selects only the *connected* diagrams. One then defines the *classical field* $\phi_c(x) \equiv \delta W/\delta J(x)$, which represents the vacuum expectation value of the field in the presence of J. The *effective action* $\Gamma[\phi_c]$ is the Legendre transform of $W[J]$: $\Gamma[\phi_c] = W[J] - \int d^4x\, J(x)\,\phi_c(x)$. The key result is that the functional derivatives of Γ evaluated at $J = 0$ generate precisely the 1PI vertex functions—the amputated, connected Green functions that cannot be disconnected by cutting a single internal line. For a standard derivation, see Peskin–Schroeder [6], Ch. 11.

4. **Renormalization:** Only the 1PI Green functions need to be renormalized. Once the divergences in 1PI diagrams are cancelled by counterterms, all higher-order diagrams (built from 1PI blocks) become automatically finite.

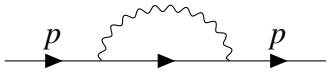
The three primitively divergent one-loop diagrams in QED. At one loop, only three classes of 1PI diagrams are UV divergent by power counting. These are the building blocks from which all higher-order divergences are constructed:

(a) Photon self-energy



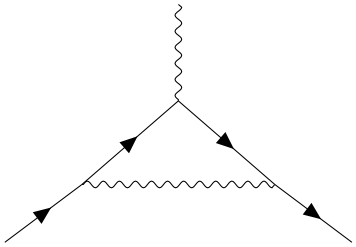
$$i\Pi^{\mu\nu}(k) = O\left(\frac{1}{\epsilon}\right)$$

(b) Electron self-energy



$$-i\Sigma(\not{p}) = O\left(\frac{1}{\epsilon}\right)$$

(c) Vertex correction



$$+ie\Gamma^{\mu} = O\left(\frac{1}{\epsilon}\right)$$

A note on signs. The relative sign in the definitions $ie_0^2\Pi^{\mu\nu}(k)$ and $-i\Sigma(\not{p})$ is a matter of convention, chosen so that the Dyson-resummed propagators take the standard forms $D^{-1}(k) \sim k^2 - e_0^2\Pi_T(k^2)$ for the photon and $S^{-1}(\not{p}) \sim \not{p} - m_0 - \Sigma(\not{p})$ for the electron. It does *not* reflect a physical sign difference between the two effects. Part of the asymmetry is also notational: for the photon we factor out an explicit e_0^2 from the insertion, whereas for the electron self-energy the coupling dependence is included directly in $\Sigma(\not{p})$.
These diagrams represent:

- **(a) Vacuum polarization** $\Pi^{\mu\nu}(k)$: a virtual e^+e^- pair modifies the photon propagator. This leads to *charge renormalization* and a running coupling $\alpha(Q^2)$, where for spacelike exchange $Q^2 \equiv -k^2 > 0$.
- **(b) Electron self-energy** $\Sigma(\not{p})$: the electron emits and reabsorbs a virtual photon. This shifts the electron mass ($m_0 \to m$) and requires *wave-function renormalization* (Z_2).
- **(c) Vertex correction** Γ^{μ}: quantum fluctuations modify the electron-photon coupling. Combined with the self-energy, this determines Z_1 and ensures gauge invariance via the Ward identity $Z_1 = Z_2$.

Naive power counting from the integrands. We can determine the degree of divergence of each diagram by inspecting the large-k behaviour of the loop integral, exactly as we did for ϕ^4 theory.

- **Photon self-energy** $\Pi^{\mu\nu}(q)$: one loop with two fermion propagators ($\sim 1/k$ each at large k, since they are linear in $\not{k}$):

$$\Pi^{\mu\nu} \sim \int \frac{d^4k}{(2\pi)^4} \frac{\mathrm{tr}[\gamma^{\mu}\not{k}\gamma^{\nu}\not{k}]}{k^2 \cdot k^2} \sim \int^{\Lambda} \frac{k^3\, dk \cdot k^2}{k^4} \sim \Lambda^2 \qquad \text{(quadratically divergent).}$$

(6.51)

- **Electron self-energy** $\Sigma(\not{p})$: one fermion and one photon propagator in the loop ($\sim 1/k$ and $\sim 1/k^2$ at large k):

$$\Sigma \sim \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\mu \not{k} \gamma_\mu}{k^2 \cdot k^2} \sim \int^\Lambda \frac{k^3\, dk \cdot k}{k^4} \sim \Lambda \qquad \text{(linearly divergent).} \tag{6.52}$$

- **Vertex correction** $\Lambda^\mu(p', p)$: two fermion and one photon propagator ($\sim 1/k^5$ in total at large k):

$$\Lambda^\mu \sim \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\nu \not{k} \gamma^\mu \not{k} \gamma_\nu}{k^2 \cdot k^2 \cdot k^2} \sim \int^\Lambda \frac{k^3\, dk \cdot k^2}{k^6} \sim \ln \Lambda \qquad \text{(logarithmically divergent).} \tag{6.53}$$

However, gauge invariance (the Ward–Takahashi identity) reduces the actual degree of divergence: the photon self-energy becomes logarithmic rather than quadratic, and the electron self-energy becomes logarithmic rather than linear. All three divergences are ultimately *logarithmic*, i.e., poles $\sim 1/\varepsilon$ in dimensional regularization. A systematic classification via the *superficial degree of divergence* will be developed in Section 6.7.

The counterterm strategy. The bare Lagrangian is rewritten in terms of renormalized fields and parameters plus counterterms:

$$\mathcal{L}_0 = \mathcal{L}_{\text{ren}} + \delta\mathcal{L}, \tag{6.54}$$

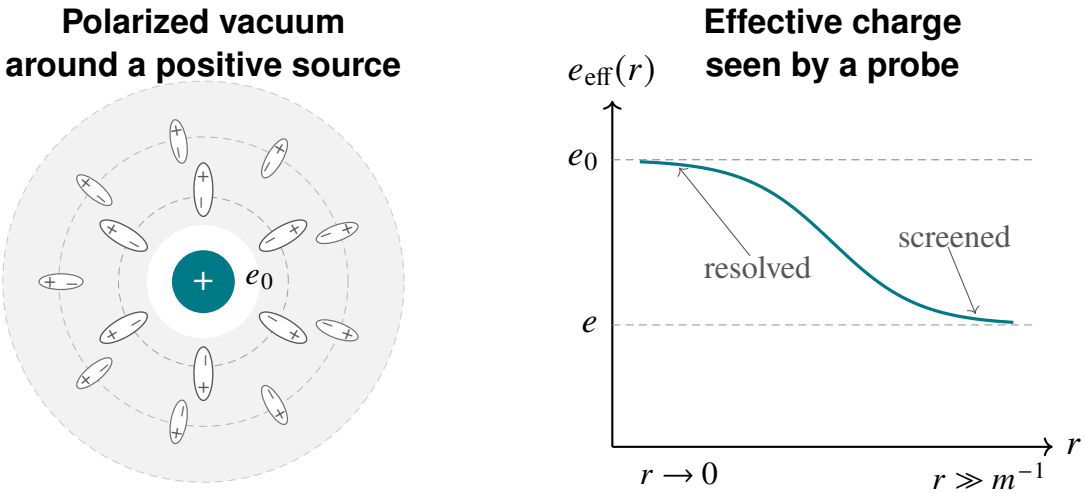
where $\delta\mathcal{L}$ contains $(Z_2 - 1)$, δm, $(Z_3 - 1)$, and $(Z_1 - 1)$ contributions that generate counterterm vertices. These counterterms are chosen so that the sum of loop diagrams plus counterterm insertions is UV finite. The following sections develop this program in detail for each of the three primitively divergent structures.

6.2 Photon self-energy in QED

Physical interpretation: vacuum polarization

The photon self-energy, also known as *vacuum polarization*, represents one of the most profound effects in quantum electrodynamics: the quantum vacuum is not "empty" but rather a boiling sea of virtual electron-positron pairs that constantly pop in and out of existence. When a photon propagates through the vacuum, it can temporarily convert into such a virtual e^+e^- pair, which then annihilates back into a photon. This process modifies the effective photon propagator and has far-reaching physical consequences:

- **Charge screening:** At large distances (low momenta $|k^2| \ll m^2$), the virtual pairs partially screen the "bare" electric charge. The observed charge e is smaller than the bare charge e_0 that would be measured at infinitesimally short distances.



The left panel illustrates how virtual e^+e^- pairs orient themselves around a positive source charge: the negative ends of the dipoles are attracted inward, partially cancelling the field seen at large distances. The right panel shows the resulting effective charge $e_{\text{eff}}(r)$: at distances $r \gg 1/m$ (the electron Compton wavelength), the full screening is in effect and one measures $e_{\text{eff}} = e$; as $r \to 0$, the screening is penetrated and $e_{\text{eff}} \to e_0 > e$.

- **Running coupling:** At short distances (equivalently, for large spacelike momentum transfer $Q^2 \equiv -k^2 \gg m^2$), less screening occurs, so the effective coupling $\alpha(Q^2)$ *increases* with Q^2. This is the famous "running of the QED coupling," verified experimentally at LEP and other high-energy experiments.
- **Lamb shift contribution:** The vacuum polarization modifies the Coulomb potential at short distances, contributing (at order $\alpha^5 m$) to the Lamb shift in hydrogen. This was one of the first precision tests of QED.
- **Muon $g-2$:** Vacuum polarization loops (including hadronic contributions) are crucial ingredients in the theoretical prediction of the anomalous magnetic moment of the muon.

The calculation below shows how the vacuum polarization tensor $\Pi^{\mu\nu}(k)$ arises from a one-loop Feynman diagram and how gauge invariance (the Ward identity) constrains it to be purely transverse: $\Pi^{\mu\nu}(k) = (k^2 g^{\mu\nu} - k^\mu k^\nu)\Pi(k^2)$.

From the Møller diagram to a self-energy insertion (explicit derivation and diagrams)

We illustrate how the photon self-energy (vacuum polarization) appears as an *insertion* into an internal photon line, starting from the tree-level (one-photon exchange) contribution to Møller scattering. For the purposes of this subsection, it

is enough to treat the external fermion lines as conserved currents; the same algebra applies to many other QED processes.

Tree-level amplitude as "current × propagator × current".

Let p_1, p_2 be the incoming electron momenta and p_3, p_4 the outgoing ones. Define the momentum transfer

$$k \equiv p_1 - p_3 = p_4 - p_2 \,. \tag{6.55}$$

The t-channel one-photon exchange amplitude can be written as

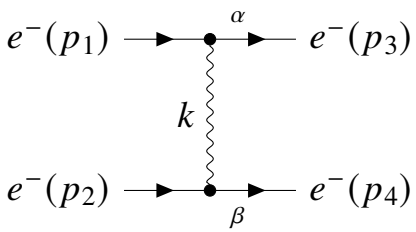
$$i\mathcal{M}^{(0)} = (-ie_0)^2 \left[\bar{u}(p_3)\gamma^\alpha u(p_1)\right] \left(iD_{F\,\alpha\beta}(k)\right) \left[\bar{u}(p_4)\gamma^\beta u(p_2)\right]. \tag{6.56}$$

It is convenient to introduce the external currents

$$J_1^\alpha(-k) \equiv \bar{u}(p_3)\gamma^\alpha u(p_1), \qquad J_2^\beta(k) \equiv \bar{u}(p_4)\gamma^\beta u(p_2)\,, \tag{6.57}$$

so that

$$i\mathcal{M}^{(0)} = (-ie_0)^2 J_1^\alpha(-k) \left(iD_{F\,\alpha\beta}(k)\right) J_2^\beta(k)\,. \tag{6.58}$$



Tree-level t-channel exchange (one photon with momentum k).

The one-loop vacuum polarization insertion.

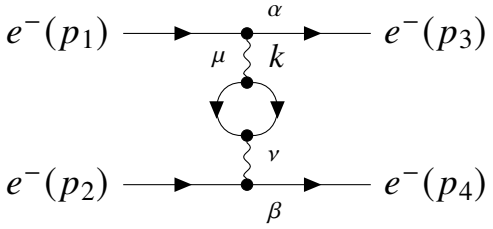
Now consider the correction where the internal photon propagator is modified by a vacuum polarization subdiagram. At order e_0^4, the corresponding amplitude is

$$i\mathcal{M}^{(1)}_{\rm VP} = (-ie_0)^2 J_1^\alpha(-k) \left(iD_{F\,\alpha\mu}(k)\right) \left(ie_0^2\Pi^{\mu\nu}(k)\right) \left(iD_{F\,\nu\beta}(k)\right) J_2^\beta(k)\,. \tag{6.59}$$

Comparing (6.58) and (6.59), we read off the *replacement rule* for an internal photon line (to this order):

$$iD_{F\,\alpha\beta}(k) \longrightarrow iD_{F\,\alpha\beta}(k) + iD_{F\,\alpha\mu}(k) \left(ie_0^2\Pi^{\mu\nu}(k)\right) iD_{F\,\nu\beta}(k)\,. \tag{6.60}$$

This is precisely the statement that the self-energy behaves like a 1PI *insertion* on the propagator.



Vacuum polarization insertion (fermion loop) on the exchanged photon.

Explicit expression for $\Pi^{\mu\nu}(k)$

The inserted blob is the 1PI photon two-point function. At one loop (electron loop), its definition in momentum space is

$$ie_0^2\Pi^{\mu\nu}(k) = (-1)\,(-ie_0)^2 \int \frac{d^dp}{(2\pi)^d}\,\mathrm{Tr}\left[\gamma^\mu \frac{i(\not{p}+m_0)}{p^2-m_0^2+i\epsilon}\gamma^\nu \frac{i(\not{p}+\not{k}+m_0)}{(p+k)^2-m_0^2+i\epsilon}\right]. \tag{6.61}$$

Collecting the factors of i and rearranging yields the more compact form

$$ie_0^2\Pi^{\mu\nu}(k) = -\,e_0^2 \int \frac{d^dp}{(2\pi)^d}\,\frac{\mathrm{Tr}\left[\gamma^\mu(\not{p}+m_0)\gamma^\nu(\not{p}+\not{k}+m_0)\right]}{\left(p^2-m_0^2+i\epsilon\right)\left((p+k)^2-m_0^2+i\epsilon\right)}. \tag{6.62}$$

The overall minus sign comes from the fermion loop.

Why the insertion modifies only the transverse propagator in physical amplitudes.

When the photon is exchanged between on-shell external fermions, the currents are conserved:

$$k_\alpha J_1^\alpha(-k) = 0, \qquad k_\beta J_2^\beta(k) = 0, \tag{6.63}$$

so any part of the corrected propagator proportional to k_α or k_β drops out from the amplitude. In this context (physical amplitudes with conserved external currents), one can decompose $\Pi^{\mu\nu}(k)$ into transverse and longitudinal pieces and keep only the transverse part; the longitudinal part does not contribute. Note that this simplification applies specifically to amplitudes sandwiched between conserved currents; for off-shell Green functions or internal subdiagrams, the full tensor structure must be retained.

Geometric-series resummation of self-energy insertions

The vacuum-polarization subdiagram is a *1PI two-point insertion* on a photon line. Algebraically, it behaves like a matrix (in Lorentz indices) inserted between two free propagators. We write $iD_0^{\mu\nu}(k)$ for the free photon propagator and define the shorthand

$$(i\Pi)^{\mu\nu}(k) \;\equiv\; ie_0^2\Pi^{\mu\nu}(k) \tag{6.64}$$

for the full 1PI vacuum-polarization insertion (i.e. the blob including the coupling factors e_0^2). The parentheses around $i\Pi$ distinguish this combined object from the stripped tensor $\Pi^{\mu\nu}$ (without e_0^2).

Step 1: write the series with repeated insertions. If we allow *any number* of vacuum-polarization insertions on the same internal photon line, the contribution to the effective propagator is the Neumann (geometric) series

$$\begin{aligned} iD^{\mu\nu}(k) = {} & iD_0^{\mu\nu}(k) + iD_0^{\mu\alpha}(k)\left(i\Pi_{\alpha\beta}(k)\right)iD_0^{\beta\nu}(k) \\ & + iD_0^{\mu\alpha}(k)\left(i\Pi_{\alpha\beta}(k)\right)iD_0^{\beta\gamma}(k)\left(i\Pi_{\gamma\delta}(k)\right)iD_0^{\delta\nu}(k) + \cdots . \end{aligned} \tag{6.65}$$

Schematically, suppressing Lorentz indices:

$$D_0 + D_0(i\Pi)D_0 + D_0(i\Pi)D_0(i\Pi)D_0 + \cdots$$

Step 2: factor out one free propagator and identify the geometric structure. Define the Lorentz-index matrix (at fixed k)

$$\big(M(k)\big)^{\mu}{}_{\nu} \equiv \big(i\Pi(k)\big)^{\mu}{}_{\alpha}\,\big(iD_0(k)\big)^{\alpha}{}_{\nu}\,. \tag{6.66}$$

Then each term in (6.65) has the form

$$iD_0\,(i\Pi\, iD_0)^n = iD_0\,M^n\,, \tag{6.67}$$

so the full series becomes

$$iD = iD_0\left(\mathbb{1} + M + M^2 + \cdots\right). \tag{6.68}$$

Formally (as a matrix series), for sufficiently small coupling this sums to

$$\mathbb{1} + M + M^2 + \cdots = (\mathbb{1} - M)^{-1}\,. \tag{6.69}$$

Substituting (6.69) into (6.68) gives

$$iD = iD_0\,(\mathbb{1} - M)^{-1}\,. \tag{6.70}$$

Step 3: rewrite the result in the standard Dyson form. Using the definition (6.66),

$$\begin{aligned}\mathbb{1} - M &= \mathbb{1} - (i\Pi)\,(iD_0) = (iD_0)^{-1}\,(iD_0) - (i\Pi)\,(iD_0)\\ &= \Big[(iD_0)^{-1} - (i\Pi)\Big]\,(iD_0)\,,\end{aligned} \tag{6.71}$$

so

$$(\mathbb{1} - M)^{-1} = (iD_0)^{-1}\Big[(iD_0)^{-1} - (i\Pi)\Big]^{-1}\,. \tag{6.72}$$

Plugging (6.72) into (6.70) yields the compact resummed propagator

$$iD^{\mu\nu}(k) = \left(\Big[(iD_0)^{-1} - (i\Pi)\Big]^{-1}\right)^{\mu\nu}(k) = \frac{iD_0^{\mu\alpha}(k)}{\delta_{\alpha}{}^{\nu} - (i\Pi)_{\alpha\beta}(k)\,iD_0^{\beta\nu}(k)}\,. \tag{6.73}$$

Here the final fraction is a compact notation for the inverse matrix in Lorentz-index space; explicitly,

$$\frac{1}{\delta_{\alpha}{}^{\nu} - (i\Pi)_{\alpha\beta}\,iD_0^{\beta\nu}} \;\equiv\; [\mathbb{1} - (i\Pi)(iD_0)]^{-1}{}_{\alpha}{}^{\nu}.$$

Equation (6.73) is the propagator-level Dyson equation. Recalling that $(i\Pi)^{\mu\nu} = ie_0^2\Pi^{\mu\nu}$, this reads

$$D^{-1} = D_0^{-1} - e_0^2\,\Pi\,. \tag{6.74}$$

Step 4: check explicitly in Feynman gauge (scalar-like geometric sum).[4] In Feynman gauge,

$$iD_0^{\mu\nu}(k) = \frac{-ig^{\mu\nu}}{k^2+i\epsilon}, \qquad (iD_0^{-1})_{\mu\nu}(k) = i(k^2+i\epsilon)\,g_{\mu\nu}\,. \tag{6.75}$$

Why the geometric series becomes scalar: dropping longitudinal pieces.

Inside a physical scattering amplitude, the photon propagator is never used as a free-standing object: it is always *sandwiched between conserved currents*. This kills any longitudinal contribution proportional to the external momentum k^μ.

1. Decompose $\Pi^{\mu\nu}(k)$ into transverse and longitudinal parts. Lorentz covariance allows

$$\Pi^{\mu\nu}(k) = -g^{\mu\nu}A(k^2) + k^\mu k^\nu B(k^2)\,. \tag{6.76}$$

Equivalently (and more transparently for gauge invariance), define the **transverse projector**

$$P_T^{\mu\nu}(k) \equiv g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\,. \tag{6.77}$$

It is called "transverse" because it projects any vector onto the subspace orthogonal to k^μ: contracting with k_μ gives $k_\mu P_T^{\mu\nu} = k^\nu - k^\nu(k^2/k^2) = 0$. In other words, $P_T^{\mu\nu}$ kills the component of a vector along k and keeps only the part transverse to k. One easily verifies that it is indeed a projector: $P_T^{\mu\alpha}P_{T\,\alpha}{}^{\nu} = P_T^{\mu\nu}$.

Any symmetric tensor built from $g^{\mu\nu}$ and k^μ can be written as

$$\Pi^{\mu\nu}(k) = P_T^{\mu\nu}(k)\,\Pi_T(k^2) + \frac{k^\mu k^\nu}{k^2}\,\Pi_L(k^2), \tag{6.78}$$

for scalar functions Π_T, Π_L. The two parametrizations are related by simple linear combinations:

$$-g^{\mu\nu}A + k^\mu k^\nu B = \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)\Pi_T + \frac{k^\mu k^\nu}{k^2}\Pi_L. \tag{6.79}$$

[4] The free photon propagator in a general covariant gauge is $iD_0^{\mu\nu}(k) = \frac{-i}{k^2+i\epsilon}\left[g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right]$, where ξ is the gauge parameter. **Feynman gauge** corresponds to $\xi = 1$, for which the propagator reduces to $-ig^{\mu\nu}/(k^2+i\epsilon)$—proportional to the metric, with no $k^\mu k^\nu$ term. This greatly simplifies calculations because the propagator acquires the same Lorentz structure as a scalar propagator (times $g^{\mu\nu}$), and the $k^\mu k^\nu$ terms that would generate complicated tensor integrals are absent. Physical results are gauge-independent (as guaranteed by the Ward identity), so we are free to choose the gauge that makes the calculation simplest.

In a gauge-invariant regularization (e.g. dimensional regularization) the Ward identity implies transversality of the vacuum polarization,

$$k_\mu \Pi^{\mu\nu}(k) = 0, \tag{6.80}$$

which forces $\Pi_L(k^2) = 0$, i.e. $\Pi^{\mu\nu}$ is purely transverse.

Box 6.2 Dictionary of notations for the vacuum polarization

Several equivalent parametrizations of the stripped tensor $\Pi^{\mu\nu}(k)$ are used in this section and in the literature. They are all related as follows.

(i) The A, B parametrization (Mandl–Shaw convention):

$$\Pi^{\mu\nu}(k) = -g^{\mu\nu} A(k^2) + k^\mu k^\nu B(k^2)\,.$$

(ii) The transverse/longitudinal decomposition:

$$\Pi^{\mu\nu}(k) = P_T^{\mu\nu}(k)\,\Pi_T(k^2) + \frac{k^\mu k^\nu}{k^2}\,\Pi_L(k^2)\,.$$

(iii) The gauge-invariant scalar function $\Pi(k^2)$, defined when $\Pi^{\mu\nu}$ is transverse:

$$\Pi^{\mu\nu}(k) = (k^2 g^{\mu\nu} - k^\mu k^\nu)\,\Pi(k^2)\,.$$

Relations. Comparing (i) and (iii) term by term ($-g^{\mu\nu}A + k^\mu k^\nu B = k^2 g^{\mu\nu}\Pi - k^\mu k^\nu \Pi$):

$$A(k^2) = -k^2\,\Pi(k^2)\,, \qquad B(k^2) = -\Pi(k^2) = \frac{A(k^2)}{k^2}\,.$$

And from (i) vs (ii): $\Pi_T = -A = k^2\Pi$, $\Pi_L = -A + k^2 B = 0$ (when the Ward identity holds).

In the text below, we use primarily $A(k^2)$ (following Mandl–Shaw) and the relation $A(0) = 0$ (from gauge invariance). The key points are:

- In a current sandwich, only A matters (the B term drops out by current conservation).
- $A(k^2)$ vanishes at $k^2 = 0$ (photon stays massless).
- $A(k^2) = k^2[A'(0) + \Pi_c(k^2)]$, where $A'(0)$ is the UV-divergent constant and Π_c the finite remainder.

2. Why longitudinal pieces drop out even without invoking (6.80). Take any amplitude in which a photon line of momentum k is exchanged between two

electromagnetic currents J_1^μ and J_2^ν. The relevant factor has the form

$$\mathcal{M} \supset J_{1\,\mu}(-k)\, D^{\mu\nu}(k)\, J_{2\,\nu}(k). \tag{6.81}$$

For on-shell external fermions, the currents are conserved:

$$k_\mu J_1^\mu(-k) = 0, \qquad k_\nu J_2^\nu(k) = 0. \tag{6.82}$$

Indeed, using $k = p_1 - p_3 = p_4 - p_2$ and the on-shell Dirac equations $(\not{p} - m)u(p) = 0$ and $\bar{u}(p)(\not{p} - m) = 0$, we find

$$\begin{aligned} k_\mu J_1^\mu(-k) &= \bar{u}(p_3)\not{k}\, u(p_1) = \bar{u}(p_3)(\not{p}_1 - \not{p}_3)u(p_1) = \bar{u}(p_3)(m-m)u(p_1) = 0, \\ k_\nu J_2^\nu(k) &= \bar{u}(p_4)\not{k}\, u(p_2) = \bar{u}(p_4)(\not{p}_4 - \not{p}_2)u(p_2) = \bar{u}(p_4)(m-m)u(p_2) = 0. \end{aligned} \tag{6.83}$$

Now consider any term in $D^{\mu\nu}(k)$ proportional to k^μ or k^ν. For example, suppose

$$D^{\mu\nu}(k) = D_T(k)\, P_T^{\mu\nu}(k) + D_L(k)\, \frac{k^\mu k^\nu}{k^2}. \tag{6.84}$$

Then the longitudinal part contributes to (6.81) as

$$J_{1\,\mu}(-k) \left[D_L(k)\, \frac{k^\mu k^\nu}{k^2} \right] J_{2\,\nu}(k) = \frac{D_L(k)}{k^2} \left(J_1(-k)\cdot k\right)\left(k \cdot J_2(k)\right) = 0, \tag{6.85}$$

by (6.82). Therefore, *in any amplitude where the photon propagator is sandwiched between conserved currents*, longitudinal pieces do not contribute—independently of the gauge choice and even if the regularization produced an unphysical longitudinal component.

3. How this simplifies the geometric series in Feynman gauge. In Feynman gauge,

$$iD_0^{\mu\nu}(k) = \frac{-ig^{\mu\nu}}{k^2 + i\epsilon}. \tag{6.86}$$

When we insert $\Pi^{\mu\nu}(k)$ between propagators and then sandwich between conserved currents, we are free to replace $\Pi^{\mu\nu}(k)$ by its transverse part only:

$$\Pi^{\mu\nu}(k) \;\rightsquigarrow\; P_T^{\mu\nu}(k)\, \Pi_T(k^2), \tag{6.87}$$

because the difference is proportional to k^μ and/or k^ν and vanishes by (6.85). If one instead uses the A, B parametrization (6.76), then in a current sandwich

$$\begin{aligned} J_{1\,\mu}\, \Pi^{\mu\nu}(k)\, J_{2\,\nu} &= -A(k^2)\, J_1 \cdot J_2 + B(k^2)\, (J_1 \cdot k)(k \cdot J_2) \\ &= -A(k^2)\, J_1 \cdot J_2, \end{aligned} \tag{6.88}$$

so the B term drops out explicitly.

With (6.87), each insertion effectively carries the same Lorentz structure (transverse projector) and the series becomes "scalar-like": the resummation acts only on the scalar coefficient of the transverse piece, while the longitudinal sector is irrelevant for the physical amplitude. Concretely, the dressed propagator in a conserved-current sandwich can be written as

$$iD^{\mu\nu}(k) \rightsquigarrow \frac{-i}{k^2 + i\epsilon - e_0^2 \Pi_T(k^2)} P_T^{\mu\nu}(k) + (\text{any longitudinal part}), \tag{6.89}$$

and the "any longitudinal part" does not contribute to (6.81) because of (6.85).

Bottom line. The phrase "drop longitudinal pieces" means: *for physical amplitudes built from conserved currents, terms proportional to k^μ or k^ν vanish upon contraction*. This makes the geometric-series resummation effectively scalar (it resums only the transverse scalar function), even though the full propagator is a tensor.

Then the n-th term in (6.65) reduces to a scalar geometric factor times $g^{\mu\nu}$:

$$iD_0\,(i\Pi\, iD_0)^n \rightsquigarrow \frac{-ig^{\mu\nu}}{k^2 + i\epsilon} \left(\frac{-e_0^2 A(k^2)}{k^2 + i\epsilon} \right)^n . \tag{6.90}$$

To see this, note that each insertion gives the scalar factor $(i\Pi)\cdot(iD_0) \rightsquigarrow (-ie_0^2 A)\cdot \frac{-i}{k^2+i\epsilon} = \frac{-e_0^2 A}{k^2+i\epsilon}$. Summing $n = 0, 1, 2, \ldots$ gives

$$\begin{aligned} iD^{\mu\nu}(k) &\rightsquigarrow \frac{-ig^{\mu\nu}}{k^2 + i\epsilon} \sum_{n=0}^{\infty} \left(\frac{-e_0^2 A(k^2)}{k^2 + i\epsilon} \right)^n \\ &= \frac{-ig^{\mu\nu}}{k^2 + i\epsilon} \frac{1}{1 + \dfrac{e_0^2 A(k^2)}{k^2 + i\epsilon}} = \frac{-ig^{\mu\nu}}{k^2 + i\epsilon + e_0^2 A(k^2)} . \end{aligned} \tag{6.91}$$

Expanding the final expression to first order in e_0^2 reproduces the first two terms $D_0 + D_0(i\Pi)D_0$ in (6.65), so the resummation is consistent order-by-order.

What this means physically. The geometric series is simply the statement that a propagator can contain *any number* of 1PI two-point insertions. Summing them reorganizes the perturbation expansion into a dressed propagator with an effective denominator, as in (6.73) (matrix form) and (6.91) (scalarized form in a conserved-current sandwich).

Regularization and gauge-invariant assumptions

The loop integral (6.62) defining the vacuum polarization is ultraviolet divergent for large loop momentum p. A simple illustrative regularization is to multiply the integrand by a convergence factor

$$f(p^2, \Lambda^2) = \left(\frac{-\Lambda^2}{p^2 - \Lambda^2} \right)^2 , \tag{6.92}$$

where Λ is a cut-off parameter. For fixed large Λ, the integral becomes convergent; sending $\Lambda \to \infty$ formally restores the original theory.

In what follows we assume the theory has been regularized in a way that is compatible with gauge invariance and preserves a massless photon in the appropriate limit. In particular, we assume that after regularization all quantities we manipulate (notably the scalar functions introduced below) are finite, and that the photon remains massless (pole at $k^2 = 0$).

Explicit algebra for the modified propagator

Applying the results of the previous subsection, we now perform the explicit algebra in Feynman gauge. Using the replacement $\Pi^{\mu\nu}(k) \rightsquigarrow -g^{\mu\nu}A(k^2)$ (valid in physical amplitudes), the single-insertion formula (6.60) yields

$$\begin{aligned}\frac{-ig_{\alpha\beta}}{k^2+i\epsilon} &\longrightarrow \frac{-ig_{\alpha\beta}}{k^2+i\epsilon} + \frac{-ig_{\alpha\mu}}{k^2+i\epsilon}\left(ie_0^2[-g^{\mu\nu}A(k^2)]\right)\frac{-ig_{\nu\beta}}{k^2+i\epsilon} \\ &= \frac{-ig_{\alpha\beta}}{k^2+i\epsilon} + \frac{i\,e_0^2 A(k^2)\,g_{\alpha\beta}}{(k^2+i\epsilon)^2} \\ &= \frac{-ig_{\alpha\beta}}{k^2+i\epsilon}\left[1 - \frac{e_0^2 A(k^2)}{k^2+i\epsilon}\right]. \end{aligned} \tag{6.93}$$

Since (after regularization) $A(k^2)$ is finite, the bracket in (6.93) can be recognized as the first terms in the expansion of a shifted denominator:

$$\begin{aligned}\frac{-ig_{\alpha\beta}}{k^2+i\epsilon}\left[1 - \frac{e_0^2 A(k^2)}{k^2+i\epsilon}\right] &= \frac{-ig_{\alpha\beta}}{k^2+i\epsilon}\,\frac{k^2+i\epsilon-e_0^2A(k^2)}{k^2+i\epsilon} \\ &= \frac{-ig_{\alpha\beta}}{k^2+i\epsilon+e_0^2A(k^2)} + O(e_0^4), \end{aligned} \tag{6.94}$$

which is the usual geometric-series resummation to this order.

Massless photon condition and the decomposition of $A(k^2)$

The bare photon propagator has a pole at $k^2 = 0$ (massless photon). After including vacuum polarization, the modified propagator (6.94) reads

$$\frac{-ig_{\alpha\beta}}{k^2+i\epsilon+e_0^2A(k^2)}\,. \tag{6.94}$$

This propagator has its pole where the denominator vanishes: $k^2 + e_0^2A(k^2) = 0$. If the photon is to remain massless (pole at $k^2 = 0$), we need $e_0^2A(0) = 0$, i.e.

$$A(0) = 0\,. \tag{6.95}$$

This condition is not an extra assumption: it follows from gauge invariance. In a gauge-invariant regularization (such as dimensional regularization), the Ward identity (6.80) forces the vacuum polarization to be purely transverse:

$$\Pi^{\mu\nu}(k) = \left(k^2 g^{\mu\nu} - k^\mu k^\nu\right)\Pi(k^2)\,.$$

Comparing with the parametrization

$$\Pi^{\mu\nu}(k) = -g^{\mu\nu}A(k^2) + k^\mu k^\nu B(k^2),$$

we see that the coefficient of $g^{\mu\nu}$ must be proportional to k^2:

$$A(k^2) = -k^2\Pi(k^2).$$

Hence $A(0) = 0$ automatically. Equivalently, if $A(0)$ were nonzero, the denominator of the propagator would behave near $k^2 = 0$ like $k^2 + e_0^2 A(0)$, which would shift the pole away from zero and act like a photon mass term. But a mass term $\frac{1}{2}m_\gamma^2 A_\mu A^\mu$ is not gauge invariant. Thus gauge invariance protects the photon mass: no photon mass term is generated by quantum corrections.

Therefore $A(k^2)$ must vanish at least linearly as $k^2 \to 0$, and we may write

$$A(k^2) = k^2 A'(0) + k^2\Pi_c(k^2), \tag{6.96}$$

where

$$A'(0) \equiv A'(k^2 = 0) = \left.\frac{dA(k^2)}{d(k^2)}\right|_{k^2=0}, \tag{6.97}$$

and $\Pi_c(k^2)$ collects the terms beyond the linear one. In particular, $\Pi_c(0) = 0$, so that $k^2\Pi_c(k^2)$ is of higher order in k^2 near the origin.

Charge renormalization extracted from the modified propagator

In scattering amplitudes, an internal photon propagator comes with two QED vertices, hence with a factor e_0^2. For this reason it is convenient to consider the combination $e_0^2 D_F$, rather than the propagator by itself, and apply (6.93) to that combination. This is the object that actually appears in the lowest-order exchange amplitude, so the constant part of the vacuum-polarization correction can be absorbed into the definition of the renormalized electric charge. Multiplying (6.93) by e_0^2 and substituting (6.96) gives

$$\begin{aligned}
\frac{-ig_{\alpha\beta}}{k^2+i\epsilon}e_0^2 &\longrightarrow \frac{-ig_{\alpha\beta}}{k^2+i\epsilon}e_0^2\left[1 - \frac{e_0^2 A(k^2)}{k^2+i\epsilon}\right] \\
&= \frac{-ig_{\alpha\beta}}{k^2+i\epsilon}e_0^2\left[1 - \frac{e_0^2\big(k^2A'(0) + k^2\Pi_c(k^2)\big)}{k^2+i\epsilon}\right] \\
&= \frac{-ig_{\alpha\beta}}{k^2+i\epsilon}e_0^2\Big[1 - e_0^2 A'(0)\Big] + \frac{ig_{\alpha\beta}}{(k^2+i\epsilon)^2}e_0^4 k^2\Pi_c(k^2).
\end{aligned} \tag{6.98}$$

The first term on the right-hand side is the original propagator multiplied by a constant factor. It is therefore natural to *define* the renormalized charge e by

$$e^2 \equiv e_0^2\Big[1 - e_0^2 A'(0)\Big], \tag{6.99}$$

so that all lowest-order amplitudes expressed in terms of e automatically include the constant part of the photon self-energy correction.

Equivalently, introduce the renormalization constant Z_3 by

$$e^2 \equiv Z_3\, e_0^2, \qquad \text{or equivalently} \qquad e \equiv Z_3^{1/2} e_0. \tag{6.100}$$

This is the four-dimensional form used in this subsection; in dimensional regularization one writes equivalently $\mu^{\epsilon/2} e = Z_3^{1/2} e_0$ once $Z_1 = Z_2$ is imposed. Comparing with (6.99), we read off

$$Z_3 = 1 - e_0^2 A'(0) + O(e_0^4). \tag{6.101}$$

Expanding the square root then gives, to this order,

$$e = e_0 \sqrt{1 - e_0^2 A'(0)} = e_0 \left[1 - \frac{1}{2} e_0^2 A'(0) + O(e_0^4) \right]. \tag{6.102}$$

This introduces no new physical content: it is simply the perturbative form of the definition of the renormalized charge. Its practical role is to show that, at one-loop order, bare and renormalized couplings differ only beyond the order to which we are working, so e_0 can be replaced by e consistently in higher-order terms.

We now want to express the result entirely in terms of the *renormalized* charge e rather than the bare charge e_0. From (6.99), $e_0^2 = e^2 + O(e^4)$, i.e. e_0 and e differ only at order e^2.

The first term of (6.98) is $e_0^2[1 - e_0^2 A'(0)]/(k^2 + i\epsilon)$, which is $e^2/(k^2 + i\epsilon)$ by definition (6.99)—here the replacement is exact.

The second term is already of order e_0^4 (it carries an explicit factor e_0^4). Replacing $e_0^4 \to e^4$ introduces an error of order $e_0^4 \times O(e^2) = O(e^6)$, which is beyond the accuracy of our one-loop calculation (order e^4). Therefore, to order e^4, we may freely write $e_0^4 \to e^4$ in the second term.

The final result for the photon propagator (times the two charges) becomes

$$\frac{-i g_{\alpha\beta}}{k^2 + i\epsilon}\, e_0^2 \;\longrightarrow\; \frac{-i g_{\alpha\beta}}{k^2 + i\epsilon}\, e^2 + \frac{i g_{\alpha\beta}}{(k^2 + i\epsilon)^2}\, e^4\, k^2\, \Pi_c(k^2) + O(e^6). \tag{6.103}$$

Interpretation of the result.
Equation (6.103) separates the photon self-energy effect into:

- a *constant* renormalization of the charge, encoded in e^2 via (6.99) (this part contains the UV divergence through $A'(0)$);
- a momentum-dependent remainder proportional to $\Pi_c(k^2)$; after the renormalization implied above, this is the non-constant part of the vacuum polarization that survives in physical radiative corrections.

The explicit evaluation of $\Pi_c(k^2)$ (and of the divergent constant $A'(0)$ in dimensional regularization) is carried out in detail in Section 7.1.

6.3 Electron propagator renormalization in QED

Physical interpretation: mass and wavefunction renormalization

The electron self-energy $\Sigma(\not{p})$ encodes how virtual photon emission and reabsorption modify the electron's propagation through the vacuum. Unlike the photon self-energy (which affects charge renormalization), the electron self-energy leads to two distinct physical effects:

- **Mass renormalization:** The "bare" mass m_0 in the Lagrangian is not the physical mass we measure. Quantum corrections shift the pole of the propagator from $\not{p} = m_0$ to $\not{p} = m$, where m is the *physical* (renormalized) mass. The mass counterterm δm (the coefficient of $-\bar{\psi}\psi$ in the counterterm Lagrangian) absorbs the infinite part of this shift.
- **Wavefunction renormalization:** The residue of the propagator pole is modified by quantum corrections. The factor Z_2 (with $Z_2 - 1$ containing a UV divergence) relates the bare field ψ_0 to the renormalized field $\psi = Z_2^{-1/2}\psi_0$. This ensures that the renormalized propagator has unit residue at the physical mass pole.

The on-shell renormalization scheme, commonly used in QED, imposes:

1. The propagator pole is at $\not{p} = m$ (physical mass).
2. The residue at the pole is i (canonical normalization).

These conditions uniquely determine δm and Z_2 in terms of the self-energy function.

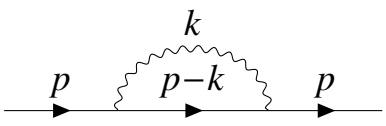
Diagrammatic overview and the need for renormalization

Free propagator. The starting point is the tree-level (free) electron propagator, represented diagrammatically as a single directed line:

$$p \xrightarrow{\quad S_0 \quad} \quad = \quad S_0(\not{p}) = \frac{i}{\not{p} - m_0 + i\epsilon}$$

where m_0 is the bare mass parameter in the Lagrangian.

Self-energy insertion. Interactions modify propagation: even in the vacuum, the electron can emit and reabsorb virtual photons. The one-loop 1PI *self-energy* correction $-i\Sigma(\not{p})$ is given by:



$$= \quad -i\Sigma(\not{p})$$

Geometric series (Dyson resummation). Inserting one self-energy correction on the fermion line, then two, then three, etc., produces a geometric series:

Algebraically, this reads:

$$S(\not p) = S_0 + S_0\big(-i\Sigma\big)S_0 + S_0\big(-i\Sigma\big)S_0\big(-i\Sigma\big)S_0 + \cdots = \frac{i}{\not p - m_0 - \Sigma(\not p) + i\epsilon}. \quad (6.104)$$

Strictly speaking, $\Sigma(\not p)$ here denotes the *one-particle irreducible* (1PI) self-energy, i.e. the sum of all self-energy diagrams that cannot be disconnected by cutting a single internal fermion line. Using the 1PI self-energy avoids double counting: a reducible insertion (two 1PI blobs joined by a bare propagator) is already generated by the geometric series itself.

Why renormalization is needed. The loop integral defining $\Sigma(\not p)$ is ultraviolet divergent. At one loop:

$$\Sigma(\not p) \sim e_0^2 \int \frac{d^4k}{(2\pi)^4} \, \frac{\gamma^\mu(\not p - \not k + m_0)\gamma_\mu}{\big(k^2 + i\epsilon\big)\big((p-k)^2 - m_0^2 + i\epsilon\big)}. \quad (6.105)$$

Power counting: the Dirac numerator $\gamma^\mu(\not p - \not k + m)\gamma_\mu$ scales as k for large k, while the two propagators supply $1/k^4$. Including the measure $d^4k \sim k^3\,dk$, the integral scales as $k^3\,dk \cdot k/k^4 = dk$, giving a superficial degree of divergence $D = 1$ (linearly divergent). In dimensional regularization the linear power divergence is absent—all UV divergences manifest as $1/\epsilon$ poles—but the integral is still divergent and requires renormalization.

Lorentz decomposition. By Lorentz covariance, $\Sigma(\not p)$ is a 4×4 matrix in Dirac space that can only depend on $\not p$ and the identity matrix $\mathbb{1}$. The most general form is therefore

$$\Sigma(\not p) = \not p\,\Sigma_V(p^2) + m_0\,\Sigma_S(p^2), \quad (6.106)$$

where Σ_V and Σ_S are Lorentz-scalar functions of p^2. The coefficient of $\mathbb{1}$ is proportional to m_0 because in the chiral limit $m_0 \to 0$ the Lagrangian has a chiral symmetry that forces Σ to be purely proportional to $\not p$. In dimensional regularization ($d = 4 - \epsilon$), each scalar function contains an ultraviolet pole:

$$\Sigma_V(p^2) = \frac{c_V}{\epsilon} + \Sigma_V^{\text{fin}}(p^2), \qquad \Sigma_S(p^2) = \frac{c_S}{\epsilon} + \Sigma_S^{\text{fin}}(p^2), \quad (6.107)$$

where c_V, c_S are constants (independent of p^2) and the superscript "fin" denotes the finite remainder.

Mass renormalization and wave-function renormalization (Mandl–Shaw approach). We now show that the divergences in $\Sigma(\not p)$ can be absorbed into a redefinition of the mass and an overall rescaling of the field, without introducing counterterms explicitly.

Start from the Dyson-resummed propagator (6.104):

$$S(\not p) = \frac{i}{\not p - m_0 - \Sigma(\not p) + i\epsilon}. \quad (6.108)$$

The key idea is to expand $\Sigma(\not{p})$ around the point $\not{p} = m$, where m is the *physical* (observed) mass, still to be determined:

$$\Sigma(\not{p}) = \underbrace{\Sigma(m)}_{\equiv\ A} + \underbrace{\left.\frac{\partial\Sigma}{\partial\not{p}}\right|_{\not{p}=m}}_{\equiv\ B}(\not{p}-m) + \Sigma_c(\not{p}), \tag{6.109}$$

where $\Sigma_c(\not{p})$ collects all higher-order terms in $(\not{p}-m)$ and satisfies

$$\Sigma_c(m) = 0, \qquad \left.\frac{\partial\Sigma_c}{\partial\not{p}}\right|_{\not{p}=m} = 0. \tag{6.110}$$

The constants A and B are divergent (they contain $1/\epsilon$ poles). In contrast, $\Sigma_c(\not{p})$ is *finite*: the superficial degree of divergence of the self-energy is $D = 1$, so its UV-divergent part is a polynomial of degree ≤ 1 in $\not{p}$. The subtraction of A (degree 0) and $B(\not{p}-m)$ (degree 1) therefore removes all divergent pieces, leaving a finite remainder.

Substituting (6.109) into (6.108):

$$\begin{aligned} S(\not{p}) &= \frac{i}{\not{p} - m_0 - A - B(\not{p}-m) - \Sigma_c(\not{p})} \\ &= \frac{i}{(1-B)(\not{p}-m) - (m_0 + A - m) - \Sigma_c(\not{p})}. \end{aligned} \tag{6.111}$$

Step 1: fix the physical mass. We *define* m by requiring that the propagator has a pole at $\not{p} = m$. Since $\Sigma_c(m) = 0$, the denominator vanishes at $\not{p} = m$ if and only if

$$\boxed{m = m_0 + A = m_0 + \Sigma(m).} \tag{6.112}$$

This determines the **mass shift**: the bare mass m_0 differs from the physical mass m by the self-energy evaluated on shell. Defining $\delta m \equiv m - m_0 = \Sigma(m)$, we see that δm is UV-divergent.

Step 2: fix the residue. With condition (6.112) satisfied, the propagator becomes

$$S(\not{p}) = \frac{i}{(1-B)\big(\not{p}-m\big) - \Sigma_c(\not{p})}. \tag{6.113}$$

Factoring out $(1-B)$ and defining the **wave-function renormalization constant**

$$\boxed{Z_2 \equiv \frac{1}{1-B} = \frac{1}{1-\Sigma'(m)},} \tag{6.114}$$

where $\Sigma'(m) \equiv \partial\Sigma/\partial\not{p}\big|_{\not{p}=m}$, we obtain

$$S(\not{p}) = \frac{iZ_2}{\not{p} - m - Z_2\,\Sigma_c(\not{p})}. \tag{6.115}$$

Since Z_2 is a momentum-independent constant and $\Sigma_c(m) = 0$ by construction, $Z_2\,\Sigma_c(\not{p}) \to Z_2 \cdot 0 = 0$ as $\not{p} \to m$, and the propagator (6.115) has a pole at $\not{p} = m$ with residue Z_2:

$$S(\not{p}) \xrightarrow{\not{p}\to m} \frac{iZ_2}{\not{p} - m}\,. \tag{6.116}$$

To restore canonical normalization (unit residue), we define the **renormalized field** $\psi = Z_2^{-1/2}\psi_0$, whose propagator is

$$\begin{aligned} S_R(\not{p}) \equiv Z_2^{-1}\,S(\not{p}) &= \frac{i}{\not{p} - m - Z_2\,\Sigma_c(\not{p})} \\ &\xrightarrow{\not{p}\to m} \frac{i}{\not{p} - m}\,. \end{aligned} \tag{6.117}$$

At the pole, the divergent Z_2 in the numerator of S is cancelled by the Z_2^{-1} prefactor, giving a residue $Z_2/Z_2 = 1$. Away from the pole, the product $Z_2\,\Sigma_c(\not{p})$ appears divergent because Z_2 has $1/\epsilon$ poles while $\Sigma_c \neq 0$. However, $Z_2 = 1 + O(\alpha)$ and $\Sigma_c = O(\alpha)$, so $Z_2\,\Sigma_c = \Sigma_c + O(\alpha^2)$: at one-loop accuracy the product reduces to Σ_c alone, which is finite. The $O(\alpha^2)$ divergent remainder is absorbed by two-loop counterterms, and so on order by order.

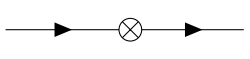
Remark (Källén–Lehmann bound). Independently of perturbation theory, the spectral representation of the exact propagator (see Section 8.6) implies $0 \le Z_2 \le 1$: the exact Z_2 is finite and measures the overlap of ψ_0 with the one-particle state. The $1/\epsilon$ poles in the perturbative expansion of Z_2 are artifacts of computing the exact (finite) quantity order by order; they are systematically removed by renormalization.

To summarize: the two renormalization conditions—pole at $\not{p} = m$ and unit residue—uniquely determine

$$\delta m = \Sigma(m)\,, \qquad Z_2 = \frac{1}{1 - \Sigma'(m)}\,.$$

In perturbation theory both quantities contain $1/\epsilon$ poles; these are absorbed by the counterterms introduced below.

Counterterm reformulation. The same result can be organized systematically via counterterms. Splitting the bare Lagrangian as $\mathcal{L}_0 = \mathcal{L}_{\text{ren}} + \delta\mathcal{L}$, the fermion counterterm $\delta\mathcal{L}_{\bar\psi\psi} = (Z_2 - 1)\,\bar\psi\, i\partial\!\!\!/\,\psi - \delta m\,\bar\psi\psi$ produces a two-point vertex (denoted by $\times$):



$$= i\Sigma_{\text{ct}}(\not{p}) = i\Big[(Z_2 - 1)\not{p} - \delta m\Big]$$

The values of $(Z_2 - 1)$ and δm are precisely those determined above from the pole structure. This vertex is added to each order of perturbation theory so that the renormalized self-energy $\Sigma_R = \Sigma + \Sigma_{\text{ct}}$ is finite.

Renormalized propagator.

The central question is: *why does the counterterm vertex end up inside the denominator of the propagator, next to the loop self-energy?* The answer is Dyson resummation, which we now spell out step by step.

(i) The counterterm is a 1PI two-point insertion. A 1PI (one-particle-irreducible) two-point diagram is an amputated graph with exactly two external fermion legs that cannot be split into two disconnected pieces by cutting a single internal line. Both contributions at order α satisfy this:

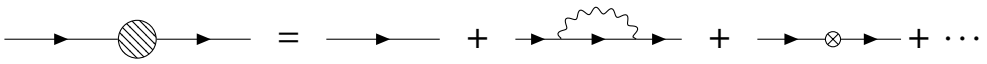
- the loop self-energy (two amputated legs, one photon bridge);
- the counterterm vertex (two amputated legs, a single insertion).

Summing all 1PI contributions defines the total **1PI two-point function**

$$-i\Sigma_R(\not p) \;=\; \underbrace{-i\Sigma(\not p)}_{\text{loop}} \;+\; \underbrace{+i\Sigma_{\rm ct}(\not p)}_{\text{counterterm}} \quad\Longrightarrow\quad \Sigma_R(\not p) \;\equiv\; \Sigma(\not p)+\Sigma_{\rm ct}(\not p). \tag{6.118}$$

The *opposite* sign of the loop's $-i\Sigma$ and the counterterm's $+i\Sigma_{\rm ct}$ is precisely what allows the counterterm to subtract the UV pole of Σ.

(ii) Dyson resummation puts 1PI insertions in the denominator. The full propagator is obtained by stringing together an arbitrary number of 1PI insertions on the bare propagator line:



Algebraically:

$$S_R \;=\; S_0 \;+\; S_0\,(-i\Sigma_R)\,S_0 \;+\; S_0\,(-i\Sigma_R)\,S_0\,(-i\Sigma_R)\,S_0 \;+\; \cdots \tag{6.119}$$

which is a geometric series. Using $S_0 = i/(\not p - m + i\epsilon)$ and summing,

$$\boxed{S_R(\not p) \;=\; \frac{i}{\not p - m - \Sigma_R(\not p) + i\epsilon}\,, \qquad \Sigma_R(\not p) \equiv \Sigma(\not p)+\Sigma_{\rm ct}(\not p)} \tag{6.120}$$

This is the reason the counterterm *appears in the denominator*: it is a 1PI two-point insertion, and Dyson resummation stacks every such insertion into the inverse propagator $S_R^{-1} = -i\,(\not p - m - \Sigma_R)$.

(iii) Explicit cancellation of the UV divergence. We now check that Σ_R is indeed finite. The one-loop self-energy admits the Dirac decomposition

$$\Sigma(\not p) = \not p\,\Sigma_V(p^2) + m\,\Sigma_S(p^2),$$

and in dimensional regularization each of Σ_V, Σ_S has the structure

$$\Sigma_V(p^2) = \Sigma_V^{\rm div} + \Sigma_V^{\rm fin}(p^2), \qquad \Sigma_S(p^2) = \Sigma_S^{\rm div} + \Sigma_S^{\rm fin}(p^2), \tag{6.121}$$

where $\Sigma_{V,S}^{\text{div}} \propto 1/\epsilon$ are p-independent constants. (They must be constants because the superficial degree of divergence of the fermion self-energy is $D = 1$: the UV-divergent part is a polynomial in $\not{p}$ of degree ≤ 1, so $\not{p}\Sigma_V$ and $m\Sigma_S$ capture all of it.)

The counterterm is, by construction, itself a polynomial of degree 1 in $\not{p}$:

$$\Sigma_{\text{ct}}(\not{p}) = (Z_2 - 1)\,\not{p} - \delta m,$$

with $(Z_2 - 1)$ and δm two *free constants* to be chosen. Adding, the two 1PI contributions combine as

$$\Sigma_R(\not{p}) \;=\; \big[\Sigma_V(p^2) + (Z_2 - 1)\big]\,\not{p} \;+\; \big[m\,\Sigma_S(p^2) - \delta m\big]. \tag{6.122}$$

Matching (6.122) against the divergent pieces of (6.121) shows that the $1/\epsilon$ poles cancel *separately* in the $\not{p}$-coefficient and in the scalar piece iff

$$\boxed{(Z_2 - 1)\big|_{\text{div}} = -\Sigma_V^{\text{div}}\,, \qquad \delta m\big|_{\text{div}} = m\,\Sigma_S^{\text{div}}} \tag{6.123}$$

With these choices,

$$\Sigma_R(\not{p}) = \big[\Sigma_V^{\text{fin}}(p^2) + \Delta Z_2\big]\not{p} + \big[m\,\Sigma_S^{\text{fin}}(p^2) - \Delta m\big] \qquad \text{(finite)},$$

where ΔZ_2 and Δm denote the finite remainders of $(Z_2 - 1)$ and δm; these finite pieces are fixed by the chosen scheme (on-shell, $\overline{\text{MS}}$, ...). The full denominator of (6.120) therefore becomes

$$\not{p} - m - \Sigma_R(\not{p}) \;=\; (\not{p} - m) - \Sigma(\not{p}) - (Z_2 - 1)\not{p} + \delta m,$$

in which every $1/\epsilon$ pole on the right-hand side has been explicitly absorbed, term by term, by the counterterm coefficients (6.123).

One-loop calculation (counterterm method)

Counterterm Feynman rule. We derive the fermion counterterm from the bare Lagrangian. The free fermion part reads

$$\mathcal{L}_0^{\text{free}} = \bar{\psi}_0\big(i\partial\!\!\!/ - m_0\big)\psi_0\,.$$

We now express everything in terms of renormalized quantities. Introduce the renormalized field ψ and the physical mass m via

$$\psi_0 = \sqrt{Z_2}\,\psi\,, \qquad m_0 = m + \delta m_0\,,$$

where Z_2 is the wave-function renormalization constant and δm_0 is the bare mass shift. Substituting:

$$\mathcal{L}_0^{\text{free}} = Z_2\,\bar{\psi}\big(i\partial\!\!\!/ - m - \delta m_0\big)\psi\,.$$

We split this into $\mathcal{L}_{\text{ren}} + \delta\mathcal{L}$ by writing $Z_2 = 1 + (Z_2 - 1)$ and expanding:

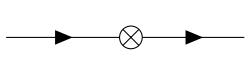
$$\mathcal{L}_0^{\text{free}} = \bar\psi\, i\partial\!\!\!/\,\psi - m\bar\psi\psi + (Z_2-1)\,\bar\psi\, i\partial\!\!\!/\,\psi - \underbrace{\big[\,m(Z_2-1) + Z_2\,\delta m_0\,\big]}_{\equiv\ \delta m}\bar\psi\psi\,. \tag{6.124}$$

The first two terms are the renormalized Lagrangian $\mathcal{L}_{\text{ren}} = \bar\psi(i\partial\!\!\!/ - m)\psi$; the remaining terms form the counterterm

$$\delta\mathcal{L}_{\bar\psi\psi} = (Z_2-1)\,\bar\psi\, i\partial\!\!\!/\,\psi - \delta m\,\bar\psi\psi\,, \tag{6.125}$$

with $\delta m \equiv m(Z_2-1) + Z_2\,\delta m_0 = Z_2 m_0 - m$. Note that δm combines the mass shift and the wave-function renormalization into a single coefficient.

In momentum space, $i\partial\!\!\!/ \to p\!\!\!/$, so the counterterm vertex gives the Feynman rule



and the renormalized self-energy becomes

$$\Sigma_R(p\!\!\!/) = \Sigma(p\!\!\!/) + (Z_2-1)p\!\!\!/ - \delta m\,. \tag{6.126}$$

One-loop self-energy integral (Feynman gauge). In dimensional regularization $d = 4-\epsilon$ and Feynman gauge $\xi = 1$:

$$D_{\mu\nu}(k) = \frac{-ig_{\mu\nu}}{k^2 + i\epsilon}, \qquad S_0(p) = \frac{i}{p\!\!\!/ - m + i\epsilon}\,. \tag{6.127}$$

Which m and e enter the loop? In the counterterm method the bare Lagrangian has been split as $\mathcal{L}_0 = \mathcal{L}_{\text{ren}} + \delta\mathcal{L}$; propagators and vertices of $\mathcal{L}_{\text{ren}}$ are written in terms of the **renormalized** quantities m and e (the physical mass and coupling). The bare $m_0 = m + \delta m_0$ and $e_0 = \mu^{\epsilon/2} Z_1 (Z_2\sqrt{Z_3})^{-1} e$ do *not* appear in (6.127): their deviations from m, e are absorbed into the counterterm coefficients (Z_2-1), δm, (Z_1-1), (Z_3-1), treated as separate perturbative insertions. *Throughout this subsection m and e always mean the renormalized quantities.*

The one-loop self-energy is

$$\Sigma(p\!\!\!/) = -e^2\mu^\epsilon \int \frac{d^dk}{(2\pi)^d}\, \frac{\gamma^\mu(p\!\!\!/ - k\!\!\!/ + m)\gamma_\mu}{\big((p-k)^2 - m^2 + i\epsilon\big)\,(k^2 + i\epsilon)}. \tag{6.128}$$

The factor μ^ϵ keeps the renormalized coupling e dimensionless in $d = 4-\epsilon$ dimensions.

Dirac algebra and Feynman parametrization. Using the d-dimensional identities $\gamma^\mu\gamma^\alpha\gamma_\mu = (2-d)\gamma^\alpha$ and $\gamma^\mu\gamma_\mu = d$:

$$\gamma^\mu(p\!\!\!/ - k\!\!\!/ + m)\gamma_\mu = (2-d)(p\!\!\!/ - k\!\!\!/) + dm\,. \tag{6.129}$$

Combining denominators with Feynman parameter x and shifting $\ell = k - (1-x)p$:

$$\Sigma(\not p) = -e^2\mu^\epsilon \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \frac{(2-d)(x\not p - \not\ell) + dm}{(\ell^2 - \Delta + i\epsilon)^2}, \tag{6.130}$$

where $\Delta = (1-x)m^2 - x(1-x)p^2$. The $\not\ell$ term vanishes by symmetry.

Scalar master integral — origin of $\Gamma(\epsilon/2)$.

After dropping $\not\ell$, the only loop integral left in (6.130) is the *standard d-dimensional scalar bubble* with a double pole at $\ell^2 = \Delta$. Wick rotating $\ell^0 \to i\ell_E^0$ (so that $d^d\ell \to i\, d^d\ell_E$ and $\ell^2 - \Delta \to -(\ell_E^2 + \Delta)$) reduces it to an ordinary Euclidean integral:

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{1}{(\ell^2 - \Delta + i\epsilon)^2} = \frac{i}{(4\pi)^{d/2}} \Gamma\!\left(2 - \frac{d}{2}\right) \Delta^{d/2-2}. \tag{6.131}$$

The Euclidean integral has been done using the well-known spherical formula $\int d^d\ell_E\, (\ell_E^2 + \Delta)^{-n} = (4\pi)^{d/2} \Delta^{d/2-n} \Gamma(n - d/2)/\Gamma(n)$. With $d = 4 - \epsilon$, $2 - d/2 = \epsilon/2$, so the master integral acquires the factor $\Gamma(\epsilon/2)$, whose Laurent expansion contains a simple pole:

$$\Gamma\!\left(\frac{\epsilon}{2}\right) = \frac{2}{\epsilon} - \gamma_E + O(\epsilon). \tag{6.132}$$

This is where $\Gamma(\epsilon/2)$ comes from: the logarithmic UV divergence that a cutoff regulator would present as $\ln(\Lambda^2/\Delta)$ is replaced, in dimensional regularization, by $\Gamma(\epsilon/2)\,\Delta^{-\epsilon/2} \sim 2/\epsilon + \ldots$.

Explicit form of $\Sigma(\not p)$ and its $1/\epsilon$ pole. Substituting (6.131) into (6.130) (and using $\not\ell \to 0$):

$$\Sigma(\not p) = -\frac{i\, e^2 \mu^\epsilon}{(4\pi)^{d/2}} \Gamma\!\left(\frac{\epsilon}{2}\right) \int_0^1 dx \left[(2-d)\, x\, \not p + d\, m\right] \Delta^{-\epsilon/2}. \tag{6.133}$$

Expanding in ϵ ($2 - d = -2 + \epsilon$, $d = 4 - \epsilon$, $(4\pi)^{-d/2} = (4\pi)^{-2} + O(\epsilon)$, $\Delta^{-\epsilon/2} = 1 + O(\epsilon)$), the leading $1/\epsilon$ pole is

$$\Sigma^{\rm div}(\not p) = -\frac{i\, e^2}{(4\pi)^2} \frac{2}{\epsilon} \int_0^1 dx \left[-2x\not p + 4m\right] = -\frac{i\, e^2}{16\pi^2} \frac{2}{\epsilon} \left[-\not p + 4m\right], \tag{6.134}$$

where we used $\int_0^1 x\, dx = \tfrac{1}{2}$, $\int_0^1 dx = 1$. The crucial structural point is: *the divergent part is a polynomial of degree 1 in $\not p$*, with p-independent coefficients, exactly matching the polynomial form of the counterterm $(Z_2 - 1)\not p - \delta m$. Writing $\Sigma^{\rm div}$ in the Dirac decomposition $\Sigma^{\rm div}(\not p) = \Sigma_V^{\rm div}\, \not p + m\, \Sigma_S^{\rm div}$, eq. (6.134) identifies

$$\Sigma_V^{\rm div} = \frac{i\, e^2}{16\pi^2} \frac{2}{\epsilon}, \qquad m\, \Sigma_S^{\rm div} = -\frac{i\, e^2}{16\pi^2} \frac{2}{\epsilon} (4m) \implies \Sigma_S^{\rm div} = -\frac{4\, i\, e^2}{16\pi^2} \frac{2}{\epsilon}.$$

Cancellation of the pole in (6.122) is the polynomial equality

$$(Z_2-1)\big|_{\text{div}}\not{p}-\delta m\big|_{\text{div}} \;=\; -\Sigma^{\text{div}}(\not{p}), \tag{6.135}$$

which is *one equation between two polynomials of degree 1 in* $\not{p}$. Two such polynomials are equal iff they agree coefficient by coefficient, so (6.135) splits into *two independent scalar equations*:

$$\underbrace{(Z_2-1)\big|_{\text{div}} = -\Sigma_V^{\text{div}}}_{\text{matching the } \not{p} \text{ coefficient}}, \qquad \underbrace{\delta m\big|_{\text{div}} = m\,\Sigma_S^{\text{div}}}_{\text{matching the } \not{p}\text{-independent part}} \quad .$$

Each determines exactly one counterterm constant, as expected.

Dirac decomposition. The one-loop self-energy has the structure

$$\Sigma(\not{p}) = \not{p}\,\Sigma_V(p^2) + m\,\Sigma_S(p^2), \tag{6.136}$$

so the renormalized self-energy becomes

$$\Sigma_R(\not{p}) = \not{p}\Big[\Sigma_V(p^2) + (Z_2-1)\Big] + m\,\Sigma_S(p^2) - \delta m\,. \tag{6.137}$$

Thus (Z_2-1) adjusts the $\not{p}$ coefficient, while δm adjusts the scalar part.

On-shell renormalization conditions

Pole condition. On-shell renormalization imposes a pole at the physical mass:

$$S_R^{-1}(\not{p})\big|_{\not{p}=m} = 0 \quad \Rightarrow \quad \delta m = m\Big[\Sigma_V(m^2) + \Sigma_S(m^2)\Big] + m(Z_2-1). \tag{6.138}$$

Residue condition. Unit residue at the pole requires:

$$\frac{\partial}{\partial \not{p}} S_R^{-1}(\not{p})\bigg|_{\not{p}=m} = 1 \quad \Rightarrow \quad Z_2 - 1 = -\,\frac{\partial}{\partial \not{p}}\Sigma(\not{p})\bigg|_{\not{p}=m}. \tag{6.139}$$

Cancellation of divergences. Combining the inverse propagator:

$$S_R^{-1}(\not{p}) = \not{p} - m - \Sigma(\not{p}) - (Z_2-1)\not{p} + \delta m, \tag{6.140}$$

the divergent parts of $\Sigma(\not{p})$ are cancelled by those of $(Z_2-1)\not{p}-\delta m$. The finite pieces are fixed by the chosen scheme (on-shell as above, or $\overline{\text{MS}}$ where one subtracts only the pole).

6.4 One-loop vertex correction in QED and its renormalization

Physical motivation: what is the vertex correction?
The QED vertex at tree level. In classical (tree-level) QED, the interaction between an electron and a photon is described by a single vertex where an electron absorbs or emits a photon. This interaction is encoded in the Lagrangian term (see Chapter 5 for the sign conventions)

$$\mathcal{L}_{\text{int}} = +e\,\bar{\psi}\gamma^{\mu}\psi\,A_{\mu}, \tag{6.141}$$

which gives the Feynman rule $+ie\gamma^{\mu}$ for the electron-photon vertex. At tree level, this coupling is simply the electric charge multiplying the tree vertex. In renormalized perturbation theory this is written in terms of the renormalized coupling e (with the usual $\mu^{\epsilon/2}$ factor in $d = 4-\epsilon$), while the difference from the bare charge e_0 is carried by counterterm constants.
Quantum corrections modify the vertex. However, quantum field theory tells us that the vacuum is not empty: virtual particle-antiparticle pairs constantly fluctuate in and out of existence. When an electron interacts with a photon, these quantum fluctuations modify the interaction. In particular, the electron can:

- emit a virtual photon just before interacting with the external photon,
- reabsorb the virtual photon just after,
- creating a "loop" of virtual particles around the vertex.

This process is called the **vertex correction**: quantum loops "dress" the bare vertex and effectively modify the strength and structure of the electron-photon coupling.
Physical consequences. The vertex correction has profound physical implications:

1. **Anomalous magnetic moment:** The electron's magnetic moment deviates from the Dirac prediction $g = 2$. The leading correction, computed by Schwinger in 1948, gives $g-2 = \alpha/\pi \approx 0.00116$, in spectacular agreement with experiment.
2. **Charge renormalization:** Together with vacuum polarization, the vertex correction contributes to the running of the electric charge with energy scale.
3. **Form factors:** The corrected vertex introduces momentum-dependent "form factors" that describe how the electron's charge and magnetic moment are distributed.

Technical aspects. The vertex correction is the most subtle of the one-loop structures because:

(i) it is a genuine three-point function with nontrivial kinematics (two independent momenta p and p');

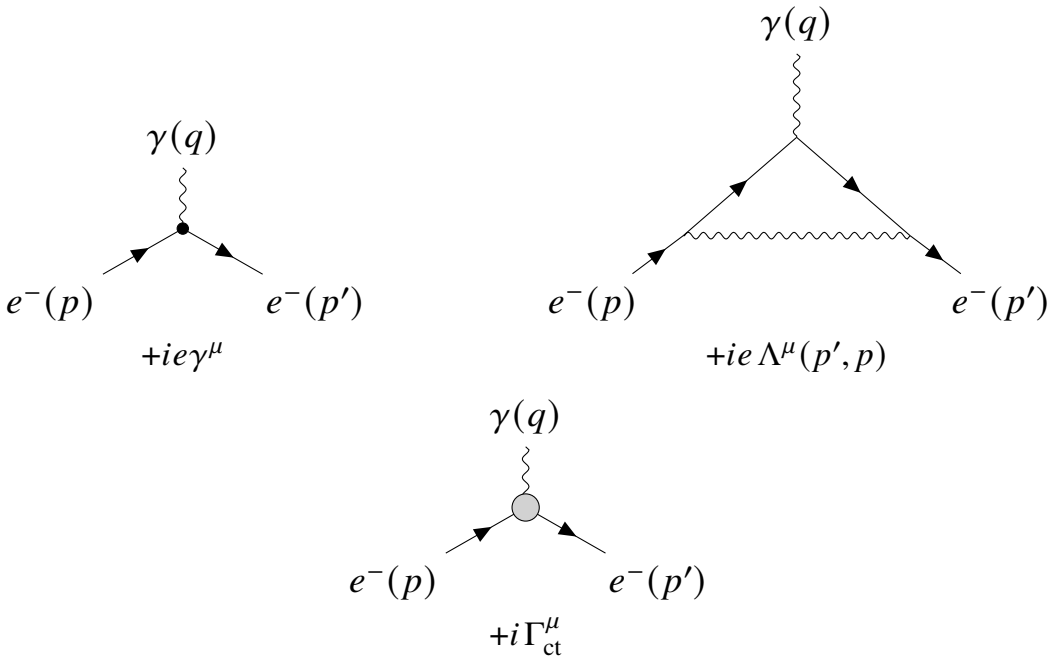
(ii) it is constrained by gauge invariance through the Ward–Takahashi identity, which relates the vertex to the electron self-energy;
(iii) for on-shell external fermions it contains infrared (IR) singularities (separate from UV divergences), which cancel only when including soft photon emission (Bloch–Nordsieck theorem).

Here we focus on the UV renormalization via counterterms; IR issues are noted but not fully treated.

Diagrams and definitions

Diagrammatically, the proper vertex is the sum of the tree vertex and the one-loop triangle, plus the counterterm vertex. The renormalized vertex function is:

$$\Gamma^{\mu}_{R}(p',p) = \gamma^{\mu} + \Lambda^{\mu}(p',p) + \Gamma^{\mu}_{\mathrm{ct}}. \tag{6.142}$$



The third diagram is the *counterterm vertex*: it represents a local insertion $\Gamma^{\mu}_{\mathrm{ct}}$, proportional to γ^{μ}, whose coefficient is an as-yet-undetermined constant. Its value will be fixed by a renormalization condition (see below, p. 291). The grey blob distinguishes it from the ordinary QED vertex.

We use the kinematic conventions

$$q \equiv p' - p, \qquad p^2 = p'^2 = m^2 \ \text{(on-shell, when stated explicitly)}. \tag{6.143}$$

The proper vertex $\Gamma^{\mu}(p',p)$ is the object that enters physical amplitudes (e.g. electron scattering in an external field).

Vertex renormalization (Mandl–Shaw approach)

Just as we derived δm and Z_2 from the pole structure of the propagator, we now determine the vertex renormalization constant Z_1 from the exact vertex at zero

momentum transfer. The key tool is the Ward–Takahashi identity, which we derive first.

Derivation of the Ward–Takahashi identity. Start from the elementary algebraic identity: for a free fermion propagator $S_0(k) = i/(\not{k} - m)$,

$$\not{q} = (\not{k} + \not{q} - m) - (\not{k} - m) = S_0^{-1}(k+q) - S_0^{-1}(k)\,. \tag{6.144}$$

Multiplying on the left by $S_0(k+q)$ and on the right by $S_0(k)$:

$$S_0(k+q)\,\not{q}\,S_0(k) = S_0(k) - S_0(k+q)\,. \tag{6.145}$$

In words: inserting $\not{q}$ between two adjacent propagators is equivalent to removing one of them and subtracting the two resulting terms.

One-loop verification (Mandl–Shaw). We verify the identity explicitly at one loop in Feynman gauge.

(Convention: renormalized parameters.) Throughout this verification we use the **renormalized** mass m and coupling e, exactly as in the one-loop self-energy calculation (6.128). This is the natural choice in the counterterm method: the tree-level propagator $S_0(q) = i/(\not{q} - m + i\epsilon)$ and the tree-level vertex $+ie\gamma^\mu$ are read off from $\mathcal{L}_{\text{ren}}$, which is built from the *renormalized* parameters m and e (not the bare m_0 and e_0). The word "free" would be misleading here: S_0 is not the propagator of the original bare theory, but the zeroth-order propagator in the perturbative expansion around $\mathcal{L}_{\text{ren}}$. All deviations of m_0, e_0 from m, e are relegated to the counterterms $(Z_2 - 1), \delta m, (Z_1 - 1), (Z_3 - 1)$, treated as separate perturbative insertions. Since the Ward–Takahashi identity in the form $q_\mu \Gamma^\mu = S^{-1}(p') - S^{-1}(p)$ involves the mass only through the *difference* $(\not{p}' - m) - (\not{p} - m) = \not{q}$, the mass choice drops out algebraically and the same identity continues to hold when eventually rewritten in bare or renormalized form.

The full vertex at one loop is $\Gamma^\mu = \gamma^\mu + \Lambda^\mu$, where Λ^μ is given by (same renormalized coupling and μ^ϵ as in (6.128), Feynman gauge)

$$\Lambda^\mu(p', p) \;=\; -e^2\mu^\epsilon \int \frac{d^d k}{(2\pi)^d}\, \gamma^\alpha\, S_0(p'-k)\,\gamma^\mu\, S_0(p-k)\,\gamma_\alpha\, D_0(k)\,, \tag{6.146}$$

with $D_0(k) = -i/(k^2 + i\epsilon)$ the (scalar) free photon propagator in Feynman gauge and $S_0(q) = i/(\not{q} - m + i\epsilon)$; the overall $-e^2 = (-ie)^2$ comes from the two virtual-photon–fermion vertices, and the central γ^μ is the amputated external photon leg. Contracting with q_μ replaces $\gamma^\mu \to \not{q}$ in the integrand. Now apply identity (6.145) with $k \to p - k$ and $k + q \to p' - k$:

$$S_0(p'-k)\,\not{q}\,S_0(p-k) = S_0(p-k) - S_0(p'-k)\,. \tag{6.147}$$

Substituting back, with the *same* overall coefficient $-e^2\mu^\epsilon$:

$$\begin{aligned} q_\mu \Lambda^\mu(p',p) &= -e^2\mu^\epsilon \int \frac{d^dk}{(2\pi)^d}\, \gamma^\alpha \big[S_0(p-k) - S_0(p'-k)\big]\gamma_\alpha\, D_0(k) \\ &= \Sigma(\not{p}) - \Sigma(\not{p}'), \end{aligned} \tag{6.148}$$

where in the last step we recognized each of the two integrals as the one-loop self-energy (6.128), since $\Sigma(\not{q}) = -e^2\mu^\epsilon \int \frac{d^dk}{(2\pi)^d}\gamma^\alpha S_0(q-k)\gamma_\alpha D_0(k)$, evaluated at momentum p and p', respectively. Adding the tree-level contribution (using renormalized m, as above) $q_\mu\gamma^\mu = \not{q} = (\not{p}' - m) - (\not{p} - m)$:

$$\begin{aligned} q_\mu \Gamma^\mu &= \not{q} + \Sigma(\not{p}) - \Sigma(\not{p}') \\ &= \big(\not{p}' - m - \Sigma(\not{p}')\big) - \big(\not{p} - m - \Sigma(\not{p})\big) \\ &= S^{-1}(p') - S^{-1}(p), \end{aligned} \tag{6.149}$$

where $S^{-1}(p) = \not{p} - m - \Sigma(\not{p})$ is the inverse one-loop (pre-counterterm) propagator built from the renormalized $\mathcal{L}_{\text{ren}}$; the m's cancel algebraically as anticipated, so the result would be identical if written with m_0.

All-orders argument. The one-loop check already contains the essential local identity. To extend it to arbitrary order, fix *one* skeleton diagram obtained from a vertex graph by removing the external photon line. What remains is a fermion line, running from the incoming momentum p to the outgoing momentum p', dressed by an arbitrary set of virtual-photon insertions.

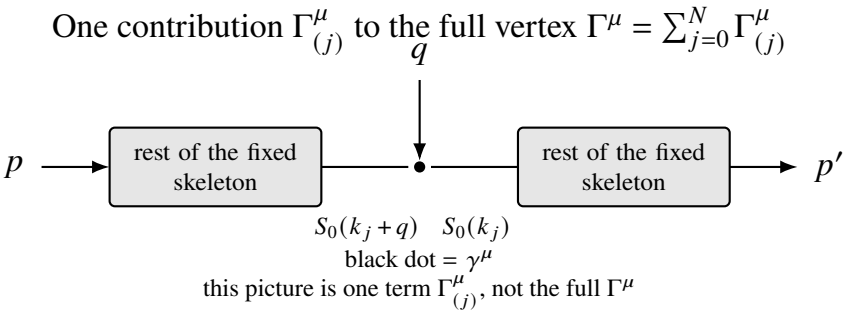
The crucial point is the following distinction:

- $\Gamma^\mu(p',p)$ is the *full proper vertex*: the sum of all amputated diagrams with two external fermions and one external photon;
- $\Gamma^\mu_{(j)}(p',p)$ is *one particular term* in that sum, namely the diagram obtained by inserting the external photon at the j-th possible position along the chosen fermion line;
- the black dot in the diagram below is only the *elementary* QED vertex γ^μ, not the whole Γ^μ.

If the external photon is attached at position j, the only local factor that depends on this insertion is

$$S_0(k_j + q)\,\gamma^\mu\, S_0(k_j),$$

while everything else in the skeleton is unchanged. A single contribution $\Gamma^\mu_{(j)}$ therefore has the schematic form shown below:



The full proper vertex is the sum over all possible insertion points:

$$\Gamma^{\mu}(p',p) = \sum_{j=0}^{N} \Gamma^{\mu}_{(j)}(p',p). \tag{6.150}$$

Now contract one fixed term with q_μ. The contraction acts only on the black-dot vertex and turns γ^μ into $\not{q}$. Therefore, inside $\Gamma^{\mu}_{(j)}$ one has the replacement

$$S_0(k_j+q)\,\gamma^{\mu}\,S_0(k_j) \xrightarrow{q_\mu} S_0(k_j+q)\,\not{q}\,S_0(k_j) = S_0(k_j) - S_0(k_j+q), \tag{6.151}$$

where we used the propagator identity (6.145).

This produces two *reduced* diagrams. We denote them by D_j and D_{j+1}: they are the same fixed skeleton, but cut at two neighboring positions along the fermion line. With this notation,

$$q_\mu \Gamma^{\mu}_{(j)} = D_j - D_{j+1}. \tag{6.152}$$

Summing over all insertion points gives a telescopic series:

$$q_\mu \Gamma^{\mu} \;=\; \sum_{j=0}^{N} \big(D_j - D_{j+1}\big) \;=\; D_0 - D_{N+1} \;=\; S^{-1}(p') - S^{-1}(p), \tag{6.153}$$

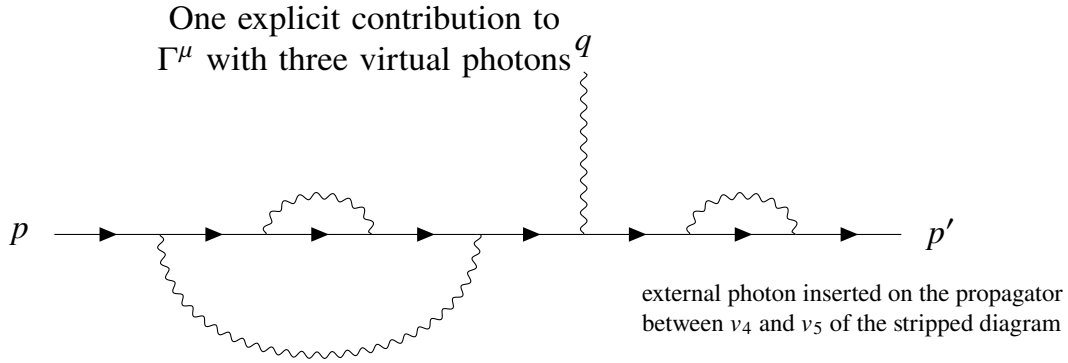
The cancellation is pairwise: every "$-D_{j+1}$" coming from insertion j cancels the "$+D_{j+1}$" coming from insertion $j+1$. Only the two boundary terms survive. By definition of the labeling, D_0 is the cut adjacent to the outgoing leg and gives $S^{-1}(p')$, while D_{N+1} is the cut adjacent to the incoming leg and gives $S^{-1}(p)$. Hence

$$\boxed{q_\mu \Gamma^{\mu}(p',p) = S^{-1}(p') - S^{-1}(p),} \tag{6.154}$$

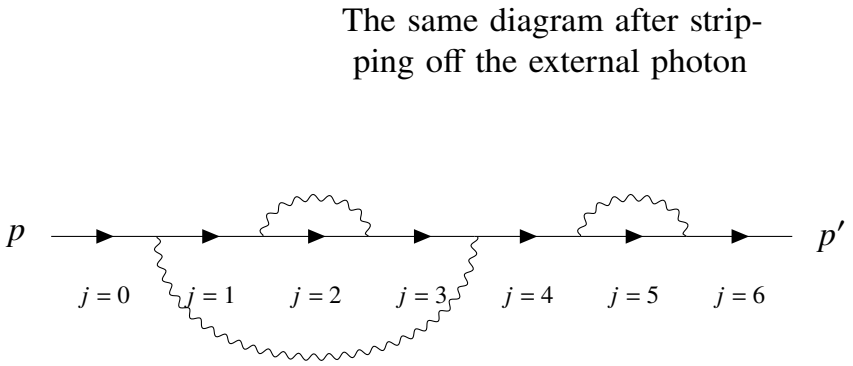
where $S^{-1}(\not{p}) = \not{p} - m - \Sigma_R(\not{p})$ is the (exact) inverse fermion propagator, written here in the renormalized convention; in bare perturbation theory one would replace $m \to m_0$ and $\Sigma_R \to \Sigma$ without changing the identity.

Concrete example with three actual virtual photons. The next picture is not the sum over all insertions. It is one single term in that sum. The three curved wavy

lines are genuine virtual photons, while the vertical line labeled q is the external photon. To keep the correspondence with the general schematic completely literal, we choose an insertion point that separates a two-photon subdiagram on the left from a one-photon subdiagram on the right. Reinserting the external photon on that propagator gives the following explicit contribution:



Now erase only the external photon q, while keeping the three virtual photons fixed. One obtains the stripped diagram below. Along its fermion line there are seven propagators, so the external photon can be reinserted in seven different places:



For this explicit three-photon pattern the full contribution to the proper vertex is therefore

$$\Gamma^\mu_{3\gamma} = \Gamma^\mu_{(0)} + \Gamma^\mu_{(1)} + \Gamma^\mu_{(2)} + \Gamma^\mu_{(3)} + \Gamma^\mu_{(4)} + \Gamma^\mu_{(5)} + \Gamma^\mu_{(6)}. \tag{6.155}$$

After contraction with q_μ, each insertion gives one difference of reduced diagrams:

$$q_\mu \Gamma^\mu_{(0)} = D_0 - D_1, \qquad q_\mu \Gamma^\mu_{(1)} = D_1 - D_2, \qquad q_\mu \Gamma^\mu_{(2)} = D_2 - D_3,$$
$$q_\mu \Gamma^\mu_{(3)} = D_3 - D_4, \qquad q_\mu \Gamma^\mu_{(4)} = D_4 - D_5, \qquad q_\mu \Gamma^\mu_{(5)} = D_5 - D_6,$$
$$q_\mu \Gamma^\mu_{(6)} = D_6 - D_7.$$

Summing them shows the telescopic cancellation with no abstraction left:

$$\begin{aligned} q_\mu \Gamma^\mu_{3\gamma} &= (D_0 - D_1) + (D_1 - D_2) + (D_2 - D_3) + (D_3 - D_4) \\ &\quad + (D_4 - D_5) + (D_5 - D_6) + (D_6 - D_7) \\ &= D_0 - D_7. \end{aligned}$$

This is the same telescopic mechanism as before, but now displayed on a genuinely more structured diagram rather than on a minimal toy picture.

Application to vertex renormalization. Taking the limit $q \to 0$ (i.e. $p' \to p$) with p on shell gives

$$\Gamma^\mu(p,p) = \left.\frac{\partial S^{-1}(\not{p})}{\partial p_\mu}\right|_{\not{p}=m}. \tag{6.156}$$

Since $S^{-1}(\not{p}) = \not{p} - m - \Sigma(\not{p})$, and $\not{p} = \gamma^\nu p_\nu$ depends on p_μ only through $\partial\not{p}/\partial p_\mu = \gamma^\mu$, the chain rule gives

$$\frac{\partial S^{-1}(\not{p})}{\partial p_\mu} = \gamma^\mu - \frac{\partial \Sigma(\not{p})}{\partial p_\mu} = \gamma^\mu - \Sigma'(\not{p})\,\gamma^\mu = \gamma^\mu\big(1 - \Sigma'(\not{p})\big). \tag{6.157}$$

Evaluating on shell and recalling that $Z_2^{-1} = 1 - \Sigma'(m)$ [eq. (6.114)]:

$$\Gamma^\mu(p,p) = \gamma^\mu\big(1 - \Sigma'(m)\big) = \frac{\gamma^\mu}{Z_2}. \tag{6.158}$$

This is a remarkable result: *gauge invariance completely determines the vertex at zero momentum transfer in terms of the electron self-energy.*

Now consider the physical amplitude for electron scattering in a weak external field. As we will show in Section 6.5, external-leg self-energy corrections reduce on shell to a purely multiplicative factor: each external fermion leg carries a factor $Z_2^{1/2}$ (the square root of the full-propagator pole residue), and each external photon carries $Z_3^{1/2}$. At zero momentum transfer the amplitude is proportional to

$$\underbrace{Z_2^{1/2}\cdot Z_2^{1/2}}_{\text{two fermion legs}}\; Z_3^{1/2}\, e_0\, \bar{u}(p)\,\Gamma^\mu(p,p)\,u(p) = Z_2\, Z_3^{1/2}\, e_0\, \bar{u}(p)\,\Gamma^\mu(p,p)\,u(p).$$

Now we use the result (6.158) just derived from the Ward–Takahashi identity: $\Gamma^\mu(p,p) = \gamma^\mu/Z_2$. Substituting:

$$Z_2\, Z_3^{1/2}\, e_0\, \frac{1}{Z_2}\, \bar{u}(p)\,\gamma^\mu\,u(p) = Z_3^{1/2}\, e_0\, \bar{u}(p)\,\gamma^\mu\,u(p).$$

The factor Z_2 from the two external fermion legs is cancelled exactly by the $1/Z_2$ that the Ward–Takahashi identity forces on the vertex at $q = 0$. The physical charge e is defined by requiring this to equal $e\,\bar{u}\gamma^\mu u$, so in $d = 4 - \epsilon$

$$\boxed{\mu^{\epsilon/2} e = Z_3^{1/2}\, e_0\,.} \tag{6.159}$$

and in four dimensions this reduces to $e = Z_3^{1/2} e_0$. The charge renormalization is controlled entirely by Z_3, the photon wave-function renormalization.

To connect with the standard notation, one introduces the **vertex renormalization constant** Z_1. The bare interaction vertex is $e_0\bar{\psi}_0\gamma^\mu\psi_0 A_{0\mu}$. Substituting $\psi_0 = Z_2^{1/2}\psi$ and $A_{0\mu} = Z_3^{1/2}A_\mu$, this becomes $e_0 Z_2 Z_3^{1/2}\bar{\psi}\gamma^\mu\psi A_\mu$. The constant Z_1 is defined as the ratio of this coefficient to the renormalized coupling:

$$e_0 Z_2 Z_3^{1/2} \equiv \mu^{\epsilon/2} Z_1 e\,, \tag{6.160}$$

where μ is the renormalization scale (needed to keep e dimensionless for $d \neq 4$). In other words, Z_1 is the overall multiplicative renormalization of the $\bar{\psi}\gamma^\mu\psi A_\mu$ vertex. The Ward identity result (6.156) then implies

$$\boxed{Z_1 = Z_2\,.} \tag{6.161}$$

Counterterm reformulation

The same result can be organized via counterterms. Start from the bare interaction term (in $d = 4-\epsilon$)

$$\mathcal{L}_{\rm int}^{(0)} = +e_0\bar{\psi}_0\gamma^\mu\psi_0 A_\mu^{(0)}. \tag{6.162}$$

Substituting $\psi_0 = Z_2^{1/2}\psi$ and $A_\mu^{(0)} = Z_3^{1/2}A_\mu$:

$$\mathcal{L}_{\rm int}^{(0)} = e_0 Z_2 Z_3^{1/2}\,\bar{\psi}\gamma^\mu\psi A_\mu = \mu^{\epsilon/2} Z_1\, e\,\bar{\psi}\gamma^\mu\psi A_\mu\,, \tag{6.163}$$

where Z_1 is defined in (6.160). Now split $Z_1 = 1 + \delta Z_1$ and write

$$\mathcal{L}_{\rm int}^{(0)} = +\mu^{\epsilon/2} e\,\bar{\psi}\gamma^\mu\psi A_\mu \;+\; \mu^{\epsilon/2}\delta Z_1\, e\,\bar{\psi}\gamma^\mu\psi A_\mu. \tag{6.164}$$

The first term gives the usual tree vertex $+ie\mu^{\epsilon/2}\gamma^\mu$. The second term is the *vertex counterterm*, with Feynman rule

$$\Gamma^\mu_{\rm ct} = +ie\,\mu^{\epsilon/2}\,\delta Z_1\,\gamma^\mu. \tag{6.165}$$

One-loop 1PI vertex function: explicit integral

The one-loop 1PI correction $\Lambda^\mu(p',p)$ is defined by

$$\Gamma^\mu(p',p) \equiv \gamma^\mu + \Lambda^\mu(p',p), \tag{6.166}$$

where Γ^μ is the regularized loop vertex before adding the local counterterm insertion. In covariant gauge, the one-loop contribution is

$$(+ie\mu^{\epsilon/2})\Lambda^\mu(p',p) = (+ie\mu^{\epsilon/2})^3 \int \frac{d^dk}{(2\pi)^d}\,\gamma^\alpha\,\frac{i(\not{p}'-\not{k}+m)}{(p'-k)^2-m^2+i\epsilon}\,\gamma^\mu \times \frac{i(\not{p}-\not{k}+m)}{(p-k)^2-m^2+i\epsilon}\,\gamma^\beta D_{\alpha\beta}(k), \tag{6.167}$$

with

$$D_{\alpha\beta}(k) = \frac{-i}{k^2 + i\epsilon}\left(g_{\alpha\beta} - (1-\xi)\frac{k_\alpha k_\beta}{k^2}\right). \tag{6.168}$$

In Feynman gauge $\xi = 1$, $D_{\alpha\beta}(k) = -ig_{\alpha\beta}/(k^2 + i\epsilon)$, so the Dirac structure simplifies to

$$\Lambda^\mu(p', p) = -e^2\mu^\epsilon \int \frac{d^d k}{(2\pi)^d} \frac{\gamma^\alpha(\not{p}' - \not{k} + m)\gamma^\mu(\not{p} - \not{k} + m)\gamma_\alpha}{\big(k^2 + i\epsilon\big)\big((p'-k)^2 - m^2 + i\epsilon\big)\big((p-k)^2 - m^2 + i\epsilon\big)}. \tag{6.169}$$

The integral is logarithmically UV divergent: for large k, the denominator scales like $k^2 \cdot k^2 \cdot k^2 \sim k^6$, the numerator scales like k^2 (two factors of k), so the integrand behaves like $d^d k\, k^2/k^6 = d^d k/k^4$, which is logarithmically divergent in $d = 4$ and becomes a $1/\epsilon$ pole in $d = 4 - \epsilon$.

Lorentz/Dirac decomposition and what is physically measurable

Including the counterterm, the **renormalized proper vertex** is

$$\Gamma^\mu_R(p', p) \;\equiv\; \Gamma^\mu(p', p) + \delta Z_1\, \gamma^\mu = \gamma^\mu + \Lambda^\mu(p', p) + \delta Z_1\, \gamma^\mu, \tag{6.170}$$

where $\delta Z_1 \equiv Z_1 - 1$ will be fixed by a renormalization condition. For on-shell electrons, $\bar{u}(p')\,\Gamma^\mu_R\, u(p)$ must be a Lorentz four-vector built from the available ingredients: the Dirac matrices γ^μ and $\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$, and the two independent four-momenta, which we can take to be $P^\mu \equiv p^\mu + p'^\mu$ and $q^\mu \equiv p'^\mu - p^\mu$. Since the vertex carries one free Lorentz index μ, the candidate structures sandwiched between $\bar{u}(p')$ and $u(p)$ are

$$\gamma^\mu, \quad P^\mu, \quad q^\mu, \quad \sigma^{\mu\nu}P_\nu, \quad \sigma^{\mu\nu}q_\nu\,.$$

(Terms like $\gamma^\mu \not{P}$ or $\gamma^\mu \not{q}$ can be reduced using the anticommutation relation $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$; higher products of gamma matrices are not independent in four dimensions.) The on-shell Dirac equation, $\not{p}\, u(p) = m\, u(p)$ and $\bar{u}(p')\not{p}' = m\,\bar{u}(p')$, allows further simplification. In particular, the Gordon identity

$$\bar{u}(p')\,\gamma^\mu\, u(p) = \bar{u}(p')\left[\frac{P^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right] u(p)$$

shows that P^μ is not independent of γ^μ and $\sigma^{\mu\nu}q_\nu$ between on-shell spinors. Similarly, contracting $\sigma^{\mu\nu}P_\nu$ with the spinors and using the Dirac equation yields a combination of γ^μ and q^μ, so it is also redundant.

After eliminating all dependent structures, three independent form factors remain:

$$\bar{u}(p')\,\Gamma^\mu_R(p', p)\, u(p) = \bar{u}(p')\left[\gamma^\mu\, F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\, F_2(q^2) + q^\mu\, F_3(q^2)\right] u(p),$$

(6.171)

where the scalar coefficients F_i can depend only on the Lorentz invariant q^2 (since $p^2 = p'^2 = m^2$ on shell). The $q^\mu F_3$ term does not contribute when the vertex is coupled to a conserved external current ($q_\mu J^\mu = 0$), and is pure-gauge in that sense. The two physically relevant form factors are: $F_1(q^2)$ (charge form factor) and $F_2(q^2)$ (Pauli form factor; $F_2(0)$ gives the anomalous magnetic moment).

Ward–Takahashi identity and why QED is special

Gauge invariance implies the Ward–Takahashi identity for the proper vertex:

$$q_\mu \Gamma^\mu(p', p) = S^{-1}(p') - S^{-1}(p), \tag{6.172}$$

where $S^{-1}(\not{p}) = \not{p} - m - \Sigma(\not{p})$ is the regularized inverse propagator. This identity has two crucial consequences:

(1) Transversality of the UV divergence. The UV divergent part of Λ^μ must be proportional to γ^μ: no new Dirac structures appear in the divergence. The argument is the following. Since $\Sigma(\not{p})$ is at most linearly divergent by power counting, its divergent part has the polynomial form $\Sigma_{\text{div}}(\not{p}) = A\not{p} + Bm$ with A, B constants, so the right-hand side of (6.172) contributes $\left[S^{-1}(p') - S^{-1}(p)\right]_{\text{div}} = -A\not{q} = -A\, q_\mu \gamma^\mu$. On the left-hand side, the WTI must hold separately for the divergent parts; comparing Dirac structures, the only Λ^μ_{div} that yields $-A\, q_\mu \gamma^\mu$ upon contraction with q_μ is $\Lambda^\mu_{\text{div}} = -A\,\gamma^\mu$ (other structures such as $\sigma^{\mu\nu} q_\nu$ contribute only to finite form factors like the anomalous magnetic moment). Hence a single counterterm $\delta Z_1\, \gamma^\mu$ suffices to absorb the vertex divergence.

(2) Equality of renormalization constants: $Z_1 = Z_2$. The Ward–Takahashi identity (6.172) is an exact consequence of gauge invariance of the bare action $\mathcal{L}_0$. In the counterterm method we split $\mathcal{L}_0 = \mathcal{L}_{\text{ren}} + \delta\mathcal{L}$ and treat $\delta\mathcal{L}$ as a perturbative insertion. The counterterms may look like new interactions, but they are not: $\mathcal{L}_{\text{ren}} + \delta\mathcal{L}$ is *identically* equal to $\mathcal{L}_0$, so the full theory is the same gauge-invariant theory we started from. Now, the renormalized Green's functions Γ^μ_R and S^{-1}_R include *both* the loop corrections computed from $\mathcal{L}_{\text{ren}}$ *and* the counterterm insertions from $\delta\mathcal{L}$. Together, they reproduce the Green's functions of the full bare theory, merely re-expressed in terms of renormalized fields and parameters. Since the bare theory is gauge invariant, the WTI must hold for the renormalized quantities as well:

$$q_\mu \Gamma^\mu_R(p', p) = S^{-1}_R(p') - S^{-1}_R(p)\,.$$

Explicit check. The bare gauge invariance of $\mathcal{L}_0$ gives, via the standard path-integral derivation, the bare WTI $q_\mu \Gamma^\mu_0 = S^{-1}_0(p') - S^{-1}_0(p)$. The bare and renormalized 1PI functions are related by external-leg rescalings: from $\psi_0 = Z_2^{1/2}\psi$ and $A_0 = Z_3^{1/2} A$ one obtains $S^{-1}_0(p) = Z_2^{-1} S^{-1}_R(p)$ and, with $e_0 = \mu^{\epsilon/2} Z_e e$, $\Gamma^\mu_0 = Z_1^{-1}\Gamma^\mu_R$, where $Z_1 = Z_e Z_2 Z_3^{1/2}$. Substituting into the bare WTI,

$$q_\mu \Gamma^\mu_R = \frac{Z_1}{Z_2}\left[S^{-1}_R(p') - S^{-1}_R(p)\right],$$

which collapses to the simple form above precisely when $Z_1 = Z_2$. In this sense the renormalized WTI is equivalent to the statement that gauge invariance survives the field rescaling. We now show, working directly in renormalized perturbation theory with counterterms, that this forces $Z_1 = Z_2$.

The counterterms shift the proper vertex and inverse propagator as

$$\Gamma^\mu_R = \Gamma^\mu + \delta Z_1\, \gamma^\mu, \qquad S_R^{-1}(\not p) = S^{-1}(\not p) + (Z_2 - 1)\not p - \delta m, \tag{6.173}$$

where $\Gamma^\mu = \gamma^\mu + \Lambda^\mu$ and $S^{-1}(\not p) = \not p - m - \Sigma(\not p)$ are the regularized one-loop quantities before counterterm insertions. Compute both sides of the renormalized identity $q_\mu \Gamma^\mu_R = S_R^{-1}(p') - S_R^{-1}(p)$ separately. On the left-hand side, using the regularized identity (6.172):

$$q_\mu \Gamma^\mu_R = q_\mu \Gamma^\mu + \delta Z_1\, q_\mu \gamma^\mu = \left[S^{-1}(p') - S^{-1}(p)\right] + \delta Z_1\, \not q\,. \tag{6.174}$$

On the right-hand side, the mass counterterms cancel in the difference:

$$\begin{aligned} S_R^{-1}(p') - S_R^{-1}(p) &= \left[S^{-1}(p') - S^{-1}(p)\right] + (Z_2 - 1)(\not p' - \not p) \\ &= \left[S^{-1}(p') - S^{-1}(p)\right] + (Z_2 - 1)\, \not q\,. \end{aligned} \tag{6.175}$$

Equating (6.174) and (6.175), the regularized Ward–Takahashi terms cancel on both sides, leaving

$$\delta Z_1\, \not q = (Z_2 - 1)\, \not q\,,$$

which, since it must hold for arbitrary q, requires

$$Z_1 = Z_2 \qquad \Longleftrightarrow \qquad \delta Z_1 = \delta Z_2\,. \tag{6.176}$$

This is the key simplification in QED: once the electron wavefunction renormalization is fixed, the vertex counterterm is fixed automatically.

Renormalization condition for the charge and determination of δZ_1

To fix the *finite* part of δZ_1 one chooses a renormalization scheme. In the on-shell (Thomson) scheme—the standard QED renormalization prescription, treated in detail in Peskin–Schroeder [6] §10.3, Schwartz [12] Ch. 19, and Itzykson–Zuber [16] §8.4—the renormalized charge is defined by demanding that the electron couples to a soft external photon with strength e, i.e. in the limit of vanishing momentum transfer:

$$\Gamma^\mu_R(p', p)\Big|_{q^2 \to 0,\; p^2 = p'^2 = m^2} = \gamma^\mu. \tag{6.177}$$

Using (6.171) and (6.165), this condition becomes

$$\gamma^\mu + \Lambda^\mu(p', p)\Big|_{\text{Thomson}} + \delta Z_1\, \gamma^\mu = \gamma^\mu, \tag{6.178}$$

so

$$\delta Z_1 = -\,\Lambda^\mu(p',p)|_{\text{Thomson}} \Big/ \gamma^\mu, \tag{6.179}$$

meaning: δZ_1 is minus the coefficient of γ^μ in the Thomson limit of Λ^μ (it cancels the UV pole). In practice Λ^μ is decomposed in the basis (6.171),

$$\bar u(p')\,\Lambda^\mu(p',p)\,u(p) = \bar u(p')\Big[\,\gamma^\mu\left(F_1(q^2)-1\right) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\,F_2(q^2)\Big]u(p), \tag{6.180}$$

(the would-be F_3 piece vanishes by current conservation), and the bare vertex γ^μ contributes 1 to F_1. The renormalized vertex is then

$$\bar u(p')\,\Gamma^\mu_R\,u(p) = \bar u(p')\Big[\,\gamma^\mu\left(F_1(q^2)+\delta Z_1\right) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2)\Big]u(p), \tag{6.181}$$

so the Thomson condition (6.177) reads $F_1(0)+\delta Z_1 = 1$, i.e.

$$\delta Z_1 = 1 - F_1(0), \qquad \text{equivalently} \qquad F_1^{\text{ren}}(0) \equiv F_1(0)+\delta Z_1 = 1. \tag{6.182}$$

Hence δZ_1 cancels the UV pole of $F_1(0)$ together with its scheme-dependent finite part, while $F_2(q^2)$ is UV finite and unaffected by the counterterm (it carries the anomalous magnetic moment, $F_2(0) = \alpha/2\pi + O(\alpha^2)$).

UV vs IR: what "the difficult part" really is

At one loop and on shell, Λ^μ contains *both* UV and IR singularities:

$$\Lambda^\mu(p',p) \sim \left(\frac{c_{\text{UV}}}{\epsilon_{\text{UV}}} + \frac{c_{\text{IR}}}{\epsilon_{\text{IR}}} + \text{finite}\right)\gamma^\mu + (\text{other finite Dirac structures}). \tag{6.183}$$

The counterterm δZ_1 is fixed to remove the UV pole (and impose the chosen renormalization condition), but *it cannot remove the IR pole*: that singularity has a completely different physical origin. The IR divergence arises because the loop integral receives contributions from virtual photons with arbitrarily small momenta. However, no real detector has infinite energy resolution: a final-state electron with energy E is experimentally indistinguishable from the same electron accompanied by a photon so soft that the total energy remains within the detector resolution ΔE. The physically measurable quantity is therefore the *inclusive* cross section, which sums:

1. the purely virtual process (with the IR-divergent vertex correction), and
2. the real-emission process in which one or more soft photons with total energy below ΔE are radiated.

The Bloch–Nordsieck theorem guarantees that the IR singularities in these two contributions cancel exactly, leaving a finite inclusive cross section that depends on ΔE only through a harmless logarithm $\ln(\Delta E/E)$. This is why the vertex correction

is "harder" than the propagator corrections: even after UV renormalization, a purely virtual on-shell vertex amplitude is not IR finite by itself—it only becomes finite once combined with real emission.

Summary. The renormalized vertex to one loop is

$$\Gamma^{\mu}_{R}(p',p) = \gamma^{\mu} + \Lambda^{\mu}(p',p) + \delta Z_1\,\gamma^{\mu} + \mathcal{O}(e^4), \tag{6.184}$$

with δZ_1 chosen to cancel the UV pole and satisfy the charge renormalization condition (e.g. (6.177)). In QED, gauge invariance further enforces $Z_1 = Z_2$ (6.176), tightly linking vertex and electron-propagator renormalization. In the on-shell (Thomson) scheme, $\delta Z_1 = 1 - F_1(0)$ from (6.182), and (6.181) reduces to a manifestly finite (in the UV) on-shell matrix element:

$$\bar{u}(p')\,\Gamma^{\mu}_{R}(p',p)\,u(p) = \bar{u}(p')\Big[\,\gamma^{\mu}\left(1 + F_1(q^2) - F_1(0)\right) + \frac{i\sigma^{\mu\nu}q_{\nu}}{2m}\,F_2(q^2)\Big]u(p) + \mathcal{O}(e^4). \tag{6.185}$$

The combination $F_1(q^2) - F_1(0)$ is the physical, UV-subtracted charge form factor (vanishing at $q^2 = 0$ and carrying the IR singularity that will cancel against soft real emission), while $F_2(q^2)$ gives the Pauli form factor and the anomalous magnetic moment $a_e = F_2(0)$.

6.5 External line renormalization

In Mandl–Shaw's treatment, the key point is that an S-matrix element is obtained from *amputated* Green functions with adiabatic switching and on-shell pole extraction. External self-energy insertions do not introduce new independent on-shell structures; they only modify the pole residues of the exact propagators. We now show this explicitly.

External fermion lines

Let $\mathcal{M}$ be the rest of an amputated diagram to which an incoming electron spinor is attached. At first order, adding one self-energy insertion on the external leg gives

$$\text{(no insertion)}: \quad \mathcal{M}\,u(p),$$

and

$$\text{(one insertion on the external leg)}: \quad \mathcal{M}\,\frac{i}{\not{p} - m + i\epsilon}\left[-i\Sigma(\not{p}) + i\delta m\right]u(p).$$

Notation choice (rigorous): in this section we are using *renormalized perturbation theory*, i.e.

$$\mathcal{L} = \bar{\psi}\,(i\partial\!\!\!/ - m)\psi + e\,\bar{\psi}\gamma^{\mu}\psi A_{\mu} \;+\; \delta\mathcal{L},$$

with $\delta\mathcal{L} \supset -\delta m\,\bar{\psi}\psi$. Therefore the zeroth-order fermion propagator is

$$S_F^{(0)}(p) = \frac{i}{\not{p} - m + i\epsilon},$$

and the mass shift appears in the insertion $+i\delta m$. Equivalently, in a bare split one writes

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m_0)\psi + e_0\bar{\psi}\gamma^{\mu}\psi A_{\mu} + \cdots$$

and expands around

$$S_{0,\mathrm{bare}}(p) = \frac{i}{\not{p} - m_0 + i\epsilon}.$$

The two organizations are identical order by order once $m_0 = m + \delta m$. Here is the origin of this structure, step by step:

1. The factor $\mathcal{M}$ is the hard amputated subamplitude.
2. The external electron line that connects $\mathcal{M}$ to the insertion contributes the zeroth-order renormalized propagator $S_F^{(0)}(p) = i/(\not{p} - m + i\epsilon)$.
3. The blob on that line is the 1PI two-point insertion plus counterterm: $-i\Sigma(\not{p}) + i\delta m$.
4. The incoming asymptotic state contributes the spinor $u(p)$.

Equivalent Dyson viewpoint (same formula, different language). We denote by $G_{\mathrm{ext}}(p)$ the *dressed external (half-)line*: the object that describes how the asymptotic incoming spinor $u(p)$, prepared at $t \to -\infty$, propagates and accumulates self-energy insertions before reaching the hard subamplitude $\mathcal{M}$ at the scattering vertex. It is *not* a full Green function: it is the truncated correlator that attaches *one* on-shell asymptotic state on one end and is left *open* (un-amputated) on the other end—ready to be inserted into $\mathcal{M}$. Diagrammatically,

$$\underbrace{u(p)}_{\text{in}} \xrightarrow{S_0(p)} \underbrace{\bullet}_{(-i\Sigma + i\delta m)} \xrightarrow{S_0(p)} \cdots \xrightarrow{S_0(p)} \mathcal{M}.$$

Expanding the geometric series of self-energy insertions reads

$$\begin{aligned} G_{\mathrm{ext}}(p) &= S_0(p)\,u(p) \\ &\quad + S_0(p)\left[-i\Sigma(\not{p}) + i\delta m\right] S_0(p)\,u(p) + \cdots, \end{aligned} \tag{6.186}$$

with

$$S_0(p) = S_F^{(0)}(p) = \frac{i}{\not{p} - m + i\epsilon}.$$

Each term has a simple meaning: the zeroth-order term $S_0(p)\,u(p)$ is the bare free propagation of the incoming spinor; the first-order term $S_0\,(-i\Sigma + i\delta m)\,S_0\,u$ inserts one renormalized self-energy blob (1PI loop $-i\Sigma$ together with its mass counterterm $+i\delta m$) between two free propagators; the higher-order terms iterate the same structure. Summing the geometric series gives the exact resummed external line

$$G_{\text{ext}}(p) = S(p)\,u(p), \qquad S(p) = \frac{i}{\not{p} - m - \Sigma(\not{p}) + \delta m + i\epsilon},$$

i.e. the exact propagator $S(p)$ of the renormalized field acting on the asymptotic spinor. Hence (6.186) is just the perturbative expansion of $S(p)\,u(p)$ around $S_0(p)\,u(p)$. For an incoming fermion, removing the external propagator factor means multiplying by $(\not{p} - m)/i$ on that leg:

$$\frac{\not{p} - m}{i}\,G_{\text{ext}}(p) = \left[1 + \frac{i}{\not{p} - m + i\epsilon}\big(-i\Sigma(\not{p}) + i\delta m\big) + \cdots\right] u(p).$$

Keeping only one insertion, the corrected leg has the form "tree + inserted leg", i.e.

$$\mathcal{M}\,u(p) \;\longrightarrow\; \mathcal{M}\left[1 + \frac{i}{\not{p} - m + i\epsilon}\big(-i\Sigma(\not{p}) + i\delta m\big)\right] u(p). \tag{6.187}$$

Using the on-shell expansion

$$\Sigma(\not{p}) = A + B(\not{p} - m) + \Sigma_c(\not{p}), \qquad \delta m = A,$$

with $\Sigma_c(\not{p}) = O\big((\not{p} - m)^2\big)$ [cf. eqs. (6.109), (6.112), (6.110)], the renormalized self-energy combination $-i\Sigma + i\delta m = -i[B(\not{p} - m) + \Sigma_c]$ enters (6.187) as

$$\mathcal{M}\,u(p) \;\longrightarrow\; \mathcal{M}\left[1 + \frac{B\,(\not{p} - m)}{\not{p} - m + i\epsilon} + \frac{\Sigma_c(\not{p})}{\not{p} - m + i\epsilon}\right] u(p). \tag{6.188}$$

The Σ_c piece vanishes on shell ($\Sigma_c = O((\not{p} - m)^2)$ divided by a linear denominator gives $O(\not{p} - m)$, which is killed by $u(p)$). The remaining B piece, however, has the genuinely *indeterminate* form $\frac{(\not{p} - m)}{\not{p} - m + i\epsilon}$: naively cancelling numerator and denominator gives B, while using $(\not{p} - m)\,u(p) = 0$ gives 0. The one-insertion calculation alone **cannot** fix the value of this $0/0$, and therefore cannot determine

the external-leg factor: an explicit (infrared, adiabatic) regulator is needed to give it a definite meaning.

The proper resolution is given in the next subsection (Mandl–Shaw viewpoint), where the adiabatic switching of the interaction selects the unique finite answer

$$1+\frac{B\,(\not p-m)}{\not p-m+i\epsilon}\;\longrightarrow\;1+\tfrac{1}{2}B=Z_2^{1/2}+O(B^2).$$

The factor of $\frac{1}{2}$ in front of B is the substantive content of the Mandl–Shaw calculation: it is what distinguishes $Z_2^{1/2}$ (the correct external-leg factor) from Z_2 (the residue of the dressed propagator after amputation).

From one insertion to the exact external-leg factor.
The exact propagator already resums all such insertions:

$$S(\not p)=\frac{i}{\not p-m-\Sigma(\not p)}\xrightarrow{\not p\to m}\frac{iZ_2}{\not p-m+i\epsilon}\quad\text{[eq. (6.116)]}.$$

When the external pole factor is removed in the S-matrix construction, the net external-leg normalization is $Z_2^{1/2}$. Hence

$$\boxed{\begin{aligned}u(p)&\to Z_2^{1/2}u(p), & \bar u(p)&\to Z_2^{1/2}\bar u(p),\\ v(p)&\to Z_2^{1/2}v(p), & \bar v(p)&\to Z_2^{1/2}\bar v(p).\end{aligned}}\tag{6.189}$$

Operationally: compute amputated diagrams and attach $Z_2^{1/2}$ to each external electron/positron leg.

Adiabatic switching (Mandl–Shaw viewpoint)

We now re-derive the external-leg factor $Z_2^{1/2}$ following step by step the treatment of Mandl–Shaw [1], §9.4. The argument is more delicate than the propagator-pole argument above, and the reason to go through it is conceptual: the apparently innocent factor $(\not p-m)/(\not p-m+i\epsilon)$ that controls the external-leg correction is in fact a $0/0$ on shell, and giving it a precise meaning requires an explicit infrared (adiabatic) regulator. The trick is that the result of the regulated calculation is finite and *independent of the choice of regulator*, so it is unambiguous.

Setup (notation matching Mandl–Shaw). We work in renormalized perturbation theory. The renormalized free fermion propagator and the renormalized self-energy are

$$S_F^{(0)}(p)=\frac{i}{\not p-m+i\epsilon},\qquad\widetilde\Sigma(\not p)\equiv\Sigma(\not p)-\delta m=B\,(\not p-m)+\Sigma_c(\not p),$$

where the on-shell expansion (6.109) has been used and $\delta m = A = \Sigma(m)$ has been subtracted by the mass counterterm; cf. properties (6.110) (Σ_c vanishes quadratically at $\not{p} = m$). The renormalized self-energy insertion contributes a Feynman factor $-i\widetilde{\Sigma}(\not{p})$.

One self-energy insertion on the incoming line [Mandl–Shaw Eq. (9.38)]. At order e^2, summing the tree term and the single insertion on the external incoming electron leg gives

$$\begin{aligned} u(p) &\longrightarrow u(p) + S_F^{(0)}(p)\left[-i\widetilde{\Sigma}(\not{p})\right]u(p) \\ &= u(p) + \frac{i}{\not{p} - m + i\epsilon}\left[-i\right]\left[B(\not{p} - m) + \Sigma_c(\not{p})\right]u(p) \\ &= \left[1 + \frac{B\,(\not{p} - m)}{\not{p} - m + i\epsilon} + \frac{\Sigma_c(\not{p})}{\not{p} - m + i\epsilon}\right]u(p). \end{aligned} \tag{6.190}$$

The Σ_c piece drops out [Mandl–Shaw Eq. (9.39)]. By (6.110) we have $\Sigma_c(p) = O\big((\not{p} - m)^2\big)$ on shell, so $\Sigma_c(p)/(\not{p} - m + i\epsilon)$ vanishes linearly as $\not{p} \to m$. On the on-shell spinor $u(p)$ this term gives zero and we are left with

$$u(p) \longrightarrow \left[1 + \frac{B\,(\not{p} - m)}{\not{p} - m + i\epsilon}\right]u(p). \tag{6.191}$$

The $0/0$ ambiguity. The factor $(\not{p} - m)/(\not{p} - m + i\epsilon)$ in (6.191) looks like 1 if one cancels numerator and denominator naively, but the Dirac equation gives $(\not{p} - m)\,u(p) = 0$, so the same expression also looks like 0. As Mandl–Shaw stress, the term *"is indeterminate, as it stands"*: extracting a definite limit requires regulating how $\not{p}$ approaches m.

The adiabatic hypothesis [Mandl–Shaw Eq. (9.40)]. The regulator is the adiabatic switching of the interaction. Introduce a smooth real function $f(t)$ with $f(t) \to 0$ as $t \to \pm\infty$ and $f(t) \simeq 1$ during a long time interval T around the scattering, and replace the (renormalized) QED interaction Hamiltonian density by

$$\mathscr{H}_I(x) \longrightarrow \mathscr{H}_I^{(f)}(x) = -e\,f(t)\,\bar{\psi}(x)\,\not{A}(x)\,\psi(x) - \delta m\,[f(t)]^2\,\bar{\psi}(x)\,\psi(x). \tag{6.192}$$

The original theory is recovered as $f(t) \to 1$. (The mass counterterm carries $[f(t)]^2$ because each occurrence of δm accompanies two powers of the coupling at second order.) *Note on conventions.* Mandl–Shaw write e_0 rather than e here because they work directly with the bare Lagrangian (not split into renormalized part plus counterterms). At the order e^2 at which we are computing, the distinction is immaterial: $e_0 = e + O(e^3)$ via $Z_e = 1 + O(e^2)$, so the second-order self-energy involves e^2 or e_0^2 interchangeably. We use the renormalized e to keep the notation aligned with the rest of the chapter.

Fourier representation [Mandl–Shaw Eqs. (9.41)–(9.42)]. Writing the Fourier transform of $f(t)$ as

$$f(t) = \int_{-\infty}^{+\infty} dE\, F(E)\, e^{iEt} = \int_{-\infty}^{+\infty} dE\, F(E)\, e^{iq\cdot x}, \qquad q \equiv (E, \mathbf{0}), \tag{6.193}$$

the normalization conditions become

$$f(0) = \int_{-\infty}^{+\infty} dE\, F(E) = 1, \qquad F(E) \to \delta(E) \text{ as } T \to \infty, \tag{6.194}$$

i.e. $F(E)$ is sharply peaked at $E = 0$ with width $\sim 1/T$. The auxiliary 4-vector $q = (E, \mathbf{0})$ has the structure of an external-field momentum: the modified interaction (6.192) conserves three-momentum at each vertex (because f depends only on t) but *not* energy.

Modified self-energy insertion [Mandl–Shaw Eq. (9.43)]. Each of the two QED vertices in the second-order self-energy carries a factor $f(t)$, hence in momentum space a factor $F(E)$ together with an "energy momentum" $q = (E, \mathbf{0})$ flowing into that vertex. Calling the two such momenta q and q', the internal fermion propagator now carries 4-momentum $\not{p} - \not{q} - \not{q}'$ (instead of $\not{p}$), and the analogue of (6.191) reads

$$u(p) \;\longrightarrow\; \left[1 + \int dE\, dE'\, F(E)\, F(E') \, \frac{B\,(\not{p} - \not{q} - m)}{\not{p} - \not{q} - \not{q}' - m + i\epsilon} \right] u(p). \tag{6.195}$$

The original ambiguity of (6.191) is gone: numerator and denominator are now genuinely different polynomials in E, E', and the integral is well defined.

First replacement: use $(\not{p} - m)\,u(p) = 0$. Acting on $u(p)$, we may freely add $-\frac{1}{2}(\not{p} - m)$ to the numerator of (6.195):

$$(\not{p} - \not{q} - m)\,u(p) = \left[(\not{p} - \not{q} - m) - \tfrac{1}{2}(\not{p} - m) \right] u(p) = \tfrac{1}{2}\left(\not{p} - 2\not{q} - m \right) u(p). \tag{6.196}$$

Second replacement: symmetrize $q \leftrightarrow q'$. The remaining factor $F(E)F(E')$ in (6.195) is symmetric under $q \leftrightarrow q'$, so in the numerator we may replace $2\not{q} \to \not{q} + \not{q}'$:

$$\tfrac{1}{2}\left(\not{p} - 2\not{q} - m \right) \;\longrightarrow\; \tfrac{1}{2}\left(\not{p} - \not{q} - \not{q}' - m \right). \tag{6.197}$$

The numerator is now exactly $\frac{1}{2}$ times the denominator: cancellation becomes trivial.

The integral becomes elementary [Mandl–Shaw Eq. (9.44)]. Substituting (6.196)–(6.197) into (6.195), the integrand reduces to $\frac{1}{2}B$, independent of E, E':

$$u(p) \longrightarrow \left[1 + \tfrac{1}{2}B \int dE\, F(E) \int dE'\, F(E')\right] u(p) = \left(1 + \tfrac{1}{2}B\right) u(p), \tag{6.198}$$

where we used the normalization (6.194) on each F. The result is finite, unambiguous, and—as Mandl–Shaw emphasize—*independent* of the precise shape of $F(E)$, hence valid in the limit $F(E) \to \delta(E)$ which restores the original theory.

Identification with $Z_2^{1/2}$ [Mandl–Shaw Eq. (9.45a)]. Recalling $Z_2 = (1 - B)^{-1}$ from (6.114), expand $Z_2^{1/2} = (1 - B)^{-1/2} = 1 + \frac{1}{2}B + O(B^2)$. Comparing with (6.198):

$$\boxed{u(p) \longrightarrow Z_2^{1/2}\, u(p) + O(e^4)\,.} \tag{6.199}$$

Although derived only at second order, the result generalizes to all orders (the exact propagator residue Z_2 remains the only new structure).

Other spinors and external photons [Mandl–Shaw Eqs. (9.45b)–(9.45c)]. Identical arguments applied to the other three external fermion polarizations give

$$\bar{u}(p) \to Z_2^{1/2} \bar{u}(p), \quad v(p) \to Z_2^{1/2} v(p), \quad \bar{v}(p) \to Z_2^{1/2} \bar{v}(p). \tag{6.200}$$

For an external physical photon, the same logic with the vacuum-polarization tensor $\Pi^{\mu\nu}(k) = (k^2 g^{\mu\nu} - k^\mu k^\nu)\,\Pi(k^2)$ contracted into a transverse polarization $\varepsilon^\nu(k)$ gives $\Pi^{\mu\nu}\varepsilon_\nu = k^2\,\Pi(k^2)\,\varepsilon^\mu = 0$ on shell; all that remains is the residue Z_3 of the exact photon propagator, split as $Z_3^{1/2}$ on the polarization vector:

$$\varepsilon^\mu(k) \longrightarrow Z_3^{1/2}\, \varepsilon^\mu(k). \tag{6.201}$$

Operational rule. The Feynman recipe extracted from the above derivation is exactly that of Mandl–Shaw §9.4:

1. compute the *amputated* 1PI Green function for the hard process using renormalized perturbation theory;
2. attach to each external leg the on-shell wavefunction $(u, \bar{u}, v, \bar{v}, \varepsilon)$ of the asymptotic particle;
3. multiply by one factor $Z_2^{1/2}$ for each external fermion line and one factor $Z_3^{1/2}$ for each external photon line.

This reproduces (6.189) and (6.202); the underlying physics is the identification of the asymptotic states (defined via adiabatic switching as bare Fock states) with the physical, dressed in/out states relevant for the S-matrix.

External photon lines

For a physical external photon ($k^2 = 0$, $k\cdot\varepsilon = 0$), an explicit vacuum-polarization insertion contributes

$$\Pi^{\mu\nu}(k)\,\varepsilon_\nu(k) = \big(k^2 g^{\mu\nu} - k^\mu k^\nu\big)\Pi(k^2)\,\varepsilon_\nu(k) = k^2\Pi(k^2)\,\varepsilon^\mu(k) = 0,$$

so again no extra on-shell structure survives.

The exact photon propagator has transverse pole residue Z_3:

$$D^{\mu\nu}(k) \xrightarrow{k^2\to 0} \frac{-iZ_3}{k^2 + i\epsilon}\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) + \text{regular}.$$

Removing this pole factor gives one factor $Z_3^{1/2}$ per external photon:

$$\boxed{\varepsilon^\mu(k) \to Z_3^{1/2}\,\varepsilon^\mu(k)\,.} \tag{6.202}$$

Combining factors and charge renormalization

The bare interaction written in renormalized fields is

$$e_0\bar\psi_0\gamma^\mu\psi_0 A_{0\mu} = e_0 Z_2 Z_3^{1/2}\bar\psi\gamma^\mu\psi A_\mu \equiv \mu^{\epsilon/2} Z_1 e\,\bar\psi\gamma^\mu\psi A_\mu$$

[eq. (6.160)]. Gauge invariance then imposes

$$Z_1 = Z_2 \qquad \text{(Ward–Takahashi identity)}.$$

Hence

$$\boxed{\mu^{\epsilon/2} e = Z_3^{1/2} e_0\,,} \tag{6.203}$$

which in four dimensions is $e = Z_3^{1/2} e_0$.

This is the Mandl–Shaw conclusion: charge renormalization is controlled by photon wave-function renormalization, while fermion external-leg normalization and vertex renormalization are tied by $Z_1 = Z_2$.

6.6 Systematic renormalization of QED

Having computed the one-loop corrections to the propagators and vertex, and understood how external legs are renormalized, we now organize the full renormalization programme starting from the bare QED Lagrangian.

The bare QED Lagrangian

We start from the bare QED Lagrangian in covariant gauge:

$$\mathcal{L}_0 = -\frac{1}{4} F_{0\mu\nu} F_0^{\mu\nu} + \bar{\psi}_0 (i\partial\!\!\!/ - m_0)\psi_0 + e_0 \bar{\psi}_0 \gamma^\mu \psi_0 A_{0\mu} - \frac{1}{2\xi_0} (\partial \cdot A_0)^2, \quad (6.204)$$

where $F_{0\mu\nu} \equiv \partial_\mu A_{0\nu} - \partial_\nu A_{0\mu}$. The subscript "0" denotes bare quantities. The last term is the *gauge-fixing* term, necessary to define the photon propagator.

Why the gauge-fixing term is needed.

The Maxwell Lagrangian $\mathcal{L}_{\text{Maxwell}} = -\frac{1}{4} F_{0\mu\nu} F_0^{\mu\nu}$ is invariant under gauge transformations $A_{0\mu}(x) \mapsto A_{0\mu}(x) + \partial_\mu \alpha(x)$. Because of this invariance, the quadratic operator $O^{\mu\nu} = g^{\mu\nu}\Box - \partial^\mu \partial^\nu$ governing the free photon field is *not invertible*: the equation of motion $O^{\mu\nu} A_\nu = 0$ is satisfied by every pure-gauge configuration $A_\mu^{\text{pg}} = \partial_\mu \alpha$, regardless of the choice of $\alpha(x)$:

$$O^{\mu\nu} A_\nu^{\text{pg}} = \left(g^{\mu\nu}\Box - \partial^\mu \partial^\nu\right)\partial_\nu \alpha = \partial^\mu \Box \alpha - \partial^\mu \Box \alpha = 0.$$

In other words, the operator cannot distinguish between field configurations that differ only by a gauge transformation—it is "blind" along pure-gauge directions. This means we cannot construct a propagator for the photon: a propagator is supposed to tell us how a source J^μ produces a field response A_ν, but if the kinetic operator annihilates an entire family of field configurations, the inverse simply does not exist. *Physically,* the gauge symmetry says that field configurations related by $A_\mu \to A_\mu + \partial_\mu \alpha$ are physically equivalent (they describe the same electromagnetic field $F_{\mu\nu}$); the path integral overcounts each physical configuration by the infinite volume of the gauge orbit, and this redundancy manifests itself precisely as the non-invertibility of the kinetic operator. The gauge-fixing term removes the redundancy by selecting one representative per orbit, lifting the zero modes and producing an invertible quadratic form. In perturbation theory we need the photon propagator, i.e. the inverse of this quadratic form.

Expanding the Maxwell term to quadratic order and integrating by parts:

$$-\frac{1}{4} F_{0\mu\nu} F_0^{\mu\nu} = \frac{1}{2} A_{0\mu} \left(g^{\mu\nu}\Box - \partial^\mu \partial^\nu\right) A_{0\nu}, \quad (6.205)$$

up to total derivatives. Similarly,

$$-\frac{1}{2\xi_0} (\partial \cdot A_0)^2 = \frac{1}{2} A_{0\mu} \left(-\frac{1}{\xi_0} \partial^\mu \partial^\nu\right) A_{0\nu}. \quad (6.206)$$

The total quadratic photon Lagrangian is therefore

$$\mathcal{L}_{\text{Maxwell}} + \mathcal{L}_{\text{gf}} = \frac{1}{2} A_{0\mu} \left[g^{\mu\nu}\Box - \left(1 - \frac{1}{\xi_0}\right) \partial^\mu \partial^\nu\right] A_{0\nu}. \quad (6.207)$$

The propagator $D_0^{\mu\nu}(x-y)$ is the response of the free photon field at x to a unit point-like source at y: it solves

$$\left[g^{\mu\alpha}\Box - \left(1 - \frac{1}{\xi_0}\right) \partial^\mu \partial^\alpha\right] D_{0\alpha\nu}(x-y) = i\,\delta_\nu^\mu\, \delta^{(4)}(x-y)\,.$$

In other words, the propagator is the Green's function—i.e. the inverse—of the quadratic differential operator that governs the free field. Physically, this is why the propagator describes how a disturbance created at one spacetime point propagates to another: it encodes all the information about the free dynamics. In position space finding this inverse requires solving a PDE, but Fourier transforming ($\partial_\mu \to ik_\mu$, so $\Box \to -k^2$ and $\partial^\mu\partial^\nu \to -k^\mu k^\nu$) turns the differential operator into an algebraic one:

$$\widetilde{O}^{\mu\nu}(k) = -k^2 g^{\mu\nu} + \left(1 - \frac{1}{\xi_0}\right) k^\mu k^\nu. \tag{6.208}$$

Without the gauge-fixing term ($\xi_0 \to \infty$, equivalently dropping $\mathcal{L}_{\text{gf}}$), this reduces to $\widetilde{O}^{\mu\nu}_{\text{Maxwell}}(k) = -k^2 g^{\mu\nu} + k^\mu k^\nu$, which is manifestly non-invertible: contracting with k_ν gives

$$\widetilde{O}^{\mu\nu}_{\text{Maxwell}}(k)\, k_\nu = -k^2 k^\mu + k^\mu k^2 = 0,$$

so k^μ is a null eigenvector for every k. (This is the momentum-space image of the position-space pure-gauge mode $A_\mu = \partial_\mu \alpha$, which becomes $\widetilde{A}_\mu(k) = ik_\mu \widetilde{\alpha}(k)$.) Equivalently, the matrix $\widetilde{O}^{\mu\nu}_{\text{Maxwell}}(k) = -k^2(g^{\mu\nu} - k^\mu k^\nu/k^2)$ is $-k^2$ times the transverse projector, which has rank 3 (not 4) and hence vanishing determinant, so its inverse does not exist. The gauge-fixing term restores the longitudinal piece via the $-k^\mu k^\nu/\xi_0$ contribution in (6.208), lifting the zero mode and making the operator invertible. We need $D_{0\alpha\nu}(k)$ such that $\widetilde{O}^{\mu\alpha} D_{0\alpha\nu} = i\,\delta^\mu_\nu$. Since the only available tensors built from k^μ are $g^{\mu\nu}$ and $k^\mu k^\nu$, we write the ansatz $D_0^{\mu\nu} = A\, g^{\mu\nu} + B\, k^\mu k^\nu$ and substitute:

$$\widetilde{O}^{\mu\alpha} D_{0\alpha\nu} = -k^2 A\ \delta^\mu_\nu + \left[-k^2 B + \left(1 - \frac{1}{\xi_0}\right)(A + Bk^2)\right] k^\mu k_\nu \stackrel{!}{=} i\,\delta^\mu_\nu.$$

Matching the δ^μ_ν and $k^\mu k_\nu$ structures separately:

$$A = \frac{-i}{k^2}, \qquad B = \frac{i\,(1-\xi_0)}{k^4},$$

which gives the covariant-gauge propagator (adding the Feynman $i\epsilon$ prescription for time ordering):

$$D_0^{\mu\nu}(k) = \frac{-i}{k^2 + i\epsilon}\left[g^{\mu\nu} - (1-\xi_0)\frac{k^\mu k^\nu}{k^2}\right]. \tag{6.209}$$

Special choices: $\xi_0 = 1$ (Feynman gauge), $\xi_0 \to 0$ (Landau gauge). Thus, the gauge-fixing term is not a physical interaction but a device needed to define the propagator and perform perturbation theory in a manifestly Lorentz-covariant way.

Field and parameter renormalization

We introduce *renormalized* fields (ψ, A_μ) and parameters (m, e, ξ) by

$$\psi_0 = \sqrt{Z_2}\,\psi, \qquad A_{0\mu} = \sqrt{Z_3}\,A_\mu, \tag{6.210}$$

$$m_0 = m + \delta m_0, \qquad e_0 = \mu^{\epsilon/2} Z_e\, e, \qquad \xi_0 = Z_3\,\xi, \tag{6.211}$$

where μ is the renormalization scale (introduced so that e remains dimensionless for $d \neq 4$). The renormalization constants Z_2, Z_3, Z_e and the bare mass shift δm_0 are determined order by order to cancel UV poles. After splitting the Lagrangian, it is convenient to trade δm_0 for the coefficient δm of the local counterterm $-\delta m\,\bar\psi\psi$.

Origin of $\xi_0 = Z_3\xi$. The bare gauge-fixing term, expressed in renormalized fields via $A_{0\mu} = \sqrt{Z_3}\,A_\mu$, becomes

$$-\frac{1}{2\xi_0}(\partial\cdot A_0)^2 = -\frac{Z_3}{2\xi_0}(\partial\cdot A)^2,$$

i.e. a gauge-fixing term for the renormalized field with effective parameter $\xi_{\text{eff}} = \xi_0/Z_3$. Defining the renormalized gauge parameter as $\xi \equiv \xi_{\text{eff}} = \xi_0/Z_3$—equivalently $\xi_0 = Z_3\xi$—ensures that the gauge-fixing term retains its canonical form $-\frac{1}{2\xi}(\partial\cdot A)^2$ in renormalized variables. This convention reflects the fact that $\mathcal{L}_{\text{gf}}$ is not gauge invariant, so it does not receive its own independent renormalization: the only Z factor that appears is the photon wavefunction renormalization Z_3, which is fully absorbed by the rescaling $\xi_0 \to \xi$. As a check, the bare and renormalized photon propagators have the same tensor structure, related simply by $D_0^{\mu\nu}(k;\xi_0) = Z_3\,D^{\mu\nu}(k;\xi)$.

It is customary to trade Z_e for the *vertex* renormalization constant Z_1, defined by

$$Z_1 \equiv Z_e\,Z_2\,Z_3^{1/2}. \tag{6.212}$$

With this definition, substituting (6.210)–(6.211) into the bare interaction gives the identity

$$\begin{aligned} e_0\,\bar\psi_0\gamma^\mu\psi_0\,A_{0\mu} &= \left(\mu^{\epsilon/2}Z_e e\right)\left(Z_2\bar\psi\gamma^\mu\psi\right)\left(\sqrt{Z_3}\,A_\mu\right) \\ &= \mu^{\epsilon/2}\underbrace{Z_e Z_2 Z_3^{1/2}}_{=Z_1}\, e\,\bar\psi\gamma^\mu\psi\,A_\mu = \mu^{\epsilon/2}\,Z_1\,e\,\bar\psi\gamma^\mu\psi\,A_\mu, \end{aligned} \tag{6.213}$$

so Z_1 is precisely the multiplicative factor that converts the bare interaction term into a renormalized one (tree-level vertex plus counterterm: $Z_1 = 1 + \delta Z_1$).

Why the renormalization scale μ is needed.

In d dimensions the action must be dimensionless: $[S] = 0$, hence $[\mathcal{L}] = d$. From the kinetic terms one reads off

$$[\psi] = \frac{d-1}{2}, \qquad [A] = \frac{d-2}{2}. \tag{6.214}$$

Requiring $[\mathcal{L}_{\text{int}}] = [e\bar{\psi}\gamma^\mu\psi A_\mu] = d$ gives

$$[e] = \frac{4-d}{2} = \frac{\epsilon}{2}. \tag{6.215}$$

Thus, in $d \neq 4$ the bare coupling is dimensionful. To keep the *renormalized* coupling e dimensionless, one writes

$$e_0 = \mu^{\epsilon/2} Z_e\, e, \tag{6.216}$$

where μ is an arbitrary mass scale. This ensures that $[e] = 0$ while $[e_0] = \epsilon/2$ as required.

This has two consequences:

- UV divergences appear as $1/\epsilon$ poles, while finite parts contain logarithms $\ln(\mu^2/m^2)$;
- changing μ changes e (at fixed e_0), which is the origin of the renormalization-group running.

Splitting the Lagrangian: $\mathcal{L}_{\text{ren}} + \delta\mathcal{L}$

Substituting the renormalization relations into $\mathcal{L}_0$, one can reorganize it as

$$\mathcal{L}_0 = \mathcal{L}_{\text{ren}} + \delta\mathcal{L}, \tag{6.217}$$

where the *renormalized Lagrangian* has the same form as QED written in terms of renormalized quantities:

$$\mathcal{L}_{\text{ren}} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\,(i\partial\!\!\!/ - m)\psi + \mu^{\epsilon/2} e\,\bar{\psi}\gamma^\mu\psi\, A_\mu - \frac{1}{2\xi}\,(\partial\cdot A)^2, \tag{6.218}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and the *counterterm Lagrangian* collects all remaining terms:

$$\begin{aligned}\delta\mathcal{L} = &-\frac{1}{4}\,(Z_3 - 1)\,F_{\mu\nu}F^{\mu\nu} + (Z_2 - 1)\,\bar{\psi}\, i\partial\!\!\!/\,\psi - \delta m\,\bar{\psi}\psi \\ &+ \mu^{\epsilon/2}(Z_1 - 1)\, e\,\bar{\psi}\gamma^\mu\psi\, A_\mu - \frac{1}{2}\left(\frac{Z_3}{\xi_0} - \frac{1}{\xi}\right)(\partial\cdot A)^2.\end{aligned} \tag{6.219}$$

With the choice $\xi_0 = Z_3\xi$, the gauge-fixing counterterm vanishes (or can be absorbed). The key point is that $\delta\mathcal{L}$ is treated as an *interaction* in perturbation theory: its terms generate additional Feynman rules (counterterm insertions) whose coefficients are fixed order by order to cancel UV poles.

Detailed derivation of $\mathcal{L}_{\text{ren.}}$ and $\delta\mathcal{L}$

Step 1: Gauge kinetic term. From $A_{0\mu} = \sqrt{Z_3}A_\mu$ we have $F_{0\mu\nu} = \sqrt{Z_3}F_{\mu\nu}$, hence

$$-\frac{1}{4} F_{0\mu\nu}F_0^{\mu\nu} = -\frac{1}{4} Z_3\, F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}\,(Z_3 - 1)\,F_{\mu\nu}F^{\mu\nu}. \tag{6.220}$$

Step 2: Fermion kinetic and mass terms. Using $\psi_0 = \sqrt{Z_2}\psi$:

$$\bar{\psi}_0 i\partial\!\!\!/ \psi_0 = Z_2 \bar{\psi} i\partial\!\!\!/ \psi = \bar{\psi} i\partial\!\!\!/ \psi + (Z_2 - 1)\bar{\psi} i\partial\!\!\!/ \psi. \tag{6.221}$$

For the mass term, with $m_0 = m + \delta m_0$:

$$-m_0 \bar{\psi}_0 \psi_0 = -(m + \delta m_0) Z_2 \bar{\psi}\psi. \tag{6.222}$$

Expanding and defining

$$\delta m \equiv m(Z_2 - 1) + Z_2 \delta m_0,$$

so that δm is the coefficient of $-\bar{\psi}\psi$ in $\delta\mathcal{L}$, one obtains

$$\bar{\psi}_0(i\partial\!\!\!/ - m_0)\psi_0 = \bar{\psi}(i\partial\!\!\!/ - m)\psi + (Z_2 - 1)\bar{\psi} i\partial\!\!\!/ \psi - \delta m \bar{\psi}\psi. \tag{6.223}$$

Step 3: Interaction term. Using (6.212):

$$\begin{aligned} +e_0 \bar{\psi}_0 \gamma^\mu \psi_0 A_{0\mu} &= +\mu^{\epsilon/2} Z_1 e \bar{\psi}\gamma^\mu \psi A_\mu \\ &= +\mu^{\epsilon/2} e \bar{\psi}\gamma^\mu \psi A_\mu + \mu^{\epsilon/2}(Z_1 - 1) e \bar{\psi}\gamma^\mu \psi A_\mu. \end{aligned} \tag{6.224}$$

Step 4: Gauge-fixing term. Since $\partial \cdot A_0 = \sqrt{Z_3}\,\partial \cdot A$:

$$-\frac{1}{2\xi_0}(\partial \cdot A_0)^2 = -\frac{Z_3}{2\xi_0}(\partial \cdot A)^2. \tag{6.225}$$

With $\xi_0 = Z_3 \xi$ this equals $-\frac{1}{2\xi}(\partial \cdot A)^2$, so the gauge-fixing term is automatically renormalized. If one keeps $\xi_0 \neq Z_3\xi$, an additional counterterm $-\frac{1}{2}\left(\frac{Z_3}{\xi_0} - \frac{1}{\xi}\right)(\partial \cdot A)^2$ appears.

Step 5: Collect results. Summing the "1" pieces gives $\mathcal{L}_{\text{ren}}$ (6.218); summing the "$(Z_i - 1)$" and "δm" pieces gives $\delta\mathcal{L}$ (6.219).

Counterterm Feynman rules

The counterterms in $\delta\mathcal{L}$ generate additional vertices in perturbation theory. In momentum space:

$$\text{Electron 2-point:} \qquad i\Sigma_{\text{ct}}(p\!\!\!/) = i\left[(Z_2 - 1)p\!\!\!/ - \delta m\right], \tag{6.226}$$

$$\text{Photon 2-point:} \qquad i\Pi^{\mu\nu}_{\text{ct}}(q) = -i(Z_3 - 1)\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right), \tag{6.227}$$

$$\text{Vertex 3-point:} \qquad i\Gamma^\mu_{\text{ct}} = +ie\,\mu^{\epsilon/2}(Z_1 - 1)\gamma^\mu. \tag{6.228}$$

These insertions are included in Feynman diagrams exactly like ordinary vertices. Their role is to cancel the UV-divergent parts of loop diagrams.

Counterterm diagrams: detailed discussion.

In perturbation theory, one computes Green functions using $\mathcal{L}_{\text{ren}}$ as the "main" Lagrangian and treats $\delta\mathcal{L}$ as additional interaction vertices. To a given loop order, one includes all diagrams built from the usual QED vertices *plus* diagrams where one (or more) vertex is replaced by a counterterm insertion.

(i) Fermion two-point counterterm. From

$$\delta\mathcal{L}_{\bar\psi\psi} = (Z_2 - 1)\,\bar\psi\, i\partial\!\!\!/\,\psi - \delta m\,\bar\psi\psi, \tag{6.229}$$

one obtains the 1PI insertion (6.226) on a fermion line. Diagrammatically:

insert (6.226).

The $(Z_2 - 1)$ term renormalizes the wavefunction (proportional to $\not{p}$), while δm renormalizes the mass (proportional to the identity).

(ii) Photon two-point counterterm. From

$$\delta\mathcal{L}_{AA} = -\frac{1}{4}\,(Z_3 - 1)\,F_{\mu\nu}F^{\mu\nu}, \tag{6.230}$$

expanding to quadratic order gives (6.227). Diagrammatically:

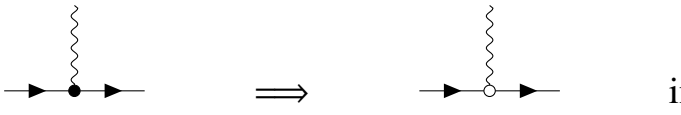
insert (6.227).

This counterterm cancels the UV-divergent vacuum polarization and implements the photon renormalization condition.

(iii) Vertex counterterm. From

$$\delta\mathcal{L}_{\bar\psi\psi A} = +(Z_1 - 1)\,e\,\bar\psi\gamma^\mu\psi\,A_\mu, \tag{6.231}$$

one obtains the vertex insertion (6.228). Diagrammatically:



insert (6.228).

This counterterm cancels the UV-divergent one-loop vertex correction. The Ward identity $Z_1 = Z_2$ (see below) relates the vertex and fermion wavefunction counterterms.

(iv) Gauge-fixing counterterm (optional). If $\xi_0 \neq Z_3\xi$, the term

$$\delta\mathcal{L}_{\text{gf}} = -\frac{1}{2}\left(\frac{1}{\xi_0} - \frac{1}{\xi}\right)(\partial\cdot A)^2 \tag{6.232}$$

modifies only the longitudinal photon propagator and does not affect physical S-matrix elements. With $\xi_0 = Z_3\xi$ this term vanishes.

One-loop divergences and renormalization conditions

At one loop, UV divergences arise from three 1PI Green functions:

$$\Sigma(\not p) \qquad \text{(electron self-energy)}, \tag{6.233}$$
$$\Pi^{\mu\nu}(q) \qquad \text{(photon vacuum polarization)}, \tag{6.234}$$
$$\Gamma^{\mu}(p',p) \qquad \text{(vertex correction)}. \tag{6.235}$$

Renormalization means choosing $(Z_2-1),(Z_3-1),(Z_1-1),\delta m$ such that, order by order in e, the sums

$$\Sigma(\not p)+\Sigma_{\rm ct}(\not p), \tag{6.236}$$
$$\Pi^{\mu\nu}(q)+\Pi^{\mu\nu}_{\rm ct}(q), \tag{6.237}$$
$$\Gamma^{\mu}(p',p)+\Gamma^{\mu}_{\rm ct}, \tag{6.238}$$

are UV finite and satisfy the chosen *renormalization conditions*.

On-shell scheme.

One imposes that renormalized propagators have poles at the physical mass with unit residues, and that the renormalized charge equals the physical charge at zero momentum transfer.

For the electron propagator

$$S_R(\not p)=\frac{i}{\not p - m - \Sigma_R(\not p)}, \qquad \Sigma_R \equiv \Sigma+\Sigma_{\rm ct}, \tag{6.239}$$

the on-shell conditions are

$$\Sigma_R(\not p)\Big|_{\not p=m}=0, \qquad \frac{\partial}{\partial \not p}\Sigma_R(\not p)\Big|_{\not p=m}=0, \tag{6.240}$$

which determine δm and Z_2.

For the photon, one requires transversality and that the renormalized vacuum polarization vanishes at $q^2=0$ (so the photon remains massless):

$$\Pi^{\mu\nu}_R(q)=\left(q^2 g^{\mu\nu}-q^\mu q^\nu\right)\Pi_R(q^2), \qquad \Pi_R(0)=0, \tag{6.241}$$

which fixes Z_3.

The charge is fixed by a condition on the vertex at $q=p'-p=0$.

$\overline{\rm MS}$ scheme

Alternatively, one chooses counterterms to subtract only the UV poles (and the standard $\overline{\rm MS}$ constants) in dimensional regularization. Then $Z_2,Z_3,Z_1,\delta m$ contain only $1/\epsilon$-type terms:

$$Z_i-1=\frac{c_i}{\epsilon}+\cdots, \qquad \delta m=\frac{c_m}{\epsilon}m+\cdots, \tag{6.242}$$

where the coefficients c_i, c_m are determined by the divergent parts of the loop integrals.

Ward identity and relations among renormalization constants

Gauge invariance implies the Ward–Takahashi identity, which enforces (to all orders)

$$Z_1 = Z_2 \,. \tag{6.243}$$

This is a powerful constraint: the vertex renormalization is not independent of the fermion wavefunction renormalization. Consequently, the charge renormalization is controlled entirely by the photon field renormalization:

$$Z_e = \frac{Z_1}{Z_2 Z_3^{1/2}} = \frac{1}{Z_3^{1/2}} \,. \tag{6.244}$$

This is one of the main simplifications of QED renormalization.

Practical workflow

To renormalize QED at a given loop order:

1. Compute the divergent parts of Σ, $\Pi^{\mu\nu}$, Γ^{μ} in $d = 4 - \epsilon$.
2. Choose a renormalization scheme (on-shell or $\overline{\text{MS}}$).
3. Solve for $(Z_2 - 1)$, $(Z_3 - 1)$, $(Z_1 - 1)$, δm so that the renormalized 1PI functions satisfy the chosen conditions.
4. Include counterterm insertions consistently in higher-loop diagrams.

Worked example: renormalization of the electron propagator

Aim. We present a complete example of renormalization focusing on the electron two-point function at one loop. We show that the counterterm method follows naturally from rewriting the bare Lagrangian in terms of renormalized quantities.

1. Setup: bare theory and the split $\mathcal{L}_0 = \mathcal{L}_{\text{ren.}} + \delta\mathcal{L}$

Start from the bare Lagrangian (6.204) and introduce renormalized fields/parameters via (6.210)–(6.211). The split $\mathcal{L}_0 = \mathcal{L}_{\text{ren}} + \delta\mathcal{L}$ was derived above. The fermion two-point counterterm is

$$\delta\mathcal{L}_{\bar\psi\psi} = (Z_2 - 1)\,\bar\psi\, i\partial\!\!\!/\,\psi - \delta m\,\bar\psi\psi \,. \tag{6.245}$$

2. The renormalized self-energy.

The full renormalized electron propagator is

$$S_R(p\!\!\!/) = \frac{i}{p\!\!\!/ - m - \Sigma_R(p\!\!\!/) + i\epsilon} \,. \tag{6.246}$$

At one loop, the renormalized self-energy is

$$\Sigma_R(p\!\!\!/) = \Sigma(p\!\!\!/) + \Sigma_{\text{ct}}(p\!\!\!/) = \Sigma(p\!\!\!/) + (Z_2 - 1)p\!\!\!/ - \delta m, \tag{6.247}$$

where $\Sigma(p\!\!\!/)$ is the loop self-energy (UV divergent in $d = 4 - \epsilon$).

3. On-shell renormalization conditions.

In the on-shell scheme we demand:

- the propagator has a pole at the physical mass m;
- the residue at the pole is 1.

Write the general Dirac decomposition of the self-energy:

$$\Sigma(\not{p}) = \not{p}\,\Sigma_V(p^2) + m\,\Sigma_S(p^2), \tag{6.248}$$

where Σ_V and Σ_S are scalar functions containing UV poles. Then

$$\Sigma_R(\not{p}) = \not{p}\Big[\Sigma_V(p^2) + (Z_2 - 1)\Big] + m\,\Sigma_S(p^2) - \delta m. \tag{6.249}$$

Pole condition. The inverse propagator vanishes at the pole:

$$S_R^{-1}(\not{p}) = \not{p} - m - \Sigma_R(\not{p}) = 0 \quad \text{at} \quad \not{p} = m\,. \tag{6.250}$$

Evaluating at $\not{p} = m$ gives

$$\delta m = m\Big[\Sigma_V(m^2) + \Sigma_S(m^2)\Big] + m(Z_2 - 1). \tag{6.251}$$

Residue condition. Near the pole, the residue-one condition requires

$$\frac{\partial}{\partial \not{p}}\Sigma_R(\not{p})\Big|_{\not{p}=m} = 0\,. \tag{6.252}$$

Using (6.247):

$$\frac{\partial}{\partial \not{p}}\Sigma_R = \frac{\partial}{\partial \not{p}}\Sigma + (Z_2 - 1), \tag{6.253}$$

so

$$Z_2 - 1 = -\frac{\partial}{\partial \not{p}}\Sigma(\not{p})\Big|_{\not{p}=m}\,. \tag{6.254}$$

Inserting (6.254) into (6.251) determines δm purely in terms of Σ evaluated on-shell.

4. $\overline{\text{MS}}$ scheme

In $\overline{\text{MS}}$, one does not impose on-shell conditions. Instead, one chooses Z_2 and δm to cancel only the $1/\epsilon$ poles. If the one-loop result has the form

$$\Sigma(\not{p}) = \frac{1}{\epsilon}\Big(A\,\not{p} + B\,m\Big) + \text{finite}, \tag{6.255}$$

then one sets

$$Z_2 - 1 = -\frac{A}{\epsilon} + \cdots, \tag{6.256}$$

$$\delta m = \frac{B\,m}{\epsilon} + \cdots, \tag{6.257}$$

so that $\Sigma_R(\not{p})$ is finite.

5. Final result.

Combining the loop self-energy and counterterm, the renormalized inverse propagator is

$$S_R^{-1}(\not{p}) = \not{p} - m - \Sigma(\not{p}) - (Z_2 - 1)\not{p} + \delta m\,. \tag{6.258}$$

By construction, the UV divergent $1/\epsilon$ parts cancel between $\Sigma(\not{p})$ and the counterterm contributions. The propagator is finite and satisfies the chosen renormalization conditions.

6.7 General renormalization theory: power counting and the classification of divergences

So far in this chapter, we have carried out the renormalization of QED diagram by diagram. It is natural to ask: *how do we know in advance which diagrams are divergent?* And more fundamentally: *what makes QED special, in that only a finite number of counterterms suffice to remove all divergences?*

This section addresses these questions by developing the general theory of ultraviolet divergences, following Weinberg [7], Chapter 12. The central result is that the degree of divergence of any Feynman diagram can be determined by simple power counting, and that the requirement of **renormalizability**—the property that only finitely many types of counterterms are needed—places severe constraints on the allowed interactions. We derive all the key formulas in detail.

The superficial degree of divergence

Consider a general quantum field theory containing fields of various types, labelled f, and interactions of various types, labelled i. Each interaction of type i involves n_{if} fields of type f and d_i derivatives. We want to determine when a given Feynman diagram produces a divergent momentum-space integral.

Definition. The **superficial degree of divergence** D of a Feynman diagram is defined as the number of powers of loop momentum in the numerator of the integrand minus the number in the denominator, plus four for each independent loop momentum. In the region where all internal momenta go to infinity together with a common factor $\kappa \to \infty$, the integral behaves as:

$$\int^{\infty} \kappa^{D-1}\, d\kappa\,. \tag{6.259}$$

Therefore:

- $D > 0$: the diagram is **power-law divergent** (the integral diverges as Λ^D);
- $D = 0$: the diagram is **logarithmically divergent** ($\sim \ln \Lambda$);
- $D < 0$: the diagram is **superficially convergent** (at least in the region where all momenta grow together).

Notation. For a given connected, one-particle-irreducible (1PI) Feynman diagram, define:

$$\begin{aligned} I_f &\equiv \text{number of internal lines of field type } f\,, \\ E_f &\equiv \text{number of external lines of field type } f\,, \\ N_i &\equiv \text{number of vertices of interaction type } i\,. \end{aligned} \tag{6.260}$$

Propagator behavior and the spin parameter s_f

The asymptotic behavior of the propagator $\Delta_f(k)$ for a field of type f at large momentum k is written as:

$$\Delta_f(k) \sim k^{-2+2s_f} \qquad \text{for } |k| \to \infty\,, \tag{6.261}$$

where s_f is a parameter related to the spin of the field.

Box 6.3 Values of s_f for common fields

- **Scalar field** (ϕ): The propagator is $\Delta(k) = i/(k^2 - m^2)$, which behaves as k^{-2} at large k. Hence $s_\phi = 0$.
- **Dirac field** (ψ): The propagator is $S(k) = i(\not{k} + m)/(k^2 - m^2)$, which behaves as k^{-1} at large k (the numerator contributes one power of k). Hence $s_\psi = 1/2$.
- **Massive vector field** (A_μ, Proca): The propagator tends to a (non-vanishing) *constant* at large $|k|$—i.e. scales as $|k|^0$, the same as k^{-2+2s_f} with $s_f = 1$—because of the $k_\mu k_\nu/m^2$ term. Hence $s_A = 1$ for a massive vector.[a]
- **Photon** (A_μ, massless gauge field): Thanks to gauge invariance, the effective propagator in any gauge is $\sim k^{-2}$. Hence $s_\gamma = 0$, the same as a scalar.

More generally, for a massive field in the Lorentz representation (A, B) one has $s_f = A + B$. Here (A, B) are the two half-integer labels of the standard classification of irreducible representations of the (complexified) Lorentz algebra $\mathfrak{so}(1,3)_{\mathbb{C}} \simeq \mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$, so each of $A, B \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \dots\}$ labels an SU(2) irrep. The familiar fields correspond to:

$(0,0)$	scalar	$s_f = 0$
$(\frac{1}{2},0)$, $(0,\frac{1}{2})$	Weyl (L, R) spinors; Dirac is their sum	$s_f = \frac{1}{2}$
$(\frac{1}{2},\frac{1}{2})$	vector A_μ	$s_f = 1$
$(1,0)\oplus(0,1)$	antisymmetric tensor $F_{\mu\nu}$	$s_f = 1$

The total spin contained in (A,B) ranges from $|A-B|$ to $A+B$; the *maximum* spin $A+B$ controls the worst high-momentum behaviour of the propagator, which is what enters power counting. The photon is special: despite being a spin-1 particle, gauge invariance ensures that it propagates like a scalar at high momenta. This is crucial for the renormalizability of QED.

[a]The Proca Lagrangian is $\mathcal{L}_{\text{Proca}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu$. The mass term breaks gauge invariance, but it makes the quadratic form invertible without any gauge-fixing term: the equations of motion $\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0$ imply $\partial_\mu A^\mu = 0$ as a consistency condition (taking ∂_ν on both sides), so the field automatically carries $4-1=3$ physical polarizations of a massive spin-1 particle. The momentum-space propagator is

$$D^{\mu\nu}_{\text{Proca}}(k) = \frac{-i}{k^2 - m^2 + i\epsilon}\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{m^2}\right).$$

The longitudinal piece $k^\mu k^\nu/m^2$ grows like $|k|^2$, which combined with the $1/(k^2-m^2) \sim |k|^{-2}$ prefactor gives a non-decaying contribution: $D^{\mu\nu}_{\text{Proca}}(k) \to (i/m^2)\,\hat{k}^\mu \hat{k}^\nu + O(|k|^{-2})$ as $|k| \to \infty$, i.e. a constant tensor. In the power-counting formula (6.261) this corresponds to $s_A = 1$. This bad UV behaviour is the reason why non-gauge-invariant massive vector theories are generically non-renormalizable; the gauge-invariant resolution (Higgs mechanism) restores the soft $\sim |k|^{-2}$ behaviour.

Derivation of the formula for D

We now compute D by counting powers of momentum in a generic Feynman diagram.

Step 1: Contribution from propagators. Each internal line of type f contributes a propagator $\Delta_f(k) \sim k^{-2+2s_f}$. With I_f such lines, the total contribution to D is:

$$\text{propagators:} \qquad \sum_f I_f(-2+2s_f)\,. \tag{6.262}$$

Step 2: Contribution from vertices. Each vertex of type i carries d_i derivatives, which in momentum space become factors of momenta. With N_i such vertices, the contribution is:

$$\text{vertices:} \qquad \sum_i N_i\, d_i\,. \tag{6.263}$$

Step 3: Number of independent loop momenta. Each of the $\sum_f I_f$ internal lines can be labelled with an independent four-momentum. However, the momentum-conserving delta function at each vertex imposes a constraint on these momenta. With $\sum_i N_i$ vertices, we have $\sum_i N_i$ delta functions, but one of them merely enforces overall momentum conservation and does not constrain internal momenta. The number of independent loop momenta is therefore:

$$L = \sum_f I_f - \sum_i N_i + 1\,, \tag{6.264}$$

which is exactly the number of independent loops in the diagram. Each loop contributes a four-dimensional integration d^4k, giving a factor of +4 per loop:

$$\text{loops:} \qquad 4L = 4\left[\sum_f I_f - \sum_i N_i + 1\right]. \tag{6.265}$$

Step 4: Combining all contributions. Adding (6.262), (6.263), and (6.265):

$$\begin{aligned} D &= \sum_f I_f(-2+2s_f) + \sum_i N_i d_i + 4\sum_f I_f - 4\sum_i N_i + 4 \\ &= \sum_f I_f(2+2s_f) + \sum_i N_i(d_i-4) + 4\,. \end{aligned} \tag{6.266}$$

This is a correct formula, but it is inconvenient: D depends on the internal structure of the diagram (the numbers I_f of internal lines), which varies from diagram to diagram even for the same physical process. We want a formula that depends only on the *external* content of the diagram.

The topological identity

To eliminate I_f, we use a fundamental identity that relates internal and external lines to vertices.

Topological identity. Consider all the lines of type f attached to vertices. Each interaction of type i has n_{if} lines of type f, so the total number of line-ends at vertices is $\sum_i N_i n_{if}$. Now, each *internal* line connects two vertices (contributing 2 line-ends), while each *external* line connects to only one vertex (contributing 1 line-end). Therefore:

$$\boxed{2I_f + E_f = \sum_i N_i n_{if}} \tag{6.267}$$

This identity holds for each field type f separately. Solving for I_f:

$$I_f = \frac{1}{2}\left(\sum_i N_i n_{if} - E_f\right). \tag{6.268}$$

The master formula for D

Substituting (6.268) into (6.266):

$$\begin{aligned} D &= \sum_f \frac{1}{2}\left(\sum_i N_i n_{if} - E_f\right)(2+2s_f) + \sum_i N_i(d_i-4)+4 \\ &= \sum_f \sum_i N_i n_{if}(1+s_f) - \sum_f E_f(1+s_f) + \sum_i N_i(d_i-4)+4 \\ &= \sum_i N_i\left[d_i + \sum_f n_{if}(1+s_f) - 4\right] - \sum_f E_f(1+s_f)+4\,. \end{aligned} \tag{6.269}$$

Defining the **index of divergence** of an interaction of type i:

$$\boxed{\Delta_i \equiv 4 - d_i - \sum_f n_{if}(1+s_f)} \tag{6.270}$$

the formula for D becomes:

$$\boxed{D = 4 - \sum_f E_f(s_f+1) - \sum_i N_i \Delta_i} \tag{6.271}$$

This is the **master formula** for the superficial degree of divergence. Its power lies in the fact that it separates the dependence on the physical process (through E_f) from the dependence on the specific diagram (through N_i).

Box 6.4 Dimensional analysis derivation of Δ_i

The parameter Δ_i has a direct interpretation in terms of dimensional analysis (in natural units $\hbar = c = 1$, where [mass] = [momentum] = [energy] = [length]$^{-1}$).
Step 1: Field dimensionality. The propagator of a field f is the Fourier transform of a vacuum expectation value of a product of two fields:

$$\Delta_f(k) = \int d^4x\, e^{ik\cdot x}\langle 0|T\{\phi_f(x)\phi_f(0)\}|0\rangle\,. \tag{6.272}$$

The integration d^4x has dimension [mass]$^{-4}$, and the propagator behaves as k^{-2+2s_f}, so:

$$[\Delta_f] = -4 + 2\mathscr{D}_f = -2 + 2s_f \quad\Longrightarrow\quad \mathscr{D}_f = 1 + s_f\,, \tag{6.273}$$

where $\mathscr{D}_f$ is the mass dimension of the field ϕ_f.
For the standard fields:

Field	s_f	$\mathscr{D}_f = 1 + s_f$
Scalar ϕ	0	1
Dirac ψ	1/2	3/2
Photon A_μ	0	1

Step 2: Coupling constant dimensionality. An interaction i with n_{if} fields and d_i derivatives has dimension $d_i + \sum_f n_{if}\mathscr{D}_f$. The action $S = \int d^4x\,\mathcal{L}$ must be dimensionless, so $\mathcal{L}$ has dimension $[\text{mass}]^4$. The coupling constant g_i must therefore have dimension:

$$[g_i] = 4 - d_i - \sum_f n_{if}(1 + s_f) = \Delta_i\,. \tag{6.274}$$

Thus, Δ_i **is nothing but the mass dimension of the coupling constant** of interaction i.

Classification of interactions: renormalizable, super-renormalizable, and non-renormalizable

The master formula (6.271) immediately leads to a classification of interactions based on Δ_i.

Case 1: $\Delta_i \geq 0$ **for all interactions (renormalizable theories).**

If all interactions satisfy $\Delta_i \geq 0$, the term $-\sum_i N_i\Delta_i$ is non-positive, and the formula gives an **upper bound** on D that depends only on the external content of the diagram:

$$D \leq 4 - \sum_f E_f(s_f + 1)\,. \tag{6.275}$$

Only a *finite* number of combinations of external lines (E_f) can give $D \geq 0$. This means that only a finite number of types of diagrams are divergent, and therefore only a finite number of counterterms are needed. Such theories are called **renormalizable**.

Within the renormalizable class, one further distinguishes:

- $\Delta_i = 0$: **marginal** (or strictly renormalizable) interactions. The coupling is dimensionless.
- $\Delta_i > 0$: **relevant** (or super-renormalizable) interactions. Adding more vertices *lowers* D; these interactions produce divergences only at low orders.

Case 2: $\Delta_i < 0$ **for some interactions (non-renormalizable theories).**

If any interaction has $\Delta_i < 0$, then adding more vertices of that type *increases* D. No matter how many external lines we have, by including enough such vertices we can make $D \geq 0$. This means that *every* amplitude eventually diverges at sufficiently high order, and infinitely many different counterterms are needed. Such interactions are called **non-renormalizable** (or **irrelevant**), and theories containing them are called non-renormalizable.

Type	Δ_i	$[g_i]$	**Effect on** D
Super-renormalizable (relevant)	> 0	positive mass dim.	lowers D
Renormalizable (marginal)	$= 0$	dimensionless	does not change D
Non-renormalizable (irrelevant)	< 0	negative mass dim.	raises D

Application to QED

Let us verify that QED is renormalizable by computing Δ_i for each interaction in the QED Lagrangian:

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(i\not{\partial} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + e\bar{\psi}\gamma^\mu\psi A_\mu + \mathcal{L}_{\text{ct}}, \tag{6.276}$$

where $\mathcal{L}_{\text{ct}}$ contains the counterterms.

Interaction	d_i	$n_{i\gamma}$	n_{ie}	$\Delta_i = 4 - d_i - n_{i\gamma} - \frac{3}{2}n_{ie}$	**Type**
$e\bar{\psi}\gamma^\mu\psi A_\mu$	0	1	2	$4-0-1-3=0$	marginal
$F_{\mu\nu}F^{\mu\nu}$ (kinetic)	2	2	0	$4-2-2-0=0$	marginal
$\bar{\psi}\not{\partial}\psi$ (kinetic)	1	0	2	$4-1-0-3=0$	marginal
$m\bar{\psi}\psi$ (mass)	0	0	2	$4-0-0-3=1$	super-ren.

Since all $\Delta_i \geq 0$, **QED is renormalizable**. The bound (6.275) gives, with $s_\gamma = 0$ and $s_e = 1/2$:

$$\boxed{D \leq 4 - E_\gamma - \frac{3}{2}E_e} \tag{6.277}$$

Box 6.5 Enumeration of superficially divergent amplitudes in QED

The bound (6.277) lists the 1PI amplitudes with $D \geq 0$. Since E_e is even (fermion lines come in pairs), the possibilities are:

E_e	E_γ	$D \leq$	**Amplitude**	**Status**
0	1	3	Photon tadpole	Vanishes (Furry)
0	2	2	Photon self-energy $\Pi^{\mu\nu}$	Divergent
0	3	1	Three-photon vertex	Vanishes (Furry)
0	4	0	Light-by-light	Finite (gauge inv.)
2	0	1	Electron self-energy Σ	Divergent
2	1	0	Vertex Γ^μ	Divergent
1	0,1,2	$\frac{5}{2}, \frac{3}{2}, \frac{1}{2}$	Odd-fermion	Absent (fermion-# cons.)

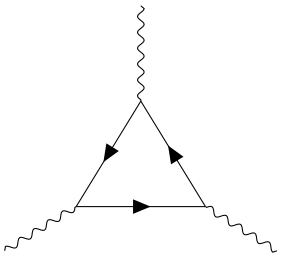
Only **three** 1PI amplitudes are genuinely divergent in QED:

- Photon self-energy $\Pi^{\mu\nu}(q)$, $D \leq 2$ — absorbed by Z_3.
- Electron self-energy $\Sigma(p)$, $D \leq 1$ — absorbed by Z_2 and δm.
- Vertex $\Gamma^\mu(p', p)$, $D \leq 0$ — absorbed by Z_1 (with $Z_1 = Z_2$ by Ward).

These are exactly the three structures renormalized in the preceding sections: power counting guarantees *no other divergences* can appear, at any order.

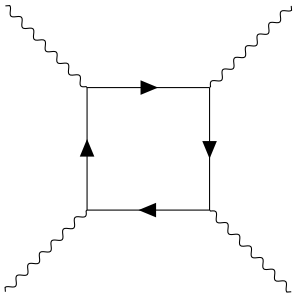
Remark: the role of symmetries. Several amplitudes that power counting allows to diverge are actually finite thanks to symmetry:

- The **three-photon vertex** ($E_\gamma = 3$, $D \leq 1$) vanishes by **Furry's theorem** (charge conjugation invariance): any closed fermion loop with an odd number of photon vertices gives zero. The lowest-order diagram is the fermion triangle:



+ (reversed orientation) = 0.

  The two orientations of the closed fermion loop differ by a relative sign (each C-conjugation reverses the loop and contributes a $(-1)^{n_\gamma}$ for n_γ photon vertices), so for odd n_γ they cancel exactly.
- **Light-by-light scattering** ($E_\gamma = 4$, $D = 0$) is rendered finite by **gauge invariance**. The divergent part would have to be a constant $K(\eta_{\mu\nu}\eta_{\rho\sigma} + \text{perms})$, but current conservation $q^\mu M_{\mu\nu\rho\sigma} = 0$ forces $K = 0$. The lowest-order diagram is the fermion box (plus its $3! = 6$ photon-leg permutations):



+ (perms).

Naive power counting gives $D = 0$ (logarithmic divergence: 4 propagators $\sim k^{-4}$ vs. $\int d^4k$). However, the WTI requires that $q_i^\mu M_{\mu\nu\rho\sigma} = 0$ for each external photon, which forces enough powers of external momenta in the numerator to make the integral UV finite.

- **Odd-fermion-number amplitudes** are absent because each QED vertex contains one ψ and one $\bar{\psi}$, so fermion number conservation forces external fermion fields to occur in pairs.

Symmetries thus play a dual role: they reduce the number of independent divergences, and they guarantee the consistency of the renormalization program (the available counterterms match the divergences exactly).

Why the superficial degree of divergence is not the whole story: subdivergences

The superficial degree of divergence D characterizes the behavior of the integral when *all* internal momenta go to infinity together. However, divergences can also arise when only the momenta of a *subgraph* go to infinity while others remain fixed. These are called **subdivergences**.

Example. Consider a two-loop contribution to Compton scattering ($E_e = 2$, $E_\gamma = 2$), for which $D \leq -1$. The overall integral is superficially convergent. However, if the diagram contains a one-loop electron self-energy insertion on an internal fermion line, that subgraph has $D = 1$ and diverges, even though the full diagram is superficially convergent.

A theorem due to Weinberg states that a Feynman integral is *actually* convergent if and only if both the full integral *and* every possible subintegration have $D < 0$. In a renormalizable theory, every divergent subgraph corresponds to one of the finitely many divergent amplitudes enumerated above, and therefore comes accompanied by its counterterm. The counterterms cancel the subdivergences, rendering the full integral convergent.

The cancellation of divergences and overlapping divergences

The renormalization program works as follows. Any graph or subgraph with $D \geq 0$ produces a divergent contribution that is a **polynomial in external momenta of**

degree D, plus a finite remainder. Such a polynomial is exactly what would be produced by a local interaction with E_f fields and at most D derivatives. For a renormalizable theory ($\Delta_i \geq 0$ for all i), these are precisely the interactions already present in the Lagrangian. Therefore:

- The divergent part of $\Sigma(p)$ ($D = 1$) is a first-order polynomial in $\not{p}$: $\Sigma^{\text{div}} = A\not{p} + Bm$, which is absorbed by $(Z_2 - 1)$ and δm.
- The divergent part of $\Pi^{\mu\nu}(q)$ ($D = 2$) is a second-order polynomial in q, but gauge invariance ($q^\mu \Pi_{\mu\nu} = 0$) reduces it to a single constant C multiplying $(q^2\eta_{\mu\nu} - q_\mu q_\nu)$, absorbed by $(Z_3 - 1)$.
- The divergent part of $\Gamma^\mu(p', p)$ ($D = 0$) is a constant proportional to γ^μ: $\Gamma^{\mu,\text{div}} = L\gamma^\mu$, absorbed by $(Z_1 - 1)$.

The Ward identity further relates these constants: $L = A$ (the vertex divergence equals the $\not{p}$-coefficient of the self-energy divergence), i.e., $Z_1 = Z_2$, reducing the number of independent renormalization constants.

Overlapping divergences. A subtlety arises when two divergent subgraphs share an internal line, so they cannot be treated as independent divergent integrals. In QED, this occurs starting at two-loop order, for instance when two vertex corrections overlap inside a photon or electron self-energy. Despite this complication, it can be shown (through the BPHZ prescription of Bogoliubov, Parasiuk, Hepp, and Zimmermann) that the counterterms still cancel all divergences. The key idea is to consider all possible "forests" of nested subgraphs and subtract their divergences systematically, starting from the innermost and working outward.

Beyond renormalizability: effective field theories

Why non-renormalizable theories are not hopeless. If we include interactions with $\Delta_i < 0$, infinitely many types of counterterms are needed, one for every interaction allowed by symmetries. This may seem catastrophic, but it is not: as long as we include *all* interactions consistent with the symmetries of the theory, every divergence can still be absorbed into a coupling constant. In this sense, non-renormalizable theories are just as "renormalizable" as renormalizable ones—they simply require infinitely many parameters.

Dimensional suppression. The key insight is that the coupling constant g_i of a non-renormalizable interaction has dimension $[\text{mass}]^{\Delta_i}$ with $\Delta_i < 0$. If all such couplings are set by a single high-energy scale M, then $g_i \sim M^{\Delta_i}$. At energies $E \ll M$, the contribution of such an interaction to any amplitude is suppressed by a factor:

$$\left(\frac{E}{M}\right)^{|\Delta_i|} \ll 1 \,. \tag{6.278}$$

This is the modern perspective of **effective field theory** (EFT): at low energies, non-renormalizable interactions are negligible, and the physics is well described by the renormalizable part of the Lagrangian alone. The success of QED is not because nature is fundamentally described by a renormalizable theory, but because the non-renormalizable corrections are suppressed by powers of E/M, where M is some very high scale (at least $\sim 10^7$ GeV, from the precision of the electron's anomalous magnetic moment).

Example: the Euler–Heisenberg Lagrangian. When the electron is "integrated out" from QED at energies $E \ll m_e$, one obtains an effective Lagrangian for photon-photon interactions:

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{2\alpha^2}{45m_e^4}\left[(F_{\mu\nu}F^{\mu\nu})^2 + 7(F_{\mu\nu}\tilde{F}^{\mu\nu})^2\right] + \cdots \tag{6.279}$$

The leading correction is a dimension-8 operator (four field strengths $F_{\mu\nu}$, each of dimension 2, giving total dimension 8), so the coupling has $[g] = 4 - 8 = -4$. Its effects are suppressed by $(E/m_e)^4$ compared to the free-field term.

Wilson's renormalization group and the floating cutoff

The traditional approach—developed in the 1960s by **Bogoliubov, Parasiuk, Hepp,** and **Zimmermann** (**BPHZ**, already mentioned in passing for overlapping divergences, §6.7)—treats the cutoff Λ as an unphysical regulator, sent to infinity at the end of the calculation while counterterms absorb all Λ-dependence into a finite number of renormalized parameters. Wilson [17] pioneered an alternative viewpoint in which Λ is kept *finite and physical*: it is the scale below which the theory is defined. Renormalization then becomes a statement about how the description of the theory must change as we lower the cutoff to focus on physics at progressively longer wavelengths.

Integrating out high-momentum shells. Imagine a Euclidean path integral with a sharp UV cutoff Λ_0:

$$Z = \int_{|k|<\Lambda_0} \mathcal{D}\phi\, e^{-S_{\Lambda_0}[\phi]}, \qquad S_{\Lambda_0}[\phi] = \int d^d x \sum_i g_i(\Lambda_0)\, \mathcal{O}_i[\phi],$$

where $\{\mathcal{O}_i\}$ is a (in principle infinite) basis of local operators compatible with the symmetries of the theory and $g_i(\Lambda_0)$ are the bare couplings. Split the field into a low-momentum and a high-momentum part, $\phi = \phi_< + \phi_>$, with $\phi_<$ supported on $|k| < \Lambda$ and $\phi_>$ on $\Lambda < |k| < \Lambda_0$. Performing the path integral over $\phi_>$ defines the **Wilsonian effective action** at the new scale Λ,

$$e^{-S_\Lambda[\phi_<]} = \int_{\Lambda<|k|<\Lambda_0} \mathcal{D}\phi_>\, e^{-S_{\Lambda_0}[\phi_<+\phi_>]},$$

which has the same form as the original action but with renormalized, Λ-dependent couplings $g_i(\Lambda)$. Iterating the procedure for infinitesimal shells $\Lambda \to \Lambda - d\Lambda$ generates a flow $g_i(\Lambda)$ in coupling space: this is the *Wilson renormalization group flow*. The whole construction is physical: the couplings $g_i(\Lambda)$ encode what an observer working at energies $\lesssim \Lambda$ can measure, with the modes $\Lambda < |k| < \Lambda_0$ already absorbed into them.

The floating cutoff. The phrase "floating cutoff" refers to the fact that Λ is not a fixed regulator but a variable dial. The same physics admits infinitely many descriptions, one for each value of Λ, related by the RG flow. Demanding that physical observables (Green functions of $\phi_<$ at momenta $\ll \Lambda$) be invariant under a shift of Λ gives a non-trivial constraint on how the couplings must evolve.

Classifying operators by dimension. In d spacetime dimensions, an operator $\mathcal{O}_i$ has a canonical mass dimension $[\mathcal{O}_i]$, and the corresponding coupling has dimension $\Delta_i = d - [\mathcal{O}_i]$. The natural dimensionless coupling is

$$\mathcal{G}_i(\Lambda) \equiv \Lambda^{-\Delta_i} g_i(\Lambda),$$

which measures the strength of $\mathcal{O}_i$ in units of the cutoff. Cutoff-independence of physical observables, combined with the flow $g_i(\Lambda)$, gives the **Wilson RG equation**

$$\Lambda \frac{d}{d\Lambda} \mathcal{G}_i(\Lambda) = \beta_i\big(\mathcal{G}(\Lambda)\big), \tag{6.280}$$

where the β_i are calculable order by order in perturbation theory and have a universal piece coming from the rescaling of Λ in the definition of $\mathcal{G}_i$, plus an interaction-induced piece from the loop integrals over the shell $\Lambda < |k| < \Lambda_0$.

Relevant, marginal, irrelevant. Linearizing (6.280) near the free (Gaussian) fixed point $\mathcal{G}_i^* = 0$, the leading behaviour is purely dimensional:

$$\Lambda \frac{d\mathcal{G}_i}{d\Lambda} = -\Delta_i \mathcal{G}_i + O(\mathcal{G}^2) \implies \mathcal{G}_i(\Lambda) \simeq \mathcal{G}_i(\Lambda_0) \left(\frac{\Lambda}{\Lambda_0}\right)^{-\Delta_i}.$$

Three cases:

- $\Delta_i > 0$ (*relevant*, super-renormalizable): $\mathcal{G}_i$ *grows* as $\Lambda \to 0$. Examples: the mass term ($\Delta = 2$ in $d = 4$), which is why fine tuning is required to keep scalar masses small (the hierarchy problem).
- $\Delta_i = 0$ (*marginal*, renormalizable): the dimensional flow vanishes; the actual evolution is logarithmic, controlled entirely by the loop-induced piece of β_i (running of α in QED, of α_s in QCD).

- $\Delta_i < 0$ (*irrelevant*, non-renormalizable): $\mathcal{G}_i$ *decays* as

$$\mathcal{G}_i(\Lambda) \simeq \mathcal{G}_i(\Lambda_0)\left(\frac{\Lambda}{\Lambda_0}\right)^{|\Delta_i|}, \qquad \Lambda \ll \Lambda_0.$$

  For $|\Delta_i| = 4$ (e.g. the Euler–Heisenberg operator $(F_{\mu\nu}F^{\mu\nu})^2$, $\Delta = -4$ in $d = 4$) and a low-energy probe at $\Lambda/\Lambda_0 \sim 10^{-2}$, the suppression is by a factor 10^{-8}.

Why renormalizable theories "work so well." Because irrelevant couplings die off as a power of Λ/Λ_0, the low-energy effective action is dominated by the finitely many relevant and marginal operators—exactly the ones admitted by traditional renormalization. This is not a tautology but a deep statement: an unknown high-scale theory ($\Lambda_0 \sim M_{\rm Pl}$, say) projects, at experimentally accessible energies, onto a renormalizable Lagrangian. The non-renormalizable operators are still there, but their effects are systematically suppressed by powers of E/Λ_0 and are usually unobservable. This is the modern *effective field theory* picture: every renormalizable QFT is to be read as the leading approximation of an EFT, valid up to some scale Λ_0 where new degrees of freedom appear.

Comparison with BPHZ. The traditional and Wilsonian viewpoints are equivalent on physical predictions but differ in interpretation:

	BPHZ / continuum	Wilson
Cutoff Λ	unphysical regulator, $\to \infty$	finite, physical
Bare couplings	infinite, fine-tuned	finite, Λ-dependent
Counterterms	cancel divergences	encode integrated-out modes
Non-renorm. ops.	forbidden	present but irrelevant

The Wilsonian language makes manifest *why* the BPHZ procedure is internally consistent and *why* ignoring non-renormalizable operators is justified at low energies.

Accidental symmetries

The requirement of renormalizability is so restrictive that it can *force* the Lagrangian to have symmetries that were not assumed from the outset. These are called **accidental symmetries**.

Example: QED with multiple lepton flavors. Consider the most general renormalizable, Lorentz- and gauge-invariant Lagrangian for photons and N species of charged spin-1/2 fields ψ_i. A priori, the kinetic terms, mass terms, and interaction terms could all involve non-diagonal matrices in flavor space. However, by field redefinitions (non-singular changes of basis $\psi_i \to S_{ij}\psi_j$), the Lagrangian can always

be brought to the diagonal form:

$$\mathcal{L} = -\frac{1}{4} Z_3 F_{\mu\nu} F^{\mu\nu} + \sum_i \bar{\psi}_i \left(i\partial\!\!\!/ + eA\!\!\!/ - m_i \right) \psi_i \,, \tag{6.281}$$

This diagonal form reveals several **accidental symmetries**:

- **Individual lepton flavor conservation**: The number of leptons of each flavor (e, μ, τ) minus the number of antileptons is separately conserved. Processes like $\mu \to e + \gamma$ are forbidden, even though they were not explicitly excluded.
- **C, P, and T invariance**: The Lagrangian (6.281) is automatically invariant under charge conjugation, parity, and time reversal. No explicit assumption of these symmetries was needed.

These symmetries are "accidental" because they are not fundamental—they emerge automatically from the constraints of renormalizability. Non-renormalizable interactions (such as a Pauli term $\bar{\psi}[\gamma_\mu, \gamma_\nu]\psi F^{\mu\nu}$, or four-fermion interactions) would violate them, but their effects are suppressed by powers of E/M.

Remark. Most of the experimentally observed symmetries of elementary particle physics—lepton flavor conservation, baryon number conservation, CP invariance of the strong and electromagnetic interactions—are accidental symmetries of the renormalizable Standard Model Lagrangian. Their violation by non-renormalizable interactions is a subject of active experimental investigation (neutrino oscillations violate lepton flavor; proton decay would violate baryon number).

Summary: renormalization constants and their physical effects

The entire one-loop renormalization of QED is controlled by four quantities—three multiplicative constants and one additive shift—each absorbing the UV divergence of one 1PI amplitude. The Ward–Takahashi identity $Z_1 = Z_2$ reduces the independent constants to three.

Const.	1PI amplitude	Bare → ren.	Physical role
Z_3	Photon s.e. $\Pi^{\mu\nu}$	$A_{0\mu} = \sqrt{Z_3}\, A_\mu$	Charge ren.: $\mu^{\epsilon/2} e = Z_3^{1/2} e_0$
Z_2	Electron s.e. Σ	$\psi_0 = \sqrt{Z_2}\, \psi$	Fermion w.f. ren.
Z_1	Vertex Γ^μ	$e_0 Z_2 Z_3^{1/2} = \mu^{\epsilon/2} Z_1 e$	Vertex ren. ($Z_1 = Z_2$, Ward)
δm	Electron s.e. Σ	$m_0 = m + \delta m$	Mass renormalization

Object	**Effect of renormalization**
Internal photon propagator	$\dfrac{-ig_{\alpha\beta}}{k^2} \longrightarrow \dfrac{-ig_{\alpha\beta}}{k^2}\left[1 - e^2\Pi_c(k^2)\right]$
Internal fermion propagator	$\dfrac{i}{\not{p} - m} \longrightarrow \dfrac{i}{\not{p} - m - \Sigma_c(\not{p})}$
QED vertex	$ie\gamma^\mu \longrightarrow ie\left[\gamma^\mu + e^2\Lambda_c^\mu(p', p)\right]$
External fermion spinor	$u(p) \longrightarrow Z_2^{1/2} u(p)$ (similarly $\bar{u}, v, \bar{v}$)
External photon polarization	$\varepsilon^\mu(k) \longrightarrow Z_3^{1/2}\varepsilon^\mu(k)$

The finite corrections $\Pi_c(k^2)$, $\Sigma_c(\not{p})$, and $\Lambda_c^\mu(p', p)$ are the physically observable radiative effects (vacuum polarization, anomalous magnetic moment, Lamb shift, running coupling). All UV divergences are absorbed into Z_2, Z_3, δm (with $Z_1 = Z_2$), and the charge renormalization depends *only* on Z_3:

$$\boxed{e = Z_3^{1/2} e_0 .} \tag{6.282}$$

7 QED Renormalization: explicit calculations

Purpose and scope of this chapter

The previous chapter introduced the conceptual framework of renormalization in QED: the identification of divergent 1PI structures, the counterterm strategy, and the physical meaning of the renormalization constants Z_1, Z_2, Z_3, and δm. This chapter provides the **explicit computational details** that turn these concepts into concrete predictions.

The calculations presented here are representative of the techniques used throughout perturbative quantum field theory:

- **Dimensional regularization** ($d = 4 - \epsilon$) to handle UV divergences in a gauge-invariant manner;
- **Feynman parametrization** to combine propagator denominators into a single quadratic form suitable for momentum integration;
- **Dirac trace technology** to evaluate spinor loops in d dimensions, including the crucial identity $\gamma^\mu \gamma^\alpha \gamma_\mu = (2-d)\gamma^\alpha$;
- **Standard loop integrals** to perform the d-dimensional momentum integrations (see Appendix or standard references like Peskin–Schroeder [6]);
- **Extraction of poles and finite parts** to separate the divergent $(1/\epsilon)$ contributions absorbed by counterterms from the finite corrections that yield physical predictions.

The three main calculations—vacuum polarization $\Pi(k^2)$, electron self-energy $\Sigma(\not{p})$, and vertex correction $\Lambda^\mu(p', p)$—are presented in full detail, with every non-obvious algebraic step spelled out. This level of detail is essential for students learning the techniques; once mastered, many intermediate steps can be performed more quickly or looked up in references.

7.1 Photon self-energy (vacuum polarization) in dimensional regularization: detailed derivation of the finite part

Conventions. We work in $d = 4 - \epsilon$ with dimensional regularization and introduce the scale μ so that e is dimensionless:

$$e_0 = \mu^{\epsilon/2} e, \qquad e_0^2 = \mu^{\epsilon} e^2 . \tag{7.1}$$

Here, exactly as in Chapter 6, the loop propagators and vertices are written in terms of the **renormalized** parameters m and e; the deviations of m_0 and e_0 from m and e are accounted for separately by counterterm insertions. Thus the relation (7.1) should be read as a *one-loop stripped convention*: the extra Z factors are postponed to the counterterm vertices instead of being carried inside the loop integral itself. We use the Minkowski metric $g^{\mu\nu}$ and the standard QED vertex $+ie\gamma^{\mu}$ (Convention A).

Step 1: the loop integral defining $\Pi^{\mu\nu}(k)$. Including the minus sign for a closed fermion loop, the one-loop vacuum polarization is

$$i\Pi^{\mu\nu}(k) = (-1)(-ie)^2 \mu^{\epsilon} \int \frac{d^d p}{(2\pi)^d} \, \mathrm{Tr}\left[\gamma^{\mu} \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon} \gamma^{\nu} \frac{i(\not{p} + \not{k} + m)}{(p+k)^2 - m^2 + i\epsilon} \right] . \tag{7.2}$$

Collect the factors of i:

$$i\Pi^{\mu\nu}(k) = -e^2 \mu^{\epsilon} \int \frac{d^d p}{(2\pi)^d} \, \frac{\mathrm{Tr}\left[\gamma^{\mu} (\not{p} + m) \gamma^{\nu} (\not{p} + \not{k} + m) \right]}{\left(p^2 - m^2 + i\epsilon \right) \left((p+k)^2 - m^2 + i\epsilon \right)} . \tag{7.3}$$

Step 2: evaluate the Dirac trace explicitly. Expand the numerator:

$$\begin{aligned} \mathrm{Tr}\left[\gamma^{\mu} (\not{p} + m) \gamma^{\nu} (\not{p} + \not{k} + m) \right] &= \mathrm{Tr}\left[\gamma^{\mu} \not{p} \gamma^{\nu} (\not{p} + \not{k}) \right] + m \, \mathrm{Tr}\left[\gamma^{\mu} \not{p} \gamma^{\nu} \right] \\ &\quad + m \, \mathrm{Tr}\left[\gamma^{\mu} \gamma^{\nu} (\not{p} + \not{k}) \right] + m^2 \mathrm{Tr}[\gamma^{\mu} \gamma^{\nu}] . \end{aligned} \tag{7.4}$$

The middle two traces vanish because they contain an odd number of γ matrices:

$$\mathrm{Tr}(\text{odd number of } \gamma\text{'s}) = 0 . \tag{7.5}$$

Hence

$$\mathrm{Tr}\left[\gamma^{\mu} (\not{p} + m) \gamma^{\nu} (\not{p} + \not{k} + m) \right] = \mathrm{Tr}\left[\gamma^{\mu} \not{p} \gamma^{\nu} (\not{p} + \not{k}) \right] + m^2 \mathrm{Tr}[\gamma^{\mu} \gamma^{\nu}] . \tag{7.6}$$

Use the standard identities (valid here since no γ^5 appears)

$$\mathrm{Tr}\left[\gamma^{\mu} \not{a} \gamma^{\nu} \not{b} \right] = 4\Big(a^{\mu} b^{\nu} + a^{\nu} b^{\mu} - g^{\mu\nu} a \cdot b \Big), \tag{7.7}$$

$$\mathrm{Tr}[\gamma^{\mu} \gamma^{\nu}] = 4 g^{\mu\nu} . \tag{7.8}$$

With $a = p$ and $b = p + k$ this gives

$$N^{\mu\nu}(p,k) \equiv \mathrm{Tr}\left[\gamma^\mu(\not{p} + m)\gamma^\nu(\not{p} + \not{k} + m)\right]$$
$$= 4\Big(p^\mu(p+k)^\nu + p^\nu(p+k)^\mu - g^{\mu\nu}p\cdot(p+k) + m^2 g^{\mu\nu}\Big). \tag{7.9}$$

Substitute into (7.3):

$$i\Pi^{\mu\nu}(k) = -4e^2\mu^\epsilon \int \frac{d^dp}{(2\pi)^d} \frac{p^\mu(p+k)^\nu + p^\nu(p+k)^\mu - g^{\mu\nu}p\cdot(p+k) + m^2 g^{\mu\nu}}{\big(p^2 - m^2 + i\epsilon\big)\big((p+k)^2 - m^2 + i\epsilon\big)}. \tag{7.10}$$

Step 3: Feynman parametrization and completing the square. Combine denominators:

$$\frac{1}{AB} = \int_0^1 dx \, \frac{1}{\big[xA + (1-x)B\big]^2}, \tag{7.11}$$

with

$$A = (p+k)^2 - m^2 + i\epsilon, \qquad B = p^2 - m^2 + i\epsilon\,. \tag{7.12}$$

Then

$$xA + (1-x)B = x\big((p+k)^2 - m^2 + i\epsilon\big) + (1-x)\big(p^2 - m^2 + i\epsilon\big)$$
$$= p^2 + 2x\,p\cdot k + xk^2 - m^2 + i\epsilon\,. \tag{7.13}$$

The important point is that the coefficient of p^2 is exactly 1, because $x + (1-x) = 1$. Now add and subtract x^2k^2 so that the first three terms become a square:

$$xA + (1-x)B = \big(p^2 + 2x\,p\cdot k + x^2k^2\big) - x^2k^2 + xk^2 - m^2 + i\epsilon$$
$$= (p + xk)^2 - \big[m^2 - x(1-x)k^2\big] + i\epsilon\,. \tag{7.14}$$

This is the desired completed square. Define the shifted momentum

$$\ell \equiv p + xk, \qquad p = \ell - xk, \qquad p + k = \ell + (1-x)k\,. \tag{7.15}$$

Introduce

$$\Delta(x,k^2) \equiv m^2 - x(1-x)k^2\,. \tag{7.16}$$

Then the denominator takes the compact form

$$\frac{1}{\big(p^2 - m^2 + i\epsilon\big)\big((p+k)^2 - m^2 + i\epsilon\big)}$$
$$= \int_0^1 dx \, \frac{1}{\Big[(p+xk)^2 - \Delta(x,k^2) + i\epsilon\Big]^2}, \tag{7.17}$$

and, after the shift $p \mapsto \ell - xk$, the denominator is simply $(\ell^2 - \Delta + i\epsilon)^2$. To verify (7.17) explicitly, expand $(p+xk)^2 = p^2 + 2x\,p\cdot k + x^2k^2$:

$$\begin{aligned}(p+xk)^2 - \Delta &= p^2 + 2x\,p\cdot k + x^2k^2 - m^2 + x(1-x)k^2 \\ &= p^2 + 2x\,p\cdot k + xk^2 - m^2 \\ &= x\big((p+k)^2 - m^2\big) + (1-x)\big(p^2 - m^2\big)\,. \end{aligned} \tag{7.18}$$

Now set $\ell = p + xk$; then the denominator is $(\ell^2 - \Delta + i\epsilon)^2$.

Step 4: rewrite the numerator in terms of ℓ and separate even/odd pieces. Using (7.15), we compute the building blocks:

$$\begin{aligned} p^\mu (p+k)^\nu &= (\ell^\mu - xk^\mu)\big(\ell^\nu + (1-x)k^\nu\big) \\ &= \ell^\mu\ell^\nu + (1-x)\ell^\mu k^\nu - xk^\mu\ell^\nu - x(1-x)k^\mu k^\nu, \end{aligned} \tag{7.19}$$

$$p^\nu (p+k)^\mu = \ell^\nu\ell^\mu + (1-x)\ell^\nu k^\mu - xk^\nu\ell^\mu - x(1-x)k^\nu k^\mu, \tag{7.20}$$

$$p\cdot(p+k) = (\ell - xk)\cdot\big(\ell + (1-x)k\big) = \ell^2 + (1-2x)\,\ell\cdot k - x(1-x)k^2\,. \tag{7.21}$$

Insert (7.19)–(7.21) into the numerator $p^\mu(p+k)^\nu + p^\nu(p+k)^\mu - g^{\mu\nu}p\cdot(p+k) + m^2 g^{\mu\nu}$. First, collect the $\ell^\mu\ell^\nu$ terms:

$$p^\mu (p+k)^\nu + p^\nu (p+k)^\mu \supset 2\ell^\mu\ell^\nu\,. \tag{7.22}$$

Next, collect all terms *linear* in ℓ (they will vanish after integration):

$$p^\mu (p+k)^\nu + p^\nu (p+k)^\mu \supset (1-2x)\big(\ell^\mu k^\nu + \ell^\nu k^\mu\big), \tag{7.23}$$

$$-g^{\mu\nu}p\cdot(p+k) \supset -(1-2x)g^{\mu\nu}\,\ell\cdot k. \tag{7.24}$$

Finally, collect the purely k-dependent terms:

$$p^\mu (p+k)^\nu + p^\nu (p+k)^\mu \supset -2x(1-x)k^\mu k^\nu, \tag{7.25}$$

$$-g^{\mu\nu}p\cdot(p+k) + m^2 g^{\mu\nu} \supset -g^{\mu\nu}\ell^2 + g^{\mu\nu}\big(x(1-x)k^2 + m^2\big)\,. \tag{7.26}$$

Therefore the numerator in braces in (7.10) becomes

$$\begin{aligned} \mathcal{N}^{\mu\nu}(\ell,k;x) = {} & 2\ell^\mu\ell^\nu - g^{\mu\nu}\ell^2 - 2x(1-x)k^\mu k^\nu + g^{\mu\nu}\big(x(1-x)k^2 + m^2\big) \\ & + (1-2x)\Big[\ell^\mu k^\nu + \ell^\nu k^\mu - g^{\mu\nu}\ell\cdot k\Big]. \end{aligned} \tag{7.27}$$

Now the full expression is

$$i\Pi^{\mu\nu}(k) = -4e^2\mu^\epsilon \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d}\,\frac{\mathcal{N}^{\mu\nu}(\ell,k;x)}{(\ell^2 - \Delta + i\epsilon)^2}\,. \tag{7.28}$$

Step 5: symmetric integration (odd terms vanish) and tensor reduction. Because the denominator depends only on ℓ^2, integrals of odd functions vanish:

$$\int \frac{d^d\ell}{(2\pi)^d}\,\frac{\ell^\mu}{(\ell^2 - \Delta + i\epsilon)^2} = 0, \qquad \int \frac{d^d\ell}{(2\pi)^d}\,\frac{\ell\cdot k}{(\ell^2 - \Delta + i\epsilon)^2} = 0\,. \tag{7.29}$$

Hence the entire $(1-2x)[\cdots]$ line in (7.27) integrates to zero.

For the even tensor integral, rotational (Lorentz) symmetry in d dimensions gives

$$\int \frac{d^d\ell}{(2\pi)^d}\,\frac{\ell^\mu\ell^\nu}{(\ell^2-\Delta+i\epsilon)^2} = \frac{g^{\mu\nu}}{d}\int \frac{d^d\ell}{(2\pi)^d}\,\frac{\ell^2}{(\ell^2-\Delta+i\epsilon)^2}\,. \tag{7.30}$$

Define the two basic scalar integrals

$$I_2(\Delta) \equiv \int \frac{d^d\ell}{(2\pi)^d}\,\frac{1}{(\ell^2-\Delta+i\epsilon)^2}, \tag{7.31}$$

$$I_1(\Delta) \equiv \int \frac{d^d\ell}{(2\pi)^d}\,\frac{1}{(\ell^2-\Delta+i\epsilon)}\,. \tag{7.32}$$

Also note the useful identity

$$\frac{\ell^2}{(\ell^2-\Delta+i\epsilon)^2} = \frac{\ell^2-\Delta+\Delta}{(\ell^2-\Delta+i\epsilon)^2} = \frac{1}{(\ell^2-\Delta+i\epsilon)} + \Delta\,\frac{1}{(\ell^2-\Delta+i\epsilon)^2}\,, \tag{7.33}$$

so

$$\int \frac{d^d\ell}{(2\pi)^d}\,\frac{\ell^2}{(\ell^2-\Delta+i\epsilon)^2} = I_1(\Delta)+\Delta I_2(\Delta)\,. \tag{7.34}$$

Using (7.30) and (7.34), the ℓ-dependent part of (7.27) integrates to

$$\begin{aligned}\int \frac{d^d\ell}{(2\pi)^d}\,\frac{2\ell^\mu\ell^\nu-g^{\mu\nu}\ell^2}{(\ell^2-\Delta+i\epsilon)^2} &= \left(\frac{2}{d}-1\right)g^{\mu\nu}\int \frac{d^d\ell}{(2\pi)^d}\,\frac{\ell^2}{(\ell^2-\Delta+i\epsilon)^2}\\ &= \left(\frac{2}{d}-1\right)g^{\mu\nu}\Big[I_1(\Delta)+\Delta I_2(\Delta)\Big]\,.\end{aligned} \tag{7.35}$$

The remaining (pure k and m) terms simply multiply $I_2(\Delta)$:

$$\begin{aligned}&\int \frac{d^d\ell}{(2\pi)^d}\,\frac{-2x(1-x)k^\mu k^\nu+g^{\mu\nu}\big(x(1-x)k^2+m^2\big)}{(\ell^2-\Delta+i\epsilon)^2}\\ &\quad= \Big[-2x(1-x)k^\mu k^\nu+g^{\mu\nu}\big(x(1-x)k^2+m^2\big)\Big]I_2(\Delta)\,.\end{aligned} \tag{7.36}$$

Putting (7.35) and (7.36) into (7.28),

$$\begin{aligned}i\Pi^{\mu\nu}(k) = -4e^2\mu^\epsilon\int_0^1 dx\,\Bigg\{&\left(\frac{2}{d}-1\right)g^{\mu\nu}\Big[I_1(\Delta)+\Delta I_2(\Delta)\Big]\\ &+\Big[-2x(1-x)k^\mu k^\nu+g^{\mu\nu}\big(x(1-x)k^2+m^2\big)\Big]I_2(\Delta)\Bigg\}\,.\end{aligned} \tag{7.37}$$

Step 6: evaluate I_1 and I_2 in dimensional regularization. The standard formulas are

$$I_2(\Delta) = \frac{i}{(4\pi)^{d/2}}\,\Gamma\!\left(2-\frac{d}{2}\right)(\Delta)^{\frac{d}{2}-2}, \tag{7.38}$$

$$I_1(\Delta) = \frac{i}{(4\pi)^{d/2}}\,\Gamma\!\left(1-\frac{d}{2}\right)(\Delta)^{\frac{d}{2}-1}. \tag{7.39}$$

They are not independent: use $\Gamma(z+1) = z\Gamma(z)$ to write

$$\Gamma\!\left(2-\frac{d}{2}\right) = \left(1-\frac{d}{2}\right)\Gamma\!\left(1-\frac{d}{2}\right), \tag{7.40}$$

which implies the identity

$$I_1(\Delta) = \frac{\Delta}{\left(1-\frac{d}{2}\right)}\,I_2(\Delta) = \frac{2\Delta}{2-d}\,I_2(\Delta). \tag{7.41}$$

Insert (7.41) into the combination $I_1 + \Delta I_2$:

$$I_1(\Delta) + \Delta I_2(\Delta) = \Delta I_2(\Delta)\left(\frac{2}{2-d}+1\right) = \Delta I_2(\Delta)\left(\frac{4-d}{2-d}\right). \tag{7.42}$$

Now simplify the prefactor $(\frac{2}{d}-1)$:

$$\left(\frac{2}{d}-1\right) = \frac{2-d}{d}. \tag{7.43}$$

Therefore the entire ℓ-dependent contribution (7.35) becomes

$$\left(\frac{2}{d}-1\right)\Big[I_1 + \Delta I_2\Big] = \frac{2-d}{d}\,\Delta I_2\,\frac{4-d}{2-d} = \frac{4-d}{d}\,\Delta I_2. \tag{7.44}$$

Substituting (7.44) into (7.37) yields

$$\begin{aligned} i\Pi^{\mu\nu}(k) = -4e^2\mu^{\epsilon}\int_0^1 dx\,\Big[&g^{\mu\nu}\frac{4-d}{d}\,\Delta I_2(\Delta) - 2x(1-x)k^\mu k^\nu I_2(\Delta) \\ &+ g^{\mu\nu}\big(x(1-x)k^2+m^2\big)I_2(\Delta)\Big]. \end{aligned} \tag{7.45}$$

Group the $g^{\mu\nu}$ terms and use $\Delta = m^2 - x(1-x)k^2$:

$$\begin{aligned} g^{\mu\nu}\left[\frac{4-d}{d}\Delta + x(1-x)k^2 + m^2\right] &= g^{\mu\nu}\left[\frac{4-d}{d}\big(m^2 - x(1-x)k^2\big) + x(1-x)k^2 + m^2\right] \\ &= g^{\mu\nu}\left[m^2\left(\frac{4-d}{d}+1\right) + x(1-x)k^2\left(1-\frac{4-d}{d}\right)\right] \\ &= g^{\mu\nu}\left[m^2\left(\frac{4}{d}\right) + x(1-x)k^2\left(\frac{2d-4}{d}\right)\right] \\ &= \tfrac{4}{d}m^2\,g^{\mu\nu} \\ &\quad + \tfrac{2(d-2)}{d}x(1-x)\,k^2\,g^{\mu\nu}. \end{aligned} \tag{7.46}$$

Thus

$$i\Pi^{\mu\nu}(k) = -4e^2\mu^\epsilon \int_0^1 dx \left[\frac{4}{d} m^2 g^{\mu\nu} I_2(\Delta) + \frac{2(d-2)}{d} x(1-x)k^2 g^{\mu\nu} I_2(\Delta) - 2x(1-x)k^\mu k^\nu I_2(\Delta)\right]. \tag{7.47}$$

Step 7: isolate the transverse structure with the standard sign convention. From this point onward we adopt the same convention used in Chapter 6:

$$k_\mu \Pi^{\mu\nu}(k) = 0, \qquad \Rightarrow \qquad \Pi^{\mu\nu}(k) = \left(k^2 g^{\mu\nu} - k^\mu k^\nu\right)\Pi(k^2). \tag{7.48}$$

The safest way to extract the scalar coefficient is to project directly onto the transverse part, instead of trying to interpret the raw $g^{\mu\nu}$ and $k^\mu k^\nu$ pieces separately. Introduce the projector

$$P_T^{\mu\nu}(k) \equiv g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}, \qquad P_{T\,\mu\nu}(k)\,\Pi^{\mu\nu}(k) = (d-1)k^2\,\Pi(k^2). \tag{7.49}$$

Two elementary identities are used repeatedly:

$$P_{T\,\mu\nu}(k)\,k^\mu k^\nu = 0, \qquad P_{T\,\mu\nu}(k)\,g^{\mu\nu} = d-1. \tag{7.50}$$

Applying $P_{T\,\mu\nu}$ to (7.28) keeps only the coefficient of the transverse tensor. After the same tensor-reduction steps as above, one obtains

$$\Pi(k^2) = \frac{2e^2}{(4\pi)^{d/2}}\mu^\epsilon\,\Gamma\left(2-\frac{d}{2}\right)\int_0^1 dx\, x(1-x)\left[\Delta(x,k^2)\right]^{\frac{d}{2}-2}. \tag{7.51}$$

Equation (7.51) is the scalar function in (7.48).

Step 8: expand around $d = 4-\epsilon$ and separate UV pole and finite pieces. Set $d = 4-\epsilon$. Then

$$\Gamma\left(2-\frac{d}{2}\right) = \Gamma\left(\frac{\epsilon}{2}\right) = \frac{2}{\epsilon} - \gamma_E + O(\epsilon), \tag{7.52}$$

and

$$\left[\Delta(x,k^2)\right]^{\frac{d}{2}-2} = \left[\Delta(x,k^2)\right]^{-\epsilon/2} = 1 - \frac{\epsilon}{2}\ln\left(\Delta(x,k^2)\right) + O(\epsilon^2). \tag{7.53}$$

Also expand the prefactor

$$\mu^\epsilon (4\pi)^{-d/2} = (4\pi)^{-2}\left[1 + \frac{\epsilon}{2}\ln(4\pi) + \epsilon\ln\mu + O(\epsilon^2)\right]. \tag{7.54}$$

Insert (7.52)–(7.54) into (7.51) and keep terms up to $O(\epsilon^0)$. The result can be written compactly as

$$\Pi(k^2) = \frac{e^2}{2\pi^2}\int_0^1 dx\, x(1-x)\left[\frac{2}{\epsilon} - \gamma_E + \ln(4\pi) - \ln\left(\frac{\Delta(x,k^2)}{\mu^2}\right)\right] + O(\epsilon)\,. \tag{7.55}$$

This displays explicitly the UV divergence as a pole $2/\epsilon$.

Step 9: extract the convergent part by on-shell subtraction $\Pi_R(0) = 0$. With the convention (7.48), the quantity that enters the Dyson-resummed propagator as $1/(1-\Pi_R)$ is

$$\Pi_R(k^2) \equiv \Pi(0) - \Pi(k^2)\,. \tag{7.56}$$

Compute $\Pi(0)$ from (7.55): since $\Delta(x,0) = m^2$,

$$\Pi(0) = \frac{e^2}{2\pi^2}\int_0^1 dx\, x(1-x)\left[\frac{2}{\epsilon} - \gamma_E + \ln(4\pi) - \ln\left(\frac{m^2}{\mu^2}\right)\right] + O(\epsilon)\,. \tag{7.57}$$

Subtracting (7.55) from (7.57), the pole and the scheme constants cancel *exactly*, leaving a manifestly finite integral:

$$\Pi_R(k^2) = \frac{e^2}{2\pi^2}\int_0^1 dx\, x(1-x)\,\ln\left(\frac{m^2 - x(1-x)k^2 - i\epsilon}{m^2}\right). \tag{7.58}$$

Equivalently, in terms of $\alpha = e^2/(4\pi)$,

$$\Pi_R(k^2) = \frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\,\ln\left(\frac{m^2 - x(1-x)k^2 - i\epsilon}{m^2}\right). \tag{7.59}$$

For space-like momentum $k^2 = -Q^2 < 0$ this becomes

$$\Pi_R(-Q^2) = \frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\,\ln\left(1 + \frac{x(1-x)Q^2}{m^2}\right), \tag{7.60}$$

which is manifestly positive, as expected from charge screening and the running of $\alpha(Q^2)$.

Step 10: reconstruct the renormalized tensor. The finite renormalized vacuum-polarization tensor is

$$\Pi_R^{\mu\nu}(k) = \left(k^2 g^{\mu\nu} - k^\mu k^\nu\right)\Pi_R(k^2)\,, \tag{7.61}$$

which is transverse and UV finite by construction.

Physical significance of vacuum polarization.

The calculation above reveals several deep aspects of quantum electrodynamics:

- **Dimensional regularization preserves the gauge-invariant structure.** Once the tensor is projected onto the transverse sector, the result reorganizes itself into $\Pi^{\mu\nu} \propto (k^2 g^{\mu\nu} - k^\mu k^\nu)$. This is one of the great virtues of dimensional regularization: it respects the Ward identity and therefore never generates a physical photon mass.
- **The photon remains massless.** The non-transverse piece (proportional to $m^2 g^{\mu\nu}$) cancels exactly, ensuring that no photon mass is generated. This is a consequence of gauge invariance (specifically, the Ward identity $k_\mu \Pi^{\mu\nu} = 0$) and would be violated by a naive cutoff regularization.
- **The pole in $1/\epsilon$ is universal.** The coefficient of the UV divergence is fixed by short-distance physics and is independent of the external momentum k^2. This is the essence of renormalizability: all the infinite parts can be absorbed into a finite number of parameters.
- **The finite part contains physical predictions.** The momentum-dependent function $\Pi_R(k^2)$ encodes the physical effect of virtual electron-positron pairs. It leads to the running of the fine structure constant $\alpha(Q^2)$ and contributes to precision observables like the Lamb shift and the muon anomalous magnetic moment.

7.2 Electron self-energy in dimensional regularization: detailed derivation of the finite part

Goal. Compute the one-loop (unrenormalized) electron self-energy $\Sigma(\not{p})$ in dimensional regularization $d = 4 - \epsilon$, isolate its UV pole, and write the *finite (convergent)* part after a clear subtraction. We work in Feynman gauge $\xi = 1$ for transparency.

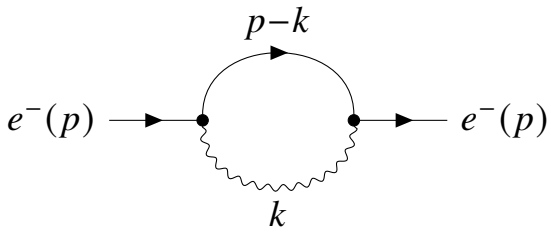
Conventions. We introduce μ so that e is dimensionless:

$$e_0 = \mu^{\epsilon/2} e, \qquad e_0^2 = \mu^\epsilon e^2 \,. \tag{7.62}$$

As in Chapter 6, the propagators and vertices appearing in the loop are built from the **renormalized** mass m and coupling e. The bare quantities m_0 and e_0 differ from them only through local counterterms, encoded in $(Z_2 - 1)$, δm, $(Z_1 - 1)$, and $(Z_3 - 1)$. In this sense, (7.62) is again the stripped one-loop convention in which the explicit Z factors are not inserted inside the loop integral. The free propagators are

$$S_0(\not{p}) = \frac{i}{\not{p} - m + i\epsilon}, \qquad D_{\mu\nu}(k) = \frac{-i g_{\mu\nu}}{k^2 + i\epsilon} \qquad (\xi = 1) \,. \tag{7.63}$$

Step 1: loop integral defining $\Sigma(\not p)$. The one-loop self-energy diagram is



(7.64)

and gives

$$-i\,\Sigma(\not p) = (-ie)^2\mu^\epsilon \int \frac{d^dk}{(2\pi)^d}\,\gamma^\mu\,\frac{i(\not p - \not k + m)}{(p-k)^2 - m^2 + i\epsilon}\,\gamma^\nu\,\frac{-ig_{\mu\nu}}{k^2+i\epsilon}\,. \tag{7.65}$$

Simplifying factors of i yields

$$\Sigma(\not p) = -e^2\mu^\epsilon \int \frac{d^dk}{(2\pi)^d}\,\frac{\gamma^\mu(\not p - \not k + m)\gamma_\mu}{\left((p-k)^2 - m^2 + i\epsilon\right)(k^2+i\epsilon)}\,. \tag{7.66}$$

Step 2: Dirac algebra in d dimensions. Use the identities (no γ^5 here)

$$\gamma^\mu\gamma^\alpha\gamma_\mu = (2-d)\gamma^\alpha, \qquad \gamma^\mu\gamma_\mu = d\,. \tag{7.67}$$

Therefore

$$\begin{aligned}\gamma^\mu(\not p - \not k + m)\gamma_\mu &= \gamma^\mu(\not p - \not k)\gamma_\mu + m\,\gamma^\mu\gamma_\mu \\ &= (2-d)(\not p - \not k) + dm\,.\end{aligned} \tag{7.68}$$

Insert into (7.66):

$$\Sigma(\not p) = -e^2\mu^\epsilon \int \frac{d^dk}{(2\pi)^d}\,\frac{(2-d)(\not p - \not k) + dm}{\left((p-k)^2 - m^2 + i\epsilon\right)(k^2+i\epsilon)}\,. \tag{7.69}$$

Step 3: Feynman parametrization and completing the square. Combine denominators with

$$\frac{1}{ab} = \int_0^1 dx\,\frac{1}{\left[xa + (1-x)b\right]^2}\,, \tag{7.70}$$

where

$$a = (p-k)^2 - m^2 + i\epsilon, \qquad b = k^2 + i\epsilon\,. \tag{7.71}$$

Compute

$$\begin{aligned}xa + (1-x)b &= x\left((p-k)^2 - m^2 + i\epsilon\right) + (1-x)(k^2+i\epsilon) \\ &= x\left(p^2 - 2p\cdot k + k^2 - m^2\right) + (1-x)k^2 + i\epsilon \\ &= x(p^2 - m^2) - 2x\,p\cdot k + k^2 + i\epsilon.\end{aligned} \tag{7.72}$$

Now shift the loop momentum to eliminate the mixed term. Define

$$\ell \equiv k - xp, \qquad k = \ell + xp, \qquad p - k = (1-x)p - \ell\,. \tag{7.73}$$

Then

$$k^2 = (\ell + xp)^2 = \ell^2 + 2x\,\ell\cdot p + x^2 p^2, \tag{7.74}$$

$$-2x\,p\cdot k = -2x\,p\cdot(\ell + xp) = -2x\,p\cdot\ell - 2x^2 p^2. \tag{7.75}$$

Insert (7.74) and (7.75) into (7.72):

$$\begin{aligned} xa + (1-x)b &= x(p^2 - m^2) + \left(\ell^2 + 2x\,\ell\cdot p + x^2 p^2\right) + \left(-2x\,p\cdot\ell - 2x^2 p^2\right) + i\epsilon \\ &= \ell^2 + x(p^2 - m^2) - x^2 p^2 + i\epsilon \\ &= \ell^2 - \Delta(x, p^2) + i\epsilon\,, \end{aligned} \tag{7.76}$$

where

$$\Delta(x, p^2) \equiv xm^2 - x(1-x)p^2\,. \tag{7.77}$$

Thus

$$\frac{1}{\left((p-k)^2 - m^2 + i\epsilon\right)(k^2 + i\epsilon)} = \int_0^1 dx\, \frac{1}{\left(\ell^2 - \Delta + i\epsilon\right)^2}\,. \tag{7.78}$$

Step 4: rewrite the numerator in terms of ℓ and drop odd terms. Using $k = \ell + xp$,

$$\not{p} - \not{k} = \not{p} - (\not{\ell} + x\not{p}) = (1-x)\not{p} - \not{\ell}\,. \tag{7.79}$$

Insert into the numerator of (7.69):

$$(2-d)(\not{p} - \not{k}) + dm = (2-d)\left((1-x)\not{p} - \not{\ell}\right) + dm\,. \tag{7.80}$$

Therefore

$$\Sigma(\not{p}) = -e^2\mu^\epsilon \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d}\, \frac{(2-d)\left((1-x)\not{p} - \not{\ell}\right) + dm}{(\ell^2 - \Delta + i\epsilon)^2}\,. \tag{7.81}$$

By symmetric integration (the denominator depends only on ℓ^2),

$$\int \frac{d^d\ell}{(2\pi)^d}\, \frac{\not{\ell}}{(\ell^2 - \Delta + i\epsilon)^2} = 0\,, \tag{7.82}$$

so the $\not{\ell}$ term drops out:

$$\Sigma(\not{p}) = -e^2\mu^\epsilon \int_0^1 dx \left[(2-d)(1-x)\not{p} + dm\right] \int \frac{d^d\ell}{(2\pi)^d}\, \frac{1}{(\ell^2 - \Delta + i\epsilon)^2}\,. \tag{7.83}$$

Step 5: evaluate the basic scalar integral. Define

$$I_2(\Delta) \equiv \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{(\ell^2 - \Delta + i\epsilon)^2}. \tag{7.84}$$

Dimensional regularization gives

$$I_2(\Delta) = \frac{i}{(4\pi)^{d/2}} \Gamma\left(2 - \frac{d}{2}\right) (\Delta)^{\frac{d}{2}-2}. \tag{7.85}$$

Thus

$$\Sigma(\not{p}) = -e^2\mu^\epsilon \int_0^1 dx \left[(2-d)(1-x)\not{p} + dm\right] \frac{i}{(4\pi)^{d/2}} \Gamma\left(2 - \frac{d}{2}\right) (\Delta)^{\frac{d}{2}-2}. \tag{7.86}$$

Step 6: expand around $d = 4 - \epsilon$ and display the UV pole. Set $d = 4 - \epsilon$ so that

$$\Gamma\left(2 - \frac{d}{2}\right) = \Gamma\left(\frac{\epsilon}{2}\right) = \frac{2}{\epsilon} - \gamma_E + O(\epsilon), \tag{7.87}$$

and

$$(\Delta)^{\frac{d}{2}-2} = (\Delta)^{-\epsilon/2} = 1 - \frac{\epsilon}{2}\ln\Delta + O(\epsilon^2). \tag{7.88}$$

Also,

$$\mu^\epsilon (4\pi)^{-d/2} = (4\pi)^{-2} \left[1 + \frac{\epsilon}{2}\ln(4\pi) + \epsilon\ln\mu + O(\epsilon^2)\right]. \tag{7.89}$$

Finally expand the Dirac prefactor in (7.86):

$$(2-d)(1-x)\not{p} + dm = (-2+\epsilon)(1-x)\not{p} + (4-\epsilon)m = -2(1-x)\not{p} + 4m + O(\epsilon). \tag{7.90}$$

Keeping terms up to $O(\epsilon^0)$ yields the standard compact form

$$\Sigma(\not{p}) = \frac{\alpha}{4\pi} \int_0^1 dx \left[-2(1-x)\not{p} + 4m\right] \left[\frac{2}{\epsilon} - \gamma_E + \ln(4\pi) - \ln\frac{\Delta(x,p^2)}{\mu^2}\right] + O(\epsilon), \tag{7.91}$$

where $\alpha = e^2/(4\pi)$ and $\Delta(x,p^2) = xm^2 - x(1-x)p^2$.

Step 7: identify the divergent structure and define a clear finite part. Write $\Sigma(\not{p})$ in the general decomposition

$$\Sigma(\not{p}) = \not{p}\,\Sigma_V(p^2) + m\,\Sigma_S(p^2). \tag{7.92}$$

From (7.91), the UV pole is proportional to

$$\Sigma(\not p)\Big|_{\rm UV} = \frac{\alpha}{4\pi}\frac{2}{\epsilon}\int_0^1 dx\,\Big[-2(1-x)\not p + 4m\Big] = \frac{\alpha}{4\pi}\frac{2}{\epsilon}\Big[-\not p + 4m\Big], \tag{7.93}$$

because

$$\int_0^1 dx\,(1-x) = \frac{1}{2}, \qquad \int_0^1 dx\;1 = 1\,. \tag{7.94}$$

So the UV divergence is a pole in $1/\epsilon$ multiplying $(-\not p + 4m)$.

To isolate the *finite (convergent) part* in the same spirit as the photon case, one must specify a subtraction condition. Two common choices are:
(A) $\overline{\rm MS}$ *finite part: subtract pole plus constants.* Define

$$\frac{1}{\bar\epsilon} \equiv \frac{1}{\epsilon} - \frac{\gamma_E}{2} + \frac{1}{2}\ln(4\pi), \tag{7.95}$$

and subtract the $2/\epsilon - \gamma_E + \ln(4\pi)$ part.
(B) On-shell finite part: subtract at the mass shell. Impose the on-shell conditions at $\not p = m$, which fix δm and Z_2. This corresponds to subtracting $\Sigma(\not p)$ and its $\not p$-derivative at $\not p = m$.

Since you asked for the "convergent part" in direct analogy with the subtracted vacuum-polarization function introduced above, the simplest explicit finite object to display *before* discussing δm and Z_2 is the $\overline{\rm MS}$-subtracted (finite) self-energy:

$$\Sigma_{\overline{\rm MS}}(\not p) \equiv \Sigma(\not p) - \frac{\alpha}{4\pi}\int_0^1 dx\,\Big[-2(1-x)\not p + 4m\Big]\left[\frac{2}{\epsilon} - \gamma_E + \ln(4\pi)\right] \tag{7.96}$$

$$= -\frac{\alpha}{4\pi}\int_0^1 dx\,\Big[-2(1-x)\not p + 4m\Big]\ln\left(\frac{\Delta(x,p^2)}{\mu^2}\right). \tag{7.97}$$

Equation (7.97) is manifestly finite for generic p^2 and exhibits the full momentum dependence through $\Delta(x,p^2) = xm^2 - x(1-x)p^2$.
Step 8 (optional, but often useful): explicit separation into $\not p$ and m parts. From (7.97),

$$\Sigma_{\overline{\rm MS}}(\not p) = \not p\;\Sigma_{V,\overline{\rm MS}}(p^2) + m\;\Sigma_{S,\overline{\rm MS}}(p^2)\,, \tag{7.98}$$

with

$$\Sigma_{V,\overline{\rm MS}}(p^2) = \frac{\alpha}{2\pi}\int_0^1 dx\;(1-x)\ln\left(\frac{\Delta(x,p^2)}{\mu^2}\right), \tag{7.99}$$

$$\Sigma_{S,\overline{\rm MS}}(p^2) = -\frac{\alpha}{\pi}\int_0^1 dx\;\ln\left(\frac{\Delta(x,p^2)}{\mu^2}\right). \tag{7.100}$$

Step 9: connect to the counterterm method (preview). In the counterterm approach, one defines the renormalized self-energy as

$$\Sigma_R(\not{p}) = \Sigma(\not{p}) + (Z_2 - 1)\not{p} - \delta m\,, \tag{7.101}$$

and chooses $(Z_2 - 1)$ and δm to cancel the UV pole (7.93) and enforce the chosen renormalization conditions (on-shell or $\overline{\text{MS}}$). The expressions (7.97)–(7.100) are precisely the finite "convergent part" that remains after subtracting the UV-divergent pieces.

Physical significance of the electron self-energy.

The electron self-energy calculation reveals important aspects of QED renormalization:

- **Two distinct divergent structures.** The self-energy has a momentum-dependent part (proportional to $\not{p}$) and a momentum-independent part (proportional to m). Both require separate counterterms: the wavefunction renormalization Z_2 and the mass counterterm δm.
- **The physical electron mass.** The counterterm δm absorbs the infinite shift in the electron mass due to electromagnetic self-interactions. The "bare mass" m_0 in the Lagrangian is unphysical; only the renormalized mass m_R (defined by the pole of the propagator) is observable.
- **Gauge dependence of intermediate quantities.** The self-energy $\Sigma(\not{p})$ depends on the gauge parameter ξ (we computed in Feynman gauge $\xi = 1$). However, physical observables like the pole mass are gauge-independent—a powerful consistency check.
- **Connection to bound-state physics.** The finite momentum-dependent part of the self-energy contributes to atomic energy levels (the self-energy contribution to the Lamb shift), demonstrating that loop effects have real physical consequences.

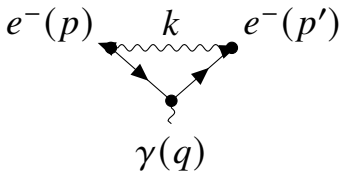
7.3 One-loop vertex correction in dimensional regularization: detailed derivation of the finite part

Goal. Compute the one-loop 1PI vertex function $\Lambda^\mu(p', p)$ in QED (Feynman gauge), display explicitly the UV pole in $d = 4 - \epsilon$, and write the *finite (convergent)* part after a clean subtraction (we use $\overline{\text{MS}}$ for definiteness). We also show in detail how the UV pole is fixed by the Ward–Takahashi identity.

Diagram and definition. The proper vertex is

$$\Gamma^\mu(p', p) = \gamma^\mu + \Lambda^\mu(p', p), \qquad q \equiv p' - p\,. \tag{7.102}$$

Diagrammatically:



(7.103)

Conventions. Dimensional regularization: $d = 4 - \epsilon$, and

$$e_0 = \mu^{\epsilon/2} e, \qquad e_0^2 = \mu^{\epsilon} e^2 \,. \tag{7.104}$$

The meaning is the same as in the previous two sections and in Chapter 6: the loop is written in terms of the **renormalized** mass m and coupling e, while the difference between bare and renormalized quantities is isolated in the counterterm vertex proportional to δZ_1 and in the other local counterterms. So (7.104) is not a new renormalization prescription, but simply the stripped one-loop convention used throughout this chapter. We work in Feynman gauge:

$$D_{\alpha\beta}(k) = \frac{-i g_{\alpha\beta}}{k^2 + i\epsilon} \,. \tag{7.105}$$

1. Loop integral for Λ^μ

Starting from the Feynman rules (including the two fermion propagators and one photon propagator), the one-loop 1PI correction is

$$\Lambda^\mu(p', p) = -e^2 \mu^\epsilon \int \frac{d^d k}{(2\pi)^d} \, \frac{\gamma^\alpha (\not{p}' - \not{k} + m) \gamma^\mu (\not{p} - \not{k} + m) \gamma_\alpha}{\big(k^2 + i\epsilon\big)\big((p'-k)^2 - m^2 + i\epsilon\big)\big((p-k)^2 - m^2 + i\epsilon\big)} \,. \tag{7.106}$$

Power counting: numerator $\sim k^2$, denominator $\sim k^6$, so $\int d^4k/k^4$ is logarithmically UV divergent $\Rightarrow$ a $1/\epsilon$ pole.

2. Three-denominator Feynman parametrization and completing the square

Introduce Feynman parameters $x, y, z \geq 0$ with $x + y + z = 1$:

$$\frac{1}{abc} = 2 \int_0^1 dx \int_0^{1-x} dy \, \frac{1}{\big[xa + yb + zc\big]^3}, \qquad z \equiv 1 - x - y \,. \tag{7.107}$$

Here we choose

$$a = k^2 + i\epsilon, \qquad b = (p' - k)^2 - m^2 + i\epsilon, \qquad c = (p - k)^2 - m^2 + i\epsilon \,. \tag{7.108}$$

Expand the combination in the denominator:

$$\begin{aligned} xa + yb + zc &= xk^2 + y\big[(p'-k)^2 - m^2\big] + z\big[(p-k)^2 - m^2\big] + i\epsilon \\ &= (x + y + z)k^2 - 2k\cdot(yp' + zp) + yp'^2 + zp^2 - (y + z)m^2 + i\epsilon \\ &= k^2 - 2k\cdot(yp' + zp) + yp'^2 + zp^2 - (y + z)m^2 + i\epsilon \,. \end{aligned} \tag{7.109}$$

Define

$$r^\mu \equiv y p'^\mu + z p^\mu, \qquad \ell^\mu \equiv k^\mu - r^\mu, \qquad k^\mu = \ell^\mu + r^\mu . \tag{7.110}$$

Since $k^2 - 2k\cdot r = (k-r)^2 - r^2 = \ell^2 - r^2$, (7.109) becomes

$$\begin{aligned} xa + yb + zc &= \ell^2 - r^2 + y p'^2 + z p^2 - (y+z)m^2 + i\epsilon \\ &= \ell^2 - \Delta(x, y; p'^2, p^2, q^2) + i\epsilon . \end{aligned} \tag{7.111}$$

The function Δ is obtained by working out r^2:

$$\begin{aligned} r^2 = (yp' + zp)^2 &= y^2 p'^2 + z^2 p^2 + 2yz\, p'\cdot p \\ &= y^2 p'^2 + z^2 p^2 + \frac{yz}{2}\left(2p'^2 + 2p^2 - 2q^2\right) \\ &= y^2 p'^2 + z^2 p^2 + yz(p'^2 + p^2 - q^2) , \end{aligned} \tag{7.112}$$

where we used

$$q^2 = (p'-p)^2 = p'^2 + p^2 - 2p'\cdot p \qquad \Longrightarrow \qquad 2p'\cdot p = p'^2 + p^2 - q^2 . \tag{7.113}$$

Therefore

$$\Delta = (y+z)m^2 - y(1-y)p'^2 - z(1-z)p^2 + 2yz\, p'\cdot p \tag{7.114}$$

or, eliminating $p'\cdot p$ in favor of q^2,

$$\Delta \equiv (y+z)m^2 - xy\, p'^2 - xz\, p^2 - yz\, q^2, \qquad z \equiv 1 - x - y . \tag{7.115}$$

This is the corrected quadratic form. It is often useful to keep both (7.114) and (7.115) in mind: the first shows the origin of each term, while the second makes the symmetry in the Feynman parameters more transparent.

3. Rewrite the numerator after the shift and classify the terms

Introduce the shifted external combinations

$$A^\mu \equiv (1-y)p'^\mu - z p^\mu, \qquad B^\mu \equiv -y p'^\mu + (1-z)p^\mu, \tag{7.116}$$

so that

$$p' - k = A - \ell, \qquad p - k = B - \ell . \tag{7.117}$$

The numerator of (7.106) becomes

$$\begin{aligned} N^\mu(\ell; x, y) &\equiv \gamma^\alpha (\not{p}' - \not{k} + m)\gamma^\mu (\not{p} - \not{k} + m)\gamma_\alpha \\ &= \gamma^\alpha (\not{A} - \not{\ell} + m)\gamma^\mu (\not{B} - \not{\ell} + m)\gamma_\alpha \\ &= N_2^\mu + N_1^\mu + N_0^\mu , \end{aligned} \tag{7.118}$$

where

$$N_2^\mu = \gamma^\alpha \not{\ell}\, \gamma^\mu \not{\ell}\, \gamma_\alpha, \tag{7.119}$$

$$N_1^\mu = -\gamma^\alpha (\not{A} + m)\gamma^\mu \not{\ell}\, \gamma_\alpha - \gamma^\alpha \not{\ell}\, \gamma^\mu (\not{B} + m)\gamma_\alpha, \tag{7.120}$$

$$N_0^\mu = \gamma^\alpha (\not{A} + m)\gamma^\mu (\not{B} + m)\gamma_\alpha. \tag{7.121}$$

This split is didactically useful: N_2^μ contains the two powers of loop momentum and therefore controls the UV pole, N_1^μ is odd in ℓ and integrates to zero, and N_0^μ is independent of ℓ and contributes only to the finite part.

4. Symmetric integration and identification of the UV-divergent part

Using (7.107) and (7.111), the vertex becomes

$$\Lambda^\mu(p', p) = -2e^2\mu^\epsilon \int_0^1 dx \int_0^{1-x} dy \int \frac{d^d\ell}{(2\pi)^d} \frac{N_2^\mu + N_1^\mu + N_0^\mu}{(\ell^2 - \Delta + i\epsilon)^3}. \tag{7.122}$$

The term linear in ℓ drops out:

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{N_1^\mu}{(\ell^2 - \Delta + i\epsilon)^3} = 0, \tag{7.123}$$

because the denominator depends only on ℓ^2.

For the quadratic term, symmetric integration gives

$$\ell_\rho \ell_\sigma \longrightarrow \frac{g_{\rho\sigma}}{d}\ell^2 \qquad \text{inside the loop integral.} \tag{7.124}$$

Therefore

$$\begin{aligned} N_2^\mu &= \ell_\rho \ell_\sigma \gamma^\alpha \gamma^\rho \gamma^\mu \gamma^\sigma \gamma_\alpha \\ &\longrightarrow \frac{\ell^2}{d} \gamma^\alpha \gamma^\rho \gamma^\mu \gamma_\rho \gamma_\alpha \\ &= \frac{\ell^2}{d} \gamma^\alpha \big[(2-d)\gamma^\mu\big]\gamma_\alpha \\ &= \frac{(d-2)^2}{d} \ell^2 \gamma^\mu. \end{aligned} \tag{7.125}$$

The vertex correction can thus be written as

$$\Lambda^\mu(p', p) = -2e^2\mu^\epsilon \int_0^1 dx \int_0^{1-x} dy \left[\frac{(d-2)^2}{d} \gamma^\mu J_2(\Delta) + N_0^\mu I_3(\Delta) \right], \tag{7.126}$$

where

$$J_2(\Delta) \equiv \int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^2}{(\ell^2 - \Delta + i\epsilon)^3}, \tag{7.127}$$

$$I_3(\Delta) \equiv \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{(\ell^2 - \Delta + i\epsilon)^3}. \tag{7.128}$$

The integral J_2 is simplified by the algebraic identity

$$\frac{\ell^2}{(\ell^2-\Delta+i\epsilon)^3} = \frac{1}{(\ell^2-\Delta+i\epsilon)^2} + \Delta\,\frac{1}{(\ell^2-\Delta+i\epsilon)^3}\,. \tag{7.129}$$

Define also

$$I_2(\Delta) \equiv \int \frac{d^d\ell}{(2\pi)^d}\,\frac{1}{(\ell^2-\Delta+i\epsilon)^2}. \tag{7.130}$$

Then

$$J_2(\Delta) = I_2(\Delta) + \Delta\, I_3(\Delta). \tag{7.131}$$

This immediately shows where the UV pole comes from: $I_2(\Delta)$ is logarithmically divergent in $d = 4-\epsilon$, while $I_3(\Delta)$ is already finite at $d = 4$. Therefore the UV divergence is local and necessarily proportional to γ^μ.

5. UV pole from the Ward–Takahashi identity

The Ward–Takahashi identity for the proper vertex reads

$$q_\mu\,\Gamma^\mu(p',p) = S^{-1}(p') - S^{-1}(p), \qquad S^{-1}(p) = \not{p} - m - \Sigma(\not{p})\,. \tag{7.132}$$

To turn this into a derivative relation, set $p' = p+q$ and expand for small q:

$$S^{-1}(p+q) - S^{-1}(p) = q_\nu\,\frac{\partial S^{-1}(p)}{\partial p_\nu} + O(q^2), \tag{7.133}$$

while

$$q_\mu\,\Gamma^\mu(p+q,p) = q_\mu\Gamma^\mu(p,p) + O(q^2). \tag{7.134}$$

Comparing the coefficients of the linear term in q gives

$$\Gamma^\mu(p,p) = \frac{\partial S^{-1}(p)}{\partial p_\mu} = \gamma^\mu - \frac{\partial\Sigma(\not{p})}{\partial p_\mu}\,, \tag{7.135}$$

hence

$$\Lambda^\mu(p,p) = -\frac{\partial}{\partial p_\mu}\Sigma(\not{p})\,. \tag{7.136}$$

From the self-energy section,

$$\Sigma(\not{p})\Big|_{\rm UV} = \frac{\alpha}{4\pi}\,\frac{2}{\epsilon}\,\big[-\not{p}+4m\big]. \tag{7.137}$$

Differentiating with respect to p_μ is now trivial because

$$\frac{\partial\not{p}}{\partial p_\mu} = \gamma^\mu, \qquad \frac{\partial m}{\partial p_\mu} = 0\,. \tag{7.138}$$

Therefore

$$\Lambda^{\mu}(p,p)\Big|_{\rm UV} = +\frac{\alpha}{4\pi}\,\frac{2}{\epsilon}\,\gamma^{\mu}\,. \tag{7.139}$$

This fixes both the sign and the coefficient of the local divergence. Since the UV pole comes from the large-ℓ region, it cannot depend on the external momenta: the same coefficient multiplies γ^{μ} for general (p',p).

6. $\overline{\rm MS}$ subtraction and the finite (convergent) part

The vertex counterterm has the form

$$\Gamma^{\mu}_{\rm ct} = +ie\,\mu^{\epsilon/2}\,\delta Z_1\,\gamma^{\mu}, \qquad \Rightarrow \qquad \Lambda^{\mu}_{R} = \Lambda^{\mu} - \delta Z_1\,\gamma^{\mu}\,. \tag{7.140}$$

In the $\overline{\rm MS}$ scheme one chooses

$$\delta Z_1^{\overline{\rm MS}} = \frac{\alpha}{4\pi}\,\frac{2}{\bar{\epsilon}}, \qquad \frac{1}{\bar{\epsilon}} \equiv \frac{1}{\epsilon} - \frac{\gamma_E}{2} + \frac{1}{2}\ln(4\pi)\,. \tag{7.141}$$

(With our conventions this equals $\delta Z_2^{\overline{\rm MS}}$, consistent with $Z_1 = Z_2$.) The equality $Z_1 = Z_2$ is therefore not just a mnemonic: in the derivation above it is the direct consequence of differentiating the self-energy pole and comparing with the vertex pole.

To write the finite remainder explicitly, note that

$$I_2(\Delta) = \frac{i}{(4\pi)^{d/2}}\Gamma\!\left(2-\frac{d}{2}\right)\Delta^{\frac{d}{2}-2}, \tag{7.142}$$

$$I_3(\Delta) = -\frac{i}{2(4\pi)^{d/2}}\Gamma\!\left(3-\frac{d}{2}\right)\Delta^{\frac{d}{2}-3}. \tag{7.143}$$

At $d = 4-\epsilon$, I_2 contains the UV pole while I_3 is finite. Since the full amplitude carries an overall factor μ^{ϵ}, the combination that is actually expanded is $\mu^{\epsilon} I_2(\Delta)$:

$$\mu^{\epsilon} I_2(\Delta) = \frac{i}{(4\pi)^2}\left[\frac{2}{\epsilon} - \gamma_E + \ln(4\pi) - \ln\frac{\Delta}{\mu^2}\right] + O(\epsilon), \tag{7.144}$$

$$I_3(\Delta) = -\frac{i}{2(4\pi)^2}\,\frac{1}{\Delta} + O(\epsilon). \tag{7.145}$$

Substituting (7.144) and (7.145) into (7.126), and then subtracting the local $\overline{\rm MS}$ counterterm, leaves a finite parameter integral. A compact way to write it is

$$\Lambda^{\mu}_{\overline{\rm MS}}(p',p) = -2e^2\int_0^1 dx\int_0^{1-x} dy\,\Big[\gamma^{\mu}\,I_{2,\rm fin}(\Delta) + \big(\Delta\gamma^{\mu} + N_0^{\mu}\big)I_3(\Delta)\Big]_{d=4}, \tag{7.146}$$

where

$$I_{2,\rm fin}(\Delta) \equiv \Big[\mu^{\epsilon} I_2(\Delta)\Big]_{\rm finite} = \frac{i}{(4\pi)^2}\left[-\ln\!\left(\frac{\Delta}{\mu^2}\right)\right]. \tag{7.147}$$

Equation (7.146) is the precise meaning of the "finite part": all $1/\epsilon$, γ_E, and $\ln 4\pi$ terms have been removed, while the full dependence on (p'^2, p^2, q^2) remains in Δ and in the Dirac structure N_0^μ.

Renormalized vertex to one loop. Finally,

$$\Gamma_R^\mu(p', p) = \gamma^\mu + \Lambda_{\overline{\text{MS}}}^\mu(p', p) + O(e^4), \tag{7.148}$$

and the counterterm choice ensures UV finiteness.

Physical significance of the vertex correction.

The vertex correction is the most directly physical of the three one-loop diagrams:

- **The anomalous magnetic moment.** At zero momentum transfer ($q^2 = 0$), the vertex correction contributes to the magnetic form factor $F_2(0)$, giving the famous Schwinger correction $a_e = \alpha/(2\pi) \approx 0.00116$ to the electron g-factor.
- **Ward identity and $Z_1 = Z_2$.** The relation $\Lambda^\mu(p, p) = -\partial\Sigma/\partial p_\mu$ is not an accident: it follows from gauge invariance and ensures that the charge renormalization is universal. This is why we need only one coupling constant e after renormalization.
- **On-shell versus infrared divergences.** When external legs are put on-shell ($p^2 = p'^2 = m^2$), the vertex develops an infrared divergence as $q^2 \to 0$. This is physical—it signals the emission of soft photons and cancels against real photon emission in physical cross sections.

7.4 Numerical size of the one-loop QED corrections and phenomenological evidence

We summarize the typical *finite* sizes of the three one-loop QED corrections (vacuum polarization, electron self-energy, vertex correction), and we point to standard experimental plots/tables showing that these effects are real and must be included in precision physics.

(i) Photon self-energy (vacuum polarization) and the running of α

In momentum space, gauge invariance implies that the renormalized vacuum polarization tensor is transverse,

$$\Pi_R^{\mu\nu}(q) = \left(q^2 g^{\mu\nu} - q^\mu q^\nu\right) \Pi_R(q^2), \tag{7.149}$$

and the transverse part of the photon propagator is dressed as

$$D^{\mu\nu}(q) = \frac{-i}{q^2 + i\epsilon}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) \frac{1}{1 - \Pi_R(q^2)} + (\text{long. part}). \tag{7.150}$$

Equivalently, one introduces an *effective charge* (in a given scheme)

$$\alpha(q^2) \equiv \frac{\alpha}{1 - \Pi_R(q^2)}. \tag{7.151}$$

For a single lepton of mass m_ℓ, at large *space-like* momentum transfer $q^2 = -Q^2$ with $Q^2 \gg m_\ell^2$, the renormalized one-loop result has the exact Feynman-parameter representation

$$\Pi_R(-Q^2) = \frac{2\alpha}{\pi} \int_0^1 dx\, x(1-x) \ln\left(1 + \frac{x(1-x)Q^2}{m_\ell^2}\right), \tag{7.152}$$

which is manifestly positive. For $Q^2 \gg m_\ell^2$ this reduces to the standard asymptotic form

$$\Pi_R(-Q^2) \simeq \frac{\alpha}{3\pi}\left[\ln\left(\frac{Q^2}{m_\ell^2}\right) - \frac{5}{3}\right], \qquad (Q^2 \gg m_\ell^2), \tag{7.153}$$

so the relative correction to the photon exchange amplitude is typically $\mathcal{O}(\Pi_R)$, i.e. logarithmically enhanced at high Q.

For the *electron loop* ($m_\ell = m_e$) this gives the indicative numerical sizes (using $\alpha \simeq 1/137.036$ and $m_e \simeq 0.511$ MeV):

Q (GeV)	$\Pi_R(-Q^2)\big\vert_{e\text{-loop}}$	size
0.1	6.9×10^{-3}	0.7%
1	1.04×10^{-2}	1.0%
10	1.40×10^{-2}	1.4%
$M_Z \simeq 91.2$	1.74×10^{-2}	1.7%

Why this is not just formal: plots and direct evidence. Direct measurements of the running of α (hence of vacuum polarization) exist both in the time-like and space-like regions:

- KLOE measures $\alpha(s)$ below 1 GeV in the time-like region via $e^+e^- \to \mu^+\mu^-\gamma$. [18]
- OPAL measures the running in the space-like region using small-angle Bhabha scattering at LEP. [19]
- MUonE is designed to extract the *space-like* hadronic vacuum polarization from μe elastic scattering. [20]

It is also useful to quote the *full* (leptonic + hadronic) effect at the Z pole: a standard compilation gives

$$\alpha^{-1}(0) \simeq 137.036 \qquad \longrightarrow \qquad \alpha^{-1}(M_Z) \simeq 128.95, \tag{7.154}$$

i.e. a $\sim 6\%$ increase of α from $q^2 = 0$ to $q^2 = M_Z^2$. [21]

(ii) Electron self-energy and the size of propagator corrections

The dressed renormalized electron propagator is

$$S_R(p) = \frac{i}{\not{p} - m - \Sigma_R(\not{p})}\,. \tag{7.155}$$

A convenient way to quote a *finite* size without infrared subtleties is to evaluate the (one-loop, MS-like subtracted) self-energy at a *space-like* Euclidean point, e.g. $p^2 = -m^2$ with $\mu = m$. At that point

$$\Delta(x, -m^2) = xm^2 - x(1-x)(-m^2) = x(2-x)m^2, \tag{7.156}$$

so the parameter integrals can be done explicitly:

$$2\int_0^1 dx\,(1-x)\ln\big[x(2-x)\big] = -1, \qquad -4\int_0^1 dx\,\ln\big[x(2-x)\big] = 8 - 8\ln 2. \tag{7.157}$$

Hence

$$\Sigma_R(\not{p})\Big|_{p^2=-m^2,\ \mu=m} = \frac{\alpha}{4\pi}\big[A\not{p} + Bm\big], \qquad A = -1, \qquad B = 8 - 8\ln 2 \simeq 2.45\,. \tag{7.158}$$

Numerically,

$$\begin{aligned} &\frac{\alpha}{4\pi} \simeq 5.80\times 10^{-4}, \\ &\Rightarrow \quad \Sigma_R(\not{p}) \simeq -(5.8\times 10^{-4})\,\not{p} + (1.43\times 10^{-3})\,m \\ &\qquad (p^2 = -m^2,\ \mu = m), \end{aligned} \tag{7.159}$$

so the finite one-loop correction is at the level of $O(10^{-3})$ (as expected on dimensional grounds: $\alpha/\pi \sim 2.3\times 10^{-3}$).

Phenomenological evidence (bound-state self-energy). While $\Sigma_R(\not{p})$ is not measured directly, its physical content appears prominently in bound-state QED: the *electron self-energy* provides a dominant contribution to the Lamb shift. [22]

(iii) Vertex correction and a clean numerical observable: $g-2$

The renormalized on-shell vertex can be parameterized by the Dirac and Pauli form factors,

$$\bar{u}(p')\,\Gamma^{\mu}(p',p)\,u(p) = \bar{u}(p')\left[\gamma^{\mu}F_1(q^2) + \frac{i\sigma^{\mu\nu}q_{\nu}}{2m}F_2(q^2)\right]u(p), \qquad q = p' - p\,. \tag{7.160}$$

The most famous finite piece is the *anomalous magnetic moment*

$$a_e \equiv \frac{g-2}{2} = F_2(0)\,. \tag{7.161}$$

At one loop (Schwinger),

$$a_e^{(1)} = \frac{\alpha}{2\pi} \simeq 1.16141 \times 10^{-3}\,. \tag{7.162}$$

This is a *pure vertex effect* (in the on-shell definition) and is experimentally established at astonishing precision. [23, 24]

Bottom line: sizes and "why they matter"

Putting the above together, at one loop you should expect:

- **vacuum polarization:** percent-level effects in typical GeV-scale photon exchange (log-enhanced), and a $\sim 6\%$ shift between $\alpha(0)$ and $\alpha(M_Z)$ once all charged species (including hadrons) are accounted for;
- **electron self-energy:** per-mille-level finite corrections to the propagator structure, with direct physical manifestation in bound-state energy shifts (Lamb shift);
- **vertex:** a clean $O(10^{-3})$ correction already at one loop (Schwinger), with higher orders required to match the observed a_e at the 10^{-12} level.

One-loop $\overline{\text{MS}}$ summary table

Quantity	Divergent one-loop part	$\overline{\text{MS}}$ counterterm(s)	Renormalized one-loop result
Photon self-energy	$\Pi^{\mu\nu}(q) = (q^2 g^{\mu\nu} - q^\mu q^\nu)\Pi(q^2)$, $\Pi(q^2)\big\|_{\text{div}} = \dfrac{2\alpha}{3\pi}\dfrac{1}{\bar{\epsilon}}$	$\delta Z_3^{\overline{\text{MS}}} = Z_3 - 1 = \frac{2\alpha}{3\pi}\frac{1}{\bar{\epsilon}}$	$\Pi^{\mu\nu}_{R,\overline{\text{MS}}}(q) = (q^2 g^{\mu\nu} - q^\mu q^\nu)\Pi_{\overline{\text{MS}}}(q^2)$, $\Pi_{\overline{\text{MS}}}(q^2) = -\dfrac{2\alpha}{\pi}\displaystyle\int_0^1 dx\, x(1-x) \times \ln\dfrac{\Delta_q(x)}{\mu^2}$
Electron self-energy	$\Sigma(\not{p})\big\|_{\text{div}} = \frac{\alpha}{4\pi}\frac{2}{\bar{\epsilon}}(-\not{p} + 4m)$	$\delta Z_2^{\overline{\text{MS}}} = Z_2 - 1 = \dfrac{\alpha}{2\pi}\dfrac{1}{\bar{\epsilon}}$, $\delta m^{\overline{\text{MS}}} = \dfrac{2\alpha}{\pi}\dfrac{m}{\bar{\epsilon}}$	$\Sigma_{\overline{\text{MS}}}(\not{p}) = -\dfrac{\alpha}{4\pi}\displaystyle\int_0^1 dx \times \left[-2(1-x)\not{p} + 4m\right]\ln\dfrac{\Delta_p(x)}{\mu^2}$
Vertex correction	$\Lambda^\mu(p',p)\big\|_{\text{div}} = \frac{\alpha}{2\pi}\frac{1}{\bar{\epsilon}}\gamma^\mu$	$\delta Z_1^{\overline{\text{MS}}} = Z_1 - 1 = \dfrac{\alpha}{2\pi}\dfrac{1}{\bar{\epsilon}}$, $Z_1^{\overline{\text{MS}}} = Z_2^{\overline{\text{MS}}}$	$\Gamma^\mu_R(p',p) = \gamma^\mu + \Lambda^\mu_{\overline{\text{MS}}}(p',p) + O(e^4)$
$\dfrac{1}{\bar{\epsilon}} \equiv \dfrac{1}{\epsilon} - \dfrac{\gamma_E}{2} + \dfrac{1}{2}\ln(4\pi)$, $\Delta_q(x) = m^2 - x(1-x)q^2 - i\epsilon$, $\Delta_p(x) = xm^2 - x(1-x)p^2$, $e = Z_3^{1/2} e_0$, $Z_e = Z_3^{-1/2}$			

8 General Methods for Interacting Fields

This chapter develops the general formalism connecting the S-matrix to vacuum expectation values of time-ordered products of field operators. The central result is the *LSZ reduction formula* [25], which provides a rigorous foundation for the Feynman diagram technique and illuminates the physical content of scattering amplitudes. A key tool is the Källén–Lehmann spectral representation [26, 27], which encodes the general structure of two-point functions. We follow primarily the treatment of Bjorken and Drell [10], supplementing it with the perturbative Green-function discussion of Mandl–Shaw [1], the pole-and-residue viewpoint emphasized by Peskin–Schroeder [6], and the compact LSZ presentation of Schwartz [12].

8.1 Introduction: from free to interacting fields

A theory of free fields alone has no physical content. The nature of the physical world is revealed to us only through the interactions between fields, to which we now turn.

In constructing general interactions, we are motivated by analogy with the electromagnetic field. The "minimal" substitution $p_\mu \to p_\mu + eA_\mu$ (in our convention with $D_\mu = \partial_\mu - ieA_\mu$ and $q_e = -e$; cf. Section 5.0 of Chapter 5), corresponding to the classical interaction of a point charge, provides the standard prescription for introducing electrodynamic couplings. This leads to the QED Lagrangian density:

$$\mathcal{L}(x) = \bar{\psi}(x)(i\partial\!\!\!/ + e_0 A\!\!\!/(x) - m_0)\psi(x) - \frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x)\,, \tag{8.1}$$

which describes the local interaction of the electron and photon fields at the same spacetime point x. The coupled field equations are:

$$(i\partial\!\!\!/ - m_0)\psi(x) = -e_0 A\!\!\!/(x)\psi(x), \tag{8.2}$$

$$\partial_\nu F^{\mu\nu}(x) = -e_0\bar{\psi}(x)\gamma^\mu\psi(x)\,. \tag{8.3}$$

It is evident from these equations that in discussing the coupling between fields we face a nonlinear problem of vast complexity—even classically, as found in the study of radiation damping and runaway solutions for classical charges under the influence of their own and each other's fields.

The subscripts on m_0 and e_0 indicate that these are the *bare* mass and charge, which differ from the physically observable values m and e due to renormalization

effects. Indeed, perturbation theory calculations show that the corrections to mass and charge are divergent, requiring careful renormalization procedures to extract finite physical predictions.

Despite these complications, a systematic approach is possible. The physical predictions of any quantum field theory are contained in the S-matrix elements, whose squared moduli give the probabilities for physical processes to occur. Alternatively, one can focus on closely related functions called **Green functions**. Although Green functions are less directly related to observables than S-matrix elements, they are easier to calculate, and S-matrix elements can be readily obtained from them. Moreover, S-matrix elements for several different processes can be derived from a single Green function, allowing relations between these processes to be inferred.

8.2 Properties of physical states

The goal of quantum field theory as a physical theory is to describe the dynamics of interacting particles observed in nature. We have seen how the properties of free particles emerge from the quantization of classical fields and how symmetries constrain the form of interaction Lagrangians. The task remaining is to construct and evaluate the general matrix elements that describe the dynamical behavior of interacting particles.

We are interested both in the amplitudes for single particles to propagate in spacetime—the one-particle Green's functions—and in the transition amplitudes between different initial and final states, that is, the S-matrix.

Assumptions on the spectrum

To study the properties of exact states and propagators, we rely on invariance arguments. In particular, displacement and Lorentz invariance play fundamental roles.

We work in the *Heisenberg picture*, where operators carry the full time dependence and states are time-independent. This is the natural framework for an exact, non-perturbative treatment of interacting fields[1]: we need to discuss the behaviour

[1] A word of caution on the interaction picture used in earlier chapters. Haag's theorem [28] states that, for a relativistic quantum field theory with infinitely many degrees of freedom, the Fock space of the free theory and the Hilbert space of the interacting theory are *unitarily inequivalent*: no unitary operator $U(t)$ can map one into the other. This means that the interaction-picture splitting $H = H_0 + H_I$, which relies precisely on such a unitary map, cannot be defined rigorously in the exact theory. In practice, however, this is not a problem: perturbative calculations are always performed within a regularised theory (with ultraviolet cutoff and finite volume), where the number of degrees of freedom is finite and Haag's theorem does not apply. The interaction picture and the Dyson

of the field operators $\varphi(x)$ at asymptotic times $t \to \pm\infty$, and this requires operators that evolve in time. Moreover, in the Heisenberg picture states do not evolve, so they can be classified once and for all by the eigenvalues of conserved quantities.

We choose the states Φ to be eigenstates of energy and momentum. This is possible because a conserved energy-momentum four-vector P^μ is guaranteed by displacement invariance. We label each state by the eigenvalues of P^μ together with those of all other mutually commuting constants of the motion (spin, internal charges, etc.), forming a complete set of commuting observables.

We assume the following physically motivated conditions on the eigenspectrum of P^μ:

1. **Forward light cone condition.** The eigenvalues of energy and momentum lie within the forward light cone:

$$P^2 = P_\mu P^\mu \geq 0, \qquad P^0 \geq 0\,. \tag{8.4}$$

2. **Unique vacuum.** There exists a nondegenerate Lorentz-invariant ground state of lowest energy, the vacuum state

$$\Phi_0 \equiv |0\rangle\,, \tag{8.5}$$

with the zero of energy chosen so that

$$P^0|0\rangle = 0\,. \tag{8.6}$$

From the forward light cone condition (8.4), it follows that $\mathbf{P}|0\rangle = 0$ as well. Lorentz invariance of the vacuum ensures that $|0\rangle$ appears as the vacuum state to all inertial observers.

3. **Stable one-particle states.** There exist stable single-particle states

$$\Phi_{1(i)} \equiv |P^{(i)}\rangle \tag{8.7}$$

with $P^{(i)}_\mu P^{(i)\mu} = m_i^2$ for each stable particle of physical mass m_i. These states transform as irreducible unitary representations of the Poincaré group, labelled by mass m_i and spin s_i.

4. **Mass gap / isolated mass shell.** The vacuum and single-particle states form a discrete part of the spectrum of P^μ. More precisely, in the sector of quantum numbers carried by the field $\varphi(x)$, the one-particle mass hyperboloid $p^2 = m^2$ is *isolated* from the rest of the spectrum: there exists a lowest continuum threshold $m_1^2 > m^2$ such that the continuous part starts only at $p^2 \geq m_1^2$. The

expansion remain perfectly valid tools within this regularised, order-by-order framework. The point here is that for an *exact*, non-perturbative formulation—such as the LSZ reduction formula—one should work directly in the Heisenberg picture, which makes no reference to a free-field Fock space.

threshold m_1 is not fixed universally by the mass of the lightest particle: it is the invariant mass of the lightest multiparticle state with the same conserved quantum numbers as the field insertion. For example, a Z_2-odd scalar field in a ϕ^4 theory couples to odd-particle states, so the first continuum threshold in that channel is typically $3m$, whereas an even composite operator can have a two-particle threshold at $2m$. This separation is crucial: as we shall see, it allows the one-particle contributions to be cleanly extracted in the asymptotic limit $t \to \pm\infty$, while the continuum contributions vanish.

5. **Vanishing vacuum expectation value.** The field $\varphi(x)$ is assumed to satisfy

$$\langle 0|\varphi(x)|0\rangle = 0\,. \tag{8.8}$$

This is not a restriction: if $\langle 0|\varphi(x)|0\rangle = v \neq 0$, one redefines $\varphi(x) \to \varphi(x) - v$. The shifted field satisfies the same equations of motion with modified couplings. This condition is necessary for the correct identification of the asymptotic fields in the LSZ formalism.

6. **Asymptotic completeness.** Intuitively, an *in-state* describes what a scattering experiment looks like in the far past ($t \to -\infty$), when the particles are widely separated and propagate freely; an *out-state* describes the same system in the far future ($t \to +\infty$), after all interactions have ceased. These states will be constructed precisely in the next sections; for now, the key assumption is that they span the full Hilbert space of the interacting theory:

$$\mathcal{H} = \mathcal{H}_{\text{in}} = \mathcal{H}_{\text{out}}\,. \tag{8.9}$$

This ensures that every physical state can be described as a scattering state, and guarantees the unitarity of the S-matrix. Asymptotic completeness is assumed in all standard treatments, although a rigorous proof remains an open problem in axiomatic quantum field theory.

Taken together, these hypotheses define the standard textbook regime in which the LSZ construction works cleanly: stable massive particles with an isolated one-particle shell. Stable bound states are not excluded in principle, provided they appear as isolated poles and admit suitable interpolating fields. The main caveat is that theories with long-range interactions, such as QED, require a more careful treatment of asymptotic states: charged sectors are accompanied by arbitrarily soft photons, so the sharp one-particle pole assumed here is replaced by infrared-sensitive dressed states or inclusive observables.

These assumptions reflect our physical expectation that the interaction does not violently alter the spectrum of states from the free-field case. This is the spirit of perturbation theory: having introduced fields for stable particles into a Lagrangian, we assume that their mutual interactions do not destroy the basic structure of the spectrum.

Notation for the vacuum. In the non-perturbative LSZ construction below, $|0\rangle$ denotes the exact physical vacuum of the interacting theory. In the later perturbative discussion of the Gell-Mann–Low formula, we switch to the standard interaction-picture notation: $|\Omega\rangle$ denotes the exact interacting vacuum and $|0\rangle$ denotes the free vacuum of H_0. This switch is made explicitly when the perturbative formula is introduced.

8.3 In-fields and the asymptotic condition

Since we are primarily concerned with scattering problems, we first construct the states that give a simple description of the physical system at the initial time $t \to -\infty$. At this time the incoming wave packets are widely separated, so no finite-range interaction can exchange momentum between them. Each object is already a physical, dressed particle; its self-energy effects are part of its mass and field normalization.

Construction of in-fields for a scalar field

Consider a Hermitian scalar field $\varphi(x)$ satisfying the wave equation

$$(\Box + m_0^2)\varphi(x) = j(x)\,, \tag{8.10}$$

where $j(x)$ is a scalar operator constructed from the fields interacting with $\varphi(x)$ locally at x. We assume that the full interacting field satisfies the same equal-time commutation relations that we derived for free fields in the canonical quantisation procedure:

$$[\varphi(\vec{x},t),\varphi(\vec{y},t)] = [\pi(\vec{x},t),\pi(\vec{y},t)] = 0, \qquad [\pi(\vec{x},t),\varphi(\vec{y},t)] = -i\delta^{(3)}(\vec{x}-\vec{y})\,, \tag{8.11}$$

where $\pi(x) = \dot{\varphi}(x)$ for theories without derivative couplings. These relations are the quantum counterpart of the classical Poisson brackets and constitute a basic postulate of canonical quantisation: we retain them as an axiom of the interacting theory.

We denote by $\varphi_{\rm in}(x)$ the operator that creates states of single physical particles. To ensure the correct interpretation, we require:

1. **Covariance.** In the Heisenberg picture, the full interacting field transforms under spacetime translations as

$$e^{iP\cdot a}\,\varphi(x)\,e^{-iP\cdot a} = \varphi(x+a)\,, \tag{8.12}$$

where P^μ is the *exact* four-momentum operator (including interactions). Let us recall how this comes about. The full Lagrangian $\mathcal{L}$ is invariant under spacetime translations $x^\mu \to x^\mu + a^\mu$. By Noether's theorem, this invariance yields a conserved four-momentum P^μ—the *total* four-momentum of the interacting theory, not just the free part. In Chapter 1, Eq. (1.201), we proved by explicit computation with mode expansions that the free four-momentum satisfies $[\psi(x), P^\mu] = +i\partial^\mu\psi(x)$, or equivalently (by antisymmetry of the commutator) $[P^\mu, \psi(x)] = -i\partial^\mu\psi(x)$. We adopt the latter form here. For the interacting theory, the same relation holds:

$$[P^\mu, \varphi(x)] = -i\partial^\mu\varphi(x)\,. \tag{8.13}$$

This can be seen directly from the canonical formalism. For the temporal component, recall that in the Heisenberg picture every operator evolves according to

$$i\frac{\partial\varphi}{\partial t} = [\varphi, H]\,, \tag{8.14}$$

where H is the *full* Hamiltonian, including interactions. Rearranging:

$$[P^0, \varphi(x)] = [H, \varphi(x)] = -i\,\partial_0\varphi(x)\,. \tag{8.15}$$

For the spatial components, using the equal-time commutation relations (8.11) and the expression $P^i = \int d^3x\,\pi(\vec{x},t)\,\partial^i\varphi(\vec{x},t)$ from Noether's theorem:

$$[P^i, \varphi(\vec{y},t)] = \int d^3x\,[\pi(\vec{x},t), \varphi(\vec{y},t)]\,\partial^i\varphi(\vec{x},t) = -i\partial^i\varphi(\vec{y},t)\,. \tag{8.16}$$

The finite translation (8.12) then follows by exponentiation of the infinitesimal result (8.13). The key identity is the *Hadamard lemma*: for any operator A and any Hermitian generator G,

$$e^{i\alpha G}\,A\,e^{-i\alpha G} = \sum_{n=0}^{\infty}\frac{(i\alpha)^n}{n!}\underbrace{[G,[G,\cdots[G}_{n},A]\cdots]]\,. \tag{8.17}$$

To prove this, define $f(\alpha) = e^{i\alpha G}\,A\,e^{-i\alpha G}$ and Taylor-expand around $\alpha = 0$. Differentiating gives $f'(\alpha) = i\,[G, f(\alpha)]$, so $f^{(n)}(0) = (i)^n\,[G,[G,\cdots[G,A]\cdots]]$ with n nested commutators.
In our case, the generator is $G = a_\mu P^\mu$ (with the role of α absorbed into a_μ), and $A = \varphi(x)$. The first few nested commutators are:

$$\begin{aligned}
[a_\mu P^\mu, \varphi(x)] &= -ia^\mu\partial_\mu\varphi(x)\,,\\
[a_\mu P^\mu, [a_\nu P^\nu, \varphi(x)]] &= [a_\mu P^\mu, -ia^\nu\partial_\nu\varphi(x)]\\
&= (-i)^2 a^\mu a^\nu\partial_\mu\partial_\nu\varphi(x)\,,
\end{aligned} \tag{8.18}$$

where in the second line we used the fact that a^ν is a c-number and that the commutation relation (8.13) holds for $\partial_\nu\varphi$ as well as for φ. Let us spell out why. Since P^μ is an integral over all space and does not depend on the spacetime point x, the derivative $\partial_\nu^{(x)}$ commutes with P^μ inside a commutator. Taking $\partial_\nu^{(x)}$ of both sides of (8.13):

$$\partial_\nu^{(x)}[P^\mu,\varphi(x)] = [P^\mu,\partial_\nu\varphi(x)] = -i\partial^\mu\partial_\nu\varphi(x)\,. \tag{8.19}$$

More generally, P^μ generates translations of *any* local operator $\mathcal{O}(x)$ constructed from products of fields and their derivatives at x. This follows by repeated application of the identity $[A,BC] = [A,B]C + B[A,C]$: for a product $\mathcal{O} = \mathcal{O}_1\mathcal{O}_2$,

$$\begin{aligned}[P^\mu,\mathcal{O}_1\mathcal{O}_2] &= [P^\mu,\mathcal{O}_1]\mathcal{O}_2 + \mathcal{O}_1[P^\mu,\mathcal{O}_2]\\ &= (-i\partial^\mu\mathcal{O}_1)\mathcal{O}_2 + \mathcal{O}_1(-i\partial^\mu\mathcal{O}_2)\\ &= -i\partial^\mu(\mathcal{O}_1\mathcal{O}_2)\,,\end{aligned} \tag{8.20}$$

which is just the Leibniz rule. By induction, $[P^\mu,\mathcal{O}(x)] = -i\partial^\mu\mathcal{O}(x)$ for any local composite operator.
The pattern is clear: the n-th nested commutator gives

$$(-i)^n a^{\mu_1}\cdots a^{\mu_n}\partial_{\mu_1}\cdots\partial_{\mu_n}\varphi(x)\,.$$

Substituting into (8.17):

$$\begin{aligned}e^{iP\cdot a}\,\varphi(x)\,e^{-iP\cdot a} &= \sum_{n=0}^{\infty}\frac{(i)^n(-i)^n}{n!}\,a^{\mu_1}\cdots a^{\mu_n}\partial_{\mu_1}\cdots\partial_{\mu_n}\varphi(x)\\ &= \sum_{n=0}^{\infty}\frac{1}{n!}(a\cdot\partial)^n\varphi(x) = \varphi(x+a)\,,\end{aligned} \tag{8.21}$$

where we used $(i)^n(-i)^n = 1$ and recognised the Taylor expansion of φ around x. We require that $\varphi_{\text{in}}(x)$ inherits this property:

$$[P^\mu,\varphi_{\text{in}}(x)] = -i\frac{\partial\varphi_{\text{in}}(x)}{\partial x_\mu}\,. \tag{8.22}$$

This is indeed the case, as we shall verify explicitly after constructing $\varphi_{\text{in}}(x)$.

2. **Free-field equation.** The spacetime development is described by a free-particle Klein–Gordon equation with the *physical* mass m:

$$(\Box + m^2)\varphi_{\text{in}}(x) = 0\,. \tag{8.23}$$

Proof that $\varphi_{\text{in}}(x)$ creates only one-particle states. Consider an arbitrary eigenstate $|n\rangle$ with $P^\mu|n\rangle = p_n^\mu|n\rangle$, and form the matrix element with the vacuum:

$$-i\frac{\partial}{\partial x_\mu}\langle n|\varphi_{\text{in}}(x)|0\rangle = \langle n|[P^\mu, \varphi_{\text{in}}(x)]|0\rangle = p_n^\mu \langle n|\varphi_{\text{in}}(x)|0\rangle\,. \tag{8.24}$$

Iterating this operation and using (8.23):

$$(\Box + m^2)\langle n|\varphi_{\text{in}}(x)|0\rangle = (m^2 - p_n^2)\langle n|\varphi_{\text{in}}(x)|0\rangle = 0\,. \tag{8.25}$$

Therefore, the only states produced from the vacuum by $\varphi_{\text{in}}(x)$ are those with $p_n^2 = m^2$, that is, the one-particle states of mass m. □

Fourier expansion of the in-field

Since $\varphi_{\text{in}}(x)$ satisfies the free Klein–Gordon equation, its Fourier expansion takes the standard form:

$$\varphi_{\text{in}}(x) = \int d^3k \left[a_{\text{in}}(k) f_k(x) + a_{\text{in}}^\dagger(k) f_k^*(x)\right], \tag{8.26}$$

with

$$f_k(x) = \frac{1}{\sqrt{(2\pi)^3 2\omega_k}} e^{-ik\cdot x}, \qquad \omega_k = \sqrt{\vec{k}^2 + m^2} = k^0, \tag{8.27}$$

and the inverse relation

$$a_{\text{in}}(k) = i\int d^3x\, f_k^*(x) \overleftrightarrow{\partial_0} \varphi_{\text{in}}(x)\,, \tag{8.28}$$

where $A \overleftrightarrow{\partial_0} B \equiv A(\partial_0 B) - (\partial_0 A)B$.

The operator coefficients satisfy

$$[P^\mu, a_{\text{in}}(k)] = -k^\mu a_{\text{in}}(k), \qquad [P^\mu, a_{\text{in}}^\dagger(k)] = +k^\mu a_{\text{in}}^\dagger(k)\,. \tag{8.29}$$

Repeated application of $a_{\text{in}}^\dagger(k)$ to the vacuum builds up general n-particle eigenstates. To see that these are eigenstates of P^μ, we commute P^μ past each creation operator one at a time using (8.29). For the one-particle state:

$$\begin{aligned} P^\mu a_{\text{in}}^\dagger(k)|0\rangle &= \big([P^\mu, a_{\text{in}}^\dagger(k)] + a_{\text{in}}^\dagger(k)\, P^\mu\big)|0\rangle \\ &= k^\mu a_{\text{in}}^\dagger(k)|0\rangle + a_{\text{in}}^\dagger(k) \underbrace{P^\mu|0\rangle}_{=0} = k^\mu |k\text{ in}\rangle\,. \end{aligned} \tag{8.30}$$

For two particles, P^μ first commutes past $a_{\text{in}}^\dagger(k_1)$, picking up k_1^μ, and the remainder $a_{\text{in}}^\dagger(k_1)P^\mu$ acts on $a_{\text{in}}^\dagger(k_2)|0\rangle$, which by (8.30) gives k_2^μ. The pattern repeats for N particles:

$$P^\mu|k_1\cdots k_N\text{ in}\rangle = P^\mu a_{\text{in}}^\dagger(k_1)\cdots a_{\text{in}}^\dagger(k_N)|0\rangle = \sum_{i=1}^{N} k_i^\mu\, |k_1\cdots k_N\text{ in}\rangle\,, \tag{8.31}$$

with $a_{\text{in}}(k)|0\rangle = 0$.

Explicit construction of $\varphi_{\text{in}}(x)$

To express $\varphi_{\text{in}}(x)$ in terms of the interacting field $\varphi(x)$, we rewrite the field equation (8.10) using the physical mass m by adding a mass counterterm:

$$(\Box + m^2)\varphi(x) = j(x) + \delta m^2 \varphi(x) \equiv \tilde{j}(x)\,, \tag{8.32}$$

where $\delta m^2 = m^2 - m_0^2$. Written in this form, the equation $(\Box + m^2)\varphi = \tilde{j}$ has exactly the structure of a *scattering problem*: $\varphi(x)$ plays the role of the total wave, and $\tilde{j}(x)$ acts as the source that generates scattered waves. The strategy for isolating the free part is the same as in non-relativistic quantum mechanics, where the **Lippmann–Schwinger equation** provides the decomposition of a scattering state into incident and scattered waves. Let us briefly recall the derivation. The full Hamiltonian is $H = H_0 + V$, and we seek an eigenstate $|\psi\rangle$ with energy E:

$$(E - H_0 - V)|\psi\rangle = 0\,. \tag{8.33}$$

Let $|\psi_0\rangle$ be the free state satisfying $(E - H_0)|\psi_0\rangle = 0$ with the same energy. Then $|\psi\rangle - |\psi_0\rangle$ satisfies

$$(E - H_0)(|\psi\rangle - |\psi_0\rangle) = V|\psi\rangle\,. \tag{8.34}$$

Inverting $(E - H_0)$ with a retarded boundary condition ($+i\epsilon$ prescription, selecting outgoing scattered waves) gives

$$|\psi\rangle = |\psi_0\rangle + G_0^{(+)}(E)\,V\,|\psi\rangle\,, \qquad G_0^{(+)}(E) = \frac{1}{E - H_0 + i\epsilon}\,. \tag{8.35}$$

The first term is the incident free wave; the second is the scattered wave, generated by the potential V acting on the full state and propagated by the free retarded Green's function. Subtracting the scattered wave from the total state recovers the free part: $|\psi_0\rangle = |\psi\rangle - G_0^{(+)}V|\psi\rangle$.

We do exactly the same here: subtracting from $\varphi(x)$ the scattered waves generated by $\tilde{j}$ via the retarded Green's function leaves just the free waves propagating with mass m. This gives:

$$\boxed{\sqrt{Z}\,\varphi_{\text{in}}(x) = \varphi(x) - \int d^4y\,\Delta_{\text{ret}}(x - y; m)\,\tilde{j}(y)}\,, \tag{8.36}$$

where $\Delta_{\text{ret}}(x - y; m)$ is the retarded Green's function satisfying

$$(\Box_x + m^2)\Delta_{\text{ret}}(x - y; m) = \delta^{(4)}(x - y), \qquad \Delta_{\text{ret}}(x - y; m) = 0 \text{ for } x^0 < y^0\,. \tag{8.37}$$

The constant $\sqrt{Z}$ normalizes $\varphi_{\text{in}}(x)$ to unit amplitude for creating one-particle states from the vacuum.

Verification of the defining properties. As defined in (8.36), $\varphi_{\text{in}}(x)$ satisfies both conditions (8.22) and (8.23). For the covariance property, we substitute $x \to x+a$ in (8.36):

$$\sqrt{Z}\,\varphi_{\text{in}}(x+a) = \varphi(x+a) - \int d^4y\,\Delta_{\text{ret}}(x+a-y)\,\tilde{j}(y)\,. \tag{8.38}$$

In the first term we use the translation property $\varphi(x+a) = e^{iP\cdot a}\varphi(x)e^{-iP\cdot a}$. In the integral we change variable $y' = y-a$, so that $\Delta_{\text{ret}}(x+a-y) = \Delta_{\text{ret}}(x-y')$ and $\tilde{j}(y) = \tilde{j}(y'+a)$:

$$= e^{iP\cdot a}\varphi(x)e^{-iP\cdot a} - \int d^4y'\,\Delta_{\text{ret}}(x-y')\,\tilde{j}(y'+a)\,. \tag{8.39}$$

Since $\tilde{j}$ is a scalar operator, $\tilde{j}(y'+a) = e^{iP\cdot a}\,\tilde{j}(y')\,e^{-iP\cdot a}$. The retarded Green's function $\Delta_{\text{ret}}(x-y')$ is a c-number, so it commutes with $e^{\pm iP\cdot a}$ and we can factor the unitary conjugation out of the integral:

$$\begin{aligned}
&= e^{iP\cdot a}\varphi(x)e^{-iP\cdot a} - e^{iP\cdot a}\left[\int d^4y'\,\Delta_{\text{ret}}(x-y')\,\tilde{j}(y')\right]e^{-iP\cdot a} \\
&= e^{iP\cdot a}\underbrace{\left[\varphi(x) - \int d^4y'\,\Delta_{\text{ret}}(x-y')\,\tilde{j}(y')\right]}_{=\sqrt{Z}\,\varphi_{\text{in}}(x)\text{ by (8.36)}}e^{-iP\cdot a} = e^{iP\cdot a}\sqrt{Z}\,\varphi_{\text{in}}(x)\,e^{-iP\cdot a}\,,
\end{aligned} \tag{8.40}$$

as required. For the free-field equation, apply $(\Box_x + m^2)$ to both sides of (8.36):

$$\begin{aligned}
\sqrt{Z}\,(\Box_x+m^2)\varphi_{\text{in}}(x) &= (\Box_x+m^2)\varphi(x) - \int d^4y\,\underbrace{(\Box_x+m^2)\Delta_{\text{ret}}(x-y;m)}_{=\,\delta^{(4)}(x-y)}\,\tilde{j}(y) \\
&= \tilde{j}(x) - \int d^4y\,\delta^{(4)}(x-y)\,\tilde{j}(y) \\
&= \tilde{j}(x) - \tilde{j}(x) = 0\,,
\end{aligned} \tag{8.41}$$

where in the first term we used the field equation (8.32), and in the second we used the defining property (8.37) of the retarded Green's function. Since $\sqrt{Z} \neq 0$, we conclude $(\Box+m^2)\varphi_{\text{in}}(x) = 0$.

The weak asymptotic condition

The form of (8.36) might suggest that as $x^0 \to -\infty$, the retarded Green's function term vanishes as it would for a source with finite time support, so that

$$\varphi(x) \overset{?}{\to} \sqrt{Z}\,\varphi_{\text{in}}(x) \quad \text{as } x^0 \to -\infty. \tag{8.42}$$

However, this *cannot* hold as an operator identity. The source $\tilde{j}(x) = j(x) + \delta m^2\,\varphi(x)$ includes the self-interaction of the field with itself, which persists at all times—even a single isolated particle is always surrounded by its own virtual cloud. There is no time at which $\varphi(x)$ literally becomes a free field.

To see the problem more concretely, consider a matrix element of $\varphi(x)$ between the vacuum and a multiparticle state carrying the same quantum numbers as φ. If (8.42) held as an operator statement, this matrix element would have to vanish for $x^0 \to -\infty$ (since φ_{in} creates only one-particle states). In an interacting theory the local field can also have overlap with continuum states in the same quantum-number sector, so the strong limit cannot hold.

The correct asymptotic condition, due to Lehmann, Symanzik, and Zimmermann [25], takes a *weak* form: the convergence holds only inside matrix elements between normalizable (wave-packet) states, and only after a spatial smearing that projects out the one-particle contribution.

Definition of the smeared field. Let $f(\vec{x},t)$ be a normalizable solution of the free Klein–Gordon equation $(\Box + m^2)f = 0$, for example a smooth wave packet:

$$f(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}}\,\tilde{f}(\vec{k})\,e^{-ik\cdot x}\,, \tag{8.43}$$

where $\tilde{f}(\vec{k})$ is chosen smooth and sufficiently rapidly decreasing, for instance a Schwartz function, so that the particle is localized in momentum space and the oscillatory integrals below are well defined. Define the **smeared field operator**:

$$\varphi^f(t) \equiv i\int d^3x\, f^*(\vec{x},t)\overleftrightarrow{\partial_0}\,\varphi(\vec{x},t)\,, \tag{8.44}$$

The purpose of this construction is to project the field onto the one-particle sector. To see why, suppose φ were a free field with mode expansion (8.26). Comparing (8.43) with (8.27), we see that $f^*(x) = \int d^3k\,\tilde{f}^*(\vec{k})\,f_k^*(x)$, so by linearity of the spatial integral and (8.44):

$$\varphi^f = \int d^3k\,\tilde{f}^*(\vec{k})\,i\int d^3x\, f_k^*(x)\overleftrightarrow{\partial_0}\,\varphi(x)\,. \tag{8.45}$$

Now substitute the mode expansion $\varphi(x) = \int d^3k'\,[a(k')f_{k'}(x) + a^\dagger(k')f_{k'}^*(x)]$ and use the orthonormality of the modes in the Klein–Gordon inner product:

$$i\int d^3x\, f_k^*\overleftrightarrow{\partial_0} f_{k'} = \delta^3(\vec{k}-\vec{k}'), \qquad i\int d^3x\, f_k^*\overleftrightarrow{\partial_0} f_{k'}^* = 0\,. \tag{8.46}$$

The first relation picks out $a(\vec{k})$; the second kills $a^\dagger(\vec{k}')$, because the Klein–Gordon inner product gives zero for this pairing of two negative-frequency modes. Therefore:

$$\varphi^f = \int d^3k\,\tilde{f}^*(\vec{k})\,a(\vec{k}) \equiv a(\tilde{f}^*)\,, \tag{8.47}$$

the annihilation operator smeared with the wave packet $\tilde{f}^*$. Since each $a(\vec{k})$ is time-independent, so is $a(\tilde{f}^*)$—the result does not depend on the time at which we evaluate the spatial integral. The key question is what happens for the interacting field.

Time independence of $\varphi^f_{\mathbf{in}}$. Since $\varphi_{\rm in}(x)$ satisfies the free Klein–Gordon equation, the analogously defined

$$\varphi^f_{\rm in} \equiv i \int d^3x\, f^*(\vec{x},t) \overleftrightarrow{\partial_0} \varphi_{\rm in}(\vec{x},t) \tag{8.48}$$

is indeed *time-independent*. To see this, compute its time derivative:

$$\begin{aligned}
\frac{d}{dt}\varphi^f_{\rm in} &= i \int d^3x \left[\dot{f}^* \overleftrightarrow{\partial_0} \varphi_{\rm in} + f^* \overleftrightarrow{\partial_0} \dot{\varphi}_{\rm in} \right] \\
&= i \int d^3x \left[\dot{f}^* \dot{\varphi}_{\rm in} - \ddot{f}^* \varphi_{\rm in} + f^* \ddot{\varphi}_{\rm in} - \dot{f}^* \dot{\varphi}_{\rm in} \right] \\
&= i \int d^3x \left[f^* \ddot{\varphi}_{\rm in} - \ddot{f}^* \varphi_{\rm in} \right] .
\end{aligned} \tag{8.49}$$

Now use the fact that both f and $\varphi_{\rm in}$ satisfy the Klein–Gordon equation: $\ddot{f}^* = (\nabla^2 - m^2) f^*$ and $\ddot{\varphi}_{\rm in} = (\nabla^2 - m^2)\varphi_{\rm in}$. Substituting:

$$\begin{aligned}
\frac{d}{dt}\varphi^f_{\rm in} &= i \int d^3x \left[f^* (\nabla^2 - m^2)\varphi_{\rm in} - (\nabla^2 - m^2) f^* \cdot \varphi_{\rm in} \right] \\
&= i \int d^3x \left[f^* \nabla^2 \varphi_{\rm in} - (\nabla^2 f^*)\varphi_{\rm in} \right] .
\end{aligned} \tag{8.50}$$

The integrand is $\nabla \cdot (f^* \nabla \varphi_{\rm in} - \varphi_{\rm in} \nabla f^*)$, a total divergence. For normalizable wave packets, the fields vanish at spatial infinity, so the integral is zero:

$$\frac{d}{dt}\varphi^f_{\rm in} = 0 . \tag{8.51}$$

This means $\varphi^f_{\rm in}$ is a time-independent operator. Since $\varphi_{\rm in}$ is a free field, the same decomposition as in (8.45) applies:

$$\varphi^f_{\rm in} = \int d^3k\, \tilde{f}^*(\vec{k})\, a_{\rm in}(\vec{k}) = a_{\rm in}(\tilde{f}^*) , \tag{8.52}$$

the annihilation operator for a particle with wave-packet profile $\tilde{f}^*$, whose value does not depend on the time t at which we evaluate the spatial integral. The Hermitian conjugate $(\varphi^f_{\rm in})^\dagger = \int d^3k\, \tilde{f}(\vec{k})\, a^\dagger_{\rm in}(\vec{k}) = a^\dagger_{\rm in}(\tilde{f})$ is the corresponding creation operator.

Why the continuum contributions vanish. For the interacting field, $\varphi^f(t)$ defined in (8.44) *is* time-dependent: the field $\varphi(x)$ does not satisfy the free Klein–Gordon

equation, so the argument above does not go through. The crucial point is what happens to the matrix element $\langle\alpha|\varphi^f(t)|\beta\rangle$ as $t \to -\infty$.

Insert a complete set of exact energy-momentum eigenstates between the operators:

$$\langle\alpha|\varphi^f(t)|\beta\rangle = i\int d^3x \sum_n f^*(\vec{x},t)\overleftrightarrow{\partial_0}\langle\alpha|\varphi(\vec{x},t)|n\rangle\langle n|\beta\rangle\,. \tag{8.53}$$

To illustrate the mechanism most transparently, take $\langle\alpha| = \langle 0|$ (the vacuum). Since $P^\mu|0\rangle = 0$, the translation property gives

$$\langle 0|\varphi(x)|n\rangle = \langle 0|e^{iP\cdot x}\varphi(0)e^{-iP\cdot x}|n\rangle = e^{-ip_n\cdot x}\langle 0|\varphi(0)|n\rangle\,. \tag{8.54}$$

Substituting $f^*(\vec{x},t) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}}\,\tilde{f}^*(\vec{k})\,e^{i\omega_k t - i\vec{k}\cdot\vec{x}}$ and computing the $\overleftrightarrow{\partial_0}$ derivative:

$$\begin{aligned} f^*(\vec{x},t)\overleftrightarrow{\partial_0}e^{-ip_n\cdot x} &= f^*\cdot(-iE_n)e^{-ip_n\cdot x} - (i\omega_k)\,f^*\cdot e^{-ip_n\cdot x} \\ &= -i(E_n+\omega_k)\,f^*\,e^{-ip_n\cdot x}\,, \end{aligned} \tag{8.55}$$

where in the second term $\partial_0 f^* = i\omega_k f^*$ for each Fourier mode. The spatial integral of the product $f_k^*\cdot e^{-ip_n\cdot x}$ gives

$$\int d^3x\, e^{-i\vec{k}\cdot\vec{x}}\,e^{i\vec{p}_n\cdot\vec{x}} = (2\pi)^3\,\delta^3(\vec{p}_n - \vec{k})\,, \tag{8.56}$$

which enforces $\vec{p}_n = \vec{k}$. After performing the d^3k integral using this delta function, the time-dependent factors combine into $e^{i\omega_k t}\cdot e^{-iE_n t}\big|_{\vec{p}_n=\vec{k}} = e^{i(\omega_k - E_n)t}$. Including the normalization and prefactors, the result is

$$\langle 0|\varphi^f(t)|\beta\rangle = \sum_\lambda \int d^3k\,\tilde{f}^*(\vec{k})\,G_\lambda(\vec{k})\,e^{i(\omega_k - E_\lambda(\vec{k}))t}\,, \tag{8.57}$$

where the sum runs over all intermediate-state types λ (one-particle, two-particle, etc.) with total momentum $\vec{k}$ and energy $E_\lambda(\vec{k}) = \sqrt{\vec{k}^2 + M_\lambda^2}$, and $G_\lambda(\vec{k})$ collects the matrix elements $\langle 0|\varphi(0)|\lambda,\vec{k}\rangle$, $\langle\lambda,\vec{k}|\beta\rangle$, and kinematic factors.

Now the mass gap assumption plays its role. For **one-particle intermediate states**, $M_\lambda = m$ and $E_\lambda(\vec{k}) = \omega_k$, so the phase vanishes identically:

$$e^{i(\omega_k - E_\lambda)t}\big|_{\text{1-particle}} = e^{i(\omega_k-\omega_k)t} = 1 \qquad \text{(time-independent)}. \tag{8.58}$$

The one-particle contribution to $\langle 0|\varphi^f(t)|\beta\rangle$ is therefore time-independent and gives the free-field result $\sqrt{Z}\,\langle 0|\varphi^f_{\text{in}}|\beta\rangle$.

For **continuum intermediate states**, the invariant mass satisfies $M_\lambda \geq m_1 > m$, so $E_\lambda(\vec{k}) = \sqrt{\vec{k}^2 + M_\lambda^2} > \omega_k$ for every $\vec{k}$. The phase $\omega_k - E_\lambda(\vec{k})$ is strictly negative, and the exponential $e^{i(\omega_k - E_\lambda)t}$ oscillates *rapidly* as $t \to -\infty$. It is essential that we integrate over a smooth wave packet $\tilde{f}^*(\vec{k})$: for sharp-momentum plane waves, the rapidly oscillating phase would simply keep oscillating forever without decaying, and the argument would fail. With a smooth $\tilde{f}^*(\vec{k})$, however, the Riemann–Lebesgue lemma[2] guarantees that these contributions vanish:

$$\int d^3k\, \tilde{f}^*(\vec{k})\, G_\lambda(\vec{k})\, e^{i(\omega_k - E_\lambda(\vec{k}))t} \xrightarrow{t\to-\infty} 0, \tag{8.59}$$

provided $\omega_k - E_\lambda(\vec{k}) \neq 0$ throughout the integration region and the product $\tilde{f}^*(\vec{k})G_\lambda(\vec{k})$ is integrable. The mass gap guarantees the first condition. If the continuum started at $M_\lambda = m$, there would be states with $E_\lambda = \omega_k$, and the oscillating phase could not eliminate them.

The argument generalises to matrix elements $\langle\alpha|\varphi^f(t)|\beta\rangle$ with arbitrary normalizable $\langle\alpha|$. One inserts completeness and uses translation invariance; the resulting phase $\omega_k + E_\alpha - E_n(\vec{k} + \vec{p}_\alpha)$ again separates one-particle from multiparticle contributions thanks to the mass gap.

Statement of the asymptotic condition. The result is the **weak asymptotic condition** of LSZ:

$$\boxed{\lim_{t\to-\infty} \langle\alpha|\varphi^f(t)|\beta\rangle = \sqrt{Z}\,\langle\alpha|\varphi^f_{\text{in}}|\beta\rangle}, \tag{8.60}$$

for any normalizable states $|\alpha\rangle$, $|\beta\rangle$. The factor $\sqrt{Z}$ accounts for the fact that the interacting field has amplitude $\sqrt{Z}$ to create a one-particle state from the vacuum. The word "weak" refers to the fact that the convergence holds only inside matrix elements between normalizable states—it is *not* an operator identity. The smearing with a wave packet f is essential: it simultaneously projects onto the one-particle sector and provides the smooth momentum-space profile needed for the Riemann–Lebesgue lemma to suppress the continuum.

8.4 Out-fields and the S-matrix

Just as we have reduced the dynamics at $t \to -\infty$ to that of free particles using the in-fields, we can do the same at $t \to +\infty$ by defining **out-fields** $\varphi_{\text{out}}(x)$.

[2]The Riemann–Lebesgue lemma states that when a smooth, integrable function is multiplied by a rapidly oscillating phase $e^{i\Omega t}$ and integrated, the result vanishes as $\Omega \to \infty$ (or, equivalently, as $t \to \infty$). The physical intuition is simple: when the oscillation is fast enough, the integrand spends equal amounts of time being positive and negative across the smooth envelope, and these contributions cancel out. In our case, $t \to -\infty$ plays the role of the large parameter, making the phase oscillate faster and faster over the integration region in $\vec{k}$.

Construction of out-fields

The out-fields are defined to satisfy

$$[P^\mu, \varphi_{\rm out}(x)] = -i\frac{\partial \varphi_{\rm out}(x)}{\partial x_\mu}, \qquad (\Box + m^2)\varphi_{\rm out}(x) = 0, \tag{8.61}$$

and therefore $\varphi_{\rm out}(x)$ produces only one-particle states from the vacuum, just as $\varphi_{\rm in}(x)$ does. The Fourier expansion is

$$\varphi_{\rm out}(x) = \int d^3k \left[a_{\rm out}(k) f_k(x) + a^\dagger_{\rm out}(k) f_k^*(x)\right]. \tag{8.62}$$

The asymptotic condition for the out-fields takes the form

$$\lim_{t\to+\infty} \langle\alpha|\varphi^f(t)|\beta\rangle = \sqrt{Z}\,\langle\alpha|\varphi^f_{\rm out}|\beta\rangle, \tag{8.63}$$

or simply $\varphi(x) \to \sqrt{Z}\,\varphi_{\rm out}(x)$ as $t \to +\infty$ (in the weak sense).

The explicit construction uses the advanced Green's function:

$$\sqrt{Z}\,\varphi_{\rm out}(x) = \varphi(x) - \int d^4y\, \Delta_{\rm adv}(x-y;m)\,\tilde{j}(y), \tag{8.64}$$

where

$$(\Box_x + m^2)\Delta_{\rm adv}(x-y;m) = \delta^{(4)}(x-y), \qquad \Delta_{\rm adv}(x-y;m) = 0 \text{ for } x^0 > y^0. \tag{8.65}$$

The same normalizing constant $\sqrt{Z}$ appears because the amplitude for $\varphi(x)$ to create a one-particle state from the vacuum is the same:

$$\langle 0|\varphi(x)|p\rangle = \sqrt{Z}\,\langle 0|\varphi_{\rm in}(x)|p\rangle = \sqrt{Z}\,\langle 0|\varphi_{\rm out}(x)|p\rangle = \sqrt{Z}\,\frac{e^{-ip\cdot x}}{\sqrt{(2\pi)^3 2\omega_p}}. \tag{8.66}$$

Definition of the S-matrix

The in- and out-states are defined by

$$|p_1\cdots p_n \text{ in}\rangle = a^\dagger_{\rm in}(p_1)\cdots a^\dagger_{\rm in}(p_n)|0\rangle, \tag{8.67}$$

$$|p_1\cdots p_m \text{ out}\rangle = a^\dagger_{\rm out}(p_1)\cdots a^\dagger_{\rm out}(p_m)|0\rangle. \tag{8.68}$$

The S-matrix element for a transition from an initial state $|\alpha \text{ in}\rangle$ to a final state $|\beta \text{ out}\rangle$ is the probability amplitude:

$$\boxed{S_{\beta\alpha} = \langle\beta \text{ out}|\alpha \text{ in}\rangle} \tag{8.69}$$

By completeness of the in- and out-states, all matrix elements define an operator S on the in-state Hilbert space:

$$\langle\beta \text{ in}|S = \langle\beta \text{ out}|, \qquad S_{\beta\alpha} = \langle\beta \text{ in}|S|\alpha \text{ in}\rangle. \tag{8.70}$$

Properties of the S-matrix

Several important properties of S follow from our initial assumptions. We now prove each one in detail.

Property 1: Stability of the vacuum.

Statement: The vacuum state is stable under scattering, and we can choose $S_{00} = 1$.

Proof. By definition of the S-matrix element:

$$S_{00} = \langle 0 \text{ out}|0 \text{ in}\rangle. \tag{8.71}$$

We show that the vacuum cannot make a transition to any other state. Consider $\langle n \text{ out}|0 \text{ in}\rangle$ for an arbitrary out-state $|n \text{ out}\rangle$ with four-momentum p_n^μ. Since P^μ is the *same* operator acting on both in- and out-states, we have

$$p_n^\mu \langle n \text{ out}|0 \text{ in}\rangle = \langle n \text{ out}|P^\mu|0 \text{ in}\rangle = 0\,, \tag{8.72}$$

where we used $P^\mu|0 \text{ in}\rangle = 0$ (the vacuum has zero energy and momentum). Therefore:

$$\text{if } p_n^\mu \neq 0 \quad \Longrightarrow \quad \langle n \text{ out}|0 \text{ in}\rangle = 0\,. \tag{8.73}$$

By our spectral assumptions, the vacuum is the *unique* state with $P^\mu = 0$: every other state (one-particle, multiparticle) has $P^0 > 0$. So the only nonvanishing overlap is with $|n\rangle = |0\rangle$. Using the completeness of out-states, $\sum_n |n \text{ out}\rangle\langle n \text{ out}| = \mathbb{1}$:

$$1 = \langle 0 \text{ in}|0 \text{ in}\rangle = \sum_n |\langle n \text{ out}|0 \text{ in}\rangle|^2 = |S_{00}|^2\,, \tag{8.74}$$

since only the $n = 0$ term survives. Hence $|S_{00}| = 1$, and the vacuum state is the same for both the in- and out-descriptions (there is nothing to scatter). The only freedom is an overall phase:

$$\langle 0 \text{ out}| = e^{i\varphi_0}\langle 0 \text{ in}|\,. \tag{8.75}$$

We *choose* the phase convention $\varphi_0 = 0$, giving:

$$\langle 0 \text{ out}| = \langle 0 \text{ in}| = \langle 0|, \qquad S_{00} = 1\,. \tag{8.76}$$

□

Property 2: Stability of one-particle states.

Statement: A single stable particle cannot decay or transform into other particles: $|p \text{ in}\rangle = |p \text{ out}\rangle$.

Proof. The argument parallels that for the vacuum. Consider $\langle n \text{ out}|p \text{ in}\rangle$ for an arbitrary out-state $|n \text{ out}\rangle$ with total four-momentum p_n^μ. Since $P^\mu|p \text{ in}\rangle = p^\mu|p \text{ in}\rangle$:

$$p_n^\mu \langle n \text{ out}|p \text{ in}\rangle = \langle n \text{ out}|P^\mu|p \text{ in}\rangle = p^\mu \langle n \text{ out}|p \text{ in}\rangle . \tag{8.77}$$

Therefore:

$$(p_n^\mu - p^\mu)\langle n \text{ out}|p \text{ in}\rangle = 0 \quad \Longrightarrow \quad \text{if } p_n^\mu \neq p^\mu, \text{ then } \langle n \text{ out}|p \text{ in}\rangle = 0 . \tag{8.78}$$

Now we ask: which out-states can have exactly $p_n^\mu = p^\mu$? Since $p^2 = m^2$ (the particle is on-shell), we need $p_n^2 = m^2$. But by the *mass gap* assumption, the multiparticle continuum starts at $p^2 \geq m_1^2 > m^2$. The only states with invariant mass m^2 are one-particle states with the same conserved quantum numbers. Therefore the sole out-state with $p_n^\mu = p^\mu$ is the one-particle state $|p \text{ out}\rangle$ itself, up to possible degeneracies such as spin or internal labels, which are understood to be included in the state label.

For normalized wave-packet states, completeness of the out-basis gives

$$1 = \langle p \text{ in}|p \text{ in}\rangle = \sum_n |\langle n \text{ out}|p \text{ in}\rangle|^2 = |\langle p \text{ out}|p \text{ in}\rangle|^2 , \tag{8.79}$$

since only the $n = |p \text{ out}\rangle$ term survives. Hence $|\langle p \text{ out}|p \text{ in}\rangle| = 1$, and $|p \text{ in}\rangle$ and $|p \text{ out}\rangle$ differ at most by a phase. In the sharp-momentum notation used below the same statement is understood distributionally, with Kronecker deltas replaced by Dirac delta functions. Choosing this phase to be zero (a convention compatible with the vacuum phase choice):

$$|p \text{ in}\rangle = |p \text{ out}\rangle \equiv |p\rangle . \tag{8.80}$$

Consequently, using the definition (8.70):

$$\langle p \text{ in}|S|p' \text{ in}\rangle = \langle p \text{ out}|p' \text{ in}\rangle = \langle p|p'\rangle = \delta^{(3)}(\vec{p} - \vec{p}') . \tag{8.81}$$

The S-matrix acts as the identity on one-particle states: a single isolated particle with momentum p at $t = -\infty$ remains a particle with the same momentum at $t = +\infty$. Note that this result depends crucially on the mass gap—without it, the one-particle state could "leak" into the multiparticle continuum. □

Property 3: Relation between in- and out-fields.

Statement: The S-matrix transforms in-fields to out-fields according to $\varphi_{\text{in}}(x) = S\,\varphi_{\text{out}}(x)\,S^{-1}$.

Proof. We must show the operator identity $\varphi_{\text{in}}(x)\,S = S\,\varphi_{\text{out}}(x)$, or equivalently

$$\langle \beta \text{ in}|\varphi_{\text{in}}(x)\,S|\alpha \text{ in}\rangle = \langle \beta \text{ in}|S\,\varphi_{\text{out}}(x)|\alpha \text{ in}\rangle \tag{8.82}$$

for all in-states $|\alpha\ \text{in}\rangle$, $|\beta\ \text{in}\rangle$ (which form a complete set). Using $\langle\beta\ \text{in}|S = \langle\beta\ \text{out}|$ from (8.70), the right-hand side becomes

$$\text{RHS} = \langle\beta\ \text{out}|\varphi_{\text{out}}(x)|\alpha\ \text{in}\rangle\,. \tag{8.83}$$

For the left-hand side, we insert the completeness relation $\sum_\gamma |\gamma\ \text{in}\rangle\langle\gamma\ \text{in}| = \mathbb{1}$ between $\varphi_{\text{in}}(x)$ and S:

$$\text{LHS} = \sum_\gamma \langle\beta\ \text{in}|\varphi_{\text{in}}(x)|\gamma\ \text{in}\rangle \underbrace{\langle\gamma\ \text{in}|S|\alpha\ \text{in}\rangle}_{=\,\langle\gamma\ \text{out}|\alpha\ \text{in}\rangle} = \sum_\gamma \langle\beta\ \text{in}|\varphi_{\text{in}}(x)|\gamma\ \text{in}\rangle\,\langle\gamma\ \text{out}|\alpha\ \text{in}\rangle\,. \tag{8.84}$$

Similarly, inserting $\sum_\gamma |\gamma\ \text{out}\rangle\langle\gamma\ \text{out}| = \mathbb{1}$ on the right-hand side:

$$\text{RHS} = \sum_\gamma \langle\beta\ \text{out}|\varphi_{\text{out}}(x)|\gamma\ \text{out}\rangle\,\langle\gamma\ \text{out}|\alpha\ \text{in}\rangle\,. \tag{8.85}$$

Comparing (8.84) and (8.85), the identity holds if and only if

$$\langle\beta\ \text{in}|\varphi_{\text{in}}(x)|\gamma\ \text{in}\rangle = \langle\beta\ \text{out}|\varphi_{\text{out}}(x)|\gamma\ \text{out}\rangle \qquad \text{for all } \beta,\gamma\,. \tag{8.86}$$

This is the key identity, and it holds because φ_{in} and φ_{out} have *identical algebraic structure*. Both are free fields of mass m with the same mode functions $f_k(x)$ [cf. (8.26) and (8.62)]:

$$\varphi_{\text{in/out}}(x) = \int d^3k \left[a_{\text{in/out}}(k)\, f_k(x) + a^\dagger_{\text{in/out}}(k)\, f_k^*(x)\right]. \tag{8.87}$$

The in- and out-creation operators satisfy the same canonical commutation relations,

$$[a_{\text{in}}(k), a^\dagger_{\text{in}}(k')] = \delta^{(3)}(\vec{k}-\vec{k}') = [a_{\text{out}}(k), a^\dagger_{\text{out}}(k')]\,, \tag{8.88}$$

and both annihilate the same vacuum: $a_{\text{in}}(k)|0\rangle = 0 = a_{\text{out}}(k)|0\rangle$. Since the in-states (8.67) and out-states (8.68) are built from $|0\rangle$ by the same algebraic operations (applying $a^\dagger$'s), and the matrix elements of a, $a^\dagger$ between Fock states are fully determined by the CCR and the vacuum condition, we have for instance

$$a_{\text{in}}(k)|k_1,\ldots,k_n\ \text{in}\rangle = \sum_{i=1}^{n} \delta^{(3)}(\vec{k}-\vec{k}_i)\,|k_1,\ldots,\hat{k}_i,\ldots,k_n\ \text{in}\rangle\,, \tag{8.89}$$

and exactly the same formula with in $\to$ out. Thus (8.86) follows, and since the in-states are complete:

$$\boxed{\varphi_{\text{in}}(x) = S\,\varphi_{\text{out}}(x)\,S^{-1}} \tag{8.90}$$

□

Property 4: Unitarity of the S-matrix.

Statement: $SS^\dagger = S^\dagger S = 1$.

Proof. From (8.70), we have $\langle\beta \text{ in}|S = \langle\beta \text{ out}|$. Taking the Hermitian conjugate:

$$S^\dagger|\beta \text{ in}\rangle = |\beta \text{ out}\rangle. \tag{8.91}$$

Now consider:

$$\langle\beta \text{ in}|SS^\dagger|\alpha \text{ in}\rangle = \langle\beta \text{ out}|S^\dagger|\alpha \text{ in}\rangle. \tag{8.92}$$

Using (8.91) with $|\alpha \text{ in}\rangle$:

$$S^\dagger|\alpha \text{ in}\rangle = |\alpha \text{ out}\rangle, \tag{8.93}$$

so:

$$\langle\beta \text{ in}|SS^\dagger|\alpha \text{ in}\rangle = \langle\beta \text{ out}|\alpha \text{ out}\rangle. \tag{8.94}$$

Both in-states and out-states form complete orthonormal sets (they describe the same Hilbert space, just with different labels). Therefore:

$$\langle\beta \text{ out}|\alpha \text{ out}\rangle = \delta_{\beta\alpha} = \langle\beta \text{ in}|\alpha \text{ in}\rangle. \tag{8.95}$$

Hence:

$$\langle\beta \text{ in}|SS^\dagger|\alpha \text{ in}\rangle = \delta_{\beta\alpha} = \langle\beta \text{ in}|\alpha \text{ in}\rangle. \tag{8.96}$$

Since this holds for all α, β:

$$SS^\dagger = 1\,. \tag{8.97}$$

Repeating the same matrix-element argument with the completeness of the out-states gives:

$$\boxed{SS^\dagger = S^\dagger S = 1}\,. \tag{8.98}$$

Physically, unitarity expresses probability conservation: the total probability for any initial state to evolve into *some* final state is unity. □

Property 5: Lorentz invariance of the S-matrix.

Statement: $U(a,b)\,S\,U^{-1}(a,b) = S$, where $U(a,b)$ is the unitary operator generating the Poincaré transformation $x'_\mu = b_\mu + a_\mu{}^\nu x_\nu$.

Proof. Under a Poincaré transformation, field operators transform as:

$$U(a,b)\,\varphi(x)\,U^{-1}(a,b) = \varphi(ax+b)\,. \tag{8.99}$$

Since $\varphi_{\text{in}}(x)$ and $\varphi_{\text{out}}(x)$ are constructed from $\varphi(x)$ through (8.36) and (8.64), and since the Green's functions $\Delta_{\text{ret/adv}}$ are Lorentz covariant, the in- and out-fields transform the same way:

$$U(a,b)\,\varphi_{\text{in}}(x)\,U^{-1}(a,b) = \varphi_{\text{in}}(ax+b)\,. \tag{8.100}$$

Now use Property 3, equation (8.90):

$$\begin{aligned}\varphi_{\text{in}}(ax+b) &= U(a,b)\,\varphi_{\text{in}}(x)\,U^{-1}(a,b)\\ &= U(a,b)\,S\,\varphi_{\text{out}}(x)\,S^{-1}\,U^{-1}(a,b).\end{aligned} \tag{8.101}$$

But we can also write, using (8.90) directly at the transformed point:

$$\varphi_{\text{in}}(ax+b) = S\,\varphi_{\text{out}}(ax+b)\,S^{-1}. \tag{8.102}$$

Using the transformation law for φ_{out}:

$$\varphi_{\text{out}}(ax+b) = U(a,b)\,\varphi_{\text{out}}(x)\,U^{-1}(a,b). \tag{8.103}$$

Substituting (8.103) into (8.102):

$$\varphi_{\text{in}}(ax+b) = S\,U(a,b)\,\varphi_{\text{out}}(x)\,U^{-1}(a,b)\,S^{-1}. \tag{8.104}$$

Comparing (8.101) and (8.104):

$$U(a,b)\,S\,\varphi_{\text{out}}(x)\,S^{-1}\,U^{-1}(a,b) = S\,U(a,b)\,\varphi_{\text{out}}(x)\,U^{-1}(a,b)\,S^{-1}. \tag{8.105}$$

This can be rewritten as:

$$\left[U(a,b)\,S\,U^{-1}(a,b)\right]\varphi_{\text{out}}(x)\left[U(a,b)\,S\,U^{-1}(a,b)\right]^{-1} = S\,\varphi_{\text{out}}(x)\,S^{-1}. \tag{8.106}$$

Since $\varphi_{\text{out}}(x)$ is an arbitrary field and the out-fields generate the full Hilbert space, we conclude:

$$\boxed{U(a,b)\,S\,U^{-1}(a,b) = S}\,. \tag{8.107}$$

Equivalently, $[S, U(a,b)] = 0$: the S-matrix commutes with all Poincaré transformations. This means that if we Lorentz-transform both the initial and final states, the S-matrix element remains unchanged—as required by the principle of relativity. □

8.5 Comparison with the interaction-picture approach

In Chapter 3, we developed the perturbative approach to scattering using the interaction picture and defined the S-matrix through the Dyson expansion. The present chapter takes a different, more fundamental approach. It is instructive to compare the two viewpoints and clarify their relationship.

Two definitions of the S-matrix

Interaction-picture definition (Chapter 3). In the interaction picture, the S-matrix is defined as the time-evolution operator that connects states at $t = -\infty$ to states at $t = +\infty$:

$$|\Phi(\infty)\rangle = S\,|\Phi(-\infty)\rangle, \qquad S = U_I(+\infty, -\infty)\,. \tag{8.108}$$

The states $|\Phi(t)\rangle$ are interaction-picture states, which evolve in time according to

$$i\,\frac{d}{dt}|\Phi(t)\rangle = V_I(t)\,|\Phi(t)\rangle\,, \tag{8.109}$$

where $V_I(t) = e^{iH_0 t} V\, e^{-iH_0 t}$ is the interaction Hamiltonian in the interaction picture. In natural units, the Dyson expansion provides a perturbative solution:

$$\begin{aligned} S &= T\exp\left[-i\int_{-\infty}^{+\infty} dt\, V_I(t)\right] \\ &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \cdots d^4x_n\, T\{\mathscr{H}_I(x_1)\cdots\mathscr{H}_I(x_n)\}. \end{aligned} \tag{8.110}$$

LSZ definition (this chapter). In the Heisenberg picture, we define in- and out-states using the asymptotic free fields $\varphi_{\text{in}}(x)$ and $\varphi_{\text{out}}(x)$:

$$|\alpha\ \text{in}\rangle = a^\dagger_{\text{in}}(p_1)\cdots a^\dagger_{\text{in}}(p_n)|0\rangle, \qquad |\beta\ \text{out}\rangle = a^\dagger_{\text{out}}(q_1)\cdots a^\dagger_{\text{out}}(q_m)|0\rangle\,, \tag{8.111}$$

and the S-matrix element is the overlap:

$$S_{\beta\alpha} = \langle\beta\ \text{out}|\alpha\ \text{in}\rangle\,. \tag{8.112}$$

Relationship between the two approaches

The two definitions are *equivalent* when properly interpreted, but they emphasize different aspects:

- **The interaction-picture approach** is *computational*: it provides an explicit perturbation expansion (Dyson series) that can be evaluated order by order using Wick's theorem and Feynman diagrams.
- **The LSZ approach** is *structural*: it relates S-matrix elements to Green's functions without relying on perturbation theory, providing a rigorous definition that holds to all orders.

To see the connection, note that in the interaction picture:

$$S_{fi} = \langle f|S|i\rangle = \langle f, t = +\infty|\Phi(\infty)\rangle = \langle f|U_I(+\infty, -\infty)|i\rangle\,, \tag{8.113}$$

where $|i\rangle$ and $|f\rangle$ are free-particle states (eigenstates of H_0). In the LSZ formalism:

$$S_{fi} = \langle f \text{ out} | i \text{ in} \rangle , \tag{8.114}$$

where the in- and out-states are constructed from the exact Heisenberg-picture vacuum using asymptotic fields.

The key insight is that the *free-particle states* $|i\rangle$, $|f\rangle$ of the interaction picture correspond to the *asymptotic states* $|i \text{ in}\rangle$, $|f \text{ out}\rangle$ of the LSZ formalism. The adiabatic hypothesis in the interaction-picture approach plays the role of the weak asymptotic condition in the LSZ formalism—both ensure that interactions can be "turned off" asymptotically.

Comparison of in/out fields with interaction-picture fields

The in-fields and out-fields defined in this chapter are *fundamentally different* from the interaction-picture fields of Chapter 2. This distinction is crucial for understanding the structure of scattering theory.

Interaction-picture fields (Chapter 2). The interaction-picture field is defined by

$$\varphi_I(x) = e^{iH_0 t}\, \varphi_S(\vec{x})\, e^{-iH_0 t} = U_0^\dagger(t,t_0)\, \varphi_S(\vec{x})\, U_0(t,t_0) , \tag{8.115}$$

where $\varphi_S(x)$ is the Schrödinger-picture field (time-independent) and H_0 is the free Hamiltonian. Key properties:

- $\varphi_I(x)$ satisfies the *free* field equation determined by the quadratic part chosen for H_0. In a bare split this mass is m_0; in renormalized perturbation theory one usually chooses a renormalized or physical mass and treats the difference as a counterterm.
- The time evolution is governed by H_0 alone.
- $\varphi_I(x)$ is related to φ_S by a unitary transformation at each time t.
- The mass appearing in the field equation is therefore a bookkeeping choice tied to the perturbative split $H = H_0 + V$.

In-fields and out-fields (this chapter). The in-field $\varphi_{\text{in}}(x)$ is defined through the asymptotic condition (8.36):

$$\sqrt{Z}\, \varphi_{\text{in}}(x) = \varphi(x) - \int d^4y\, \Delta_{\text{ret}}(x-y;m)\, \tilde{j}(y), \tag{8.116}$$

where $\varphi(x)$ is the full *Heisenberg-picture* interacting field. Key properties:

- $\varphi_{\text{in}}(x)$ satisfies the free field equation with the *physical* mass: $(\Box + m^2)\varphi_{\text{in}} = 0$.

- $\varphi_{\text{in}}(x)$ is the asymptotic limit of the interacting field as $t \to -\infty$ (in the weak sense).
- The normalization involves the wave function renormalization constant $\sqrt{Z}$.
- The physical mass m (not the bare mass m_0) appears in the field equation.

Summary of differences.

Property	**Int. picture φ_I**	**In/out fields $\varphi_{\text{in/out}}$**
Picture	Interaction picture	Heisenberg picture
Defined via	Unitary transform of φ_S	Asymptotic limit of $\varphi(x)$
Evolution gen.	Free Hamiltonian H_0	Free equation, mass m
Mass parameter	Mass chosen in H_0	Physical mass m
Normalization	Same as φ_S	Involves $\sqrt{Z}$
States created	Fock states of H_0	Physical particle states

Physical interpretation. The interaction-picture field $\varphi_I(x)$ is a mathematical tool for perturbation theory. It satisfies the free equations associated with the chosen H_0 and creates the corresponding Fock states. In contrast, $\varphi_{\text{in}}(x)$ and $\varphi_{\text{out}}(x)$ are the physical asymptotic fields that create actual particles with the observed mass m. The factor $\sqrt{Z}$ accounts for the overlap between the local interacting field $\varphi(x)$ and the isolated one-particle state; the remaining spectral weight, when present, belongs to continuum states.

The precise relation between φ_{in} and φ_I. The two fields live in different pictures and are *not* equal, but they are related by a precise chain of identities. Recall from Chapter 2, Eq. (2.10), that a Schrödinger-picture operator is promoted to a Heisenberg-picture operator by

$$A_H(t) = U^\dagger(t,t_0)\, A_S\, U(t,t_0).$$

Applying this to the field operator $\varphi_S(\vec{x})$ with $t_0 = 0$ and $U(t,0) = e^{-iHt}$ gives

$$\varphi(x) = e^{iHt}\, \varphi_S(\vec{x})\, e^{-iHt}\,, \tag{8.117}$$

where $H = H_0 + V$ is the full Hamiltonian. The interaction-picture field is obtained by evolving φ_S with the free part H_0 only:

$$\varphi_I(x) = e^{iH_0t}\, \varphi_S(\vec{x})\, e^{-iH_0t}\,. \tag{8.118}$$

Combining these, we can express the Heisenberg field in terms of the interaction-picture field:

$$\varphi(x) = e^{iHt}\, e^{-iH_0t}\, \varphi_I(x)\, e^{iH_0t}\, e^{-iHt} = U^\dagger(t)\, \varphi_I(x)\, U(t)\,, \tag{8.119}$$

where $U(t) \equiv e^{iH_0 t} e^{-iHt}$ is the time-evolution operator relating the two pictures.

Now consider the limit $t \to -\infty$. In the interaction-picture approach, the adiabatic prescription identifies the Fock states of H_0 with the asymptotic particle states, after the pole position and residue have been matched by renormalization. In the LSZ approach, the weak asymptotic condition gives instead $\varphi(x) \to \sqrt{Z}\,\varphi_{\text{in}}(x)$ inside suitably smeared matrix elements. Comparing these asymptotic matrix elements motivates the schematic perturbative identification

$$\varphi_{\text{in}}(x) \sim \frac{1}{\sqrt{Z}}\varphi_I(x) \quad \text{inside asymptotic, renormalized matrix elements}. \tag{8.120}$$

The two fields differ in two ways: (i) φ_I satisfies the free equation associated with the chosen perturbative mass parameter, while φ_{in} uses the physical pole mass m; (ii) the relative normalisation is fixed by the pole residue. Both differences are handled by renormalization. After mass and field-strength renormalization, the perturbative Green's functions built from interaction-picture fields have the same external one-particle poles and residues required by LSZ, so the perturbative S-matrix from the Dyson expansion agrees with the LSZ S-matrix order by order.

8.6 The spectral representation

Before deriving the reduction formula, we study the vacuum expectation value of the field commutator. This fixes the normalization constant Z.

Derivation of the Källén–Lehmann representation

We now derive the spectral representation for the vacuum expectation value of the field commutator, showing all intermediate steps.

Step 1: Definition and completeness. Consider the vacuum expectation value of the commutator:

$$i\Delta'(x,x') \equiv \langle 0|[\varphi(x),\varphi(x')]|0\rangle = \langle 0|\varphi(x)\varphi(x')|0\rangle - \langle 0|\varphi(x')\varphi(x)|0\rangle\,. \tag{8.121}$$

Insert a complete set of energy-momentum eigenstates $\sum_n |n\rangle\langle n| = 1$ between the field operators. For the first term:

$$\langle 0|\varphi(x)\varphi(x')|0\rangle = \sum_n \langle 0|\varphi(x)|n\rangle\langle n|\varphi(x')|0\rangle\,. \tag{8.122}$$

Step 2: Use of displacement invariance. The field operator at point x is related to the field at the origin by the displacement operator:

$$\varphi(x) = e^{iP\cdot x}\varphi(0)e^{-iP\cdot x}\,. \tag{8.123}$$

For a matrix element between eigenstates $|n\rangle$ and $|m\rangle$ with $P^\mu|n\rangle = p_n^\mu|n\rangle$:

$$\begin{aligned}\langle n|\varphi(x)|m\rangle &= \langle n|e^{iP\cdot x}\varphi(0)e^{-iP\cdot x}|m\rangle \\ &= e^{ip_n\cdot x}\langle n|\varphi(0)|m\rangle e^{-ip_m\cdot x} \\ &= e^{i(p_n-p_m)\cdot x}\langle n|\varphi(0)|m\rangle\,. \end{aligned} \tag{8.124}$$

In particular, since $P^\mu|0\rangle = 0$:

$$\langle 0|\varphi(x)|n\rangle = e^{-ip_n\cdot x}\langle 0|\varphi(0)|n\rangle, \qquad \langle n|\varphi(x')|0\rangle = e^{ip_n\cdot x'}\langle n|\varphi(0)|0\rangle\,. \tag{8.125}$$

Step 3: Evaluation of the two terms. Substituting (8.125) into (8.122):

$$\begin{aligned}\langle 0|\varphi(x)\varphi(x')|0\rangle &= \sum_n e^{-ip_n\cdot x}\langle 0|\varphi(0)|n\rangle\cdot e^{ip_n\cdot x'}\langle n|\varphi(0)|0\rangle \\ &= \sum_n |\langle 0|\varphi(0)|n\rangle|^2 e^{-ip_n\cdot(x-x')}\,. \end{aligned} \tag{8.126}$$

Similarly, for the second term:

$$\begin{aligned}\langle 0|\varphi(x')\varphi(x)|0\rangle &= \sum_n \langle 0|\varphi(x')|n\rangle\langle n|\varphi(x)|0\rangle \\ &= \sum_n |\langle 0|\varphi(0)|n\rangle|^2 e^{ip_n\cdot(x-x')}\,. \end{aligned} \tag{8.127}$$

Step 4: The commutator. Combining (8.126) and (8.127):

$$\begin{aligned}i\Delta'(x-x') &= \langle 0|[\varphi(x),\varphi(x')]|0\rangle \\ &= \sum_n |\langle 0|\varphi(0)|n\rangle|^2\left(e^{-ip_n\cdot(x-x')} - e^{ip_n\cdot(x-x')}\right)\,. \end{aligned} \tag{8.128}$$

Note that the result depends only on the difference $x-x'$, as required by translation invariance.

Step 5: Definition of the spectral amplitude. To convert the discrete sum into a more useful form, we group together all states with the same four-momentum q. Introduce the identity:

$$1 = \int d^4q\,\delta^{(4)}(p_n - q)\,. \tag{8.129}$$

Define the **spectral amplitude**:

$$\rho(q) \equiv (2\pi)^3 \sum_n \delta^{(4)}(p_n - q)\,|\langle 0|\varphi(0)|n\rangle|^2\,. \tag{8.130}$$

The factor $(2\pi)^3$ is conventional and matches the normalization of momentum states. Then:

$$\sum_n |\langle 0|\varphi(0)|n\rangle|^2 e^{-ip_n\cdot y} = \frac{1}{(2\pi)^3}\int d^4q\,\rho(q)\,e^{-iq\cdot y}\,, \tag{8.131}$$

where $y = x - x'$.

Step 6: Lorentz invariance of $\rho(q)$. We now prove that $\rho(q)$ depends only on q^2. Under a Lorentz transformation Λ:

- The vacuum is invariant: $U(\Lambda)|0\rangle = |0\rangle$.

- The scalar field transforms as $U(\Lambda)\varphi(x)U^{-1}(\Lambda) = \varphi(\Lambda x)$;[3] at the origin this reduces to $U(\Lambda)\varphi(0)U^{-1}(\Lambda) = \varphi(0)$.
- The states transform as: $|n\rangle \longrightarrow |\Lambda n\rangle$ with $p^{\mu}_{\Lambda n} = \Lambda^{\mu}{}_{\nu} p^{\nu}_{n}$.

[3]This is the defining transformation rule of a scalar field in the active convention adopted here, where $U(\Lambda)|n\rangle = |\Lambda n\rangle$ with $p_{\Lambda n} = \Lambda p_n$ (third bullet).

Derivation. Starting from the translation property $\varphi(x) = e^{iP\cdot x}\varphi(0)e^{-iP\cdot x}$ and the four-vector rule for the momentum operator, written in terms of Λ with one index lowered and one raised,

$$U(\Lambda)\, P^{\mu}\, U^{-1}(\Lambda) \;=\; \Lambda_{\nu}{}^{\mu}\, P^{\nu}\,,$$

where $\Lambda_{\nu}{}^{\mu} \equiv \eta_{\nu\rho}\,\Lambda^{\rho}{}_{\sigma}\,\eta^{\sigma\mu}$ is the matrix Λ with indices "shifted across" (lower–upper instead of upper–lower). For a Lorentz transformation, this coincides with $(\Lambda^{-1})^{\mu}{}_{\nu}$, so the inverse the reader might expect from the four-vector rule is hidden in the position of the indices on Λ. The rule itself is enforced by the requirement $P^{\mu}|\Lambda n\rangle = (\Lambda p_n)^{\mu}|\Lambda n\rangle$.

Action on the exponential. Conjugation by $U(\Lambda)$ acts term by term in the Taylor expansion of an exponential of operators, because for any A, $UA^nU^{-1} = (UAU^{-1})^n$ (insert $\mathbb{1} = U^{-1}U$ between each factor). Since x_{μ} is a c-number that commutes through U, only P^{μ} is conjugated:

$$\begin{aligned} U(\Lambda)\, e^{iP\cdot x}\, U^{-1}(\Lambda) \;&=\; \sum_{n=0}^{\infty} \frac{1}{n!} \Big[i\, U(\Lambda)\, (P\cdot x)\, U^{-1}(\Lambda) \Big]^n \\ &=\; \exp\Big[i\, U(\Lambda)\, P^{\mu}\, U^{-1}(\Lambda)\, x_{\mu} \Big] \;=\; \exp\big(i\, \Lambda_{\nu}{}^{\mu}\, P^{\nu} x_{\mu} \big)\,. \end{aligned}$$

The same applies to $e^{-iP\cdot x}$ with the opposite sign in the exponent. Therefore

$$\begin{aligned} U(\Lambda)\, \varphi(x)\, U^{-1}(\Lambda) \;&=\; U(\Lambda)\, e^{iP\cdot x}\, U^{-1}(\Lambda)\; U(\Lambda)\, \varphi(0)\, U^{-1}(\Lambda)\; U(\Lambda)\, e^{-iP\cdot x}\, U^{-1}(\Lambda) \\ &=\; e^{i\Lambda_{\nu}{}^{\mu} P^{\nu} x_{\mu}}\, \varphi(0)\, e^{-i\Lambda_{\nu}{}^{\mu} P^{\nu} x_{\mu}}\,, \end{aligned}$$

where in the last step we used $U(\Lambda)\varphi(0)U^{-1}(\Lambda) = \varphi(0)$ (scalar at the origin) and inserted three copies of $\mathbb{1} = U^{-1}(\Lambda)U(\Lambda)$ between the three factors.

Index manipulation. The key step is now the contraction $\Lambda_{\nu}{}^{\mu}\, x_{\mu} = (\Lambda x)_{\nu}$: the upper index μ of Λ contracts with x_{μ}, and the lower index ν becomes the index of a covector identified with the boosted Λx. Indeed, raising and lowering with η,

$$\Lambda_{\nu}{}^{\mu}\, x_{\mu} \;=\; \eta_{\nu\rho}\,\Lambda^{\rho}{}_{\sigma}\,\eta^{\sigma\mu}\, x_{\mu} \;=\; \eta_{\nu\rho}\,\Lambda^{\rho}{}_{\sigma}\, x^{\sigma} \;=\; \eta_{\nu\rho}\,(\Lambda x)^{\rho} \;=\; (\Lambda x)_{\nu}\,.$$

Substituting into the exponent, $\Lambda_{\nu}{}^{\mu} P^{\nu} x_{\mu} = P^{\nu}(\Lambda x)_{\nu} = P\cdot(\Lambda x)$, and finally

$$U(\Lambda)\, \varphi(x)\, U^{-1}(\Lambda) \;=\; e^{iP\cdot(\Lambda x)}\, \varphi(0)\, e^{-iP\cdot(\Lambda x)} \;=\; \varphi(\Lambda x)\,.$$

Alternative passive convention. The same physical content can be written with Λ^{-1} in the argument: defining $\varphi'(x) \equiv U^{-1}(\Lambda)\,\varphi(x)\,U(\Lambda)$ gives $\varphi'(x) = \varphi(\Lambda^{-1}x)$, obtained from the active form by replacing Λ with Λ^{-1}. This is the convention used in Chapter 1 for the general field transformation $\phi'_r(x) = \phi_r(\Lambda^{-1}x) + \frac{1}{2}\omega_{\mu\nu}(S^{\mu\nu})_r{}^s\phi_s(\Lambda^{-1}x)$, with $S^{\mu\nu} = 0$ in the scalar case.

Therefore:

$$\begin{aligned}\rho(q) &= (2\pi)^3 \sum_n \delta^{(4)}(p_n - q)\, |\langle 0|\varphi(0)|n\rangle|^2 \\ &= (2\pi)^3 \sum_n \delta^{(4)}(p_n - q)\, |\langle 0|U(\Lambda)\varphi(0)U^{-1}(\Lambda)|\Lambda n\rangle|^2 \end{aligned} \tag{8.132}$$

In the second equality we used $\langle 0|U(\Lambda) = \langle 0|$ and $U^{-1}(\Lambda)|\Lambda n\rangle = |n\rangle$, so the matrix element is unchanged. We now apply the scalar transformation $U(\Lambda)\varphi(0)U^{-1}(\Lambda) = \varphi(0)$ and relabel the dummy index $m := \Lambda n$ (a bijection over the spectrum). Under this relabel $p_m = \Lambda p_n$, hence $p_n = \Lambda^{-1} p_m$, and

$$\delta^{(4)}(p_n - q) = \delta^{(4)}(\Lambda^{-1}p_m - q) = \delta^{(4)}\big(\Lambda^{-1}(p_m - \Lambda q)\big) = \delta^{(4)}(p_m - \Lambda q)\,,$$

where the last step uses the Lorentz invariance of the four-dimensional delta, $\delta^{(4)}(\Lambda y) = |\det\Lambda|^{-1}\delta^{(4)}(y) = \delta^{(4)}(y)$ for proper Lorentz transformations. Therefore:

$$\begin{aligned}\rho(q) &= (2\pi)^3 \sum_m \delta^{(4)}(p_m - \Lambda q)\, |\langle 0|\varphi(0)|m\rangle|^2 \\ &= \rho(\Lambda q)\,. \end{aligned} \tag{8.133}$$

Since $\rho(q) = \rho(\Lambda q)$ for all Lorentz transformations, and the only Lorentz-invariant function of a four-vector is a function of its square, we have:

$$\rho(q) = \rho(q^2)\,\theta(q^0)\,, \tag{8.134}$$

where $\theta(q^0)$ restricts to the forward light cone (since all physical states have $p^0 > 0$).

Step 7: Properties of $\rho(q^2)$. From the definition (8.130):

- $\rho(q^2) \geq 0$ (it is a sum of absolute squares).
- $\rho(q^2) = 0$ for $q^2 < 0$ (no physical states have spacelike momentum).
- $\rho(q^2)$ is real (obvious from the definition).

Step 8: Conversion to the spectral integral. Using (8.134), we can write:

$$\int d^4q\,\rho(q)\,e^{-iq\cdot y} = \int_0^\infty d\sigma^2\,\rho(\sigma^2)\int d^4q\,\delta(q^2-\sigma^2)\,\theta(q^0)\,e^{-iq\cdot y}\,. \tag{8.135}$$

The inner integral is recognized as the free-field positive-frequency Wightman function:

$$\Delta^{(+)}(y;\sigma) = \int \frac{d^4q}{(2\pi)^3}\,\delta(q^2-\sigma^2)\,\theta(q^0)\,e^{-iq\cdot y} = \int \frac{d^3q}{(2\pi)^3 2\omega_q}\,e^{-iq\cdot y}\,, \tag{8.136}$$

where $\omega_q = \sqrt{\vec{q}^2 + \sigma^2}$.

Step 9: Final result. Substituting back into (8.128):

$$\begin{aligned} i\Delta'(x-x') &= \int_0^\infty d\sigma^2\, \rho(\sigma^2)\left[\Delta^{(+)}(x-x';\sigma) - \Delta^{(+)}(x'-x;\sigma)\right] \\ &= \int_0^\infty d\sigma^2\, \rho(\sigma^2)\, i\Delta(x-x';\sigma)\,, \end{aligned} \tag{8.137}$$

Here $\Delta^{(\pm)}(y;\sigma)$ are the positive- and negative-frequency Wightman two-point functions of a *free* scalar field $\varphi^{(\sigma)}$ of mass σ,

$$\begin{aligned} \Delta^{(+)}(y;\sigma) &= \langle 0|\varphi^{(\sigma)}(y)\varphi^{(\sigma)}(0)|0\rangle\,, \\ \Delta^{(-)}(y;\sigma) &= \langle 0|\varphi^{(\sigma)}(0)\varphi^{(\sigma)}(y)|0\rangle = \Delta^{(+)}(-y;\sigma)\,, \end{aligned} \tag{8.138}$$

each given by the on-shell mode integral $\int d^3q/((2\pi)^3 2\omega_q)\, e^{\mp iq\cdot y}$ with $\omega_q = \sqrt{\vec{q}^2+\sigma^2}$ (cf. (8.136)). The combination

$$\Delta(y;\sigma) \;=\; -i\left[\Delta^{(+)}(y;\sigma) - \Delta^{(-)}(y;\sigma)\right] \tag{8.139}$$

is the *free-field commutator function* (Pauli–Jordan function) with mass σ, characterized by the c-number identity

$$[\varphi^{(\sigma)}(x),\, \varphi^{(\sigma)}(x')] \;=\; i\,\Delta(x-x';\sigma)\,\mathbb{1}\,. \tag{8.140}$$

It is real, Lorentz-invariant, antisymmetric ($\Delta(-y;\sigma) = -\Delta(y;\sigma)$), and vanishes for spacelike separations (microcausality). See Chapter 1 for the explicit mode-expansion derivation, where $\Delta(y;\sigma)$ emerges directly from the canonical commutation relations applied to the free-field expansion.

The **Källén–Lehmann spectral representation** then takes the form:

$$\boxed{\Delta'(x-x') = \int_0^\infty d\sigma^2\; \rho(\sigma^2)\,\Delta(x-x';\sigma)} \tag{8.141}$$

where $\Delta(x-x';\sigma)$ is the free-field commutator function with mass parameter σ.

Spectral representation for the propagator

The same derivation applies to the Feynman propagator, defined as the time-ordered two-point function (cf. Chapter 1, where $i\Delta_F = \langle 0|T(\varphi\varphi)|0\rangle$). Writing the T-product explicitly and inserting the complete set $\mathbb{1} = |0\rangle\langle 0| + \sum_n |n\rangle\langle n|$ (the vacuum

drops out since $\langle 0|\varphi(x)|0\rangle = 0$):

$$\begin{aligned} i\Delta'_F(x-x') &\equiv \langle 0|T\{\varphi(x)\varphi(x')\}|0\rangle \\ &= \theta(t-t')\langle 0|\varphi(x)\varphi(x')|0\rangle + \theta(t'-t)\langle 0|\varphi(x')\varphi(x)|0\rangle \\ &= \theta(t-t')\sum_n e^{-ip_n\cdot(x-x')}|\langle 0|\varphi(0)|n\rangle|^2 \\ &\quad + \theta(t'-t)\sum_n e^{+ip_n\cdot(x-x')}|\langle 0|\varphi(0)|n\rangle|^2 , \end{aligned} \tag{8.142}$$

where in the last line we used $\varphi(x) = e^{iP\cdot x}\varphi(0)e^{-iP\cdot x}$ and $P^\mu|n\rangle = p_n^\mu|n\rangle$. Both Wightman functions are precisely those evaluated in Steps 5–6. Substituting the spectral form (8.131) and the identification of $\Delta^{(+)}(y;\sigma)$ from (8.136), the sums become the spectral integrals

$$\sum_n e^{\mp ip_n\cdot y}|\langle 0|\varphi(0)|n\rangle|^2 = \int_0^\infty d\sigma^2\,\rho(\sigma^2)\,\Delta^{(+)}(\pm y;\sigma)\,, \tag{8.143}$$

hence

$$\begin{aligned} i\Delta'_F(x-x') &= \int_0^\infty d\sigma^2\,\rho(\sigma^2)\,\big[\theta(t-t')\,\Delta^{(+)}(x-x';\sigma) \\ &\qquad\qquad + \theta(t'-t)\,\Delta^{(+)}(x'-x;\sigma)\big] \\ &= \int_0^\infty d\sigma^2\,\rho(\sigma^2)\,i\Delta_F(x-x';\sigma)\,, \end{aligned} \tag{8.144}$$

where in the last step we recognized the standard expression for the free Feynman propagator of mass σ in terms of the positive-frequency Wightman function,

$$i\Delta_F(y;\sigma) \;=\; \theta(y^0)\,\Delta^{(+)}(y;\sigma) + \theta(-y^0)\,\Delta^{(+)}(-y;\sigma)\,. \tag{8.145}$$

Momentum space. To pass to momentum space, take the Fourier transform of both sides of (8.144). *By definition*, the momentum-space propagator $i\Delta'_F(p)$ is the Fourier transform of the position-space propagator $i\Delta'_F(y)$ (where $y = x - x'$):

$$i\Delta'_F(p) \;\equiv\; \int d^4y\, e^{ip\cdot y}\, i\Delta'_F(y)\,. \tag{8.146}$$

This is just notation/convention — not an identity to be proven. Now substitute the spectral form $i\Delta'_F(y) = \int_0^\infty d\sigma^2\,\rho(\sigma^2)\,i\Delta_F(y;\sigma)$ from (8.144); the only y-dependence sits inside $\Delta_F(y;\sigma)$, so the σ^2-integral commutes with the y-integral

and we may swap their order:

$$
\begin{aligned}
i\Delta'_F(p) &= \int d^4y\, e^{ip\cdot y} \int_0^\infty d\sigma^2\, \rho(\sigma^2)\, i\Delta_F(y;\sigma) \\
&= \int_0^\infty d\sigma^2\, \rho(\sigma^2) \underbrace{\int d^4y\, e^{ip\cdot y}\, i\Delta_F(y;\sigma)}_{= \dfrac{i}{p^2-\sigma^2+i\epsilon}} \\
&= \int_0^\infty d\sigma^2\, \rho(\sigma^2)\, \frac{i}{p^2-\sigma^2+i\epsilon}\,,
\end{aligned}
\tag{8.147}
$$

where in the second line we recognized the Fourier transform of the free Feynman propagator of mass σ — the standard result derived in Chapter 1:

$$
\int d^4y\, e^{ip\cdot y}\, i\Delta_F(y;\sigma) \;=\; \frac{i}{p^2-\sigma^2+i\epsilon}\,. \tag{8.148}
$$

Spectral density and the physical pole

The spectral representation makes explicit what, in the Peskin–Schroeder language ([6, Section 7.1, Eqs. (7.4)–(7.9)]), is often stated as a pole condition on the full propagator. The structure of $\rho(\sigma^2)$ follows directly from the decomposition of the eigenstates of P^μ used in Step 4: the vacuum $|0\rangle$ drops out because $\langle 0|\varphi(0)|0\rangle = 0$, and the remaining intermediate states split into

- the **stable one-particle states** $|p\rangle$ of mass m, whose four-momentum lies on the hyperboloid $p^2 = m^2$. Their contribution to $\rho(q)$ is concentrated on a single mass shell, giving a $\delta(\sigma^2 - m^2)$ in the spectral integral. The weight is fixed by the field-particle overlap $|\langle 0|\varphi(0)|p\rangle|^2$ and is conventionally called Z (the *wave-function renormalization constant*);
- the **multi-particle continuum**, made of states with $p^2 \geq m_1^2$, where m_1 is the threshold for the lowest multi-particle channel into which $\varphi(0)|0\rangle$ has nonzero overlap (e.g. $m_1 = 3m$ in a $\mathbb{Z}_2$-symmetric φ^4 theory, $m_1 = m + m_\pi$ in nucleon–pion theory, etc.). These states span a continuum of invariant masses, producing a smooth $\rho_{\text{cont.}}(\sigma^2)$ supported on $[m_1^2, \infty)$.

Assuming the one-particle state is *separated from the continuum by a mass gap*, i.e. $m_1 > m$, the spectral density therefore has the form

$$
\rho(\sigma^2) = Z\,\delta(\sigma^2 - m^2) + \rho_{\text{cont.}}(\sigma^2)\,\theta(\sigma^2 - m_1^2), \qquad m_1^2 > m^2. \tag{8.149}
$$

Derivation of the coefficient Z. The factor Z in front of $\delta(\sigma^2 - m^2)$ is not a free parameter — it is fixed by the field–particle matrix element. Restricting the

sum in (8.130) to one-particle states $|p\rangle$ with covariant normalization $\langle p|p'\rangle = (2\pi)^3\, 2E_p\, \delta^{(3)}(\vec{p}-\vec{p}')$,

$$\sum_n^{(1\text{-part})} |n\rangle\langle n| = \int \frac{d^3p}{(2\pi)^3\, 2E_p}\, |p\rangle\langle p|\,, \tag{8.150}$$

and using Lorentz invariance of $\varphi(0)$ together with $U(\Lambda)|p\rangle = |\Lambda p\rangle$, the matrix element $\langle 0|\varphi(0)|p\rangle$ is independent of p on the mass shell. We *define*

$$\langle 0|\varphi(0)|p\rangle = \sqrt{Z}\,, \qquad |\langle 0|\varphi(0)|p\rangle|^2 = Z\,, \tag{8.151}$$

which is the standard definition of the wave-function renormalization constant.

The one-particle contribution to the spectral function is now obtained by substituting these ingredients into the definition (8.130),

$$\rho(q) \;\equiv\; (2\pi)^3 \sum_n \delta^{(4)}(p_n - q)\, |\langle 0|\varphi(0)|n\rangle|^2\,.$$

Restricting the sum to the one-particle sector: $\sum_n \to \int d^3p/[(2\pi)^3\, 2E_p]$, the eigenvalue p_n^μ becomes the on-shell four-momentum $p^\mu = (E_p, \vec{p})$ with $E_p = \sqrt{\vec{p}^{\,2} + m^2}$, and the matrix-element factor reduces to the constant $|\langle 0|\varphi(0)|p\rangle|^2 = Z$. Therefore

$$\begin{aligned}
\rho^{(1\text{-part})}(q) &= (2\pi)^3 \int \frac{d^3p}{(2\pi)^3\, 2E_p}\, \delta^{(4)}(p-q)\, Z \\
&= Z \int d^4p\, \delta(p^2 - m^2)\, \theta(p^0)\, \delta^{(4)}(p-q) \\
&= Z\, \delta(q^2 - m^2)\, \theta(q^0)\,,
\end{aligned} \tag{8.152}$$

where in the second line we used the Lorentz-invariant form of the on-shell measure $d^3p/(2\pi)^3\, 2E_p = (2\pi)^{-3}\, d^4p\, \delta(p^2 - m^2)\theta(p^0)$ (this rewrites the three-momentum integral over the mass-shell as a covariant four-momentum integral restricted to the positive-energy hyperboloid). Comparing with $\rho(q) = \rho(\sigma^2 = q^2)\, \theta(q^0)$ identifies the δ-coefficient as Z.

This structure is illustrated in Fig. 8.1.

Thus the exact Feynman propagator is

$$i\Delta_F'(p) = \frac{iZ}{p^2 - m^2 + i\epsilon} + \int_{m_1^2}^{\infty} d\sigma^2\, \rho_{\text{cont.}}(\sigma^2)\, \frac{i}{p^2 - \sigma^2 + i\epsilon}\,. \tag{8.153}$$

Near the one-particle shell the continuum term is regular, so

$$i\Delta_F'(p) \xrightarrow[p^2 \to m^2]{} \frac{iZ}{p^2 - m^2 + i\epsilon} + \text{terms regular at } p^2 = m^2,$$

$$\lim_{p^2 \to m^2} (p^2 - m^2)\, i\Delta_F'(p) = iZ. \tag{8.154}$$

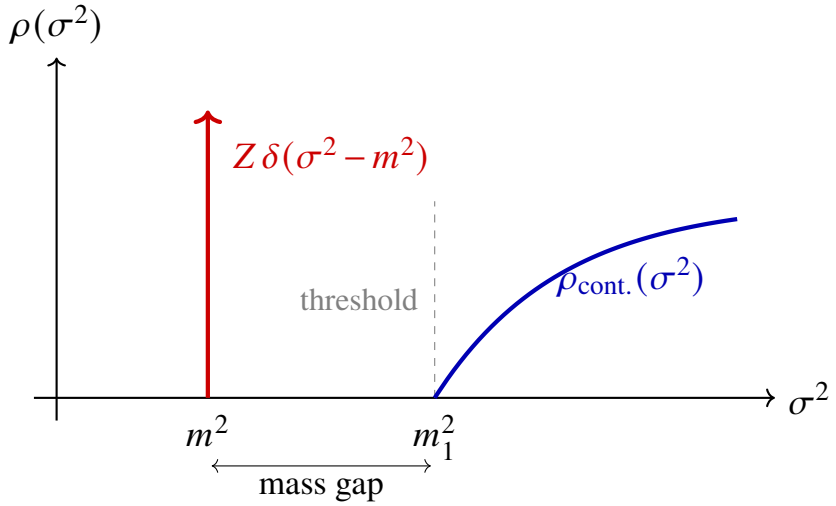


Figure 8.1. Schematic structure of the spectral density $\rho(\sigma^2)$ for a theory with a stable one-particle state of mass m separated from the multi-particle continuum by a mass gap. The isolated δ-function at $\sigma^2 = m^2$ carries weight equal to the wave-function renormalization constant Z; the smooth distribution starting at the threshold $\sigma^2 = m_1^2$ encodes the multi-particle intermediate states.

This is precisely the form of the propagator that enters the LSZ reduction formula, derived later in this chapter. There, the procedure of amputating external legs and taking the on-shell limit $p^2 \to m^2$ for each external particle picks out exactly the residue of the isolated one-particle pole: the multi-particle continuum and any other analytic background are less singular on the mass shell and are killed by the $(p^2 - m^2)$ amputation factor. As we shall see, each external line of an n-particle S-matrix element therefore acquires a wave-function renormalization factor $\sqrt{Z}$, which is the practical role played by Z in scattering amplitudes.

Perturbative (1PI) characterization. In perturbation theory the same propagator is more conveniently re-expressed in terms of the *one-particle-irreducible* (1PI) self-energy $\Pi(p^2)$, defined as the sum of all 1PI diagrams contributing to the two-point function (the systematic treatment for QED is given in Chapter 6, where the analogous object is $\Pi^{\mu\nu}(k)$ for the photon and $\Sigma(\not{p})$ for the electron). Resumming the chain of 1PI insertions via the Dyson equation yields

$$i\Delta'_F(p) \;=\; \frac{i}{p^2 - m_0^2 - \Pi(p^2) + i\epsilon}\,, \tag{8.155}$$

where m_0 is the bare mass appearing in the Lagrangian (8.1). Requiring that this expression match the spectral form near the physical pole gives two on-shell renormalization conditions:

$$\underbrace{m^2 - m_0^2 - \Pi(m^2) = 0}_{\text{mass renormalization}}\,, \qquad \underbrace{Z^{-1} = 1 - \Pi'(m^2)}_{\text{field-strength renormalization}}\,. \tag{8.156}$$

Small derivation. Define the inverse propagator $D(p^2) \equiv p^2 - m_0^2 - \Pi(p^2)$, so that $i\Delta'_F(p) = i/D(p^2)$. For the propagator to have a simple pole at the physical mass $p^2 = m^2$, the function $D(p^2)$ must vanish at $p^2 = m^2$:

$$D(m^2) = m^2 - m_0^2 - \Pi(m^2) = 0\,, \tag{8.157}$$

which is the mass renormalization condition. Taylor-expanding $D(p^2)$ around the physical pole,

$$\begin{aligned} D(p^2) &= (p^2 - m^2)\, D'(m^2) + O\big((p^2 - m^2)^2\big) \\ &= (p^2 - m^2)\left[1 - \Pi'(m^2)\right] + O\big((p^2 - m^2)^2\big)\,, \end{aligned} \tag{8.158}$$

the propagator near the pole reads

$$i\Delta'_F(p) \xrightarrow[p^2 \to m^2]{} \frac{i}{(p^2 - m^2)\,[1 - \Pi'(m^2)]} = \frac{i\,/\,[1 - \Pi'(m^2)]}{p^2 - m^2}\,. \tag{8.159}$$

Comparing with the spectral form $i\Delta'_F(p) \to iZ/(p^2 - m^2)$ from (8.154) identifies the residue as $Z = 1/[1 - \Pi'(m^2)]$, i.e. the field-strength renormalization condition $Z^{-1} = 1 - \Pi'(m^2)$.

The first condition states that the radiative correction $\Pi(m^2)$ shifts the bare mass m_0 to the physical mass m; the second states that the slope of the self-energy at the on-shell point rescales the field–particle overlap, determining Z. Different sign conventions for Π (e.g. $\Pi \leftrightarrow -\Pi$, or absorbing factors of i) change the signs in (8.155)–(8.156), but not the invariant content: the exact propagator has a simple pole at $p^2 = m^2$ and Z is its residue.

In an on-shell renormalization scheme one chooses the renormalized field $\varphi_R = Z^{-1/2}\varphi$ so that the renormalized propagator has unit residue at $p^2 = m^2$; in a generic (e.g. minimal-subtraction) scheme the finite residue remains and reappears as an external-leg factor in scattering amplitudes. The perturbative constant Z defined by (8.156) and the non-perturbative spectral weight in (8.149) should not be identified automatically: they coincide only once a normalization convention for the field has been specified.

The wave function renormalization constant Z

By the assumed structure of the spectrum, $\rho(\sigma^2)$ at $\sigma^2 = m^2$ receives contributions only from the one-particle states. Using (8.66):

$$(2\pi)^3 \int d^3p\, \delta^{(4)}(p - q)\, \frac{Z}{(2\pi)^3 2\omega_p} = Z\,\delta(q^2 - m^2)\,\theta(q^0)\,. \tag{8.160}$$

Separating this contribution:

$$\Delta'(x-x') = Z\,\Delta(x-x';m) + \int_{m_1^2}^{\infty} d\sigma^2\,\rho(\sigma^2)\,\Delta(x-x';\sigma)\,, \tag{8.161}$$

where m_1^2 is the threshold of the continuum in the quantum-number sector selected by the field. If a two-particle state in that sector is allowed, then $m_1^2 = 4m^2$; if symmetries forbid that channel, the first threshold is higher.

Taking a time derivative and setting $t' = t$, we use the canonical commutation relations:

$$\lim_{t'\to t} i\frac{\partial}{\partial t}\Delta'(x-x') = \langle 0|[\dot\varphi(\vec{x},t),\varphi(\vec{x}',t)]|0\rangle = -i\delta^{(3)}(\vec{x}-\vec{x}')\,. \tag{8.162}$$

Since the same relation holds for the free Δ function:

$$\boxed{1 = Z + \int_{m_1^2}^{\infty} \rho(\sigma^2)\,d\sigma^2}\,. \tag{8.163}$$

Together with the positivity of $\rho(\sigma^2)$, this proves:

$$\boxed{0 < Z \le 1}\,. \tag{8.164}$$

The equality $Z = 1$ occurs only when the continuum integral in (8.163) vanishes in this channel. If the field has nonzero overlap with continuum states, then $Z < 1$. Physically, the amplitude for $\varphi(x)$ to produce the isolated one-particle state is reduced because some spectral weight belongs to multiparticle states.

Analytic structure and dispersion relation

The spectral representation (8.153) reveals the analytic structure of the exact propagator as a function of the complex variable p^2, a viewpoint emphasized by Peskin–Schroeder [6]:

- An **isolated simple pole** at $p^2 = m^2$, with residue Z, corresponding to the stable one-particle state.
- A **branch cut** along the real axis for $p^2 \ge m_1^2$, arising from the continuous integral of poles in the second term of (8.153). The threshold is set by the lightest multiparticle state that has the same conserved quantum numbers as the *field insertion*, i.e. the operator placed inside the correlation function (here $\varphi(x)$ in $\langle 0|T(\varphi(x)\varphi(x'))|0\rangle$). Only states that $\varphi(0)|0\rangle$ can actually produce

contribute to the spectral sum; more generally, replacing φ with a composite local operator $\mathcal{O}(x)$ would change the threshold according to the quantum numbers of $\mathcal{O}$ (for example, an even operator $\mathcal{O} = \varphi^2$ in a $\mathbb{Z}_2$-symmetric theory has a two-particle threshold at $2m$, while φ itself has a three-particle threshold at $3m$).

The role of the $i\epsilon$ prescription. There are two equivalent ways to view the propagator as a function of p^2, and it is useful to be explicit about both since they share the symbol "$i\epsilon$" in different roles.

View (a): singularities just below the real axis. Take the spectral form (8.153) literally, with $\epsilon > 0$ small but finite in each denominator $1/(p^2 - \sigma^2 + i\epsilon)$. The singularities of this expression sit at $p^2 = \sigma^2 - i\epsilon$, i.e. infinitesimally *below* the real p^2 axis. For real p^2 the propagator is then a well-defined distribution; no further shift is needed.

View (b): boundary value of an analytic function with singularities on the axis. Treat Δ'_F as an analytic function of a complex variable $z = p^2$ defined by

$$\Delta'_F(z) = \int_0^\infty d\sigma^2 \, \frac{\rho(\sigma^2)}{z - \sigma^2}, \tag{8.165}$$

without any $i\epsilon$ in the denominator. Its singularities now sit *on* the real axis: the pole at $z = m^2$ and the branch cut $z \in [m_1^2, \infty)$. The function is analytic in the upper half-plane and in the lower half-plane separately, but not on the real axis at the singularities. The physical Feynman propagator is then defined as the boundary value of $\Delta'_F(z)$ on the real axis approached *from above*,

$$\Delta'_F(p^2)\big|_{\text{physical}} = \lim_{\epsilon \to 0^+} \Delta'_F(p^2 + i\epsilon), \tag{8.166}$$

where the $+i\epsilon$ now appears in the *argument* of Δ'_F rather than in the denominator: it is the prescription "evaluate at a point slightly in the upper half-plane and shrink towards the axis".

Equivalence of the two views. Plugging $z = p^2 + i\epsilon$ into the formula of view (b) immediately reproduces view (a):

$$\Delta'_F(p^2 + i\epsilon) = \int_0^\infty d\sigma^2 \, \frac{\rho(\sigma^2)}{(p^2 + i\epsilon) - \sigma^2} = \int_0^\infty d\sigma^2 \, \frac{\rho(\sigma^2)}{p^2 - \sigma^2 + i\epsilon}. \tag{8.167}$$

So the two "$+i\epsilon$"s denote the same physical content — in view (a) it sits inside the denominator regulating each pole, in view (b) it parametrizes the direction of approach to the real axis from above. Throughout we shall use the same symbol ϵ for both. Approaching the real axis from *below* ($-i\epsilon$ in the argument, equivalently in the denominators) would instead produce the complex conjugate of the Feynman

propagator, called the *Dyson* or *anti-time-ordered* propagator,[4] named so because its time-ordering convention is reversed. The two boundary values differ on the cut by the *discontinuity*

$$\operatorname{Disc}\Delta'_F(p^2) \;\equiv\; \Delta'_F(p^2+i\epsilon)-\Delta'_F(p^2-i\epsilon)\,. \tag{8.168}$$

Recovering ρ from the discontinuity. The key analytic identity is the *Sokhotski–Plemelj formula* applied to a real variable x:

$$\frac{1}{x\pm i\epsilon} \;=\; \mathcal{P}\frac{1}{x} \;\mp\; i\pi\,\delta(x)\,, \tag{8.169}$$

where $\mathcal{P}$ denotes the Cauchy principal value.[5] Subtracting the two prescriptions cancels the principal value and leaves only the δ:

$$\frac{1}{x+i\epsilon}-\frac{1}{x-i\epsilon} \;=\; -2\pi i\,\delta(x)\,. \tag{8.170}$$

[4]The Feynman propagator is the vacuum expectation value of the *time-ordered* product

$$i\Delta_F(x-y)=\langle 0|T[\varphi(x)\varphi(y)]|0\rangle\,,\qquad T[\varphi(x)\varphi(y)]=\theta(t_x-t_y)\,\varphi(x)\varphi(y)+\theta(t_y-t_x)\,\varphi(y)\varphi(x)\,,$$

which orders the operators with the latest time on the left. The *anti-time-ordered* propagator is defined dually, with the *earliest* time on the left:

$$i\Delta_{\bar F}(x-y)=\langle 0|\bar T[\varphi(x)\varphi(y)]|0\rangle\,,\qquad \bar T[\varphi(x)\varphi(y)]=\theta(t_y-t_x)\,\varphi(x)\varphi(y)+\theta(t_x-t_y)\,\varphi(y)\varphi(x)\,.$$

Physically, T encodes forward propagation of positive-frequency modes (particles) and backward propagation of negative-frequency modes (antiparticles); $\bar T$ does the opposite. In momentum space this swaps the $+i\epsilon$ in each Feynman propagator for $-i\epsilon$, which is why the anti-time-ordered propagator is the complex conjugate of the Feynman one along the real p^2 axis. Both propagators appear naturally in the closed-time-path (Schwinger–Keldysh) formalism for non-equilibrium QFT, where the time contour runs forward and then backward in time, attaching T to the forward branch and $\bar T$ to the backward one. The name "Dyson" refers to F. J. Dyson's early use of this object in covariant perturbation theory.

[5]The principal-value distribution $\mathcal{P}(1/x)$ is *not* a function (the function $1/x$ diverges at $x=0$) but a tempered distribution, defined by its action on a test function f as

$$\langle \mathcal{P}\tfrac{1}{x},f\rangle \;\equiv\; \lim_{\delta\to 0^+}\left[\int_{-\infty}^{-\delta}\frac{f(x)}{x}\,dx+\int_{\delta}^{\infty}\frac{f(x)}{x}\,dx\right] \;=\; \int_{-\infty}^{\infty}\frac{f(x)-f(0)}{x}\,dx\,.$$

The two separate integrals diverge as $-\ln\delta$ but with opposite signs, so their sum is finite: the $1/x$ singularity is removed by the symmetric subtraction. Likewise, the Sokhotski–Plemelj identity (8.169) is an equality between *distributions*, meaning that for every test function f,

$$\lim_{\epsilon\to 0^+}\int\frac{f(x)}{x\pm i\epsilon}\,dx \;=\; \mathcal{P}\int\frac{f(x)}{x}\,dx \;\mp\; i\pi\,f(0)\,.$$

Heuristically, writing $1/(x\pm i\epsilon)=(x\mp i\epsilon)/(x^2+\epsilon^2)$ splits the limit into a real part $x/(x^2+\epsilon^2)\to\mathcal{P}(1/x)$ and an imaginary part $\mp\epsilon/(x^2+\epsilon^2)\to\mp\pi\,\delta(x)$ (a Lorentzian of width ϵ collapsing to a π-normalized delta).

Applying this to every denominator in the spectral integral (8.153), and noting that for p^2 on the cut ($p^2 \geq m_1^2 > m^2$) the isolated pole term contributes $\delta(p^2 - m^2) = 0$,

$$\begin{aligned} \operatorname{Disc} \Delta'_F(p^2) &= \int_{m_1^2}^{\infty} d\sigma^2 \, \rho_{\text{cont.}}(\sigma^2) \left[\frac{1}{p^2 - \sigma^2 + i\epsilon} - \frac{1}{p^2 - \sigma^2 - i\epsilon} \right] \\ &= \int_{m_1^2}^{\infty} d\sigma^2 \, \rho_{\text{cont.}}(\sigma^2) \, (-2\pi i) \, \delta(p^2 - \sigma^2) \\ &= -2\pi i \, \rho(p^2), \qquad p^2 \geq m_1^2 . \end{aligned} \tag{8.171}$$

Furthermore, because ρ is real, the propagator satisfies the Schwarz reflection $\Delta'_F(p^2 - i\epsilon) = [\Delta'_F(p^2 + i\epsilon)]^*$,[67] so the discontinuity is purely imaginary:

$$\operatorname{Disc} \Delta'_F(p^2) = \Delta'_F(p^2 + i\epsilon) - [\Delta'_F(p^2 + i\epsilon)]^* = 2i \operatorname{Im} \Delta'_F(p^2 + i\epsilon) . \tag{8.172}$$

Equating the two expressions for $\operatorname{Disc} \Delta'_F$ gives the announced inversion formula:

$$\boxed{\rho(p^2) \;=\; -\frac{1}{\pi} \operatorname{Im} \Delta'_F(p^2 + i\epsilon), \qquad p^2 \geq m_1^2} \, . \tag{8.173}$$

Dispersion-relation reading. The Källén–Lehmann formula (8.153) can now be read as a *dispersion relation*. Factoring out the conventional overall i in the time-ordered two-point function, the analytic function Δ'_F is expressed at arbitrary complex p^2 as an integral over its own imaginary part along the cut, plus the isolated pole,

$$\Delta'_F(p^2) \;=\; \frac{Z}{p^2 - m^2 + i\epsilon} \;+\; \int_{m_1^2}^{\infty} \frac{d\sigma^2}{\pi} \, \frac{-\operatorname{Im} \Delta'_F(\sigma^2 + i\epsilon)}{p^2 - \sigma^2 + i\epsilon} \, . \tag{8.174}$$

[6] The *Schwarz reflection principle* is a theorem of complex analysis: if a function $f(z)$ is analytic in a domain D containing a segment I of the real axis, and is *real on* I, then f extends analytically to the mirror image of D across the real axis via $f(\bar{z}) = \overline{f(z)}$. Applied to our propagator: from the spectral form $\Delta'_F(z) = \int_0^{\infty} d\sigma^2 \, \rho(\sigma^2)/(z - \sigma^2)$ with ρ real and supported on $\sigma^2 \geq m^2$, $\Delta'_F(p^2)$ is manifestly *real* for real $p^2 < m^2$ (every denominator $p^2 - \sigma^2$ is real and negative there). Hence the reflection principle applies on the segment $(-\infty, m^2)$, and analytic continuation yields $\Delta'_F(\bar{z}) = \overline{\Delta'_F(z)}$ throughout the analyticity domain. Specializing to $z = p^2 + i\epsilon$ gives $\Delta'_F(p^2 - i\epsilon) = [\Delta'_F(p^2 + i\epsilon)]^*$.

[7] *Uniqueness of the analytic continuation.* The reflection above relies on the *identity theorem* for analytic functions: if two functions are analytic on a connected domain D and agree on a set with an accumulation point in D (e.g. a segment of $\mathbb{R}$), they coincide on all of D. In our setup, $D = \mathbb{C} \setminus \big(\{m^2\} \cup [m_1^2, \infty)\big)$ is connected, and the candidate reflection $\tilde{f}(z) := \overline{\Delta'_F(\bar{z})}$ is analytic on D and agrees with Δ'_F on the real segment $(-\infty, m^2) \subset D$ where the latter is real; the identity theorem then forces $\tilde{f} \equiv \Delta'_F$ throughout D, which is the Schwarz reflection. The same theorem also implies an important *negative* statement: analytic continuation is unique *only as long as one stays inside* D. The cut $[m_1^2, \infty)$ is precisely where analyticity fails, so one cannot continue *across* the cut as a single-valued function: going from $p^2 + i\epsilon$ to $p^2 - i\epsilon$ *around* the cut (through D) gives the complex conjugate value, while going *straight through* the cut would land on a different Riemann sheet. The mismatch between the two routes is exactly the discontinuity $\operatorname{Disc} \Delta'_F = -2\pi i \rho$.

This is the same conclusion that Cauchy's integral formula would give by deforming the contour around the analytic structure of Δ'_F in the complex p^2 plane: the full function is determined by its singularities — the pole position and residue (m^2, Z) and the discontinuity (i.e. the imaginary part) along the cut.

Physical interpretation: cuts come from real intermediate states. The inversion formula $\rho(p^2) = -\pi^{-1}\,\mathrm{Im}\,\Delta'_F(p^2 + i\epsilon)$ has a very transparent physical meaning, which we now unpack.

In words. The imaginary part of the exact propagator is non-zero *only* at invariant masses $\sqrt{p^2}$ for which physical, on-shell, multi-particle intermediate states actually exist. Every time we cross a new kinematical threshold $\sqrt{p^2} = m_a + m_b + \cdots$ at which a multi-particle channel $|X\rangle = |a, b, \dots\rangle$ becomes accessible, a new piece switches on in $\mathrm{Im}\,\Delta'_F(p^2)$ and therefore in $\rho(p^2)$. Equivalently: **branch cuts of the propagator are built out of real intermediate states**.

Where this comes from: unitarity. This is not coincidence — it follows directly from S-matrix unitarity. Let us derive it in three short steps.

Step 1: from unitarity to an operator identity for T. Write the S matrix as $S = \mathbb{1} + iT$, where T encodes all genuine scattering. Unitarity $S^\dagger S = \mathbb{1}$ becomes

$$\mathbb{1} = (\mathbb{1} - iT^\dagger)(\mathbb{1} + iT) = \mathbb{1} + i(T - T^\dagger) + T^\dagger T\,,$$

that is, after cancelling the $\mathbb{1}$,

$$-i\big(T - T^\dagger\big) = T^\dagger T \qquad \Longleftrightarrow \qquad 2\,\mathrm{Im}\,T = T^\dagger T\,. \tag{8.175}$$

Step 2: take matrix elements between identical states. Sandwiching (8.175) between the same asymptotic state $|i\rangle$ on both sides, and inserting a complete set $\sum_X |X\rangle\langle X|$ in the right-hand side:

$$2\,\mathrm{Im}\langle i|T|i\rangle \;=\; \langle i|T^\dagger T|i\rangle \;=\; \sum_X |\langle X|T|i\rangle|^2\,.$$

Recognizing $\langle f|T|i\rangle$ as the invariant scattering amplitude $\mathcal{M}(i \to f)$ (up to an overall $(2\pi)^4\delta^{(4)}(\cdot)$ enforcing energy–momentum conservation), this is the *optical theorem*:

$$\boxed{2\,\mathrm{Im}\,\mathcal{M}(\mathrm{i} \to \mathrm{i}) \;=\; \sum_X \int d\Pi_X \,\big|\mathcal{M}(\mathrm{i} \to X)\big|^2\,,} \tag{8.176}$$

the imaginary part of any *forward* amplitude equals the sum over *all* final states X into which the system can transition, integrated over their Lorentz-invariant phase space $d\Pi_X$.

Step 3: specialize to the two-point function. The same unitarity argument applied to a single field operator $\varphi(0)$ acting on the vacuum (rather than to a full

scattering amplitude) gives precisely the relation (8.130): inserting $\sum_X |X\rangle\langle X|$ between $\varphi(0)$ and $\varphi(0)^\dagger$ in the spectral integrand produces the spectral density $\rho(p^2)$. One thus arrives at the equivalence

$$\mathrm{Im}\,\Delta'_F(p^2) \neq 0 \quad \Longleftrightarrow \quad \text{there exists a physical state } |X\rangle \text{ with } p_X^2 = p^2\,. \tag{8.177}$$

Putting it together. The "analytic structure" picture of the propagator (poles + cuts on the real p^2 axis) is simply unitarity expressed in the language of complex analysis:

- The isolated pole at $p^2 = m^2$ comes from the one-particle intermediate state $|p\rangle$ — a $\delta(\sigma^2 - m^2)$ in ρ.
- The branch cut starting at $p^2 = m_1^2$ comes from multi-particle intermediate states. Below m_1^2 no on-shell multi-particle state exists in this channel, so $\rho_{\text{cont.}}(\sigma^2) = 0$ and the propagator is analytic there.
- Each higher threshold $(m_a + m_b)^2$, $(m_a + m_b + m_c)^2$, ... where a heavier channel opens adds another onset to $\mathrm{Im}\,\Delta'_F$, and hence to $\rho(\sigma^2)$.

Perturbative version. In perturbation theory the same content is recovered diagrammatically: the imaginary part of the exact propagator is controlled by the imaginary part of the 1PI self-energy via the Dyson form (8.155), and $\mathrm{Im}\,\Pi(p^2)$ at a given order is computed by "cutting" the contributing Feynman diagrams along all sets of internal propagators that can be simultaneously put on shell. The systematic statement is provided by the *Cutkosky cutting rules*, the diagrammatic incarnation of the optical theorem; their full statement, mathematical justification, and worked examples (bubble and sunset diagrams) are developed in Appendix H.

Box 8.1 The propagator encodes the spectrum

The exact propagator, viewed as a function of complex p^2, is an analytic function whose singularities encode the spectrum of the theory in the relevant quantum-number sector. The pole gives the particle mass and the overlap of the field with the one-particle state (through Z); the branch cut gives the multiparticle thresholds and, through $\rho(\sigma^2)$, the density of accessible intermediate states weighted by their coupling to the field. This "pole-plus-cut" picture, central to the Peskin–Schroeder presentation, provides a powerful organizing principle: statements about propagation and scattering can often be traced back to the analytic structure of two-point functions.

Spectral representation for the Dirac field

The Källén–Lehmann analysis extends to the Dirac field [1, 6]. Consider the exact fermion two-point function

$$iS'_F(x-y) \equiv \langle 0|T\left(\psi(x)\bar{\psi}(y)\right)|0\rangle\,. \tag{8.178}$$

Inserting a complete set of states and using translation invariance as in the scalar case, Lorentz covariance constrains the momentum-space propagator of a parity-conserving theory to the general form

$$iS'_F(p) = i\int_0^\infty d\sigma^2\,\frac{\rho_V(\sigma^2)\,\not{p} + \rho_S(\sigma^2)\,\sigma}{p^2-\sigma^2+i\epsilon}, \tag{8.179}$$

where ρ_V and ρ_S obey positivity constraints inherited from Hilbert-space positivity, but they should not be treated as two arbitrary independent positive scalar densities. The two Lorentz structures—$\not{p}$ and $\sigma\cdot\mathbb{1}$—are the only ones available for the two-point function of a parity-conserving Dirac field. In a parity-violating theory, additional chiral structures can appear.

Just as in the scalar case, the one-particle state contributes a delta-function to both spectral densities, producing an isolated pole:

$$iS'_F(p) \xrightarrow[p^2\to m^2]{} \frac{iZ_2(\not{p}+m)}{p^2-m^2+i\epsilon} + \text{terms regular at } p^2=m^2, \tag{8.180}$$

where m is the physical fermion mass and Z_2 is the residue of the isolated fermion pole. In a massive theory with canonical normalization one has $0 < Z_2 \le 1$, with strict inequality only when continuum states carry spectral weight in the same channel.

The perturbative expression for the exact fermion propagator is most usefully written in terms of the one-particle-irreducible self-energy $\Sigma(p)$:

$$iS'_F(p) = \frac{i}{\not{p} - m_0 - \Sigma(p) + i\epsilon}. \tag{8.181}$$

Lorentz covariance and the discrete symmetries of a parity-conserving theory (such as massive QED with an infrared regulator) restrict Σ to the form $\Sigma(p) = A(p^2)\not{p} + B(p^2)$. It is useful to write the inverse propagator as

$$\begin{aligned} S'^{-1}_F(p) &= \not{p}\,C(p^2) - D(p^2), \\ C(p^2) &= 1 - A(p^2), \qquad D(p^2) = m_0 + B(p^2). \end{aligned} \tag{8.182}$$

The physical mass and the residue are then fixed by

$$D(m^2) = m\,C(m^2), \qquad Z_2^{-1} = C(m^2) + 2m^2C'(m^2) - 2mD'(m^2), \tag{8.183}$$

where primes denote derivatives with respect to p^2. The first condition fixes the physical mass as the pole of the full propagator. To see the second, write $\not{p} = m + \delta$ near the positive-energy pole, so that $p^2 - m^2 = (\not{p}-m)(\not{p}+m) = 2m\,\delta + O(\delta^2)$ on the pole projector. Expanding

$$\not{p}\,C(p^2) - D(p^2) = \big[C(m^2) + 2m^2C'(m^2) - 2mD'(m^2)\big](\not{p}-m) + O\big((\not{p}-m)^2\big)$$

after using $D(m^2) = mC(m^2)$ gives the coefficient that inverts to the pole residue Z_2. Together with a charge-renormalization condition, these equations define the on-shell renormalization scheme for a massive theory (see Chapter 6).

In a massive theory with a mass gap, the analytic structure mirrors the scalar case: the exact fermion propagator has an isolated pole at $p^2 = m^2$ and a branch cut starting at the lowest multiparticle threshold in the same charge sector. In QED the photon is massless, so the charged fermion is accompanied by arbitrarily soft photons and the branch point reaches the would-be pole at $p^2 = m^2$. This is the infrared reason why the simple isolated-pole LSZ formalism must be modified for charged external states in QED.

8.7 The LSZ reduction formula for scalar fields

We now develop the central result of this chapter: the **reduction formula** of Lehmann, Symanzik, and Zimmermann, which expresses S-matrix elements in terms of vacuum expectation values of time-ordered products of field operators.

Removing a particle from the in-state

Consider the S-matrix element

$$S_{\beta,\alpha p} = \langle \beta \text{ out} | \alpha p \text{ in} \rangle , \tag{8.184}$$

where $|\alpha p \text{ in}\rangle$ is the in-state with an assemblage α of particles plus an additional particle with momentum p. Using (8.67):

$$\begin{aligned} \langle \beta \text{ out} | \alpha p \text{ in} \rangle &= \langle \beta \text{ out} | a^\dagger_{\text{in}}(p) | \alpha \text{ in} \rangle \\ &= \langle \beta \text{ out} | a^\dagger_{\text{out}}(p) | \alpha \text{ in} \rangle \\ &\quad + \langle \beta \text{ out} | a^\dagger_{\text{in}}(p) - a^\dagger_{\text{out}}(p) | \alpha \text{ in} \rangle . \end{aligned} \tag{8.185}$$

The first term equals $\langle \beta - p \text{ out} | \alpha \text{ in} \rangle$, representing an out-state with particle p removed (if present in β); otherwise this term vanishes.

For the second term, we use (8.26)–(8.28) and the asymptotic conditions. Recall from (8.28) that the annihilation operator is extracted via

$$a_{\text{in}}(k) = i \int d^3x \, f_k^*(x) \overleftrightarrow{\partial_0} \varphi_{\text{in}}(x) . \tag{8.186}$$

Taking the Hermitian conjugate (and using $f_k^{**} = f_k$):

$$a^\dagger_{\text{in}}(k) = -i \int d^3x \, f_k(x) \overleftrightarrow{\partial_0} \varphi_{\text{in}}(x) , \tag{8.187}$$

with the same relation holding for the out-operators with $\varphi_{\text{in}} \to \varphi_{\text{out}}$. Subtracting:

$$\begin{aligned}&\langle\beta \text{ out}|a^\dagger_{\text{in}}(p) - a^\dagger_{\text{out}}(p)|\alpha \text{ in}\rangle \\ &\quad = -i\int d^3x\, f_p(x) \overleftrightarrow{\partial_0} \langle\beta \text{ out}|[\varphi_{\text{in}}(x) - \varphi_{\text{out}}(x)]|\alpha \text{ in}\rangle\,. \end{aligned} \tag{8.188}$$

Applying the weak asymptotic condition, we replace $\varphi_{\text{in}}(\vec{x},x^0)$ by $(1/\sqrt{Z})\varphi(\vec{x},x^0)$ in the limit $x^0 \to -\infty$, and $\varphi_{\text{out}}(\vec{x},x^0)$ by $(1/\sqrt{Z})\varphi(\vec{x},x^0)$ as $x^0 \to +\infty$:

$$\begin{aligned}\langle\beta \text{ out}|\alpha p \text{ in}\rangle &= \langle\beta - p \text{ out}|\alpha \text{ in}\rangle \\ &\quad + \frac{i}{\sqrt{Z}}\Big(\lim_{x^0\to+\infty} - \lim_{x^0\to-\infty}\Big) \\ &\qquad \times \int d^3x\, f_p(\vec{x},x^0) \overleftrightarrow{\partial_0} \langle\beta \text{ out}|\varphi(\vec{x},x^0)|\alpha \text{ in}\rangle\,. \end{aligned} \tag{8.189}$$

Converting to a four-dimensional integral

The time limits can be incorporated into a four-dimensional volume integral using the identity:

$$\begin{aligned}&\Big(\lim_{x^0\to+\infty} - \lim_{x^0\to-\infty}\Big)\int d^3x\, g_1(x) \overleftrightarrow{\partial_0} g_2(x) \\ &\quad = \int_{-\infty}^{+\infty} d^4x \frac{\partial}{\partial x^0}\Big[g_1(x) \overleftrightarrow{\partial_0} g_2(x)\Big] \\ &\quad = \int d^4x \left[g_1(x)\frac{\partial^2}{\partial(x^0)^2}g_2(x) - \frac{\partial^2 g_1(x)}{\partial(x^0)^2}g_2(x)\right]. \end{aligned} \tag{8.190}$$

Detailed derivation of (8.190). The first equality is the fundamental theorem of calculus applied to the time integration:

$$\Big[\lim_{x^0\to+\infty} - \lim_{x^0\to-\infty}\Big] F(x^0) = \int_{-\infty}^{+\infty} dx^0 \frac{\partial}{\partial x^0} F(x^0)\,, \tag{8.191}$$

with $F(x^0) = \int d^3x\, g_1(x) \overleftrightarrow{\partial_0} g_2(x)$. The second equality is the *temporal integration by parts*, obtained by expanding the bidirectional derivative $g_1 \overleftrightarrow{\partial_0} g_2 = g_1\,\partial_0 g_2 - (\partial_0 g_1)\, g_2$ and applying Leibniz:

$$\begin{aligned}\frac{\partial}{\partial x^0}\big[g_1 \overleftrightarrow{\partial_0} g_2\big] &= \frac{\partial}{\partial x^0}\big[g_1\,\partial_0 g_2 - (\partial_0 g_1)\, g_2\big] \\ &= \underbrace{(\partial_0 g_1)(\partial_0 g_2)}_{\text{cross term}} + g_1\,\partial_0^2 g_2 \\ &\quad - (\partial_0^2 g_1)\, g_2 - \underbrace{(\partial_0 g_1)(\partial_0 g_2)}_{\text{cross term}} \\ &= g_1\,\partial_0^2 g_2 - (\partial_0^2 g_1)\, g_2\,. \end{aligned} \tag{8.192}$$

The cross terms $(\partial_0 g_1)(\partial_0 g_2)$ cancel: this is precisely the Wronskian structure encoded by $\overleftrightarrow{\partial_0}$, and it is what makes the resulting volume integral amenable to the Klein–Gordon equation in the next step. Observe that, after this manipulation, both time derivatives have been concentrated on a single factor at a time, which can then be traded for spatial derivatives via $\partial_0^2 = \nabla^2 - m^2$ (for a free wave function) or kept on the matrix element.

Applying (8.190) to (8.189), with $g_1 = f_p$ and $g_2 = \langle \beta \text{ out}|\varphi(x)|\alpha \text{ in}\rangle$, we obtain:

$$\begin{aligned}\langle \beta \text{ out}|\alpha p \text{ in}\rangle = \langle \beta - p \text{ out}|\alpha \text{ in}\rangle \\ + \frac{i}{\sqrt{Z}} \int d^4x \left[f_p(x) \frac{\partial^2}{\partial (x^0)^2} \langle \beta \text{ out}|\varphi(x)|\alpha \text{ in}\rangle \right. \\ \left. - \frac{\partial^2 f_p(x)}{\partial (x^0)^2} \langle \beta \text{ out}|\varphi(x)|\alpha \text{ in}\rangle \right].\end{aligned} \tag{8.193}$$

Since $f_p(x)$ satisfies the Klein–Gordon equation:

$$\frac{\partial^2 f_p(x)}{\partial (x^0)^2} = (\nabla^2 - m^2) f_p(x)\,, \tag{8.194}$$

we substitute (8.194) into the second term of (8.193):

$$\begin{aligned}= \langle \beta - p \text{ out}|\alpha \text{ in}\rangle \\ + \frac{i}{\sqrt{Z}} \int d^4x \left[f_p(x) \frac{\partial^2}{\partial (x^0)^2} \langle \beta \text{ out}|\varphi(x)|\alpha \text{ in}\rangle \right. \\ \left. - (\nabla^2 - m^2) f_p(x) \cdot \langle \beta \text{ out}|\varphi(x)|\alpha \text{ in}\rangle \right].\end{aligned} \tag{8.195}$$

Now we integrate the $\nabla^2 f_p$ term by parts twice in $\vec{x}$ (boundary terms vanish since the fields fall off at spatial infinity):

$$-\int d^3x\, (\nabla^2 f_p)\, \langle \beta \text{ out}|\varphi|\alpha \text{ in}\rangle = -\int d^3x\, f_p\, \nabla^2 \langle \beta \text{ out}|\varphi|\alpha \text{ in}\rangle\,. \tag{8.196}$$

Substituting back into (8.195):

$$= \langle \beta - p \text{ out}|\alpha \text{ in}\rangle + \frac{i}{\sqrt{Z}} \int d^4x\, f_p(x) \left[\frac{\partial^2}{\partial (x^0)^2} - \nabla^2 + m^2 \right] \langle \beta \text{ out}|\varphi(x)|\alpha \text{ in}\rangle\,. \tag{8.197}$$

Recognising the d'Alembertian $\Box = \partial^2/\partial (x^0)^2 - \nabla^2$, we arrive at:

$$\boxed{\langle \beta \text{ out}|\alpha p \text{ in}\rangle = \langle \beta - p \text{ out}|\alpha \text{ in}\rangle + \frac{i}{\sqrt{Z}} \int d^4x\, f_p(x) (\Box + m^2) \langle \beta \text{ out}|\varphi(x)|\alpha \text{ in}\rangle\,.} \tag{8.198}$$

Removing a particle from the out-state

Let us now remove a particle p' from the assemblage $\beta = \gamma p'$. Since $|\gamma p' \text{ out}\rangle = a^\dagger_{\text{out}}(p')|\gamma \text{ out}\rangle$, taking the Hermitian conjugate gives $\langle \gamma p' \text{ out}| = \langle \gamma \text{ out}|a_{\text{out}}(p')$. Therefore:

$$\langle \gamma p' \text{ out}|\varphi(x)|\alpha \text{ in}\rangle = \langle \gamma \text{ out}|a_{\text{out}}(p')\,\varphi(x)|\alpha \text{ in}\rangle\,. \tag{8.199}$$

We now add and subtract a term with $a_{\text{in}}(p')$ acting to the right:

$$\begin{aligned} &= \langle \gamma \text{ out}|a_{\text{out}}(p')\,\varphi(x)|\alpha \text{ in}\rangle - \langle \gamma \text{ out}|\varphi(x)\,a_{\text{in}}(p')|\alpha \text{ in}\rangle \\ &\quad + \langle \gamma \text{ out}|\varphi(x)\,a_{\text{in}}(p')|\alpha \text{ in}\rangle\,. \end{aligned} \tag{8.200}$$

In the last term, $a_{\text{in}}(p')$ acts on $|\alpha \text{ in}\rangle$. If particle p' is present in the assemblage α, then $a_{\text{in}}(p')|\alpha \text{ in}\rangle = |\alpha - p' \text{ in}\rangle$ (up to normalisation); this gives the *disconnected* contribution $\langle \gamma \text{ out}|\varphi(x)|\alpha - p' \text{ in}\rangle$. Hence:

$$\begin{aligned} &\langle \gamma p' \text{ out}|\varphi(x)|\alpha \text{ in}\rangle \\ &= \langle \gamma \text{ out}|\varphi(x)|\alpha - p' \text{ in}\rangle + \langle \gamma \text{ out}|\big[a_{\text{out}}(p')\,\varphi(x) - \varphi(x)\,a_{\text{in}}(p')\big]|\alpha \text{ in}\rangle\,. \end{aligned} \tag{8.201}$$

Expressing the second term via the asymptotic condition. Using the analogue of (8.187) for the annihilation operator:

$$a_{\text{out}}(p') = i\int d^3y\, f^*_{p'}(y) \overleftrightarrow{\partial_0}\, \varphi_{\text{out}}(y)\,, \qquad a_{\text{in}}(p') = i\int d^3y\, f^*_{p'}(y) \overleftrightarrow{\partial_0}\, \varphi_{\text{in}}(y)\,, \tag{8.202}$$

the second term in (8.201) becomes:

$$\begin{aligned} &\langle \gamma \text{ out}|a_{\text{out}}(p')\,\varphi(x) - \varphi(x)\,a_{\text{in}}(p')|\alpha \text{ in}\rangle \\ &\quad = i\int d^3y\, f^*_{p'}(y) \overleftrightarrow{\partial_0}\, \langle \gamma \text{ out}|\big[\varphi_{\text{out}}(y)\,\varphi(x) - \varphi(x)\,\varphi_{\text{in}}(y)\big]|\alpha \text{ in}\rangle\,. \end{aligned} \tag{8.203}$$

Now we apply the weak asymptotic condition: $\varphi_{\text{out}}(y) \to \varphi(y)/\sqrt{Z}$ as $y^0 \to +\infty$ and $\varphi_{\text{in}}(y) \to \varphi(y)/\sqrt{Z}$ as $y^0 \to -\infty$, inside matrix elements. Consider the ordering, with x held *fixed* (it is an arbitrary but finite spacetime point in the Green function): when $y^0 \to +\infty$, $\varphi_{\text{out}}(y)\,\varphi(x) = T(\varphi(y)\varphi(x))$ since $y^0 > x^0$ (with x fixed); when $y^0 \to -\infty$, $\varphi(x)\,\varphi_{\text{in}}(y) = T(\varphi(y)\varphi(x))$ since $x^0 > y^0$ (again with x fixed). Therefore:

$$\begin{aligned} &\langle \gamma \text{ out}|\big[\varphi_{\text{out}}(y)\,\varphi(x) - \varphi(x)\,\varphi_{\text{in}}(y)\big]|\alpha \text{ in}\rangle \\ &\quad = \frac{1}{\sqrt{Z}}\left(\lim_{y^0\to+\infty} - \lim_{y^0\to-\infty}\right)\langle \gamma \text{ out}|T(\varphi(y)\varphi(x))|\alpha \text{ in}\rangle\,. \end{aligned} \tag{8.204}$$

Converting to a four-dimensional integral. Combining (8.203) and (8.204), and using the same identity (8.190), now with

$$g_1 = f^*_{p'}, \qquad g_2 = Z^{-1/2}\langle \gamma \text{ out}|T(\varphi(y)\varphi(x))|\alpha \text{ in}\rangle \ :$$

$$= \frac{i}{\sqrt{Z}}\int d^4y \left[f^*_{p'}(y)\frac{\partial^2}{\partial(y^0)^2}\langle \gamma \text{ out}|T(\varphi(y)\varphi(x))|\alpha \text{ in}\rangle \right. \\ \left. - \frac{\partial^2 f^*_{p'}(y)}{\partial(y^0)^2}\langle \gamma \text{ out}|T(\varphi(y)\varphi(x))|\alpha \text{ in}\rangle \right]. \tag{8.205}$$

Since $f^*_{p'}(y)$ also satisfies the Klein–Gordon equation:

$$\frac{\partial^2 f^*_{p'}(y)}{\partial(y^0)^2} = (\nabla_y^2 - m^2) f^*_{p'}(y)\,, \tag{8.206}$$

we substitute into the second term of (8.205):

$$= \frac{i}{\sqrt{Z}}\int d^4y \left[f^*_{p'}(y)\frac{\partial^2}{\partial(y^0)^2}\langle \gamma \text{ out}|T(\varphi(y)\varphi(x))|\alpha \text{ in}\rangle \right. \\ \left. - (\nabla_y^2 - m^2) f^*_{p'}(y)\cdot\langle \gamma \text{ out}|T(\varphi(y)\varphi(x))|\alpha \text{ in}\rangle \right]. \tag{8.207}$$

The $\nabla_y^2 f^*_{p'}$ piece is now *integrated by parts twice* in $\vec{y}$, transferring the Laplacian from the wave function onto the matrix element. Denoting $G(y,x) \equiv \langle \gamma \text{ out}|T(\varphi(y)\varphi(x))|\alpha \text{ in}\rangle$ for brevity:

$$\int d^3y\,(\nabla_y^2 f^*_{p'})\,G = \underbrace{\oint_{\partial V} d\mathbf{S}\cdot(\nabla_y f^*_{p'})\,G}_{=0} - \int d^3y\,(\nabla_y f^*_{p'})\cdot(\nabla_y G)$$

$$= -\underbrace{\oint_{\partial V} d\mathbf{S}\cdot f^*_{p'}\,(\nabla_y G)}_{=0} + \int d^3y\, f^*_{p'}(\nabla_y^2 G)$$

$$= \int d^3y\, f^*_{p'}(y)\,\nabla_y^2 G(y,x)\,. \tag{8.208}$$

Both surface terms at $|\vec{y}| \to \infty$ vanish: for normalizable wave packets this is automatic, and for the sharp plane-wave choice it is understood distributionally, with the usual adiabatic/$i\epsilon$ damping. Substituting (8.208) back into (8.207):

$$= \frac{i}{\sqrt{Z}}\int d^4y\, f^*_{p'}(y)\left[\frac{\partial^2}{\partial(y^0)^2} - \nabla_y^2 + m^2\right]\langle \gamma \text{ out}|T(\varphi(y)\varphi(x))|\alpha \text{ in}\rangle$$

$$= \frac{i}{\sqrt{Z}}\int d^4y\, f^*_{p'}(y)\,(\Box_y + m^2)\,\langle \gamma \text{ out}|T(\varphi(y)\varphi(x))|\alpha \text{ in}\rangle\,. \tag{8.209}$$

It is useful to keep the arrow notation explicit: in the following formula the operator $(\Box_y + m^2)$ acts on the time-ordered matrix element, not on the external wave function $f^*_{p'}(y)$. Only later, when one rewrites the result in momentum space and continues the external momentum off shell before taking the pole residue, does the same differential operator become the usual amputation factor. The final result is:

$$\begin{aligned}\langle \gamma p' \text{ out}|\varphi(x)|\alpha \text{ in}\rangle &= \langle \gamma \text{ out}|\varphi(x)|\alpha - p' \text{ in}\rangle \\ &+ \frac{i}{\sqrt{Z}} \int d^4y\, f^*_{p'}(y)\,(\Box_y + m^2)\,\langle \gamma \text{ out}|T(\varphi(y)\varphi(x))|\alpha \text{ in}\rangle\,. \end{aligned} \tag{8.210}$$

The general reduction formula

Repeatedly applying this technique to remove all particles from the states, we arrive at the vacuum expectation value of a time-ordered product:

$$\begin{aligned}&\langle p_1 \cdots p_n \text{ out}|q_1 \cdots q_m \text{ in}\rangle \\ &= \left(\frac{i}{\sqrt{Z}}\right)^{m+n} \left[\prod_{i=1}^{m} \int d^4x_i\, f_{q_i}(x_i)\overrightarrow{(\Box_{x_i} + m^2)}\right] \\ &\quad \times \langle 0|T(\varphi(y_1)\cdots\varphi(y_n)\varphi(x_1)\cdots\varphi(x_m))|0\rangle \\ &\quad \times \left[\prod_{j=1}^{n} \int d^4y_j\, \overleftarrow{(\Box_{y_j} + m^2)} f^*_{p_j}(y_j)\right], \end{aligned} \tag{8.211}$$

valid when all $p_j \neq q_i$ (forward-scattering terms have been dropped for simplicity).

A note on arrow notation. Throughout this chapter, in LSZ-type formulas with differential operators sandwiched between wave functions and Green functions, the arrows indicate the *direction in which the operator acts*:

- $\overrightarrow{(\Box + m^2)}$ (right-pointing) acts on whatever stands to its *right* — in the formulas above, it acts on the Green function $\langle 0|T(\cdots)|0\rangle$.
- $\overleftarrow{(\Box + m^2)}$ (left-pointing) acts on whatever stands to its *left* — again the Green function.

This convention will be used identically in the Dirac and photon LSZ formulas below, with $(\Box + m^2)$ replaced by the appropriate free wave operator $(i\not{\partial} - m$ for Dirac fields, $\Box$ for the massless photon). The arrows are purely a bookkeeping device: by integration by parts (boundary terms vanish for normalisable wave packets or under the standard $i\epsilon$ prescription), one can freely move the operator onto the external wave function, where it produces the on-shell amputation factor.

The LSZ reduction formula can be written more compactly. The derivation above uses on-shell wave packets, but the momentum-space formula is understood

as a residue prescription: first Fourier transform the Green function with an arbitrary external four-momentum, multiply by the inverse free wave operator, and only then take the limit $p^2 \to m^2$. The normalisation factors are evaluated on the physical mass shell. With this standard convention, inserting the explicit plane waves (8.27) for $f_p(x)$ and $f^*_{p'}(y)$ gives:

$$f_p(x) = \frac{e^{-ip\cdot x}}{\sqrt{(2\pi)^3 2\omega_p}}, \qquad f^*_{p'}(y) = \frac{e^{+ip'\cdot y}}{\sqrt{(2\pi)^3 2\omega_{p'}}}, \tag{8.212}$$

we note the key identity:

$$(\Box + m^2)e^{\pm ip\cdot x} = (-p^2 + m^2)e^{\pm ip\cdot x}, \tag{8.213}$$

which follows from $\partial_\mu e^{\pm ip\cdot x} = \pm ip_\mu e^{\pm ip\cdot x}$ and hence $\Box\, e^{\pm ip\cdot x} = (\pm ip_\mu)(\pm ip^\mu)e^{\pm ip\cdot x} = -p^2\, e^{\pm ip\cdot x}$.

Let us show explicitly how this works. Consider a typical *incoming* factor from (8.211):

$$\int d^4x_i\, f_{q_i}(x_i)\,(\Box_{x_i} + m^2)\,\langle 0|T(\cdots\varphi(x_i)\cdots)|0\rangle\,. \tag{8.214}$$

We integrate by parts *twice*, transferring $\Box_{x_i}$ from the Green's function onto $f_{q_i}(x_i)$. Let us denote $G(x_i) \equiv \langle 0|T(\cdots\varphi(x_i)\cdots)|0\rangle$ for brevity. Writing $\Box = \partial_\mu\partial^\mu$, the first integration by parts gives:

$$\int d^4x_i\, f_{q_i}\,(\partial_\mu\partial^\mu G) = \big[\text{boundary}\big] - \int d^4x_i\,(\partial_\mu f_{q_i})\,(\partial^\mu G)\,, \tag{8.215}$$

and the second:

$$-\int d^4x_i\,(\partial_\mu f_{q_i})\,(\partial^\mu G) = -\big[\text{boundary}\big] + \int d^4x_i\,(\partial_\mu\partial^\mu f_{q_i})\,G = \int d^4x_i\,(\Box f_{q_i})\,G\,. \tag{8.216}$$

For normalizable wave packets the boundary terms vanish; for the sharp plane-wave formula this step is understood distributionally, with the usual adiabatic/$i\epsilon$ damping. Calling the left-hand side of (8.214) I_i, and including the mass term (which requires no integration by parts), we obtain:

$$\begin{aligned} I_i &= \int d^4x_i\,\big[(\Box_{x_i} + m^2)\,f_{q_i}(x_i)\big]\,\langle 0|T(\cdots\varphi(x_i)\cdots)|0\rangle \\ &= \int d^4x_i\,\frac{(\Box_{x_i} + m^2)\,e^{-iq_i\cdot x_i}}{\sqrt{(2\pi)^3 2\omega_{q_i}}}\,\langle 0|T(\cdots\varphi(x_i)\cdots)|0\rangle \\ &= \frac{m^2 - q_i^2}{\sqrt{(2\pi)^3 2\omega_{q_i}}}\int d^4x_i\, e^{-iq_i\cdot x_i}\,\langle 0|T(\cdots\varphi(x_i)\cdots)|0\rangle\,. \end{aligned}$$

The remaining integral over x_i is precisely the Fourier transform of the Green's function at momentum q_i. Similarly, for a typical *outgoing* factor, the arrow notation means that the Klein–Gordon operator initially acts on the Green function to its left; integrating by parts moves it onto the outgoing plane wave:

$$\begin{aligned}&\int d^4y_j\,\langle 0|T(\cdots\varphi(y_j)\cdots)|0\rangle\,(\Box_{y_j}+m^2)\,f^*_{p_j}(y_j)\\&\quad=\frac{m^2-p_j^2}{\sqrt{(2\pi)^3 2\omega_{p_j}}}\int d^4y_j\,e^{+ip_j\cdot y_j}\,\langle 0|T(\cdots\varphi(y_j)\cdots)|0\rangle\,.\end{aligned}\tag{8.217}$$

This gives the same amputation factor $(m^2-p_j^2)$, now associated with the outgoing external pole.

Combining all m incoming and n outgoing factors, the reduction formula (8.211) becomes:

$$\begin{aligned}&\langle p_1\cdots p_n\ \text{out}|q_1\cdots q_m\ \text{in}\rangle\\&=\left(\frac{i}{\sqrt{Z}}\right)^{m+n}\prod_{i=1}^{m}\frac{m^2-q_i^2}{\sqrt{(2\pi)^3 2\omega_{q_i}}}\prod_{j=1}^{n}\frac{m^2-p_j^2}{\sqrt{(2\pi)^3 2\omega_{p_j}}}\\&\quad\times\widetilde{G}(p_1,\ldots,p_n;q_1,\ldots,q_m)\,,\end{aligned}\tag{8.218}$$

where $\widetilde{G}$ is the Fourier transform of the full $(m+n)$-point Green's function, with the external momenta kept off shell until the final *on-shell limit* $p_j^2,q_i^2\to m^2$ is taken (see below). Each factor (m^2-p^2) *removes one external propagator*: since the Green's function in momentum space contains a pole $\sim i/(p^2-m^2)$ for each external leg, the product $(m^2-p^2)\cdot i/(p^2-m^2)=-i$ yields a finite, non-zero result on shell. The order of operations is essential:

1. keep $p^2\neq m^2$ (off-shell), so that $\widetilde{G}(p,\ldots)$ is a regular function with the one-particle pole isolated;
2. multiply by the amputation factor (m^2-p^2), which cancels the pole and leaves a finite residue;
3. *only now* take the on-shell limit $p^2\to m^2$, i.e.

$$\lim_{p^2\to m^2}(m^2-p^2)\,\widetilde{G}(p,\ldots)\;=\;-i\cdot\big[\text{amputated residue}\big]\,.\tag{8.219}$$

Reversing this order would give either a divergence (setting $p^2=m^2$ before amputating) or an indeterminate $0\cdot\infty$ (amputating with p already on shell). Hence the LSZ formula is properly understood as a residue prescription: amputate first, mass-shell limit last.

Why the connected Green's function gives the scattering amplitude. The full Green's function $\widetilde{G}$ in (8.218) contains connected and disconnected parts. LSZ reduction applied recursively separates the same structures in the S-matrix: disconnected Green-function components reproduce the lower-particle overlaps that appeared explicitly in the reduction steps, while the fully connected component gives the genuine transition amplitude.

What are "lower-cluster terms"? By *lower-cluster terms* (or "disconnected lower-cluster contributions") we mean those pieces of the N-point Green's function ($N = m + n$) that *factorise* into products of correlators on fewer points. They are "lower" because each factor in the product involves strictly less than N external fields. The general decomposition reads

$$\widetilde{G}^{(N)}_{\text{full}} = \widetilde{G}^{(N)}_{\text{conn}} + \sum_{\substack{\text{partitions of} \\ \{x_1,\dots,x_N\} \\ \text{with } \geq 2 \text{ blocks}}} \prod_{\text{blocks } B} \widetilde{G}^{(|B|)}_{\text{conn}}\,, \tag{8.220}$$

where each non-trivial partition groups the N external points into two or more disjoint subsets ("clusters"), each cluster fully connected internally but disconnected from the others. Every block has $|B| < N$ external points, hence the term "lower".

Paradigmatic example: $2 \to 2$. For the four-point function the non-trivial partitions of $\{x_1, x_2, y_1, y_2\}$ into clusters of size at least one are: the three pairings $\{(x_1y_1)(x_2y_2)\}$, $\{(x_1y_2)(x_2y_1)\}$, $\{(x_1x_2)(y_1y_2)\}$, plus partitions involving singletons. Singleton-clusters yield factors of $\langle 0|\varphi|0\rangle = 0$ after the field shift, so the surviving disconnected piece is a sum of products of two two-point functions:

$$\widetilde{G}^{(4)}_{\text{disc}} = \sum_{\text{pairings}} \widetilde{G}^{(2)}_{\text{conn}}(\cdot,\cdot)\,\widetilde{G}^{(2)}_{\text{conn}}(\cdot,\cdot)\,. \tag{8.221}$$

Physically these are the configurations in which the two "in" particles propagate independently to the two "out" particles *without ever interacting* — they are precisely the forward-scattering pieces that build the identity part of the S-matrix, $S = \mathbb{1} + iT$, and that have to be subtracted to isolate the non-trivial scattering amplitude $\mathcal{M}$.

Where they enter the LSZ recursion. The lower-cluster terms are exactly the contributions that appeared explicitly at each step of the reduction in the form of overlaps with one fewer particle, e.g.

$$\langle \gamma p' \text{ out}|\varphi(x)|\alpha \text{ in}\rangle = \underbrace{\langle \gamma \text{ out}|\varphi(x)|\alpha - p' \text{ in}\rangle}_{\text{lower cluster}} + (\text{piece feeding the connected } \widetilde{G}_{\text{conn}})\,. \tag{8.222}$$

Applying LSZ recursively to all particles collects all such lower-cluster pieces, while the genuinely new physics at each step lives in the connected component. *Stripping the disconnected lower-cluster terms* thus means isolating $\widetilde{G}^{(N)}_{\text{conn}}$ from $\widetilde{G}^{(N)}_{\text{full}}$ in (8.220); the result is what the LSZ formula relates to the non-trivial scattering amplitude.

Single isolated external insertion. If an external field insertion is completely disconnected from all the others, its contribution factorises through the one-point function,

$$\begin{aligned}&\langle 0|T(\varphi(x_1)\cdots\varphi(x_i)\cdots\varphi(x_{m+n}))|0\rangle\big|_{x_i \text{ isolated}}\\ &\quad=\langle 0|\varphi(x_i)|0\rangle\,\langle 0|T(\varphi(x_1)\cdots\hat{\varphi}(x_i)\cdots\varphi(x_{m+n}))|0\rangle\,,\end{aligned}\tag{8.223}$$

where the hat over $\hat{\varphi}(x_i)$ is the standard *omission notation*: $T(\varphi(x_1)\cdots\hat{\varphi}(x_i)\cdots\varphi(x_{m+n}))$ denotes the time-ordered product of all the fields *except* $\varphi(x_i)$, i.e. the $(m+n-1)$-point Green's function on the remaining points $\{x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{m+n}\}$. The same convention is used throughout the rest of this chapter.

Our field has been shifted/renormalised so that $\langle 0|\varphi|0\rangle = 0$, hence such terms vanish.

Disconnected groups of external insertions. For $2\to 2$ scattering, the disconnected part of the exact four-point function contains products of exact two-point functions, schematically

$$\begin{aligned}\widetilde{G}_{\text{disc.}} &= (2\pi)^4\delta^{(4)}(q_1-p_1)\,i\Delta'_F(q_1)\,(2\pi)^4\delta^{(4)}(q_2-p_2)\,i\Delta'_F(q_2)\\ &\quad+(p_1\leftrightarrow p_2)+\cdots\,.\end{aligned}\tag{8.224}$$

These terms describe particles that propagate from the in-state to the out-state without participating in a common connected interaction. In the recursive LSZ derivation they are precisely the forward/no-scattering overlaps such as $\langle\beta - p \text{ out}|\alpha \text{ in}\rangle$; in the operator language they belong to the identity part and, more generally, to products of lower-point S-matrix elements.

The fully connected scattering amplitude is therefore defined after subtracting these lower-cluster contributions:

$$S_{\beta\alpha} = S^{\text{disc.}}_{\beta\alpha} + i(2\pi)^4\delta^{(4)}(P_\beta - P_\alpha)\,\mathcal{M}_{\beta\alpha}\,,\tag{8.225}$$

and $\mathcal{M}_{\beta\alpha}$ is obtained from the connected Green's function. This is the precise sense in which the physically non-trivial scattering amplitude is determined by $\widetilde{G}_{\text{conn.}}$.

Box 8.2 The LSZ Reduction Formula

After stripping the disconnected lower-cluster terms, the connected part of the S-matrix element is obtained from the connected Green's function by removing the external propagators and then putting the external momenta on shell. Suppressing the standard external state normalisation factors displayed in (8.218), this is written schematically as

$$\boxed{S_{fi}^{\text{conn.}} = \mathcal{A}_{\text{LSZ}}\,\widetilde{G}_{\text{conn.}}(p_1,\ldots,p_r)\Big|_{p_i^2=m^2}} \tag{8.226}$$

where

$$\mathcal{A}_{\text{LSZ}} = \prod_{\text{ext. legs}} \left[\frac{i}{\sqrt{Z}}\,(m^2-p^2)\right]. \tag{8.227}$$

Here $\widetilde{G}_{\text{conn.}}$ is the Fourier transform of the connected time-ordered Green's function, and the momenta are kept off shell until after the factors in $\mathcal{A}_{\text{LSZ}}$ have cancelled the external poles. Disconnected terms give the forward-scattering and lower-cluster delta functions already separated in the reduction steps.

Box 8.3 Renormalisation scheme and the fate of $\sqrt{Z}$

The factor $1/\sqrt{Z}$ for each external leg in (8.226) originates from the LSZ asymptotic condition $\varphi(x) \to \sqrt{Z}\,\varphi_{\text{in/out}}(x)$, with Z the residue of the *exact* propagator at the physical one-particle pole:

$$i\Delta_F'(p^2) \xrightarrow{\;p^2\to m^2_{\text{phys}}\;} \frac{iZ}{p^2-m^2_{\text{phys}}+i\epsilon} + (\text{regular}). \tag{8.228}$$

Whether a finite external-leg factor appears explicitly in scattering amplitudes depends on the *renormalisation scheme*, i.e. on how the finite parts of the counterterms and the normalization of the renormalized field are chosen. It is important to distinguish the pole residue of the propagator from the renormalization constant that relates bare and renormalized fields.

On-shell (OS) scheme. The field renormalisation $\varphi_{\text{bare}} = Z_\phi^{1/2}\varphi_R$ is fixed order by order so that the renormalised propagator has unit residue at the physical pole:

$$\lim_{p^2\to m^2_{\text{phys}}} (p^2-m^2_{\text{phys}})\, i\Delta^{\text{OS}}_{F,R}(p^2) = i\,. \tag{8.229}$$

Consequence: when the Green's functions in (8.226) are computed with the OS-renormalized field φ_R, the LSZ residue factor for that field is 1 and no

separate $\sqrt{Z_{\text{pole}}}$ multiplies the amplitude. The mass parameter is automatically the physical pole mass. The counterterm δZ_ϕ absorbs both the UV divergence *and* the finite part required to enforce unit residue. This is the natural scheme whenever external states are physical on-shell particles.

Minimal subtraction ($\overline{\text{MS}}$ and similar). Counterterms are chosen to remove *only the UV divergences* (poles in ϵ, with the universal constant $\gamma_E - \ln 4\pi$ subtracted in $\overline{\text{MS}}$). No finite-part tuning is imposed to make the physical pole residue equal to one. Thus the renormalized propagator generally has

$$i\Delta_{F,R}^{\overline{\text{MS}}}(p^2) \xrightarrow{p^2 \to m^2_{\text{phys}}} \frac{iZ_{\text{pole}}^{\overline{\text{MS}}}(\mu)}{p^2 - m^2_{\text{phys}} + i\epsilon} + (\text{regular}), \qquad Z_{\text{pole}}^{\overline{\text{MS}}}(\mu) \neq 1 \quad (8.230)$$

in general. Consequence: the LSZ residue factors must be kept explicitly as finite external-leg factors multiplying the amputated connected Green function. Equivalently, the finite derivative of the self-energy at the pole enters through the LSZ residue; one should not add separate amputated diagrams with self-energy insertions on external on-shell legs in addition to this factor, or the effect is double counted. The advantage is that the scheme is mass-independent and well suited to studying the renormalisation-group running of couplings with μ – hence its ubiquity in perturbative QCD.

Same physics, different bookkeeping. The physical S-matrix is scheme-independent. The choice OS vs. $\overline{\text{MS}}$ only reshuffles where the finite parts live: in the OS scheme they are buried inside the field counterterm so that the renormalized propagator has unit pole residue; in $\overline{\text{MS}}$ the finite pole residue remains visible as an LSZ factor. Numerical predictions agree once self-energies and external-leg factors are accounted for consistently.

Scheme	Z	Effect on LSZ
On-shell (OS)	pole residue of φ_R fixed to 1	no separate $\sqrt{Z_{\text{pole}}}$ in amplitudes
$\overline{\text{MS}}$, MOM, …	pole residue generally finite and μ-dep.	LSZ residue remains as a finite external-leg factor

The LSZ formula at work: tree-level ϕ^4 scattering

To see how the reduction formula operates in practice, consider the simplest non-trivial example: $2 \to 2$ scattering in the theory $\mathcal{L}_{\text{int}} = -(\lambda/4!)\,\varphi^4$, at lowest order in λ.

The connected four-point Green's function at tree level consists of a single vertex with four propagators attached to it:

$$\widetilde{G}_{\text{conn.}}(p_1,p_2,p_3,p_4) = (2\pi)^4\delta^{(4)}\left(\sum_{i=1}^{4} p_i\right)\prod_{i=1}^{4}\frac{i}{p_i^2-m^2+i\epsilon}\cdot(-i\lambda)\,, \tag{8.231}$$

where all four momenta are taken as incoming, subject to $\sum_i p_i = 0$. At tree level there are no self-energy corrections, so $Z = 1 + O(\lambda)$ and the full and free propagators coincide.

Applying the LSZ prescription (8.226), and suppressing the overall momentum-conserving delta function and external normalisation factors, we multiply by $i(m^2 - p_i^2)$ for each external leg and take $p_i^2 \to m^2$:

$$\begin{aligned} S_{fi}^{\text{conn.}} &= \prod_{i=1}^{4}\left[i(m^2-p_i^2)\frac{i}{p_i^2-m^2+i\epsilon}\right](-i\lambda)\Bigg|_{p_i^2=m^2} \\ &= \prod_{i=1}^{4}(+1)\cdot(-i\lambda) = -i\lambda\,. \end{aligned} \tag{8.232}$$

Each factor $(m^2 - p_i^2)$ cancels the corresponding propagator pole, leaving the finite stripped result $i\mathcal{M} = -i\lambda$, hence $\mathcal{M} = -\lambda$. This is precisely the Feynman rule for the vertex read off directly from the Lagrangian: the LSZ machinery, though seemingly elaborate, reproduces the intuitive result.

The power of the formalism becomes apparent at higher orders: when self-energy insertions dress the external two-point function, the propagator poles acquire a finite residue. The LSZ factors then extract precisely that residue from the full Green function. Whether this finite residue is visible as an explicit external-leg factor or absorbed into the field counterterm depends on the renormalisation scheme (see the box "Renormalisation scheme and the fate of $\sqrt{Z}$" above): in an on-shell scheme the renormalized field is normalized to unit pole residue, while in $\overline{\text{MS}}$ the finite pole residue remains and must be included through LSZ [1, 6].

8.8 The reduction formula for Dirac fields

We now carry out the full LSZ reduction for Dirac spinor fields, following the same logic as for the scalar case but keeping careful track of the spinor structure.

Asymptotic fields and the Yang–Feldman equation

The interacting Dirac field $\psi(x)$ satisfies the Dirac equation with the *physical* mass m:

$$(i\partial\!\!\!/ - m)\psi(x) = \tilde{j}(x), \qquad \tilde{j}(x) = j(x) - (m - m_0)\psi(x)\,, \tag{8.233}$$

where $j(x)$ is the interaction current (e.g. $j = e_0 \not{A}\psi$ in the bare QED equation, later rewritten in terms of renormalized parameters and counterterms) and the mass counterterm $(m - m_0)\psi$ ensures the physical mass is m. The formal solution is:

$$\psi(x) = \sqrt{Z_2}\,\psi_{\text{in}}(x) + \int d^4y\, S_{\text{ret}}(x-y;m)\,\tilde{j}(y)\,, \tag{8.234}$$

where $S_{\text{ret}}(x-y;m)$ is the retarded Dirac Green's function satisfying

$$(i\not{\partial}_x - m)S_{\text{ret}}(x-y;m) = \delta^{(4)}(x-y), \qquad S_{\text{ret}}(x-y) = 0 \text{ for } x^0 < y^0\,. \tag{8.235}$$

The field renormalisation constant Z_2 and the free in-field ψ_{in} are defined so that ψ_{in} satisfies the free Dirac equation $(i\not{\partial} - m)\psi_{\text{in}} = 0$ and creates/annihilates single-particle states with unit norm. The **weak asymptotic condition** for Dirac fields states that, inside matrix elements between physical states:

$$\psi(x) \xrightarrow{x^0 \to -\infty} \sqrt{Z_2}\,\psi_{\text{in}}(x)\,, \qquad \psi(x) \xrightarrow{x^0 \to +\infty} \sqrt{Z_2}\,\psi_{\text{out}}(x)\,. \tag{8.236}$$

Fourier expansion of the in-field

Since $\psi_{\text{in}}(x)$ satisfies the free Dirac equation, its Fourier expansion is:

$$\psi_{\text{in}}(x) = \int d^3p \sum_{s=\pm} \Big[b_{\text{in}}(p,s)\,U_{ps}(x) + d^{\dagger}_{\text{in}}(p,s)\,V_{ps}(x) \Big]\,, \tag{8.237}$$

with the positive- and negative-frequency solutions:

$$U_{ps}(x) = \frac{1}{(2\pi)^{3/2}}\sqrt{\frac{m}{E_p}}\,u(p,s)\,e^{-ip\cdot x}, \qquad V_{ps}(x) = \frac{1}{(2\pi)^{3/2}}\sqrt{\frac{m}{E_p}}\,v(p,s)\,e^{ip\cdot x}\,. \tag{8.238}$$

The inverse relations that extract the operators from the field are:

$$b_{\text{in}}(p,s) = \int d^3x\, U^{\dagger}_{ps}(x)\,\psi_{\text{in}}(x)\,, \quad d^{\dagger}_{\text{in}}(p,s) = \int d^3x\, V^{\dagger}_{ps}(x)\,\psi_{\text{in}}(x)\,. \tag{8.239}$$

Proof of (8.239). We verify the first relation. Using the orthogonality of the spinor solutions at equal time:

$$\int d^3x\, U^{\dagger}_{ps}(x)\,U_{p's'}(x) = \delta_{ss'}\delta^{(3)}(\vec{p}-\vec{p}\,')\,, \qquad \int d^3x\, U^{\dagger}_{ps}(x)\,V_{p's'}(x) = 0\,, \tag{8.240}$$

The first identity follows from the spinor normalisation $u^{\dagger}(p,s)u(p,s') = (E_p/m)\delta_{ss'}$ and the prefactor m/E_p in the modes. For the mixed term, the spatial integral gives $\delta^{(3)}(\vec{p}+\vec{p}\,')$ because $U^{\dagger} \propto e^{+ip\cdot x}$ while $V \propto e^{+ip'\cdot x}$; the spinor factor is then

$u^\dagger(p,s)v(-p,s') = 0$. Substituting (8.237) into $\int d^3x\, U^\dagger_{ps}\psi_{\rm in}$ and using (8.240), only the $b_{\rm in}$ term survives, giving $b_{\rm in}(p,s)$. The second relation follows analogously.

The conjugate relations for $\bar\psi_{\rm in} = \psi^\dagger_{\rm in}\gamma^0$ are:

$$\begin{aligned} b^\dagger_{\rm in}(p,s) &= \int d^3x\, \bar\psi_{\rm in}(x)\,\gamma^0\, U_{ps}(x)\,, \\ d_{\rm in}(p,s) &= \int d^3x\, \bar\psi_{\rm in}(x)\,\gamma^0\, V_{ps}(x)\,. \end{aligned} \tag{8.241}$$

General in-states with fermions and antifermions are constructed as:

$$\begin{aligned} &|(\bar p_k \bar s_k)\cdots(\bar p_1 \bar s_1);(p_j s_j)\cdots(p_1 s_1)\ \text{in}\rangle \\ &\qquad = d^\dagger_{\rm in}(\bar p_k,\bar s_k)\cdots b^\dagger_{\rm in}(p_1,s_1)|0\rangle\,. \end{aligned} \tag{8.242}$$

Removing an incoming electron from the in-state

Consider the S-matrix element with an electron of momentum p and spin s in the in-state:

$$S_{\beta,\alpha(ps)} = \langle \beta\ \text{out}|\alpha(ps)\ \text{in}\rangle\,, \tag{8.243}$$

where $|\alpha(ps)\ \text{in}\rangle = b^\dagger_{\rm in}(p,s)|\alpha\ \text{in}\rangle$. Proceeding as in the scalar case, we split:

$$\begin{aligned} \langle \beta\ \text{out}|\alpha(ps)\ \text{in}\rangle &= \langle \beta\ \text{out}|b^\dagger_{\rm in}(p,s)|\alpha\ \text{in}\rangle \\ &= \langle \beta\ \text{out}|b^\dagger_{\rm out}(p,s)|\alpha\ \text{in}\rangle \\ &\quad + \langle \beta\ \text{out}|\big[b^\dagger_{\rm in}(p,s) - b^\dagger_{\rm out}(p,s)\big]|\alpha\ \text{in}\rangle\,. \end{aligned} \tag{8.244}$$

The first term gives the disconnected contribution $\langle \beta - (ps)\ \text{out}|\alpha\ \text{in}\rangle$.

For the second term, we need the expression for $b^\dagger_{\rm in}(p,s)$ in terms of the field. From (8.241):

$$b^\dagger_{\rm in}(p,s) = \int d^3x\, \bar\psi_{\rm in}(x)\,\gamma^0\, U_{ps}(x)\,. \tag{8.245}$$

Note that unlike the scalar case, the inverse relation involves $\bar\psi\gamma^0$ rather than $\overleftrightarrow{\partial_0}$, because the Dirac equation is first order. For free fields the corresponding equal-time Dirac inner product is time-independent:

$$\frac{d}{dx^0}\int d^3x\, \bar\psi_{\rm in}(x)\,\gamma^0\, U_{ps}(x) = 0\,, \tag{8.246}$$

since both $\psi_{\rm in}$ and U_{ps} satisfy the same free Dirac equation.

Proof of (8.246). The statement (8.246) is the conservation of the Dirac inner product. We show that the current

$$j^\mu(x) \;\equiv\; \bar\psi_{\rm in}(x)\,\gamma^\mu\, U_{ps}(x) \tag{8.247}$$

is divergenceless, $\partial_\mu j^\mu = 0$; integrating j^0 over $\vec{x}$ then gives the conserved "charge" (8.246).

Both $\psi_{\rm in}$ and U_{ps} satisfy the free Dirac equation of the physical mass m:

$$(i\gamma^\mu\partial_\mu - m)\,\psi_{\rm in} = 0\,, \qquad (i\gamma^\mu\partial_\mu - m)\,U_{ps} = 0\,. \tag{8.248}$$

Taking the Dirac adjoint of the first equation (Hermitian conjugate, then multiplication by γ^0 from the right) gives

$$\bar\psi_{\rm in}\big(i\overleftarrow{\partial}_\mu\gamma^\mu + m\big) = 0 \quad\Longleftrightarrow\quad i\,(\partial_\mu\bar\psi_{\rm in})\,\gamma^\mu = -m\,\bar\psi_{\rm in}\,. \tag{8.249}$$

Now apply Leibniz to $\partial_\mu j^\mu$:

$$\begin{aligned}\partial_\mu(\bar\psi_{\rm in}\gamma^\mu U_{ps}) &= \underbrace{(\partial_\mu\bar\psi_{\rm in})\gamma^\mu}_{=im\,\bar\psi_{\rm in}}\,U_{ps} \;+\; \bar\psi_{\rm in}\,\underbrace{\gamma^\mu(\partial_\mu U_{ps})}_{=-im\,U_{ps}} \\ &= im\,\bar\psi_{\rm in}\,U_{ps} - im\,\bar\psi_{\rm in}\,U_{ps} = 0\,,\end{aligned} \tag{8.250}$$

where the first identity uses (8.249) and the second uses (8.248) rewritten as $\gamma^\mu\partial_\mu U_{ps} = -im\,U_{ps}$. The two mass terms cancel exactly: $\partial_\mu j^\mu = 0$.

Separating temporal and spatial parts and integrating over $\vec{x}$:

$$\begin{aligned}\frac{d}{dx^0}\int d^3x\,\bar\psi_{\rm in}\gamma^0 U_{ps} &= -\int d^3x\,\nabla\cdot(\bar\psi_{\rm in}\vec\gamma\,U_{ps}) \\ &= -\oint_{\partial V} d\mathbf{S}\cdot(\bar\psi_{\rm in}\vec\gamma\,U_{ps}) \;=\; 0\,,\end{aligned} \tag{8.251}$$

the spatial boundary term vanishing for normalisable wave packets (or, for the sharp plane-wave choice, with the standard adiabatic / $i\epsilon$ prescription). This proves (8.246).

Geometric interpretation. The bilinear $\langle\phi,\chi\rangle_D \equiv \int d^3x\,\bar\phi\gamma^0\chi = \int d^3x\,\phi^\dagger\chi$ is the ordinary L^2 inner product on Dirac spinors. The statement (8.246) says that this inner product is *conserved* by the free Dirac flow: the time evolution is unitary on $L^2(\mathbb{R}^3)\otimes\mathbb{C}^4$, and orthonormal mode functions remain orthonormal at every time. This is the spinorial counterpart of the Klein–Gordon identity used in the scalar reduction with $\overleftrightarrow{\partial_0}$.

Subtracting the in and out expressions:

$$\begin{aligned}&\langle\beta\ \text{out}|b^\dagger_{\rm in}(p,s) - b^\dagger_{\rm out}(p,s)|\alpha\ \text{in}\rangle \\ &\quad= \int d^3x\,\langle\beta\ \text{out}|\big[\bar\psi_{\rm in}(x) - \bar\psi_{\rm out}(x)\big]|\alpha\ \text{in}\rangle\,\gamma^0 U_{ps}(x)\,.\end{aligned} \tag{8.252}$$

Applying the weak asymptotic condition. Using (8.236), $\bar{\psi}_{\text{in}}(x) \to \bar{\psi}(x)/\sqrt{Z_2}$ as $x^0 \to -\infty$ and $\bar{\psi}_{\text{out}}(x) \to \bar{\psi}(x)/\sqrt{Z_2}$ as $x^0 \to +\infty$:

$$\begin{aligned}\langle \beta \text{ out}|\alpha(ps) \text{ in}\rangle = \langle \beta - (ps) \text{ out}|\alpha \text{ in}\rangle \\ + \frac{1}{\sqrt{Z_2}}\Big(\lim_{x^0\to-\infty} - \lim_{x^0\to+\infty}\Big) \\ \int d^3x \, \langle \beta \text{ out}|\bar{\psi}(x)|\alpha \text{ in}\rangle \, \gamma^0 U_{ps}(x)\,. \end{aligned} \tag{8.253}$$

Note the sign: (in) − (out) here, opposite to the scalar case, because we are computing $b^\dagger_{\text{in}} - b^\dagger_{\text{out}}$ rather than $a^\dagger_{\text{in}} - a^\dagger_{\text{out}}$.

Converting to a four-dimensional integral. Define $\bar{g}(x) \equiv \langle \beta \text{ out}|\bar{\psi}(x)|\alpha \text{ in}\rangle$ (a 1×4 row spinor). We prove the following identity, which is the Dirac analogue of (8.190):

$$\Big(\lim_{x^0\to-\infty} - \lim_{x^0\to+\infty}\Big) \int d^3x \, \bar{g}\,\gamma^0 U_{ps} = i \int d^4x \, \bar{g}\,(i\overleftarrow{\not\partial} + m)\, U_{ps}\,, \tag{8.254}$$

where $\bar{g}\, i\overleftarrow{\not\partial} \equiv i(\partial_\mu \bar{g})\gamma^\mu$ and U_{ps} satisfies $(i\not\partial - m)U_{ps} = 0$.

Proof. We start from the right-hand side. Multiplying out:

$$i\int d^4x \, \bar{g}\,(i\overleftarrow{\not\partial} + m)\, U_{ps} = -\int d^4x \, (\partial_\mu \bar{g})\,\gamma^\mu\, U_{ps} + im \int d^4x \, \bar{g}\, U_{ps}\,. \tag{8.255}$$

Splitting into temporal and spatial parts:

$$\begin{aligned} = &-\int d^4x \, (\partial_0 \bar{g})\,\gamma^0\, U_{ps} \\ &-\int d^4x \, (\partial_k \bar{g})\,\gamma^k\, U_{ps} \\ &+ im \int d^4x \, \bar{g}\, U_{ps}\,. \end{aligned} \tag{8.256}$$

Step 1: Integrate the spatial term by parts (boundary terms vanish at spatial infinity):

$$-\int d^4x \, (\partial_k \bar{g})\,\gamma^k\, U_{ps} = +\int d^4x \, \bar{g}\,\gamma^k\,(\partial_k U_{ps})\,. \tag{8.257}$$

Step 2: Use the free Dirac equation. From $(i\gamma^\mu\partial_\mu - m)U_{ps} = 0$ we have $i\gamma^0\partial_0 U_{ps} + i\gamma^k\partial_k U_{ps} = mU_{ps}$, hence:

$$\gamma^k \partial_k U_{ps} = -\gamma^0 \partial_0 U_{ps} - imU_{ps}\,. \tag{8.258}$$

Step 3: Substitute (8.258) into (8.257):

$$+\int d^4x\,\bar{g}\,\gamma^k\,(\partial_k U_{ps}) = -\int d^4x\,\bar{g}\,\gamma^0\,(\partial_0 U_{ps}) - im\int d^4x\,\bar{g}\,U_{ps}\,. \tag{8.259}$$

Step 4: Substitute (8.259) back into (8.256):

$$\begin{aligned}
&= -\int d^4x\,(\partial_0\bar{g})\,\gamma^0\,U_{ps} - \int d^4x\,\bar{g}\,\gamma^0\,(\partial_0 U_{ps}) \\
&\quad \underbrace{-im\int d^4x\,\bar{g}\,U_{ps} + im\int d^4x\,\bar{g}\,U_{ps}}_{=\,0} \\
&= -\int d^4x\,\left[(\partial_0\bar{g})\,\gamma^0\,U_{ps} + \bar{g}\,\gamma^0\,(\partial_0 U_{ps})\right] = -\int d^4x\,\partial_0\left[\bar{g}\,\gamma^0\,U_{ps}\right],
\end{aligned} \tag{8.260}$$

where the last step is just the product rule in reverse. Finally:

$$-\int_{-\infty}^{+\infty} dx^0\,\partial_0\int d^3x\,\bar{g}\,\gamma^0\,U_{ps} = \left(\lim_{x^0\to-\infty} - \lim_{x^0\to+\infty}\right)\int d^3x\,\bar{g}\,\gamma^0\,U_{ps}\,. \quad \square \tag{8.261}$$

Assembling the final result. Substituting (8.254) into (8.253):

$$\begin{aligned}
\langle\beta\ \text{out}|\alpha(ps)\ \text{in}\rangle &= \langle\beta - (ps)\ \text{out}|\alpha\ \text{in}\rangle \\
&\quad + \frac{i}{\sqrt{Z_2}}\int d^4x\,\langle\beta\ \text{out}|\bar{\psi}(x)|\alpha\ \text{in}\rangle\,(i\overleftarrow{\not\partial} + m)\,U_{ps}(x)\,.
\end{aligned} \tag{8.262}$$

Writing $(i\overleftarrow{\not\partial} + m) = -(-i\overleftarrow{\not\partial} - m)$:

Box 8.4

$$\begin{aligned}
\langle\beta\ \text{out}|\alpha(ps)\ \text{in}\rangle &= \langle\beta - (ps)\ \text{out}|\alpha\ \text{in}\rangle \\
&\quad - \frac{i}{\sqrt{Z_2}}\int d^4x\,\langle\beta\ \text{out}|\bar{\psi}(x)|\alpha\ \text{in}\rangle\,(-i\overleftarrow{\not\partial} - m)\,U_{ps}(x)\,.
\end{aligned} \tag{8.263}$$

This is the Dirac analogue of (8.198). The operator $(-i\overleftarrow{\not\partial} - m)$ plays the role of $(\Box + m^2)$ in the scalar case: acting on the Green function it supplies the inverse external fermion propagator, while after integration by parts it vanishes on the free external wave function by the Dirac equation.

Removing an incoming positron from the in-state

For an incoming positron with momentum $\bar{p}$ and spin $\bar{s}$, we have $|\alpha(\bar{p}\bar{s})\text{ in}\rangle = d^\dagger_{\text{in}}(\bar{p},\bar{s})|\alpha\text{ in}\rangle$. From (8.239):

$$d^\dagger_{\text{in}}(\bar{p},\bar{s}) = \int d^3x\, V^\dagger_{\bar{p}\bar{s}}(x)\,\psi_{\text{in}}(x)\,. \tag{8.264}$$

Proceeding as before — splitting into $d^\dagger_{\text{out}} + (d^\dagger_{\text{in}} - d^\dagger_{\text{out}})$, applying the weak asymptotic condition, and converting to a 4D integral — we obtain:

$$\begin{aligned}&\langle\beta\text{ out}|d^\dagger_{\text{in}}(\bar{p},\bar{s}) - d^\dagger_{\text{out}}(\bar{p},\bar{s})|\alpha\text{ in}\rangle \\ &\quad= \int d^3x\, V^\dagger_{\bar{p}\bar{s}}(x)\,\langle\beta\text{ out}|\big[\psi_{\text{in}}(x) - \psi_{\text{out}}(x)\big]|\alpha\text{ in}\rangle\,.\end{aligned} \tag{8.265}$$

Note that here ψ (not $\bar{\psi}$) appears, because $d^\dagger$ is extracted from ψ (the negative-frequency part). Using the asymptotic condition and converting to a 4D integral with the free Dirac equations

$$(i\partial\!\!\!/ - m)U_{ps} = 0, \qquad (i\partial\!\!\!/ - m)V_{ps} = 0\,, \tag{8.266}$$

where the second equation is equivalent to $(p\!\!\!/ + m)v(p,s) = 0$ because $V_{ps} \propto e^{+ip\cdot x}$, gives:

Box 8.5

$$\begin{aligned}\langle\beta\text{ out}|\alpha(\bar{p}\bar{s})\text{ in}\rangle &= \langle\beta - (\bar{p}\bar{s})\text{ out}|\alpha\text{ in}\rangle \\ &\quad + \frac{i}{\sqrt{Z_2}}\int d^4x\,\bar{V}_{\bar{p}\bar{s}}(x)\,(i\partial\!\!\!/ - m)\,\langle\beta\text{ out}|\psi(x)|\alpha\text{ in}\rangle\,.\end{aligned} \tag{8.267}$$

Here $(i\partial\!\!\!/ - m)$ acts to the right on the matrix element. The sign difference with respect to the electron case (8.263) arises from the column/row placement of ψ versus $\bar{\psi}$ and from the orientation of the asymptotic time limits.

Removing an outgoing electron from the out-state

To remove an outgoing electron with momentum p', spin s' from $\langle\gamma(p's')\text{ out}|$, we use $\langle\gamma(p's')\text{ out}| = \langle\gamma\text{ out}|b_{\text{out}}(p',s')$. From (8.239):

$$b_{\text{out}}(p',s') = \int d^3x\, U^\dagger_{p's'}(x)\,\psi_{\text{out}}(x)\,. \tag{8.268}$$

We write:

$$\begin{aligned}&\langle\gamma(p's')\ \text{out}|\varphi(y)\cdots|\alpha\ \text{in}\rangle = \langle\gamma\ \text{out}|b_{\text{out}}(p',s')\,\varphi(y)\cdots|\alpha\ \text{in}\rangle\\&= \langle\gamma\ \text{out}|\varphi(y)\cdots b_{\text{in}}(p',s')|\alpha\ \text{in}\rangle\\&\quad+\langle\gamma\ \text{out}|\big[b_{\text{out}}(p',s')\,\varphi(y)\cdots-\varphi(y)\cdots b_{\text{in}}(p',s')\big]|\alpha\ \text{in}\rangle\,.\end{aligned}\tag{8.269}$$

The first term gives the disconnected piece. For the second term, the asymptotic condition gives:

- As $x^0\to+\infty$: $b_{\text{out}}(p',s')$ involves $\psi_{\text{out}}(x)\to\psi(x)/\sqrt{Z_2}$, and $\psi_{\text{out}}(x)\,\varphi(y)\cdots = T(\psi(x)\varphi(y)\cdots)$ since $x^0>y^0$.
- As $x^0\to-\infty$: $b_{\text{in}}(p',s')$ involves $\psi_{\text{in}}(x)\to\psi(x)/\sqrt{Z_2}$, and $\varphi(y)\cdots\psi_{\text{in}}(x) = T(\psi(x)\varphi(y)\cdots)$ since $x^0<y^0$.

If O contains other fermionic fields, the time-ordered product includes the usual minus signs from permuting fermion operators; throughout this section a fixed external-operator ordering is understood. In both limits, the result involves $T(\psi(x)\cdots)$, so:

$$\begin{aligned}&\langle\gamma\ \text{out}|\big[\psi_{\text{out}}(x)\,O - O\,\psi_{\text{in}}(x)\big]|\alpha\ \text{in}\rangle\\&\quad=\frac{1}{\sqrt{Z_2}}\Big(\lim_{x^0\to+\infty}-\lim_{x^0\to-\infty}\Big)\langle\gamma\ \text{out}|T(\psi(x)\,O)|\alpha\ \text{in}\rangle\,.\end{aligned}\tag{8.270}$$

Converting to a 4D integral and using $(i\partial\!\!\!/_x - m)U_{p's'}(x) = 0$, after integration by parts on the Dirac operator we obtain:

Box 8.6

$$\begin{aligned}&\langle\gamma(p's')\ \text{out}|O|\alpha\ \text{in}\rangle = \langle\gamma\ \text{out}|O|\alpha-(p's')\ \text{in}\rangle\\&\quad-\frac{i}{\sqrt{Z_2}}\int d^4x\,\bar U_{p's'}(x)\,(i\partial\!\!\!/-m)\\&\qquad\times\langle\gamma\ \text{out}|T(\psi(x)\,O)|\alpha\ \text{in}\rangle\,.\end{aligned}\tag{8.271}$$

Here $(i\partial\!\!\!/-m)$ acts on the matrix element to its right.

Removing an outgoing positron from the out-state

For an outgoing positron, $\langle\gamma(\bar p'\bar s')\ \text{out}| = \langle\gamma\ \text{out}|d_{\text{out}}(\bar p',\bar s')$, and from (8.241):

$$d_{\text{out}}(\bar p',\bar s') = \int d^3x\,\bar\psi_{\text{out}}(x)\,\gamma^0\,V_{\bar p'\bar s'}(x)\,.\tag{8.272}$$

The same procedure (split, asymptotic condition, 4D integral, integration by parts) yields:

Box 8.7

$$\langle \gamma(\bar{p}'\bar{s}')\ \text{out}|O|\alpha\ \text{in}\rangle = \langle \gamma\ \text{out}|O|\alpha - (\bar{p}'\bar{s}')\ \text{in}\rangle + \frac{i}{\sqrt{Z_2}}\int d^4x\,\langle \gamma\ \text{out}|T(\bar{\psi}(x)\,O)|\alpha\ \text{in}\rangle \times(-i\overleftarrow{\not\partial} - m)\,V_{\bar{p}'\bar{s}'}(x)\,. \tag{8.273}$$

The operator $(-i\overleftarrow{\not\partial} - m)$ acts to the left on the matrix element.

Summary and comparison with the scalar case

Collecting the four cases, the pattern is:

Particle	Operator	Free-field solution
Incoming e^- $(b^\dagger_{\text{in}})$	$\langle\bar{\psi}\,|\,(-i\overleftarrow{\not\partial} - m)$	$U_{ps}(x)$
Incoming e^+ $(d^\dagger_{\text{in}})$	$\bar{V}\,(i\not\partial - m)\,\|\psi\rangle$	$V_{\bar{p}\bar{s}}(x)$
Outgoing e^- (b_{out})	$\bar{U}\,(i\not\partial - m)\,\|T\psi\rangle$	$U_{p's'}(x)$
Outgoing e^+ (d_{out})	$\langle T\bar{\psi}\,|\,(-i\overleftarrow{\not\partial} - m)$	$V_{\bar{p}'\bar{s}'}(x)$

The correspondence with the scalar case is:

$$\underbrace{(\Box + m^2)}_{\text{scalar}} \longleftrightarrow \underbrace{(i\not\partial - m)}_{\text{Dirac}}, \qquad \underbrace{f_p(x)}_{\text{scalar}} \longleftrightarrow \underbrace{U_{ps}(x),\ V_{\bar{p}\bar{s}}(x)}_{\text{Dirac}}\,. \tag{8.274}$$

The close correspondence between the electron formula (8.263) and the positron formula (8.273) reflects the field-theoretic statement: positron processes are calculated by propagating negative-energy electrons backward in time.

The general Dirac reduction formula

Repeatedly applying the reduction steps to remove all fermions from the in- and out-states, we arrive at the Dirac analogue of (8.211), written in the same bracketed style ([LEFT block] $\langle 0|T(\ldots)|0\rangle$ [RIGHT block]) used for the scalar and photon

master formulas:

$$
\begin{aligned}
&\langle (p_1' s_1') \cdots (p_n' s_n');\, (\bar{p}_1' \bar{s}_1') \cdots (\bar{p}_{\bar{n}}' \bar{s}_{\bar{n}}') \text{ out}| \\
&\qquad (p_1 s_1) \cdots (p_m s_m);\, (\bar{p}_1 \bar{s}_1) \cdots (\bar{p}_{\bar{m}} \bar{s}_{\bar{m}}) \text{ in}\rangle \\
&= \left(\frac{-i}{\sqrt{Z_2}}\right)^{m+n} \left(\frac{i}{\sqrt{Z_2}}\right)^{\bar{m}+\bar{n}} \\
&\quad \times \left[\prod_{j=1}^{n} \int d^4 y_j \; \bar{U}_{p_j' s_j'}(y_j) \overrightarrow{(i\partial\!\!\!/_{y_j} - m)} \prod_{k=1}^{\bar{m}} \int d^4 \bar{x}_k \; \bar{V}_{\bar{p}_k \bar{s}_k}(\bar{x}_k) \overrightarrow{(i\partial\!\!\!/_{\bar{x}_k} - m)} \right] \\
&\quad \times \langle 0 | T \big(\psi(y_1) \cdots \psi(y_n) \bar{\psi}(\bar{x}_1) \cdots \bar{\psi}(\bar{x}_{\bar{m}}) \\
&\qquad\qquad \bar{\psi}(x_1) \cdots \bar{\psi}(x_m) \bar{\psi}(\bar{y}_1) \cdots \bar{\psi}(\bar{y}_{\bar{n}}) \big) | 0 \rangle \\
&\quad \times \left[\prod_{i=1}^{m} \int d^4 x_i \; \overleftarrow{(-i\partial\!\!\!/_{x_i} - m)} U_{p_i s_i}(x_i) \prod_{l=1}^{\bar{n}} \int d^4 \bar{y}_l \; \overleftarrow{(-i\partial\!\!\!/_{\bar{y}_l} - m)} V_{\bar{p}_l' \bar{s}_l'}(\bar{y}_l) \right] .
\end{aligned}
\tag{8.275}
$$

The displayed ordering fixes the overall fermionic sign convention. If one chooses a different ordering of external fermion operators, the formula must be multiplied by the sign of the corresponding permutation.

In momentum space, after inserting the explicit forms of U_{ps} and $V_{\bar{p}\bar{s}}$ and integrating by parts (exactly as in the scalar case discussed above; cf. (8.218)), the Dirac operators become inverse external fermion propagators, with the sign fixed by the momentum flow. For example,

$$
(i\partial\!\!\!/ - m)\, e^{-ip\cdot x}\, u(p,s) = (p\!\!\!/ - m)\, u(p,s)\, e^{-ip\cdot x} = 0 \quad \text{on shell,} \tag{8.276}
$$

whereas for the negative-frequency wave function,

$$
(i\partial\!\!\!/ - m)\, e^{+ip\cdot x}\, v(p,s) = -(p\!\!\!/ + m)\, v(p,s)\, e^{+ip\cdot x} = 0 \quad \text{on shell.} \tag{8.277}
$$

Off shell, the corresponding inverse Dirac operator cancels the external propagator pole. For an electron line with momentum p,

$$
(p\!\!\!/ - m)\, S_F(p) = (p\!\!\!/ - m) \frac{i(p\!\!\!/ + m)}{p^2 - m^2 + i\epsilon} \xrightarrow[p^2 \to m^2]{} i\,, \tag{8.278}
$$

up to the standard $i\epsilon$ prescription. Thus the LSZ operator removes the external fermion propagator and leaves the appropriate external spinor, just as $(m^2 - p^2) \cdot i/(p^2 - m^2) = -i$ removes an external scalar propagator.

The reduction formula for the photon field A^μ

The construction for the electromagnetic field A^μ proceeds in exact parallel with the scalar and Dirac cases, with two minor adjustments specific to massless gauge bosons.

Asymptotic condition and free field. The interacting electromagnetic field has, by the same arguments as in the scalar case, a weak asymptotic limit

$$A^\mu(x) \xrightarrow{x^0\to-\infty} \sqrt{Z_3}\,A^\mu_{\text{in}}(x)\,, \qquad A^\mu(x) \xrightarrow{x^0\to+\infty} \sqrt{Z_3}\,A^\mu_{\text{out}}(x)\,, \tag{8.279}$$

where Z_3 is the photon wave-function renormalisation constant and the in/out fields are massless free vector fields satisfying $\Box A^\mu_{\text{in/out}}(x) = 0$ (in Feynman–like covariant gauges; gauge-fixing terms contribute only to unphysical polarisations and drop out of physical S-matrix elements via the Ward identity). The mode expansion is

$$A^\mu_{\text{in}}(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \sum_\lambda \Big[a_{\text{in}}(k,\lambda)\,\varepsilon^\mu(k,\lambda)\,e^{-ik\cdot x} + a^\dagger_{\text{in}}(k,\lambda)\,\varepsilon^{\mu*}(k,\lambda)\,e^{+ik\cdot x}\Big], \tag{8.280}$$

with $\omega_k = |\vec{k}|$ and the polarisation 4-vectors $\varepsilon^\mu(k,\lambda)$ from Chapter 1.

Inverting the mode expansion. For a positive-frequency photon plane wave $A^\mu_p(x) = \varepsilon^\mu(p,\lambda)\,e^{-ip\cdot x}/\sqrt{(2\pi)^3 2\omega_p}$, the same Klein–Gordon-style overlap as in the scalar case (massless, applied component by component) gives

$$a_{\text{in}}(p,\lambda) = i\int d^3x\, A^{\mu*}_p(x) \overleftrightarrow{\partial_0} A^{\text{in}}_\mu(x)\,, \qquad a^\dagger_{\text{in}}(p,\lambda) = -i\int d^3x\, A^\mu_p(x) \overleftrightarrow{\partial_0} A^{\text{in}}_\mu(x)\,. \tag{8.281}$$

Single-photon reduction. Removing an outgoing photon with momentum p and polarisation λ from the state, and following step-by-step the scalar derivation (8.201)–(8.210) with $(\Box + m^2) \to \Box$ and $f_p \to \varepsilon^\mu(p,\lambda)\,e^{-ip\cdot x}/\sqrt{(2\pi)^3 2\omega_p}$, one obtains

$$\begin{aligned}\langle \gamma\, p\lambda \text{ out}|\alpha \text{ in}\rangle &= \langle \gamma \text{ out}|\alpha - p\lambda \text{ in}\rangle \\ &\quad + \frac{i}{\sqrt{Z_3}}\int d^4y\, \frac{\varepsilon^{\mu*}(p,\lambda)\,e^{+ip\cdot y}}{\sqrt{(2\pi)^3 2\omega_p}}\, \Box_y \langle \gamma \text{ out}|T(A_\mu(y)\cdots)|\alpha \text{ in}\rangle\,.\end{aligned} \tag{8.282}$$

General photon LSZ formula. Iterating the reduction on all photons in the in- and out-states, for a process with m incoming photons (momenta q_i, polarisations λ_i) and n outgoing photons (momenta p_j, polarisations λ'_j) one finds

$$\begin{aligned}&\langle p_1\lambda'_1\cdots p_n\lambda'_n \text{ out}|q_1\lambda_1\cdots q_m\lambda_m \text{ in}\rangle \\ &= \left(\frac{i}{\sqrt{Z_3}}\right)^{m+n}\left[\prod_{i=1}^m \int d^4x_i\, \frac{\varepsilon^{\mu_i}(q_i,\lambda_i)\,e^{-iq_i\cdot x_i}}{\sqrt{(2\pi)^3 2\omega_{q_i}}}\overrightarrow{\Box_{x_i}}\right] \\ &\quad \times \langle 0|T\big(A_{\mu_1}(x_1)\cdots A_{\mu_m}(x_m)\,A_{\nu_1}(y_1)\cdots A_{\nu_n}(y_n)\big)|0\rangle \\ &\quad \times \left[\prod_{j=1}^n \int d^4y_j\, \overleftarrow{\Box_{y_j}}\, \frac{\varepsilon^{\nu_j*}(p_j,\lambda'_j)\,e^{+ip_j\cdot y_j}}{\sqrt{(2\pi)^3 2\omega_{p_j}}}\right],\end{aligned} \tag{8.283}$$

valid for all $p_j \neq q_i$. In momentum space (Fourier transforming the Green's function), each $\Box$ becomes $-p^2$ and the amputation factor for a photon leg is simply $-p^2$, evaluated in the on-shell limit $p^2 \to 0$. Inserting the explicit plane waves,

$$S_{fi}^{\text{conn.}} = \left(\frac{i}{\sqrt{Z_3}}\right)^{m+n} \prod_{i=1}^{m} \frac{\varepsilon^{\mu_i}(q_i,\lambda_i)\,(-q_i^2)}{\sqrt{(2\pi)^3 2\omega_{q_i}}} \prod_{j=1}^{n} \frac{\varepsilon^{\nu_j *}(p_j,\lambda'_j)\,(-p_j^2)}{\sqrt{(2\pi)^3 2\omega_{p_j}}} \Bigg|_{q_i^2,p_j^2\to 0}$$
$$\times \widetilde{G}_{\mu_1\cdots\mu_m\nu_1\cdots\nu_n}(q_1,\ldots,q_m;\,p_1,\ldots,p_n)\,, \tag{8.284}$$

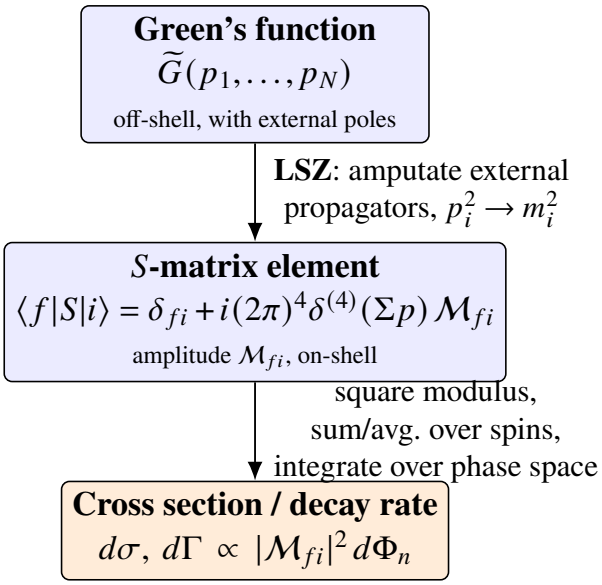
where $\widetilde{G}$ is the Fourier-transformed connected Green function of $m+n$ photon fields, with external indices left uncontracted until the polarisation vectors are attached.

Practical content. The structure is identical to the scalar formula (8.218): each external photon leg contributes a factor of $i/\sqrt{Z_3}$, an amputation factor $-p^2$ that kills the external pole, an on-shell polarisation ε^μ (or $\varepsilon^{\mu *}$ for outgoing), and a normalisation $1/\sqrt{(2\pi)^3 2\omega_p}$. As in the scalar case, the procedure is best understood as a residue prescription: continue the external photon momenta off-shell, multiply by $(-p^2)$, and only at the end take $p^2 \to 0$ with the appropriate transverse polarisation. The Ward identity guarantees that unphysical (longitudinal/scalar) polarisations decouple from the physical S-matrix, so the sum over λ in practical applications runs only over the two transverse helicities.

8.9 Green's functions and the S-matrix

In the previous sections we derived the LSZ reduction formula, which expresses S-matrix elements in terms of vacuum expectation values of time-ordered products of field operators. These vacuum expectation values are called **Green's functions** (or correlation functions). In this section we explain precisely what Green's functions are, how to compute them using perturbation theory, and how the LSZ formula connects them to physical scattering amplitudes.

The overall logical flow from correlation functions to observables can be summarised as follows:



The first arrow is the content of the LSZ formula derived in this chapter; the second arrow is the standard transition from amplitudes to observable rates, treated separately when computing cross sections.

Why are they called "Green's functions"? Connection with PDE theory

Before defining the QFT object precisely, it is worth pausing on the name. The term *Green's function* is borrowed from the classical theory of linear partial differential equations, where given a linear operator L one defines $G(x,y)$ by

$$L_x\, G(x,y) = \delta^{(d)}(x-y)\,, \tag{8.285}$$

i.e. the response of the system $L\phi = 0$ to a point-like source at y. The crucial fact is that the QFT object (8.295) is *not* merely homonymous with this classical notion: in the free case the two coincide *literally*, and in the interacting case the QFT version is a coherent generalisation through the Schwinger–Dyson equations and the generating functional.

Free case: literal identification. For the free scalar field, the two-point time-ordered VEV is precisely the Feynman Green's function of the Klein–Gordon operator:

$$i\Delta_F(x-y) \;\equiv\; \langle 0|T\big(\varphi(x)\varphi(y)\big)|0\rangle\,, \qquad (\Box_x + m^2)\, i\Delta_F(x-y) \;=\; -i\,\delta^{(4)}(x-y)\,. \tag{8.286}$$

The left-hand identity is a VEV of operators, the right-hand one is the classical defining equation $LG = \delta$ for $L = \Box + m^2$. The only non-trivial choice is which fundamental solution of $(\Box + m^2)$ one takes (retarded, advanced, Wightman, Feynman, …): different choices differ by homogeneous solutions, i.e. by how the

poles at $p^0 = \pm\omega_p$ are closed in the complex plane. The time-ordered product naturally selects the Feynman $i\epsilon$ prescription:

$$i\Delta_F(x-y) \;=\; \int \frac{d^4p}{(2\pi)^4}\,\frac{i\,e^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon}\,. \tag{8.287}$$

Same mathematical object as in the PDE theory; the QFT vacuum simply fixes the contour.

Interacting case: Schwinger–Dyson equations. For an interacting Lagrangian $\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2 - V(\varphi)$, the full two-point function

$$G^{(2)}(x-y) \;=\; \langle\Omega|T\big(\varphi(x)\varphi(y)\big)|\Omega\rangle \tag{8.288}$$

no longer satisfies the free $L\,G = \delta$ equation. Instead, it obeys the *Schwinger–Dyson* relation

$$(\Box_x + m^2)\,G^{(2)}(x-y) \;=\; -i\,\delta^{(4)}(x-y) \;-\; \langle\Omega|T\big(V'(\varphi(x))\,\varphi(y)\big)|\Omega\rangle\,, \tag{8.289}$$

which is the operator version of the classical equation of motion sourced by a δ, with the time-ordering producing the extra interaction term on the right. Iterating (8.289) couples $G^{(2)}$ to higher correlators. In perturbation theory the same content is organized by the *Dyson equation* for the exact scalar propagator,

$$i\Delta_F^{\text{full}}(p) = i\Delta_F^0(p) + i\Delta_F^0(p)\,\big[-i\Pi(p^2)\big]\,i\Delta_F^{\text{full}}(p), \tag{8.290}$$

where $i\Delta_F^0(p) = i/(p^2-m_0^2+i\epsilon)$ and $-i\Pi(p^2)$ is the scalar 1PI self-energy insertion. Equivalently,

$$i\Delta_F^{\text{full}}(p) = \frac{i}{p^2 - m_0^2 - \Pi(p^2) + i\epsilon}\,,$$

so the dressed inverse kinetic operator is shifted by the self-energy. The pole moves from the bare to the physical mass, and the pole residue is the wave-function renormalisation factor already encountered in the LSZ formula.

Synthesis: generating functional and path integral. The link is made fully systematic by the generating functional

$$\mathcal{Z}[J] = \int \mathcal{D}\varphi\; e^{iS[\varphi] + i\int d^4x\,J(x)\varphi(x)}\,, \qquad W[J] = -i\ln\mathcal{Z}[J]\,, \tag{8.291}$$

where $\mathcal{Z}[J]$ generates the full Green functions, while $W[J]$ generates the connected ones:

$$\begin{aligned} G^{(n)}(x_1,\dots,x_n) &= \frac{1}{i^n\,\mathcal{Z}[0]}\,\frac{\delta^n\mathcal{Z}[J]}{\delta J(x_1)\cdots\delta J(x_n)}\bigg|_{J=0}\,, \\ G_c^{(n)}(x_1,\dots,x_n) &= \frac{1}{i^{n-1}}\,\frac{\delta^n W[J]}{\delta J(x_1)\cdots\delta J(x_n)}\bigg|_{J=0}\,. \end{aligned} \tag{8.292}$$

For the free theory the path integral is Gaussian and yields explicitly

$$\mathcal{Z}_{\text{free}}[J] \;\propto\; \exp\left[-\frac{1}{2}\int d^4x\, d^4y\; J(x)\, i\Delta_F(x-y)\, J(y)\right], \tag{8.293}$$

whose kernel is exactly the classical Klein–Gordon Feynman Green's function, with our convention $i\Delta_F = -i(\Box + m^2)^{-1}_{\text{Feynman}}$. Wick's theorem then says that every free n-point function is a sum of products of $i\Delta_F$'s — diagrammatically, the free Green's functions are built entirely out of the classical PDE Green's function. In the interacting theory one writes

$$\mathcal{Z}[J] = \exp\left[i\, S_{\text{int}}\left(-i\frac{\delta}{\delta J}\right)\right]\mathcal{Z}_{\text{free}}[J]\,, \tag{8.294}$$

so the perturbative expansion is a series of interaction operators applied to the free Gaussian, producing the familiar Feynman diagrams: vertices joined by lines, where each line is the Feynman Green's function $i\Delta_F$.

Physical reading. The classical Green's function $G(x,y)$ represents the response of a linear system at x to a point-like source placed at y. The QFT Green's function $G^{(2)}(x,y)$ represents the quantum amplitude for a field quantum to propagate from y to x. In the free theory these two statements are mathematically the same; in the interacting theory the QFT Green's function generalises the classical one to include all virtual processes (self-interactions, loops, …) that dress the propagation. The terminology is therefore not metaphorical: it is a genuine extension of the PDE notion to the operator-valued, multi-point setting of QFT.

Summary table.

Aspect	Classical PDE	QFT
Defining equation	$L\,G(x,y) = \delta(x-y)$	$(\Box + m^2)i\Delta_F = -i\delta$ (free); Schwinger–Dyson (8.289) (interacting)
Meaning	response to point source	quantum propagation amplitude
Free 2-point function	literal identification	literal identification
Interacting 2-pt.	—	Green's function of the self-energy-dressed inverse kinetic operator
$n > 2$-point functions	—	generated by $\mathcal{Z}[J]$; perturbatively built from free $i\Delta_F$ connected by vertices

What is a Green's function?

A Green's function is the vacuum expectation value of a time-ordered product of *interacting* (Heisenberg-picture) field operators. Physically, it describes the amplitude for creating particles at certain spacetime points and detecting them at others, with all possible interactions included.

For a scalar field theory, the **n-point Green's function** is:

$$G(x_1,\dots,x_n) = \langle\Omega|T\big(\varphi(x_1)\cdots\varphi(x_n)\big)|\Omega\rangle\,, \tag{8.295}$$

where $|\Omega\rangle$ is the true vacuum of the interacting theory (the ground state of the full Hamiltonian H) and $\varphi(x)$ is the Heisenberg-picture field. The time-ordering T places operators at later times to the left.

For QED, the Green's function contains one field operator for each external particle:

$$G^{\mu\cdots}(x,y,z,\dots) = \langle\Omega|T\{A^\mu(x)\cdots\psi(y)\cdots\bar\psi(z)\cdots\}|\Omega\rangle\,, \tag{8.296}$$

with A^μ for each photon, ψ for each electron created (or positron destroyed), and $\bar\psi$ for each electron destroyed (or positron created).

Why are Green's functions useful?

- They are the natural objects produced by Feynman diagrams.
- They can be computed order-by-order in perturbation theory using Wick's theorem.
- The LSZ formula tells us exactly how to extract the physical S-matrix from them.
- They encode all information about the interacting theory: masses, coupling constants, bound states, etc.

Computing Green's functions: the interaction-picture formula

The Green's function (8.295) is defined in the Heisenberg picture, where states are time-independent and operators evolve with the full Hamiltonian. To compute it using perturbation theory, we need to express it in the *interaction picture*, where operators evolve with the free Hamiltonian and the interaction appears explicitly in the time-evolution operator.

The result (derived in detail in Peskin–Schroeder Ch. 4, or Mandl–Shaw Ch. 6) is:

$$\boxed{G(x_1,\dots,x_n) = \frac{\langle 0|T\big\{S\,\varphi_I(x_1)\cdots\varphi_I(x_n)\big\}|0\rangle}{\langle 0|S|0\rangle}}\,, \tag{8.297}$$

where:

- $|0\rangle$ is the *free* vacuum (ground state of H_0),
- $\varphi_I(x)$ is the field in the interaction picture (evolves with H_0, so it's a free field),
- $S = T\exp\left(-i\int d^4x\,\mathscr{H}_I(x)\right)$ is the interaction-picture Dyson operator, i.e. the perturbative representative of the S-matrix.

Where does this formula come from? The key steps are:

1. The Heisenberg field is related to the interaction-picture field by $\varphi_H(x) = U^\dagger(t,t_0)\,\varphi_I(x)\,U(t,t_0)$, where $U(t,t_0) = T\exp\left(-i\int_{t_0}^{t} dt'\,H_I(t')\right)$ is the time-evolution operator in the interaction picture.
2. The true vacuum $|\Omega\rangle$ is related to the free vacuum $|0\rangle$ by the Gell-Mann–Low formula: $|\Omega\rangle \propto U(0,-\infty)|0\rangle$, up to the adiabatic $i\epsilon$ prescription, normalisation and phase discussed in Appendix I.
3. After inserting these relations and simplifying, all the U operators combine into the S-matrix, and the phase factors produce the denominator $\langle 0|S|0\rangle$.

Box 8.8 Role of the denominator $\langle 0|S|0\rangle$

The denominator has a simple diagrammatic interpretation. When we expand the numerator of (8.297) using Wick's theorem, we get Feynman diagrams of two types:

1. Diagrams *connected* to the external field insertions $\varphi_I(x_1),\ldots,\varphi_I(x_n)$.
2. *Vacuum bubbles*: closed diagrams with no external legs, multiplying the connected diagrams.

The expansion of $\langle 0|S|0\rangle$ in the denominator generates *exactly the same* vacuum-bubble factor. Dividing cancels all vacuum bubbles. What remains are diagrams with no vacuum-bubble component: every connected component contains at least one external insertion. The full Green function may still factorise into disconnected pieces involving disjoint groups of external points; the fully connected part is selected separately by passing to G_c. This is the **linked-cluster theorem**.

Perturbative evaluation: Feynman diagrams

Expanding the S-matrix to order n:

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d^4z_1 \cdots d^4z_n\, T\{\mathscr{H}_I(z_1)\cdots\mathscr{H}_I(z_n)\}, \tag{8.298}$$

and substituting into (8.297), we apply Wick's theorem to evaluate the time-ordered product. Each contraction between two fields produces a Feynman propagator, and

each uncontracted interaction vertex produces a coupling constant. The systematic rules that emerge are the **Feynman rules** for the theory.

The Green's function $G(x_1,\dots,x_n)$ is therefore the sum of all Feynman diagrams with n external points at $x_1,\dots,x_n$, with all vacuum bubbles removed. In momentum space (after Fourier transforming each external coordinate) it is useful to separate the overall conservation delta function:

$$\tilde{G}(p_1,\dots,p_n) = (2\pi)^4\delta^{(4)}\left(\sum_i p_i\right) G_{\text{strip}}(p_1,\dots,p_n), \tag{8.299}$$

where the delta function enforces overall energy-momentum conservation and G_{strip} denotes the remaining momentum-dependent distribution.

Structure of the n-point function

The momentum-space Green's function has the following pole structure near the external one-particle shells. For a scalar field, with $n \geq 3$ external points and all external momenta conventionally taken as incoming, its most singular part is the product of one factor $iZ/(p_i^2 - m^2 + i\epsilon)$ for each external leg, multiplied by the amputated Green function $G_{\text{amp.}}(p_1,\dots,p_n)$; the remainder $R(p_1,\dots,p_n)$ is less singular in at least one external invariant $p_i^2 - m^2$. The two-point function is the exceptional case: it is itself the full propagator and contains only one one-particle pole. For $n \geq 3$, in perturbative language one says that the connected Green function is an amputated kernel multiplied by one full propagator for each external leg. Here $iD_F^{\text{full}}(p)$ is the *full* (exact, interacting) Feynman propagator, i.e. the Fourier transform of the two-point connected Green function (8.288) of the field $\varphi(x)$, including all self-energy corrections. Concretely, the Dyson resummation gives $iD_F^{\text{full}}(p) = i/[p^2 - m_0^2 - \Pi(p^2) + i\epsilon]$, with m_0 the bare mass and $\Pi(p^2)$ the 1PI self-energy (the systematic derivation appears in (8.301) below); near the physical mass shell, iD_F^{full} exhibits the pole $iZ/(p^2 - m_{\text{phys}}^2)$ familiar from the Källén–Lehmann representation. The factor $G_{\text{amp.}}$ is the truncated, or amputated, connected Green function obtained by stripping one full propagator from each external line. For two- and three-point proper functions this object is closely related to the usual one-particle-irreducible (1PI) vertex $\Gamma^{(n)}$. For a general scattering process, however, the amputated connected Green function may still contain internal propagators and several 1PI subgraphs; "amputated" only means that the propagators attached to the external field insertions have been removed.

This gives the useful dictionary between the Mandl–Shaw and Peskin–Schroeder presentations. Mandl–Shaw build the Green functions from the interaction-picture expansion and then identify the external legs diagrammatically. Peskin–Schroeder emphasize that the same external legs are the isolated poles of the exact Green function. LSZ is the operation that takes the residue of those poles.

The S-matrix from Green's functions

The LSZ reduction formula shows that the S-matrix element is obtained by:

1. Taking the n-point Green's function.
2. Isolating the external one-particle poles.
3. Amputating the external propagators: multiply by $(m^2 - p_i^2)$, with the sign fixed by the Fourier convention used above, for each scalar leg.
4. Putting the external momenta on-shell: $p_i^2 \to m^2$.
5. Multiplying by wave function factors: $u(p,s)$ and $\bar{u}(p,s)$ for external electrons, $v(p,s)$ and $\bar{v}(p,s)$ for external positrons, with the precise placement fixed by the incoming/outgoing assignment.
6. Including the wave function renormalization factors, e.g. $(i/\sqrt{Z})^n$ for n scalar external legs before accounting for convention-dependent signs.

This procedure isolates the physical scattering amplitude from the full Green's function. The factors $(\Box + m^2)$ in the LSZ formula become $(m^2 - p^2)$ in momentum space. They are applied before the on-shell limit: their role is to cancel the poles $1/(p^2 - m^2)$ from the external propagators, leaving a finite residue as $p^2 \to m^2$.

Box 8.9 Physical Interpretation

The Green's function describes the propagation and interaction of virtual particles at all values of p^2. The S-matrix describes the scattering of *real, on-shell* particles with $p^2 = m^2$.

The amputation procedure "removes" the external propagators because real particles don't need to propagate from infinity—they are already there as asymptotic states. The on-shell condition selects only those contributions where the external particles satisfy the relativistic dispersion relation.

The factor $\sqrt{Z}$ for each external leg accounts for the overlap between the local interacting field $\varphi(x)$ and the isolated one-particle state. With canonical normalization this overlap satisfies $0 < Z \leq 1$, and it is strictly smaller than one only when continuum states carry some of the spectral weight.

The key insight of LSZ is that we don't need to distinguish initial from final states when computing the Green's function—the time-ordered product handles everything automatically. The projection onto specific initial and final states is achieved by the $(\Box + m^2)$ factors and the exponentials $e^{\pm ip\cdot x}$.

Connected and disconnected diagrams

A Feynman diagram is **connected** if every vertex can be reached from every other vertex by following internal lines. The Green's function contains both connected

and disconnected contributions:

$$G(x_1,\dots,x_n) = G_c(x_1,\dots,x_n) + \text{disconnected pieces}. \tag{8.300}$$

For S-matrix elements, the disconnected pieces correspond to processes where some particles don't interact at all (they pass through unchanged). The interesting physics—the actual scattering—is contained in the connected part.

Cluster decomposition says that the full Green function factorizes when subsets of points are separated by large spacelike distances. Equivalently, the connected part tends to zero in that limit. This property is essential for the physical interpretation of scattering amplitudes.

The Dyson equation and one-particle-irreducible diagrams

The perturbative evaluation of Green's functions is organized systematically by introducing the concept of **one-particle-irreducible** (1PI) diagrams—those connected diagrams that remain connected after cutting any single internal line. Every connected diagram is built from 1PI blocks joined by full propagators; this observation leads to powerful resummation techniques [1, 6].

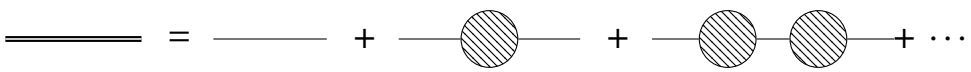
For the two-point function the construction is particularly transparent. The full (connected) propagator can be decomposed as a geometric series of 1PI self-energy insertions:

$$iD_F^{\text{full}}(p) = iD_F(p) + iD_F(p)\big[-i\Pi(p^2)\big]iD_F(p) + \cdots = \frac{i}{p^2 - m_0^2 - \Pi(p^2) + i\epsilon}, \tag{8.301}$$

where $D_F(p) = 1/(p^2 - m_0^2 + i\epsilon)$ is the free propagator, m_0 is the bare mass, and $-i\Pi(p^2)$ is the sum of all 1PI two-point diagrams (the self-energy). This is the **Dyson equation** for the scalar propagator, already encountered in (8.155). Its strength is that the full propagator—an object involving diagrams of arbitrarily high order—is determined once the 1PI self-energy is known.

Diagrammatically, the Dyson equation takes the recursive form (full = free + free × 1PI × full):

where the double line represents the full propagator, the single line the free propagator, and the shaded blob the 1PI self-energy $-i\Pi(p^2)$. Iterating the recursion generates the geometric series:



which sums to the closed-form expression (8.301).

The same structure holds for the Dirac field. The Dyson equation for the full fermion propagator is

$$iS_F^{\text{full}}(p) = \frac{i}{\not{p} - m_0 - \Sigma(p) + i\epsilon}, \tag{8.302}$$

where $-i\Sigma(p)$ is the sum of all 1PI insertions in the fermion line, in agreement with (8.181).

From full to connected to 1PI: hierarchy of Green's functions

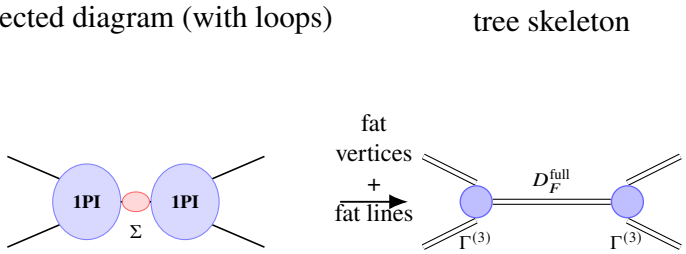
The classification of diagrams as full, connected, or 1PI defines a hierarchy of Green's functions that plays a central role in both the Mandl–Shaw and Peskin–Schroeder treatments. For an n-point function in a scalar theory:

1. The **full Green's function** $G(x_1, \ldots, x_n) = \langle \Omega | T(\varphi(x_1) \cdots \varphi(x_n)) | \Omega \rangle$ includes all diagrams, connected and disconnected, with vacuum bubbles removed by the denominator in (8.297) when it is evaluated perturbatively.
2. The **connected Green's function** G_c is the subset of diagrams in which all external points are linked. It is obtained from the full Green's function by subtracting all factorisations into disjoint groups.
3. The **1PI vertex function** $\Gamma^{(n)}$ (also called the proper vertex) is obtained from the connected Green's function by amputating all external propagators and restricting to diagrams that cannot be split into two pieces by cutting a single internal line.

Idea: replace loops with "fat" blocks. The decomposition that connects the three levels above is best understood pictorially. Take any connected Feynman diagram for G_c. We can systematically identify:

- *1PI sub-diagrams*: maximal sub-graphs that *cannot* be disconnected by cutting a single internal line. These contain all the loops and resum into the proper vertices $\Gamma^{(n)}$.
- *Self-energy sub-graphs*: 1PI two-point insertions. Resumming them along any line turns a free propagator D_F into a *full* propagator D_F^{full}.

If we represent each 1PI sub-diagram by a single "fat vertex" $\bullet$ and each chain of self-energy insertions by a single "fat line" $\Longleftrightarrow$, what is left is a graph with no loops — a **tree**, in the standard graph-theoretic sense:



The **tree skeleton** on the right is the same diagram in which every loop and every self-energy chain has been absorbed into a fat vertex ($\Gamma^{(n)}$) or a fat propagator (D_F^{full}). What remains is, by construction, loop-free: that is what "tree" means here. The skeleton has the same topology as a tree graph, even though each fat element internally encodes infinitely many ordinary diagrams.

Decomposition formula. For $n \geq 3$, the general statement is:

$$G_c(p_1,\ldots,p_n) = \sum_{\substack{\text{tree skeletons}\\ \text{with fat 1PI vertices}}} \prod_{\text{ext. legs } i} iD_F^{\text{full}}(p_i) \times \prod_{\text{int. lines } j} iD_F^{\text{full}}(k_j) \prod_{\text{vertices } v} i\Gamma^{(n_v)}(k_{v,1},\ldots,k_{v,n_v}), \quad (8.303)$$

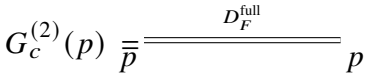
with momentum conservation at each fat vertex. "Tree skeleton" means a graph topology in which:

- each node is a 1PI vertex $\Gamma^{(n_v)}$ (any number of legs $n_v \geq 3$),
- each edge is a full propagator D_F^{full},
- cutting any single edge disconnects the graph (no closed loops at the skeleton level).

All the loop content lives inside the fat vertices and the full propagators; the sum in (8.303) only runs over the few *tree topologies* that connect the n external points.

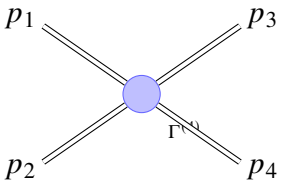
Concrete examples.

$n = 2$. The only possible tree topology with two external points has no fat vertices and a single full propagator joining them:



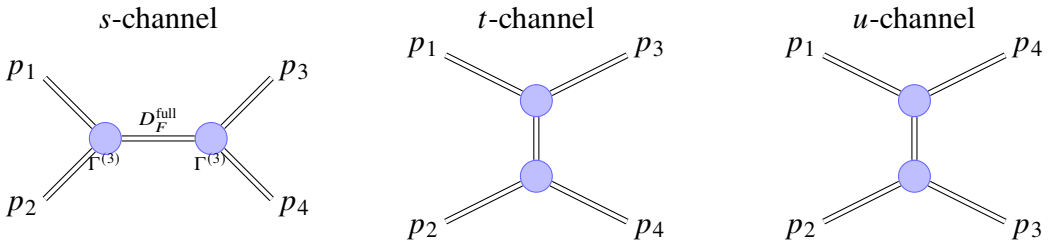
This is just the Dyson-resummed propagator: $G_c^{(2)}(p) = iD_F^{\text{full}}(p)$. This is the special case excluded from the product formula (8.303): there is no 1PI vertex with external propagators attached; the whole connected two-point function is the single full line whose resummation is given by the Dyson equation (8.301).

$n = 4$ *in* ϕ^4 *theory.* The only 1PI vertex with the right symmetry is $\Gamma^{(4)}$ (at tree level $i\Gamma^{(4)} = -i\lambda$). The tree skeleton has a single fat vertex with four full external propagators:

$G_c^{(4)} =$ 

Notice that the usual s-, t-, u-channel diagrams of the perturbative expansion of $G_c^{(4)}$ are *not* separate skeletons here: they all live *inside* the loop expansion of the single 1PI vertex $\Gamma^{(4)}$.

$n = 4$ in a theory with a cubic interaction (e.g. ϕ^3 or $\phi^4 + \phi^3$). Two distinct tree topologies contribute. The first is the contact topology, exactly as above. The second is the *exchange topology*: two 1PI three-vertices joined by one internal full propagator. There are three such graphs (s, t, u channels), corresponding to the three ways of partitioning $\{p_1, p_2, p_3, p_4\}$ into two pairs:



Each of these is a single skeleton with two fat vertices, four fat external legs, and one fat internal line. Cutting the internal line disconnects the graph (that is precisely the "tree" property).

What is and is not a tree skeleton. The key word is "skeleton": all loops have already been absorbed inside the fat blocks. Therefore the following ordinary diagrams *do not* appear in the skeleton sum:

- self-energy bubbles on an internal or external line — they are already absorbed into the corresponding D_F^{full};
- triangle, box, or higher loops with $n \le n_v$ external lines — they are already absorbed into the corresponding $\Gamma^{(n_v)}$.

The only freedom left is the choice of how the fat blocks are linked together at the level of the skeleton, and that information is fully captured by the tree topology.

Why this rearrangement is useful. (8.303) makes the structure of an interacting QFT modular: all genuine loop physics is encapsulated in a finite set of 1PI vertices and one full propagator; the skeletons are then trees, of which there are only finitely many for each n. This is the starting point of the *effective action* formalism [6]: the generating functional $\Gamma[\varphi_c]$ of 1PI vertices is the Legendre transform of the generating functional $W[J]$ of connected Green functions, and tree-level diagrams of Γ reproduce the full connected G_c. This formalism, though beyond our scope

here, provides the most systematic framework for renormalisation, spontaneous symmetry breaking and the analysis of vacuum stability.

Box 8.10 Why 1PI diagrams?

The decomposition into 1PI building blocks separates two conceptually distinct aspects of the dynamics. The 1PI vertices encode the "hard" interaction physics—the irreducible scattering amplitudes—while the full propagators encode propagation, including the effects of mass and wave function renormalization. In the renormalization programme (Chapters 6–7), this separation is essential: the divergences of the theory can be absorbed into a finite number of 1PI functions (the primitively divergent vertices), and the Dyson equation then propagates these renormalized quantities to all orders. The counting of primitively divergent diagrams is the content of the power-counting theorem; for a renormalizable theory, only a finite number of $\Gamma^{(n)}$ are superficially divergent.

8.10 Summary and physical interpretation

The key results of this chapter can be summarized as follows:

1. **Asymptotic states.** At $t \to \pm\infty$, the interacting fields approach free fields $\varphi \to \sqrt{Z}\,\varphi_{\text{in/out}}$ in the weak sense of matrix elements.
2. **Spectral density.** The exact two-point function contains an isolated one-particle pole plus a multiparticle continuum. The residue of the pole is the spectral weight Z.
3. **The S-matrix.** The transition amplitude $S_{\beta\alpha} = \langle \beta \text{ out} | \alpha \text{ in} \rangle$ connects asymptotic free-particle states.
4. **The LSZ reduction formula.** S-matrix elements are residues of time-ordered Green functions at their external one-particle poles, equivalently Green functions with external legs amputated and momenta put on-shell.
5. **Wave function renormalization.** For a canonically normalized scalar field with an isolated one-particle pole, the spectral weight satisfies $0 < Z \leq 1$. It is the probability weight for the local field to create the single physical particle from the vacuum; any remaining weight belongs to continuum states.

The LSZ formalism provides a rigorous foundation for the Feynman diagram technique. It shows that:

- The perturbation expansion of $\langle 0|T(S\,\text{fields})|0\rangle / \langle 0|S|0\rangle$ generates Feynman diagrams without vacuum bubbles.

- External leg factors arise from taking the residues of the exact propagator poles.
- The connection between Green's functions and S-matrix elements is exact under the LSZ assumptions; weak-coupling perturbation theory is only one way of evaluating the Green functions.

The physical picture is clear: scattering experiments involve preparing well-separated wave packets at $t \to -\infty$, letting them interact in some finite spacetime region, and observing well-separated final states at $t \to +\infty$. The LSZ formalism provides the mathematical framework for computing the transition amplitudes between such asymptotic states.

Part IV

Supersymmetry

9 Poincaré and Lorentz Symmetries

In this chapter, we examine the symmetry structure underlying relativistic quantum field theory. The invariance of the Lagrangian under the Poincaré group, which comprises Lorentz transformations and spacetime translations, is the foundation of all relativistic field theories. For a comprehensive treatment of group theory in physics, see Tung [29]; a detailed discussion of the Lorentz group and relativistic symmetry can be found in Sexl and Urbantke [30].

Roadmap. This chapter has two connected, but logically distinct, goals. First we study the Lorentz and Poincaré groups as spacetime symmetry groups, derive their Lie algebras, and identify the Casimir operators that classify one-particle states by mass and spin. Then we switch from states to fields: we construct the finite-dimensional Lorentz representations carried by spinor fields, introduce the sigma-matrix technology that links Weyl spinors to four-vectors, and finally translate the two-component language into the four-component Dirac and Majorana formalisms. The main thread is therefore

$$\text{Lorentz algebra} \longrightarrow \text{Poincaré Casimirs}$$
$$\longrightarrow \text{Weyl representations} \longrightarrow \text{covariant fermion Lagrangians.}$$

9.1 The Lorentz group and its Lie algebra

Definition and fundamental properties

Under an **active** Lorentz transformation $\Lambda^{\mu}{}_{\nu}$, the spacetime coordinates x^{μ} transform as

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}, \tag{9.1}$$

where there is an implicit sum over the repeated index ν. One can regard $\Lambda^{\mu}{}_{\nu}$ as the matrix element located in row μ and column ν of the 4×4 matrix Λ. The condition that the spacetime interval $g_{\mu\nu}x^{\mu}x^{\nu}$ is invariant under Lorentz transformations implies that

$$\boxed{\Lambda^{\mu}{}_{\rho}\, g_{\mu\nu}\, \Lambda^{\nu}{}_{\sigma} = g_{\rho\sigma}}. \tag{9.2}$$

In matrix notation, this reads $\Lambda^\top g \Lambda = g$, where $g = \mathrm{diag}(+1,-1,-1,-1)$ is the Minkowski metric.

The set of all Λ that satisfy eq. (9.2) forms a Lie group $O(1,3)$. Equation (9.2) implies that Λ has the following two properties:

Proposition 9.1 (Properties of Lorentz transformations). For any Λ satisfying (9.2):

(i) $\det \Lambda = \pm 1$
(ii) $|\Lambda^0{}_0| \geq 1$

Proof. (i) Taking the determinant of both sides of (9.2):

$$\det\left(\Lambda^\top g \Lambda\right) = \det(g) \quad \Longrightarrow \quad (\det \Lambda)^2 \det(g) = \det(g)$$

Since $\det(g) = -1 \neq 0$, we can divide to obtain $(\det \Lambda)^2 = 1$, hence $\det \Lambda = \pm 1$.

(ii) The $(0,0)$ component of (9.2) with $\rho = \sigma = 0$ gives:

$$\Lambda^\mu{}_0 \, g_{\mu\nu} \, \Lambda^\nu{}_0 = g_{00} = 1$$

Expanding:

$$(\Lambda^0{}_0)^2 g_{00} + \sum_{i=1}^{3} (\Lambda^i{}_0)^2 g_{ii} = (\Lambda^0{}_0)^2 - \sum_{i=1}^{3} (\Lambda^i{}_0)^2 = 1$$

Therefore $(\Lambda^0{}_0)^2 = 1 + \sum_{i=1}^{3} (\Lambda^i{}_0)^2 \geq 1$, which implies $|\Lambda^0{}_0| \geq 1$. □

These two properties divide the Lorentz group into four disconnected classes, characterized by a pair of signs $(\mathrm{sgn}(\det \Lambda), \mathrm{sgn}(\Lambda^0{}_0))$:

Class	$\det \Lambda$	$\Lambda^0{}_0$	Description
$(+,+)$	$+1$	$\geq +1$	Proper orthochronous (connected to $\mathbb{1}$)
$(+,-)$	$+1$	≤ -1	Proper non-orthochronous
$(-,+)$	-1	$\geq +1$	Improper orthochronous
$(-,-)$	-1	≤ -1	Improper non-orthochronous

The **proper orthochronous** Lorentz transformations $(+,+)$ are continuously connected to the identity and form a subgroup denoted $SO^+(1,3)$. If $\Lambda \in SO^+(1,3)$, we can generate all elements of the other classes by introducing:

- Space inversion (parity): $\Lambda_P = \mathrm{diag}(1,-1,-1,-1)$
- Time inversion: $\Lambda_T = \mathrm{diag}(-1,1,1,1)$
- Spacetime inversion: $\Lambda_P \Lambda_T = -\mathbb{1}_4$

Then $\{\Lambda, \Lambda_P \Lambda, \Lambda_T \Lambda, \Lambda_P \Lambda_T \Lambda\}$ spans the full Lorentz group.

Remark. [Physical significance] Infinitesimal Lorentz transformations must be proper and orthochronous, since only these are continuously connected to the identity. Space inversion and time inversion are *discrete* symmetries that may or may not be respected by physical interactions (famously, parity is violated in weak interactions).

Infinitesimal Lorentz transformations

The most general proper orthochronous Lorentz transformation matrix Λ is characterized by:

- A rotation angle θ about an axis $\hat{\boldsymbol{n}}$, encoded in $\vec{\theta} \equiv \theta\hat{\boldsymbol{n}}$
- A boost vector $\vec{\zeta} \equiv \hat{\boldsymbol{\beta}} \tanh^{-1}\beta$, where $\hat{\boldsymbol{\beta}}$ is the unit vector along the direction of motion and $\beta = |\vec{v}|/c$

The 4×4 matrix Λ is given by

$$\Lambda = \exp\left(-\tfrac{i}{2}\theta_{\rho\sigma}s^{\rho\sigma}\right) = \exp\left(-i\vec{\theta}\cdot\vec{s} - i\vec{\zeta}\cdot\vec{k}\right), \tag{9.3}$$

where $\theta_{\rho\sigma} = -\theta_{\sigma\rho}$, and the $s^{\rho\sigma} = -s^{\sigma\rho}$ are six independent 4×4 antisymmetric matrices that satisfy the commutation relations of the Lie algebra $\mathfrak{so}(1,3)$:

$$[s^{\alpha\beta}, s^{\rho\sigma}] = i\left(g^{\beta\rho}s^{\alpha\sigma} - g^{\alpha\rho}s^{\beta\sigma} - g^{\beta\sigma}s^{\alpha\rho} + g^{\alpha\sigma}s^{\beta\rho}\right). \tag{9.4}$$

The matrix elements of $s^{\rho\sigma}$ are given explicitly by:

$$(s^{\rho\sigma})^{\mu}{}_{\nu} = i\left(\delta^{\sigma}_{\nu}g^{\mu\rho} - \delta^{\rho}_{\nu}g^{\mu\sigma}\right). \tag{9.5}$$

Derivation of the commutation relations. We compute $[s^{\alpha\beta}, s^{\rho\sigma}]$ using the explicit form (9.5). The matrix product gives:

$$\begin{aligned}(s^{\alpha\beta}s^{\rho\sigma})^{\mu}{}_{\lambda} &= (s^{\alpha\beta})^{\mu}{}_{\nu}(s^{\rho\sigma})^{\nu}{}_{\lambda}\\ &= i(\delta^{\beta}_{\nu}g^{\mu\alpha} - \delta^{\alpha}_{\nu}g^{\mu\beta})\cdot i(\delta^{\sigma}_{\lambda}g^{\nu\rho} - \delta^{\rho}_{\lambda}g^{\nu\sigma})\\ &= -(\delta^{\beta}_{\nu}g^{\mu\alpha} - \delta^{\alpha}_{\nu}g^{\mu\beta})\\ &\qquad\times(\delta^{\sigma}_{\lambda}g^{\nu\rho} - \delta^{\rho}_{\lambda}g^{\nu\sigma})\end{aligned} \tag{9.6}$$

Expanding and using $\delta^{\beta}_{\nu}g^{\nu\rho} = g^{\beta\rho}$:

$$\begin{aligned}(s^{\alpha\beta}s^{\rho\sigma})^{\mu}{}_{\lambda} = -(&g^{\beta\rho}g^{\mu\alpha}\delta^{\sigma}_{\lambda} - g^{\beta\sigma}g^{\mu\alpha}\delta^{\rho}_{\lambda}\\ &- g^{\alpha\rho}g^{\mu\beta}\delta^{\sigma}_{\lambda} + g^{\alpha\sigma}g^{\mu\beta}\delta^{\rho}_{\lambda})\end{aligned} \tag{9.7}$$

Computing the commutator by antisymmetrizing in $(\alpha\beta) \leftrightarrow (\rho\sigma)$:

$$\begin{aligned}[s^{\alpha\beta}, s^{\rho\sigma}]^{\mu}{}_{\lambda} = &-g^{\beta\rho}(g^{\mu\alpha}\delta^{\sigma}_{\lambda} - g^{\mu\sigma}\delta^{\alpha}_{\lambda}) \\ &+ g^{\alpha\rho}(g^{\mu\beta}\delta^{\sigma}_{\lambda} - g^{\mu\sigma}\delta^{\beta}_{\lambda}) \\ &+ g^{\beta\sigma}(g^{\mu\alpha}\delta^{\rho}_{\lambda} - g^{\mu\rho}\delta^{\alpha}_{\lambda}) \\ &- g^{\alpha\sigma}(g^{\mu\beta}\delta^{\rho}_{\lambda} - g^{\mu\rho}\delta^{\beta}_{\lambda})\end{aligned} \tag{9.8}$$

Recognizing the structure of (9.5) in each term:

$$[s^{\alpha\beta}, s^{\rho\sigma}] = i(g^{\beta\rho}s^{\alpha\sigma} - g^{\alpha\rho}s^{\beta\sigma} - g^{\beta\sigma}s^{\alpha\rho} + g^{\alpha\sigma}s^{\beta\rho})$$

□

In (9.3), we have defined:

$$\begin{aligned}\theta^i &\equiv \tfrac{1}{2}\epsilon^{ijk}\theta^{jk}, \quad \zeta^i \equiv \theta^{i0} = -\theta^{0i}, \\ s^i &\equiv \tfrac{1}{2}\epsilon^{ijk}s^{jk}, \quad k^i \equiv s^{0i} = -s^{i0},\end{aligned} \tag{9.9}$$

where indices $i, j, k \in \{1, 2, 3\}$ and $\epsilon^{123} = +1$. The s^i generate infinitesimal three-dimensional rotations in space, and the k^i generate infinitesimal Lorentz boosts.

An infinitesimal orthochronous Lorentz transformation is given by:

$$\begin{aligned}\Lambda^{\mu}{}_{\nu} &\simeq \delta^{\mu}_{\nu} - \tfrac{i}{2}\theta_{\rho\sigma}(s^{\rho\sigma})^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \theta^{\mu}{}_{\nu} \\ &\simeq \left(\mathbb{1}_4 - i\vec{\theta}\cdot\vec{s} - i\vec{\zeta}\cdot\vec{k}\right)^{\mu}{}_{\nu}.\end{aligned} \tag{9.10}$$

Verification of infinitesimal form. Using (9.5):

$$\begin{aligned}-\frac{i}{2}\theta_{\rho\sigma}(s^{\rho\sigma})^{\mu}{}_{\nu} &= -\frac{i}{2}\theta_{\rho\sigma}\cdot i(\delta^{\sigma}_{\nu}g^{\mu\rho} - \delta^{\rho}_{\nu}g^{\mu\sigma}) \\ &= \frac{1}{2}\theta_{\rho\sigma}(\delta^{\sigma}_{\nu}g^{\mu\rho} - \delta^{\rho}_{\nu}g^{\mu\sigma}) \\ &= \frac{1}{2}(\theta_{\rho\nu}g^{\mu\rho} - \theta_{\sigma\nu}g^{\mu\sigma}) = \frac{1}{2}(\theta^{\mu}{}_{\nu} + \theta^{\mu}{}_{\nu}) = \theta^{\mu}{}_{\nu}\end{aligned} \tag{9.11}$$

where we used the antisymmetry $\theta_{\sigma\nu} = -\theta_{\nu\sigma}$ and $\theta^{\mu}{}_{\nu} = g^{\mu\rho}\theta_{\rho\nu} = -\theta_{\nu}{}^{\mu}$. □

Physical generators: rotations and boosts

For a field of spin s, the general transformation law under a proper orthochronous Lorentz transformation reads:

$$\psi'_{\alpha}(x') = \exp\left(-\tfrac{i}{2}\theta_{\mu\nu}S^{\mu\nu}\right)_{\alpha}{}^{\beta}\psi_{\beta}(x), \tag{9.12}$$

where the $S^{\mu\nu}$ are (finite-dimensional) irreducible matrix representations of the Lorentz algebra satisfying (9.4), and α, β label the components of the representation space.

In analogy with s^i and k^i defined in (9.9), we identify:

$$\boxed{S^i \equiv \frac{1}{2}\epsilon^{ijk}S^{jk}, \qquad K^i \equiv S^{0i}}, \tag{9.13}$$

where $i, j, k \in \{1, 2, 3\}$. The inverse relation reads $S^{jk} = \epsilon^{jki}S^i$. Since $S^{\mu\nu}$ is antisymmetric, it can be arranged as a 4×4 *matrix of matrices* (each entry $S^{\mu\nu}$ being itself a matrix acting on the representation space):

$$\left(S^{\mu\nu}\right) = \begin{pmatrix} 0 & K^1 & K^2 & K^3 \\ -K^1 & 0 & S^3 & -S^2 \\ -K^2 & -S^3 & 0 & S^1 \\ -K^3 & S^2 & -S^1 & 0 \end{pmatrix}, \tag{9.14}$$

where the row/column index runs over $\mu, \nu = 0, 1, 2, 3$. Thus the boosts K^i occupy the time–space entries (mixing x^0 with x^i), while the rotations S^i occupy the purely spatial 3×3 block.

Using (9.4), it follows that S^i and K^i satisfy:

$$\boxed{\begin{aligned} [S^i, S^j] &= i\epsilon^{ijk}S^k \\ [S^i, K^j] &= i\epsilon^{ijk}K^k \\ [K^i, K^j] &= -i\epsilon^{ijk}S^k \end{aligned}}. \tag{9.15}$$

Derivation of rotation-boost commutators. **For $[S^i, S^j]$:** Using $S^i = \frac{1}{2}\epsilon^{imn}S^{mn}$:

$$[S^i, S^j] = \frac{1}{4}\epsilon^{imn}\epsilon^{jpq}[S^{mn}, S^{pq}] \tag{9.16}$$

From (9.4) with only spatial indices (where $g^{mn} = -\delta^{mn}$):

$$[S^{mn}, S^{pq}] = i(-\delta^{np}S^{mq} + \delta^{mp}S^{nq} + \delta^{nq}S^{mp} - \delta^{mq}S^{np})$$

Using the identity $\epsilon^{imn}\epsilon^{jpq}\delta^{np} = \delta^{ij}\delta^{mq} - \delta^{iq}\delta^{mj}$ and careful index manipulation yields $[S^i, S^j] = i\epsilon^{ijk}S^k$.

For $[S^i, K^j]$: Using $S^i = \frac{1}{2}\epsilon^{imn}S^{mn}$ and $K^j = S^{0j}$:

$$[S^i, K^j] = \frac{1}{2}\epsilon^{imn}[S^{mn}, S^{0j}] \tag{9.17}$$

From (9.4):

$$[S^{mn}, S^{0j}] = i(g^{n0}S^{mj} - g^{m0}S^{nj} - g^{nj}S^{m0} + g^{mj}S^{n0})$$

Since $g^{n0} = g^{m0} = 0$ for spatial indices m, n:

$$[S^{mn}, S^{0j}] = i(\delta^{nj}S^{m0} - \delta^{mj}S^{n0}) = i(-\delta^{nj}K^m + \delta^{mj}K^n)$$

Therefore:

$$\begin{aligned}[S^i, K^j] &= \frac{i}{2}\epsilon^{imn}(-\delta^{nj}K^m + \delta^{mj}K^n) \\ &= \frac{i}{2}(-\epsilon^{imj}K^m + \epsilon^{ijn}K^n) = i\epsilon^{ijk}K^k\end{aligned}$$

For $[K^i, K^j]$: Using $K^i = S^{0i}$ and $K^j = S^{0j}$:

$$[K^i, K^j] = [S^{0i}, S^{0j}] = i(g^{i0}S^{0j} - g^{00}S^{ij} - g^{ij}S^{00} + g^{0j}S^{0i}) \tag{9.18}$$

Since $g^{i0} = g^{0j} = 0$, $g^{00} = 1$, and $S^{00} = 0$:

$$[K^i, K^j] = -iS^{ij} = -i\epsilon^{ijk}S^k$$

The crucial minus sign reflects the Lorentzian signature of spacetime. □

Remark. [Physical interpretation] The commutation relations (9.15) encode fundamental physics:

- $[S^i, S^j] = i\epsilon^{ijk}S^k$: This is the familiar $\mathfrak{su}(2)$ algebra of angular momentum.
- $[S^i, K^j] = i\epsilon^{ijk}K^k$: The boosts K^i transform as a 3-vector under rotations.
- $[K^i, K^j] = -i\epsilon^{ijk}S^k$: Two boosts in different directions do not commute—their commutator yields a rotation. This is the origin of *Thomas precession*.

The $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ structure (on the complexification)

Box 9.1 A subtlety: real vs. complex Lie algebra

The decomposition we are about to derive is a statement about the *complexified* Lorentz algebra, not about the real one. As a *real* Lie algebra, the Lorentz algebra $\mathfrak{so}(1,3)$ is *not* isomorphic to $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ (the latter being the compact algebra of $SO(4)$, with different signature). The correct real isomorphism is instead

$$\mathfrak{so}(1,3)_{\mathbb{R}} \simeq \mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}},$$

viewed as a six-dimensional real algebra. Only after **complexification**—i.e. allowing complex linear combinations of the generators—does one obtain

$$\mathfrak{so}(1,3)_{\mathbb{C}} \;\simeq\; \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) \;\simeq\; \mathfrak{su}(2)_{\mathbb{C}} \oplus \mathfrak{su}(2)_{\mathbb{C}}\,.$$

The reason is visible in the very definition (9.19) below: the combinations $\vec{S}_{\pm} = \frac{1}{2}(\vec{S} \pm i\vec{K})$ require the imaginary unit i as a coefficient, so they do *not* belong to the real span of $\{S^i, K^i\}$. Equivalently, the boost generators K^i are anti-Hermitian (non-compact directions), whereas in $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ all generators would be Hermitian. Working with $\vec{S}_{\pm}$ is nevertheless extremely convenient, because finite-dimensional irreducible representations of a real Lie algebra are in one-to-one correspondence with those of its complexification.

It is convenient to define the following *complex* linear combinations of the generators:

$$\boxed{\vec{S}_+ \equiv \frac{1}{2}(\vec{S} + i\vec{K}), \qquad \vec{S}_- \equiv \frac{1}{2}(\vec{S} - i\vec{K})}\,, \tag{9.19}$$

with inverse relations $\vec{S} = \vec{S}_+ + \vec{S}_-$ and $\vec{K} = -i(\vec{S}_+ - \vec{S}_-)$. Note that $\vec{S}_{\pm}$ live in $\mathfrak{so}(1,3)_{\mathbb{C}}$, not in the real algebra.

Then, equations (9.15) decouple and yield two independent $\mathfrak{su}(2)$ Lie algebras (in the complexified sense discussed above):

$$\boxed{\begin{aligned} [S^i_+, S^j_+] &= i\epsilon^{ijk} S^k_+ \\ [S^i_-, S^j_-] &= i\epsilon^{ijk} S^k_- \\ [S^i_+, S^j_-] &= 0 \end{aligned}}\,. \tag{9.20}$$

Derivation of decoupled algebra. We compute:

$$\begin{aligned} 4[S^i_+, S^j_+] &= [S^i + iK^i, S^j + iK^j] \\ &= [S^i, S^j] + i[S^i, K^j] + i[K^i, S^j] - [K^i, K^j] \\ &= i\epsilon^{ijk} S^k + i \cdot i\epsilon^{ijk} K^k + i \cdot i\epsilon^{ijk} K^k - (-i\epsilon^{ijk} S^k) \\ &= 2i\epsilon^{ijk} S^k - 2\epsilon^{ijk} K^k = 2i\epsilon^{ijk}(S^k + iK^k) = 4i\epsilon^{ijk} S^k_+ \end{aligned} \tag{9.21}$$

Similarly, $4[S^i_-, S^j_-] = [S^i - iK^i, S^j - iK^j] = 4i\epsilon^{ijk} S^k_-$.

For the mixed commutator:

$$\begin{aligned} 4[S^i_+, S^j_-] &= [S^i + iK^i, S^j - iK^j] \\ &= [S^i, S^j] - i[S^i, K^j] + i[K^i, S^j] + [K^i, K^j] \\ &= i\epsilon^{ijk} S^k + \epsilon^{ijk} K^k - \epsilon^{ijk} K^k - i\epsilon^{ijk} S^k = 0 \end{aligned} \tag{9.22}$$

□

The finite-dimensional irreducible representations of $\mathfrak{su}(2)$ are well known: these are the $(2s+1)\times(2s+1)$ representation matrices corresponding to spin s, where $s = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. Hence, the irreducible representations of the Lorentz group can be characterized by two numbers (s_+, s_-), where $s_\pm$ is nonnegative and either an integer or a half-integer. The eigenvalues of $\vec{S}_\pm^2$ are given by $s_\pm(s_\pm+1)$. The dimension of the representation (s_+, s_-) is $(2s_+ + 1)(2s_- + 1)$.

Remark. [Common representations]

(s_+, s_-)	Dimension	Name	Examples
$(0,0)$	1	Scalar	Higgs boson
$(\frac{1}{2},0)$	2	Left-handed Weyl spinor	ν_L, e_L
$(0,\frac{1}{2})$	2	Right-handed Weyl spinor	e_R
$(\frac{1}{2},\frac{1}{2})$	4	Four-vector	Photon A^μ
$(1,0)\oplus(0,1)$	6	Antisymmetric tensor	$F^{\mu\nu}$

9.2 The Poincaré group and its Lie algebra

Definition of the Poincaré group

The **Poincaré group** is a semidirect product of the group of spacetime translations and the Lorentz group. Under a Poincaré transformation, the spacetime coordinates transform as:

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu\,, \tag{9.23}$$

where Λ satisfies (9.2) and a^μ is a constant four-vector representing the translation.

Under a Poincaré transformation, a field of spin s transforms according to (9.12) after identifying x' with the Poincaré transformed spacetime coordinate. It is convenient to rewrite this transformation law as:

$$\psi'_\alpha(x) = \exp\left(-\tfrac{i}{2}\theta_{\mu\nu}S^{\mu\nu}\right)_\alpha{}^\beta \psi_\beta\left(\Lambda^{-1}(x-a)\right), \tag{9.24}$$

where we have used $x = \Lambda^{-1}(x'-a)$ and redefined the dummy variable x' by removing the prime.

The Poincaré algebra

Under an infinitesimal Poincaré transformation, we expand (9.24) about $\Lambda = \mathbb{1}_4$ and $a = 0$ to obtain:

$$\psi'_\alpha(x) \simeq \left[\mathbb{1} + ia_\mu P^\mu - \tfrac{i}{2}\theta_{\mu\nu}(L^{\mu\nu} + S^{\mu\nu})\right]_\alpha{}^\beta \psi_\beta(x)\,, \tag{9.25}$$

where P^μ and $L^{\mu\nu}$ are the differential operators:

$$\boxed{P^\mu \equiv i\partial^\mu, \qquad L^{\mu\nu} \equiv i(x^\mu\partial^\nu - x^\nu\partial^\mu)}\,, \tag{9.26}$$

and the **total angular momentum tensor** is $J^{\mu\nu} \equiv L^{\mu\nu} + S^{\mu\nu}$.

The four generators of spacetime translations (P^μ) and the six generators of Lorentz transformations ($J^{\mu\nu}$, $\mu < \nu$) satisfy the following commutation relations of the **Poincaré algebra**:

$$\boxed{\begin{aligned} [P^\mu, P^\nu] &= 0 \\ [J^{\mu\nu}, P^\rho] &= i(g^{\nu\rho}P^\mu - g^{\mu\rho}P^\nu) \\ [J^{\mu\nu}, J^{\rho\sigma}] &= i(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} \\ &\qquad - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}) \end{aligned}}\,. \tag{9.27}$$

Remark. [Physical interpretation of $[J^{\mu\nu}, P^\rho]$] Before turning to the algebraic derivation, we motivate the structure of the most informative commutator, $[J^{\mu\nu}, P^\rho] = i(g^{\nu\rho}P^\mu - g^{\mu\rho}P^\nu)$. It is not a convention: it is forced by the requirement that P^μ transform as a four-vector under Lorentz transformations. The argument compares two ways of writing the infinitesimal action of a Lorentz transformation on P^ρ.

Classical side. A four-vector transforms as $P^\rho \to \Lambda^\rho{}_\nu P^\nu$. For an infinitesimal transformation $\Lambda^\rho{}_\nu = \delta^\rho_\nu + \theta^\rho{}_\nu + O(\theta^2)$ with $\theta_{\mu\nu} = -\theta_{\nu\mu}$,

$$\Lambda^\rho{}_\nu P^\nu = P^\rho + \theta^\rho{}_\nu P^\nu + O(\theta^2)\,. \qquad (\star) \tag{9.28}$$

For an active boost, the rapidity vector used later is related to the antisymmetric parameter by $\zeta^i = \theta_{0i} = \theta^0{}_i$. Hence

$$P'^0 = P^0 + \vec{\zeta}\cdot\vec{P} + O(\zeta^2), \qquad P'^i = P^i + \zeta^i P^0 + O(\zeta^2),$$

which is the infinitesimal form of an active boost of the physical four-momentum.

Quantum side. In quantum theory the same transformation is realised by the unitary operator $U(\Lambda) = \exp\!\left(-\tfrac{i}{2}\theta_{\mu\nu}J^{\mu\nu}\right)$, and operators transform by conjugation:

$$U^\dagger(\Lambda)\, P^\rho\, U(\Lambda) = P^\rho + \tfrac{i}{2}\theta_{\mu\nu}\,[J^{\mu\nu}, P^\rho] + O(\theta^2)\,. \qquad (\star\star) \tag{9.29}$$

The commutator $[J^{\mu\nu}, P^\rho]$ *is* therefore the generator of Lorentz transformations acting on P^ρ.

Matching. Setting $(\star) = (\star\star)$ and rewriting the right-hand side of $(\star)$ in manifestly antisymmetric form (using $\theta_{\mu\nu} = -\theta_{\nu\mu}$),

$$\theta_{\mu\nu}\, g^{\rho\mu} P^\nu = \tfrac{1}{2}\theta_{\mu\nu}\big(g^{\rho\mu}P^\nu - g^{\rho\nu}P^\mu\big)\,,$$

and matching coefficients of the arbitrary antisymmetric $\theta_{\mu\nu}$ fixes the commutator uniquely. The structure is dictated by Lorentz covariance, not by choice; the algebraic proof below verifies the same result by direct computation with the differential operator representation $L^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu)$.

Component examples.

- $[J^{ij}, P^k] = i(\delta^{ik}P^j - \delta^{jk}P^i)$: rotations mix the spatial components of $\vec{P}$ exactly as a 3-vector. Equivalently, $[J^i, P^k] = i\epsilon^{ik\ell}P^\ell$ for $J^i = \frac{1}{2}\epsilon^{ijk}J^{jk}$.
- $[J^{0i}, P^0] = -iP^i$ and $[J^{0i}, P^j] = -i\delta^{ij}P^0$: boosts mix energy with momentum — with the active boost convention used below, this corresponds infinitesimally to $E' = E + \vec{\zeta}\cdot\vec{P} + O(\zeta^2)$, or finitely to $E' = \gamma(E + \vec{v}\cdot\vec{p})$.

Derivation of the Poincaré algebra. **For $[P^\mu, P^\nu] = 0$:** Since $P^\mu = i\partial^\mu$, and partial derivatives commute:

$$[P^\mu, P^\nu] = [i\partial^\mu, i\partial^\nu] = -[\partial^\mu, \partial^\nu] = 0$$

For $[J^{\mu\nu}, P^\rho]$: We use the orbital part $L^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu)$ and $P^\rho = i\partial^\rho$. Since $[S^{\mu\nu}, P^\rho] = 0$ (the spin generators act on internal indices, not coordinates), we have $[J^{\mu\nu}, P^\rho] = [L^{\mu\nu}, P^\rho]$.

Acting on a test function $f(x)$:

$$\begin{aligned}[L^{\mu\nu}, P^\rho]f &= L^{\mu\nu}(P^\rho f) - P^\rho(L^{\mu\nu}f)\\ &= i(x^\mu\partial^\nu - x^\nu\partial^\mu)(i\partial^\rho f)\\ &\quad - i\partial^\rho\big[i(x^\mu\partial^\nu - x^\nu\partial^\mu)f\big]\end{aligned} \tag{9.30}$$

Using $\partial^\rho(x^\mu\partial^\nu f) = g^{\rho\mu}\partial^\nu f + x^\mu\partial^\rho\partial^\nu f$ and the commutativity of partial derivatives:

$$\begin{aligned}[L^{\mu\nu}, P^\rho]f &= -(x^\mu\partial^\nu\partial^\rho f - x^\nu\partial^\mu\partial^\rho f)\\ &\quad + (g^{\rho\mu}\partial^\nu f + x^\mu\partial^\rho\partial^\nu f - g^{\rho\nu}\partial^\mu f - x^\nu\partial^\rho\partial^\mu f)\\ &= g^{\mu\rho}\partial^\nu f - g^{\nu\rho}\partial^\mu f = \tfrac{1}{i}(g^{\mu\rho}P^\nu - g^{\nu\rho}P^\mu)f\end{aligned} \tag{9.31}$$

Therefore $[J^{\mu\nu}, P^\rho] = i(g^{\nu\rho}P^\mu - g^{\mu\rho}P^\nu)$.

For $[J^{\mu\nu}, J^{\rho\sigma}]$: This is the Lorentz algebra (9.4), which holds for any representation including $J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu}$. □

The Pauli-Lubanski vector and Casimir operators

It is convenient to introduce the **Pauli-Lubanski pseudovector**:

$$\boxed{w^\mu \equiv -\tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma} J_{\nu\rho} P_\sigma = \left(\vec{J}\cdot\vec{P};\, P^0\vec{J} + \vec{K}\times\vec{P}\right)}, \tag{9.32}$$

where $\epsilon^{0123} = +1$, $J^i \equiv \frac{1}{2}\epsilon^{ijk}J^{jk}$, and $K^i \equiv J^{0i}$.

Proposition 9.2 (Properties of the Pauli-Lubanski vector).

(i) $w_\mu P^\mu = 0$
(ii) $[w^\mu, P^\nu] = 0$
(iii) $[w^\mu, J^{\rho\sigma}] = i(g^{\mu\rho}w^\sigma - g^{\mu\sigma}w^\rho)$

Proof. (i) $w_\mu P^\mu = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}J_{\nu\rho}P_\sigma P_\mu = 0$ since $\epsilon^{\mu\nu\rho\sigma}$ is antisymmetric in $\mu \leftrightarrow \sigma$ while $P_\sigma P_\mu$ is symmetric.

(ii) Using $[J_{\nu\rho}, P^\alpha] = i(g_\rho{}^\alpha P_\nu - g_\nu{}^\alpha P_\rho)$ and $[P_\sigma, P^\alpha] = 0$:

$$\begin{aligned}[w^\mu, P^\alpha] &= -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\left([J_{\nu\rho}, P^\alpha]P_\sigma + J_{\nu\rho}[P_\sigma, P^\alpha]\right) \\ &= -\frac{i}{2}\epsilon^{\mu\nu\rho\sigma}(g_\rho{}^\alpha P_\nu - g_\nu{}^\alpha P_\rho)P_\sigma = 0\end{aligned} \tag{9.33}$$

The last step follows because $\epsilon^{\mu\nu\rho\sigma}P_\nu P_\sigma = 0$ (antisymmetric times symmetric).

(iii) This follows from the fact that w^μ transforms as a four-vector under Lorentz transformations. □

The Poincaré algebra possesses two independent **Casimir operators**:

$$\boxed{C_1 = P^2 \equiv P_\mu P^\mu, \qquad C_2 = w^2 \equiv w_\mu w^\mu}, \tag{9.34}$$

which commute with all generators P^μ and $J^{\mu\nu}$.

Proof that P^2 and w^2 are Casimir operators.
$[P^2, P^\mu] = 0$ trivially since $[P_\nu, P^\mu] = 0$. For Lorentz generators:

$$\begin{aligned}[P^2, J^{\mu\nu}] &= P_\rho[P^\rho, J^{\mu\nu}] + [P_\rho, J^{\mu\nu}]P^\rho \\ &= -iP_\rho(g^{\rho\nu}P^\mu - g^{\rho\mu}P^\nu) + \text{h.c.} = 0\end{aligned}$$

Similarly, $[w^2, P^\mu] = 0$ follows from $[w^\nu, P^\mu] = 0$, and $[w^2, J^{\mu\nu}] = 0$ follows from the vector transformation property of w^μ. □

Physical interpretation: mass and spin

The representations of the Poincaré group corresponding to particle states of nonnegative energy P^0 with definite mass and spin can be labeled by the eigenvalues of the Casimir operators P^2 and w^2.

The eigenvalue of P^2 is m^2, where m is the **mass**. To see the physical interpretation of w^2, we consider two cases:

Massive particles ($m \neq 0$)**:** We evaluate w^2 in the particle rest frame (since w^2 is a Lorentz scalar). In this frame, $P^\mu = (m,0,0,0)$ and:

$$w^\mu = (0; m\vec{S})\,,$$

where S^i is defined in (9.13). Hence:

$$w^2 = -m^2\vec{S}^2 \quad \text{with eigenvalues} \quad -m^2 s(s+1)\,, \tag{9.35}$$

where $s = 0, \frac{1}{2}, 1, \ldots$ is the **spin**. Massive positive-energy states are labeled by (m,s).

Massless particles ($m = 0$)**:** We cannot use the rest frame. For the ordinary finite-helicity massless representations one has $P^2 = 0$, $w_\mu P^\mu = 0$, and $w^2 = 0$. In a frame where $P = (P^0;0,0,P^0)$ with $P^0 > 0$, these conditions imply $w = (w^0;0,0,w^0)$, excluding the continuous-spin representations. That is, in any Lorentz frame:

$$w^\mu = hP^\mu\,, \tag{9.36}$$

where h is called the **helicity operator**. From (9.32):

$$h = \frac{w^0}{P^0} = \frac{\vec{J}\cdot\vec{P}}{P^0} = \frac{\vec{S}\cdot\vec{P}}{P^0}\,. \tag{9.37}$$

The last equality follows because $\vec{L}\cdot\vec{P} = (\vec{x}\times\vec{P})\cdot\vec{P} = 0$.

Eigenvalues of h are called the helicity λ. For massless states, $\vec{S}\cdot\hat{P}$ (where $\hat{P} = \vec{P}/|\vec{P}|$) corresponds to the projection of spin along the direction of motion, with spectrum $\lambda = 0, \pm\frac{1}{2}, \pm 1, \ldots$. Under a CPT transformation, $\lambda \to -\lambda$, so both $\pm|\lambda|$ helicity states must appear in any CPT-invariant theory.

9.3 Spin-1/2 representations of the Lorentz group

The previous section used the Poincaré Casimirs to label physical one-particle states. We now change perspective: a relativistic field must also carry a finite-dimensional representation of the Lorentz group on its spinor or tensor indices. These finite-dimensional (non-unitary) representations acting on the field indices are distinct from the Hilbert-space unitary representations used to classify particles; they tell the field components how to mix when spacetime is Lorentz transformed. For spin-$\frac{1}{2}$ fields the relevant finite-dimensional representations are the two inequivalent Weyl representations.

The $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ representations

The simplest nontrivial irreducible representations of the Lorentz algebra are the two-dimensional representations $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$. The corresponding generators are:

$$\boxed{\begin{aligned}(\tfrac{1}{2},0):\quad & \vec{S}_+ = \tfrac{1}{2}(\vec{S}+i\vec{K}) = \tfrac{1}{2}\vec{\sigma}, \quad \vec{S}_- = 0 \\ & \Longrightarrow \quad \vec{S} = \tfrac{1}{2}\vec{\sigma}, \quad \vec{K} = -\tfrac{i}{2}\vec{\sigma}\end{aligned}} \,. \tag{9.38}$$

$$\boxed{\begin{aligned}(0,\tfrac{1}{2}):\quad & \vec{S}_+ = 0, \quad \vec{S}_- = \tfrac{1}{2}(\vec{S}-i\vec{K}) = \tfrac{1}{2}\vec{\sigma} \\ & \Longrightarrow \quad \vec{S} = \tfrac{1}{2}\vec{\sigma}, \quad \vec{K} = +\tfrac{i}{2}\vec{\sigma}\end{aligned}} \,. \tag{9.39}$$

Here, $\vec{\sigma} = (\sigma^1,\sigma^2,\sigma^3)$ are the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{9.40}$$

We also define $\sigma^0 = \mathbb{1}_2$ and the four-component notation:

$$\sigma^\mu = (\mathbb{1}_2, \vec{\sigma}), \qquad \bar{\sigma}^\mu = (\mathbb{1}_2, -\vec{\sigma})\,. \tag{9.41}$$

Lorentz transformation matrices

There are two kinds of two-component spinors, distinguished by their transformation character under boosts. Under an infinitesimal three-dimensional rotation parameterized by $\vec{\epsilon} = (\epsilon_1,\epsilon_2,\epsilon_3)$, a four-vector transforms (in the active convention of (9.3)) as $E \to E$, $\vec{p} \to \vec{p} + \vec{\epsilon}\times\vec{p}$, while under an infinitesimal boost parameterized by $\vec{\eta} = (\eta_1,\eta_2,\eta_3)$, it transforms as $E \to E + \vec{\eta}\cdot\vec{p}$, $\vec{p} \to \vec{p} + \vec{\eta}\,E$.

Infinitesimal transformations of two-component spinors

Under such Lorentz transformations, the two types of Weyl spinors transform as (**active convention**, consistent with (9.3)):

$$\boxed{\begin{aligned}(\tfrac{1}{2},0):\quad & \chi \to \chi' = \big(1 - \tfrac{i}{2}\vec{\epsilon}\cdot\vec{\sigma} - \tfrac{1}{2}\vec{\eta}\cdot\vec{\sigma}\big)\chi \\ (0,\tfrac{1}{2}):\quad & \psi \to \psi' = \big(1 - \tfrac{i}{2}\vec{\epsilon}\cdot\vec{\sigma} + \tfrac{1}{2}\vec{\eta}\cdot\vec{\sigma}\big)\psi\end{aligned}} \,. \tag{9.42}$$

The $(\frac{1}{2},0)$ spinor χ (left-handed, L-type) and the $(0,\frac{1}{2})$ spinor ψ (right-handed, R-type) transform *identically* under rotations (they both have spin-$\frac{1}{2}$), but *differently* under boosts—the sign of the $\vec{\eta}\cdot\vec{\sigma}$ term distinguishes them.

Box 9.2 Active vs. passive convention: a frequent source of sign confusion

We adopt the *active* convention throughout: a rotation parameter $\vec{\theta} = \theta\,\hat{z}$ rotates the *physical system* by $+\theta$ about z, and a boost parameter $\vec{\zeta} = \zeta\,\hat{z}$ ($\zeta > 0$) boosts the system in the $+z$ direction, giving the particle positive z-momentum. A four-vector therefore transforms as

$$\vec{p} \;\to\; \vec{p} + \vec{\epsilon}\times\vec{p} \quad \text{(rotation)}, \qquad E \;\to\; E + \vec{\eta}\cdot\vec{p} \quad \text{(boost)},$$

consistently with the statement at (9.42) above.
The spinor matrix M is obtained by inserting the $(\frac{1}{2},0)$ representation matrices $\vec{S} = \vec{\sigma}/2$, $\vec{K} = -i\vec{\sigma}/2$ from (9.38) into the representation exponential $D = \exp\!\left(-i\vec{\theta}\cdot\vec{S} - i\vec{\zeta}\cdot\vec{K}\right)$:

$$M = \exp\!\left(-\tfrac{i}{2}\vec{\theta}\cdot\vec{\sigma} - \tfrac{1}{2}\vec{\zeta}\cdot\vec{\sigma}\right).$$

The minus signs on *both* terms are a direct consequence of this substitution. The physical content — that this M produces the *active* boost — is verified by the sigma-matrix identity $P' = MPM^{\dagger}$ (established below in (9.57)), which yields $E' = E + \vec{\eta}\cdot\vec{p}$ and $\vec{p}\,' = \vec{p} + \vec{\eta}\,E$.
Sign subtlety. With the definitions in (9.9), the four-vector exponential (9.3) and the spinor construction give the same active boost. For a boost along $+z$,

$$\Lambda^{0}{}_{3} = \Lambda^{3}{}_{0} = +\sinh\zeta,$$

so a rest momentum $(m,0,0,0)$ is sent to $(m\cosh\zeta, 0, 0, m\sinh\zeta)$. Many textbooks write instead the passive coordinate transformation, or equivalently the inverse active transformation, with off-diagonal entries $-\sinh\zeta$. This is a convention change $\vec{\zeta} \to -\vec{\zeta}$, not a different covering map.
Passive convention. Several classical references (Aitchison–Hey vol. 1, parts of Weinberg) adopt the opposite overall sign: Λ is viewed as the transformation rotating the *frame*, so $\vec{p} \to \vec{p} - \vec{\epsilon}\times\vec{p}$ and the spinor law reads $\chi' = (1 + \frac{i}{2}\vec{\epsilon}\cdot\vec{\sigma} + \frac{1}{2}\vec{\eta}\cdot\vec{\sigma})\chi$. The two conventions are related by $\vec{\epsilon} \leftrightarrow -\vec{\epsilon}$, $\vec{\eta} \leftrightarrow -\vec{\eta}$ and produce identical physical predictions.

Finite transformations with explicit spinor indices: M and $(M^{-1})^{\dagger}$

Before writing the finite versions of (9.42), we install the index conventions that will be used throughout the rest of the chapter — conventions that immediately become necessary as soon as we have to track which representation each Weyl field belongs to.

Spinor indices: bar and dotted notation. When we attach explicit spinor indices to a Weyl field, we follow two parallel conventions:

- Left-handed $(\frac{1}{2},0)$ spinors carry *undotted* indices $\alpha,\beta,\ldots$ and are written without decoration: $\chi_\alpha, \eta_\alpha, \psi_\alpha, \ldots$
- Right-handed $(0,\frac{1}{2})$ spinors carry *dotted* indices $\dot\alpha,\dot\beta,\ldots$ and are written with an overline: $\bar\chi^{\dot\alpha}, \bar\eta^{\dot\alpha}, \bar\psi^{\dot\alpha}, \ldots$.

The bar is not decorative: it records the fact that complex conjugation maps the $(\frac{1}{2},0)$ representation into the inequivalent $(0,\frac{1}{2})$ representation. Operationally, for any L-type spinor χ_α we *define* its R-type partner by

$$\boxed{\bar\chi_{\dot\alpha} \equiv (\chi_\alpha)^*, \qquad \bar\chi^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\bar\chi_{\dot\beta}}, \tag{9.43}$$

where $\epsilon^{\dot\alpha\dot\beta}$ is the antisymmetric spinor metric used to raise dotted indices, with $\epsilon^{\dot1\dot2} = +1$; its undotted analogue $\epsilon^{\alpha\beta}$ raises undotted indices in the same way (the full conventions, including how the metric is used to form Lorentz-invariant bilinears, are collected in (9.97) below). Whenever an independent R-type field is introduced without an explicit L-type "parent" (such as the ψ of (9.42) above, written below with explicit dotted index as $\bar\psi^{\dot\alpha}$), it is understood to be of this form for some implicit underlying L-type field.

Cross-references with older notations. Older texts — notably Aitchison's Lorentz chapter, early Weinberg, and many original QFT references — denote $\bar\chi^{\dot\alpha}$ by $\chi^{\dagger\dot\alpha}$ ("dagger notation"), with the dagger emphasising the Hermitian-conjugate origin of the dotted-index partner. The two notations are interchangeable; we have chosen the bar form throughout this chapter and Chapter 10 to align with modern SUSY conventions (Wess–Bagger, Srednicki, Aitchison's SUSY chapter).

A forward warning. The same overline symbol will reappear later in the four-component Dirac formalism as the Dirac adjoint $\bar\Psi \equiv \Psi^\dagger\gamma^0$, which is a *distinct* operation acting on a four-component spinor (see (9.141) and the dedicated remark accompanying it). The two usages are disambiguated by the type of the parent field — lowercase Greek two-component spinors versus uppercase Dirac four-spinors — but it is worth flagging the collision once, at the moment the bar notation is introduced, rather than discovering it 800 lines later.

The Weyl transformation matrices M and $(M^{-1})^\dagger$. For the $(\frac{1}{2},0)$ representation, we adopt the active convention (cf. box 9.2) for the L-type Weyl spinor. The finite transformation matrix is:

$$\boxed{M = \exp\!\left(-\tfrac{i}{2}\vec\theta\cdot\vec\sigma - \tfrac{1}{2}\vec\zeta\cdot\vec\sigma\right) \simeq \mathbb{1}_2 - \tfrac{i}{2}\vec\theta\cdot\vec\sigma - \tfrac{1}{2}\vec\zeta\cdot\vec\sigma}, \tag{9.44}$$

where $\vec{\theta}$ is the rotation angle vector and $\vec{\zeta} = \hat{\boldsymbol{\beta}} \tanh^{-1}\beta$ is the rapidity vector; at first order these are the $\vec{\epsilon}$ and $\vec{\eta}$ of (9.42).

Remark. [The rapidity vector and the non-relativistic limit] The choice $\vec{\zeta} = \hat{\boldsymbol{\beta}} \tanh^{-1}\beta$ is not arbitrary: it is the unique boost parameter that is *additive* for collinear boosts, in the same sense in which the rotation angle $\vec{\theta} = \theta\hat{n}$ is additive for rotations about a fixed axis. The velocity itself fails this property: the relativistic composition law gives $\vec{v}_1 \oplus \vec{v}_2 \neq \vec{v}_1 + \vec{v}_2$, whereas collinear rapidities simply satisfy $\zeta_1 + \zeta_2$.

Setting $\beta = \tanh\zeta$ gives $\gamma = \cosh\zeta$ and $\gamma\beta = \sinh\zeta$, so an active boost along x with rapidity ζ takes the form

$$\Lambda(M) = \begin{pmatrix} \cosh\zeta & +\sinh\zeta & 0 & 0 \\ +\sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

formally a "hyperbolic rotation" (the standard textbook form has $-\sinh\zeta$ on the off-diagonal, corresponding to the passive convention; see box 9.2 for the relation between the two). Promoting $\zeta \to \vec{\zeta} = \hat{\boldsymbol{\beta}} \tanh^{-1}\beta$ extends this to an arbitrary boost direction; the corresponding spinor matrix is $M = \exp\left(-\frac{1}{2}\vec{\zeta}\cdot\vec{\sigma}\right)$ as in (9.44), and the four-vector transformation $\Lambda(M)$ is determined by the sigma identity (9.57).

Non-relativistic limit. For $|\vec{v}| \ll c$, the Taylor expansion $\tanh^{-1}\beta = \beta + \frac{\beta^3}{3} + O(\beta^5)$ gives

$$\vec{\zeta} \simeq \hat{\boldsymbol{\beta}}\beta = \frac{\vec{v}}{c} \equiv \vec{\eta},$$

recovering the infinitesimal parameter of (9.42). Analogously $\vec{\theta} \simeq \vec{\epsilon}$ at first order. Summarising:

finite parameter	infinitesimal	type
$\vec{\theta} = \theta\hat{n}$	$\vec{\epsilon}$	rotation
$\vec{\zeta} = \hat{\boldsymbol{\beta}} \tanh^{-1}\beta$	$\vec{\eta} = \vec{v}/c$	boost

The finite parameters $\vec{\theta}, \vec{\zeta}$ encode the full Lie-group exponential; the infinitesimal $\vec{\epsilon}, \vec{\eta}$ are their linearisations and label tangent vectors at the identity. The non-linearity of $\tanh^{-1}\beta$ encodes all the relativistic content of boost composition.

A two-component $(\frac{1}{2}, 0)$ spinor field $\chi_\alpha(x)$ (left-handed Weyl spinor) transforms as:

$$\chi'_\alpha(x') = M_\alpha{}^\beta \chi_\beta(x), \qquad \alpha, \beta \in \{1, 2\}\,. \tag{9.45}$$

By definition, M carries undotted spinor indices: $M_\alpha{}^\beta$.

For the $(0, \frac{1}{2})$ representation, the transformation matrix is:

$$\boxed{(M^{-1})^\dagger = \exp\!\left(-\tfrac{i}{2}\vec{\theta}\cdot\vec{\sigma} + \tfrac{1}{2}\vec{\zeta}\cdot\vec{\sigma}\right) \simeq \mathbb{1}_2 - \tfrac{i}{2}\vec{\theta}\cdot\vec{\sigma} + \tfrac{1}{2}\vec{\zeta}\cdot\vec{\sigma}}\,. \tag{9.46}$$

A two-component $(0, \frac{1}{2})$ spinor field $\bar{\psi}^{\dot{\alpha}}(x)$ (right-handed Weyl spinor, see (9.43)) transforms as:

$$\bar{\psi}'^{\dot{\alpha}}(x') = \left[(M^{-1})^\dagger\right]^{\dot{\alpha}}{}_{\dot{\beta}} \bar{\psi}^{\dot{\beta}}(x)\,. \tag{9.47}$$

The matrix $(M^{-1})^\dagger$ carries dotted spinor indices.

Verification of the relation between M and $(M^{-1})^\dagger$. From (9.44), the inverse (at first order) is obtained by reversing the transformation parameters:

$$M^{-1} \simeq \mathbb{1}_2 + \tfrac{i}{2}\vec{\theta}\cdot\vec{\sigma} + \tfrac{1}{2}\vec{\zeta}\cdot\vec{\sigma}$$

Using the hermiticity of the Pauli matrices ($\vec{\sigma}^\dagger = \vec{\sigma}$):

$$M^\dagger \simeq \mathbb{1}_2 + \tfrac{i}{2}\vec{\theta}\cdot\vec{\sigma} - \tfrac{1}{2}\vec{\zeta}\cdot\vec{\sigma}$$

Hence $(M^{-1})^\dagger = (M^\dagger)^{-1} \simeq \mathbb{1}_2 - \frac{i}{2}\vec{\theta}\cdot\vec{\sigma} + \frac{1}{2}\vec{\zeta}\cdot\vec{\sigma}$, confirming (9.46). □

Remark. [Index conventions] We adopt the convention that:

- Undotted indices $\alpha, \beta, \ldots$ label $(\frac{1}{2}, 0)$ spinors and are contracted as ${}^\alpha{}_\alpha$ (northwest to southeast)
- Dotted indices $\dot{\alpha}, \dot{\beta}, \ldots$ label $(0, \frac{1}{2})$ spinors and are contracted as ${}_{\dot{\alpha}}{}^{\dot{\alpha}}$ (southwest to northeast)

The homomorphism $SL(2,\mathbb{C}) \to SO^+(1,3)$

We now justify why the two Weyl matrices M and $(M^{-1})^\dagger$ of (9.44)–(9.46) are the *correct* objects to call "Lorentz transformations of spinors": they are the two

inequivalent two-dimensional irreducible representations of the *universal covering group* of the restricted Lorentz group, $SL(2,\mathbb{C})$.

A direct calculation from (9.44) gives $\det M = 1$ (the exponential of a traceless matrix has unit determinant), so the Weyl matrices live in the special linear group

$$SL(2,\mathbb{C}) \;\equiv\; \{\, M \in \mathrm{Mat}(2,\mathbb{C}) \;:\; \det M = 1 \,\}.$$

This is a 6-real-dimensional Lie group, exactly the same dimension as the restricted Lorentz group $SO^{+}(1,3)$. The connection between the two is more than a counting coincidence: there is an explicit two-to-one group homomorphism $SL(2,\mathbb{C}) \to SO^{+}(1,3)$, with $SL(2,\mathbb{C})$ playing the role of the *universal covering group*. The construction (following Müller-Kirsten [31], Section 1.3.3) proceeds in three steps.

Step 1 — The four-vector / Hermitian matrix correspondence. Define the map

$$\rho : \mathbb{M}_4 \longrightarrow \mathbb{H}(2,\mathbb{C}), \qquad x^\mu \longmapsto X \equiv x^\mu \sigma_\mu = \begin{pmatrix} x^0 - x^3 & -x^1 + i x^2 \\ -x^1 - i x^2 & x^0 + x^3 \end{pmatrix}, \tag{9.48}$$

sending each Minkowski four-vector to a 2×2 Hermitian matrix (the Hermiticity is immediate from $\sigma_\mu^\dagger = \sigma_\mu$ and the reality of x^μ). The map ρ is a real-linear bijection: any $X \in \mathbb{H}(2,\mathbb{C})$ has the form (9.48) for a unique real four-vector

$$x^\mu \;=\; \tfrac{1}{2}\,\mathrm{Tr}\big[X\,\bar\sigma^\mu\big], \tag{9.49}$$

as one verifies using the trace identity $\mathrm{Tr}[\sigma_\mu \bar\sigma^\nu] = 2\,\delta_\mu^\nu$. A direct computation yields the key identity

$$\det X \;=\; (x^0)^2 - |\vec{x}|^2 \;=\; x_\mu x^\mu, \tag{9.50}$$

i.e. *the determinant of the Hermitian matrix equals the Minkowski square of the four-vector*. This is the algebraic engine of the whole construction.

Step 2 — The adjoint action of $SL(2,\mathbb{C})$ on $\mathbb{H}(2,\mathbb{C})$. For any $M \in SL(2,\mathbb{C})$ and $X \in \mathbb{H}(2,\mathbb{C})$ define

$$\mathrm{ad}(M)\,X \;\equiv\; M\,X\,M^\dagger. \tag{9.51}$$

Box 9.3 Why this is an "adjoint" action

The transformation $X \mapsto MXM^\dagger$ is called the *adjoint action* of $SL(2,\mathbb{C})$ on Hermitian matrices. The name carries two layers of physical meaning, both worth unpacking.

The quantum-mechanical analogy. Every Hermitian observable $\hat{O}$ on a quantum Hilbert space transforms under a symmetry $\hat{U}$ as

$$\hat{O} \longmapsto \hat{U}\,\hat{O}\,\hat{U}^{\dagger},$$

the same rule by which density matrices transform: $\rho \mapsto U\rho U^{\dagger}$. The matrix $X = x^{\mu}\sigma_{\mu}$ *is* literally a 2×2 Hermitian matrix encoding the four-vector x^{μ}, so it behaves like a quantum observable on a two-dimensional state space, and $MXM^{\dagger}$ is the natural extension of the QM operator rule to the symmetry group at hand. The conjugation by $\dagger$ in place of M^{-1} is not an exotic device — it is what quantum mechanics has been doing all along to transformations of observables.

From group adjoint to algebra adjoint: the universal pattern. For any Lie group G acting on its own Lie algebra $\mathfrak{g}$, the group adjoint action is the conjugation

$$\mathrm{Ad}(g)\,X \;=\; g\,X\,g^{-1}, \qquad g \in G,\;\; X \in \mathfrak{g}. \tag{9.52}$$

Differentiating this at the identity along the one-parameter subgroup $g(t) = e^{tA}$, with $A \in \mathfrak{g}$, gives

$$\left.\frac{d}{dt}\right|_{t=0} \left(e^{tA}\,X\,e^{-tA}\right) \;=\; AX - XA \;=\; [A,X] \;\equiv\; \mathrm{ad}(A)\,X\,.$$

This is the *algebra adjoint* representation: $\mathrm{ad}(A) : \mathfrak{g} \to \mathfrak{g}$ is the linear map $X \mapsto [A,X]$, obtained by differentiating the group conjugation gXg^{-1} at the identity. The relation is universal across Lie theory:

$$\boxed{\text{group adjoint } \mathrm{Ad}(g) \;\xrightarrow{\;d/dt\big|_{t=0}\;}\; \text{algebra adjoint } \mathrm{ad}(A) = [A,\cdot\,]\,.}$$

In quantum mechanics this is the statement that infinitesimal symmetry transformations of an observable are generated by commutators with the Hermitian generator: $\delta_A \hat{O} = -i\,[A,\hat{O}]$ for unitary $U = e^{-iA}$.

Specialising to $SL(2,\mathbb{C})$ on Hermitian matrices. We repeat the differentiation for our slightly modified action $MXM^{\dagger}$. Writing $M = \mathbb{1}_2 + tA + O(t^2)$ with $A \in \mathfrak{sl}(2,\mathbb{C})$:

$$M\,X\,M^{\dagger} \;=\; X \;+\; t\big(AX + XA^{\dagger}\big) \;+\; O(t^2)\,.$$

The generator is $AX + XA^\dagger$ — not a pure commutator, because of the †. To compare with the universal pattern above, split A into anti-Hermitian and Hermitian parts:

$$A = A_- + A_+, \qquad A_-^\dagger = -A_- \text{ (rotations)}, \quad A_+^\dagger = +A_+ \text{ (boosts)}.$$

Then

$$AX + XA^\dagger = \underbrace{[A_-, X]}_{\text{rotation: Lie bracket}} + \underbrace{\{A_+, X\}}_{\text{boost: anticommutator}}.$$

The **rotation part recovers the standard algebra adjoint** $\mathrm{ad}(A_-)\,X = [A_-, X]$ exactly as for $\mathrm{Ad}(g)$ in (9.52): on the rotation subgroup, our modified action is literally the universal Lie-group adjoint. The **boost part is new**: a symmetric (anticommutator) product, reflecting the fact that boosts in finite-dimensional spinor representations are Hermitian rather than anti-Hermitian, and therefore *rescale* X rather than rotate it.

Physical pictures.

- *Rotations → the familiar $SU(2) \to SO(3)$ story.* For pure rotations $M = U \in SU(2)$ is unitary, $U^\dagger = U^{-1}$, and $\mathrm{ad}(U)\,X = UXU^{-1}$ literally is the standard group adjoint. The infinitesimal action on $\vec{x}\cdot\vec{\sigma}$ is the spin-$\frac{1}{2}$ commutator $\mathrm{ad}(\vec{\sigma})\,\vec{x}\cdot\vec{\sigma} \sim \vec{\sigma}\times\vec{x}$ that underlies Larmor precession, Stern–Gerlach splitting, and every angular-momentum coupling calculation. We recover the 2:1 covering $SU(2) \to SO(3)$ as the spatial sector of the present construction.
- *Boosts → finite-dimensional non-unitarity.* Boosts in $SL(2,\mathbb{C})$ are realised by Hermitian matrices $M = e^{-\vec{\zeta}\cdot\vec{\sigma}/2}$, not unitary ones. Finite-dimensional Lorentz representations *cannot* be unitary — the same obstruction that, in Wigner's one-particle classification, forces massive states into infinite-dimensional Hilbert-space representations. The † in $MXM^\dagger$ is what accommodates this non-unitarity at the level of the finite-dimensional matrix realisation: it rescales the diagonal entries of X by $e^{\pm\zeta}$, mixing energy and momentum (cf. the explicit boost calculation below (9.57)). The Jordan-like anticommutator $\{A_+, X\}$ is the infinitesimal echo of this rescaling.
- *Why † and not −1? Hermiticity = physical content.* The matrix $X = x^\mu\sigma_\mu$ carries a real four-vector precisely because X is Hermitian. Conjugation by M^{-1} would not preserve Hermiticity for non-unitary M: $(MXM^{-1})^\dagger = (M^{-1})^\dagger XM^\dagger \neq MXM^{-1}$. The substitution $M^{-1} \to M^\dagger$ is the same fix

quantum mechanics imposes to keep density matrices density matrices, and observables observables.

Summary. The map $X \mapsto MXM^\dagger$ is the quantum-operator rule applied to a Hermitian matrix realisation of a four-vector. Differentiating at the identity (the universal recipe $\mathrm{Ad}(g) \to \mathrm{ad}(A) = [A, \cdot]$) recovers the ordinary algebra adjoint $[A_-, X]$ on the rotation sector, supplemented by a Jordan-like anticommutator $\{A_+, X\}$ on the boost sector. The first is universal Lie theory; the second is the algebraic signature of the non-compactness of the Lorentz group — the same fact that distinguishes special relativity from purely spatial rotations.

Two properties are immediate:

(i) $\mathrm{ad}(M)X$ is again Hermitian: $(MXM^\dagger)^\dagger = MX^\dagger M^\dagger = MXM^\dagger$ since $X^\dagger = X$;
(ii) $\det\big(MXM^\dagger\big) = |\det M|^2 \det X = \det X$, using $\det M = 1$.

By (9.50), the second property says that the four-vector $x'^\mu \equiv \rho^{-1}(MXM^\dagger)$ has the *same* Minkowski square as x^μ:

$$x'_\mu x'^\mu \;=\; x_\mu x^\mu .$$

The map $x^\mu \mapsto x'^\mu$ is real-linear (since ρ and $\mathrm{ad}(M)$ are), preserves the quadratic form, and is continuously connected to the identity (taking $M = \mathbb{1}_2$). It is therefore an element of the restricted Lorentz group $SO^+(1,3)$. We may write

$$\boxed{x'^\mu \;=\; \Lambda^\mu{}_\nu(M)\, x^\nu, \qquad \Lambda^\mu{}_\nu(M) \;=\; \tfrac{1}{2}\,\mathrm{Tr}\big[\bar\sigma^\mu\, M\, \sigma_\nu\, M^\dagger\big]}\,. \tag{9.53}$$

This is the *explicit* formula for the homomorphism $SL(2,\mathbb{C}) \to SO^+(1,3)$. The right-hand side is obtained by combining (9.49) with $X' = MXM^\dagger$ from (9.51); the component-level version of the same statement is the sigma-matrix transformation law $M\sigma_\mu M^\dagger = \sigma_\nu \Lambda^\nu{}_\mu$, established below as eqs. (9.57)–(9.58).

Step 3 — Homomorphism, image, and kernel. We now prove three propositions that establish (9.53) as a two-to-one group homomorphism onto $SO^+(1,3)$.

(a) Λ is a group homomorphism: for all $M_1, M_2 \in SL(2,\mathbb{C})$,

$$\Lambda(M_1 M_2) \;=\; \Lambda(M_1)\,\Lambda(M_2), \qquad \Lambda(M^{-1}) \;=\; \Lambda(M)^{-1}.$$

Proof. Apply $\mathrm{ad}(M_2)$ followed by $\mathrm{ad}(M_1)$ to an arbitrary $X \in \mathbb{H}(2,\mathbb{C})$:

$$\mathrm{ad}(M_1)\,\mathrm{ad}(M_2)\,X \;=\; M_1\big(M_2 X M_2^\dagger\big)M_1^\dagger \;=\; (M_1M_2)\,X\,(M_1M_2)^\dagger \;=\; \mathrm{ad}(M_1M_2)\,X\,,$$

using $(M_1M_2)^\dagger = M_2^\dagger M_1^\dagger$. Reading both sides through ρ^{-1} converts the equality of adjoint actions on $\mathbb{H}(2,\mathbb{C})$ into the equality of the induced Lorentz transformations on $\mathbb{M}_4$, hence $\Lambda(M_1)\Lambda(M_2) = \Lambda(M_1M_2)$. Setting $M_2 = M^{-1}$ in this identity and noting $\Lambda(\mathbb{1}_2) = \mathbb{1}$ gives $\Lambda(M^{-1}) = \Lambda(M)^{-1}$. □

(b) $\Lambda(M)$ lies in the restricted Lorentz group:

$$\det\Lambda(M) \;=\; +1\,, \qquad \Lambda^0{}_0(M) \;\geq\; 1\,.$$

Proof. We already know that $\Lambda(M)$ preserves the quadratic form $x_\mu x^\mu$ (Step 2), so $\Lambda(M) \in O(1,3)$ and $\det\Lambda(M) \in \{+1,-1\}$, $\Lambda^0{}_0(M) \in (-\infty,-1]\cup[1,+\infty)$. Both functions $M \mapsto \det\Lambda(M)$ and $M \mapsto \Lambda^0{}_0(M)$ are continuous (polynomial in the entries of $M, M^\dagger$ by (9.53)). The group $SL(2,\mathbb{C})$ is connected (it is homeomorphic to $\mathbb{R}^3 \times S^3$, see paragraph after (9.54) below), and at $M = \mathbb{1}_2$ both quantities take the value $+1$. By the intermediate value theorem, they cannot jump to a value in the disconnected complementary branches, so $\det\Lambda(M) = +1$ and $\Lambda^0{}_0(M) \geq +1$ throughout $SL(2,\mathbb{C})$.

A direct algebraic version of the second statement: from (9.53) with $\mu = \nu = 0$ (and $\sigma_0 = \bar\sigma^0 = \mathbb{1}_2$),

$$\Lambda^0{}_0(M) \;=\; \tfrac{1}{2}\mathrm{Tr}\big[MM^\dagger\big]\,.$$

The singular value decomposition $M = UDV^\dagger$ with $U, V \in U(2)$ and $D = \mathrm{diag}(s_1, s_2)$, $s_i > 0$, gives $\mathrm{Tr}[MM^\dagger] = s_1^2 + s_2^2$, while $\det M = 1$ forces $s_1 s_2 = 1$. By AM–GM,

$$s_1^2 + s_2^2 \;\geq\; 2\,s_1 s_2 \;=\; 2\,,$$

so $\Lambda^0{}_0(M) = \frac{1}{2}(s_1^2 + s_2^2) \geq 1$, with equality iff $s_1 = s_2 = 1$, i.e. M unitary — a pure rotation. □

(c) The kernel of Λ is $\{\pm\mathbb{1}_2\}$:

$$\ker\Lambda \;=\; \{\,M \in SL(2,\mathbb{C}) \,:\, \Lambda(M) = \mathbb{1}\,\} \;=\; \{\pm\mathbb{1}_2\}.$$

Proof. Suppose $\Lambda(M) = \mathbb{1}$. By (9.58) (which expresses (9.53) at the level of basis sigma matrices),

$$M\sigma_\mu M^\dagger \;=\; \sigma_\nu \Lambda^\nu{}_\mu \;=\; \sigma_\mu \quad \text{for all } \mu = 0,1,2,3\,.$$

Taking $\mu = 0$: $M\mathbb{1}_2 M^\dagger = MM^\dagger = \mathbb{1}_2$, i.e. $M^\dagger = M^{-1}$ — M is *unitary*. For $\mu = i$ the same equation then reads $M\sigma_i M^{-1} = \sigma_i$, that is, M commutes with every Pauli matrix:

$$[M, \sigma_i] \;=\; 0\,, \quad i = 1,2,3\,.$$

A 2×2 matrix that commutes with all three Pauli matrices commutes in particular with the basis $\{\mathbb{1}_2, \sigma_1, \sigma_2, \sigma_3\}$ of $\mathrm{Mat}(2,\mathbb{C})$, hence belongs to the centre of $\mathrm{Mat}(2,\mathbb{C})$,

which by Schur's lemma is $\mathbb{C}\,\mathbb{1}_2$. Thus $M = c\,\mathbb{1}_2$ for some $c \in \mathbb{C}$. The constraint $\det M = c^2 = 1$ forces $c = \pm 1$, i.e. $M = \pm\mathbb{1}_2$. Conversely $\Lambda(\pm\mathbb{1}_2)$ acts on X as $X \mapsto (\pm\mathbb{1}_2)X(\pm\mathbb{1}_2) = X$, so both $\pm\mathbb{1}_2$ lie in $\ker\Lambda$. □The first isomorphism theorem then yields

$$\boxed{SO^+(1,3) \;\cong\; SL(2,\mathbb{C})/\mathbb{Z}_2\,,} \tag{9.54}$$

with the $\mathbb{Z}_2$ kernel generated by $-\mathbb{1}_2$. Since $SL(2,\mathbb{C})$ is simply connected (as a 6-dimensional real Lie group it is homeomorphic to $\mathbb{R}^3 \times S^3$), (9.54) identifies $SL(2,\mathbb{C})$ as the *universal cover* of $SO^+(1,3)$.

Inverse construction $M(\Lambda)$. The two-valuedness can be made explicit: given any $\Lambda \in SO^+(1,3)$, the element M that maps to it under (9.53) is

$$M(\Lambda) \;=\; \pm\frac{1}{\sqrt{\det\!\left[\Lambda^{\mu}{}_{\nu}\sigma_{\mu}\bar{\sigma}^{\nu}\right]}}\,\Lambda^{\mu}{}_{\nu}\,\sigma_{\mu}\bar{\sigma}^{\nu}, \tag{9.55}$$

unique up to the overall sign ± [31].

Physical consequence. The two-valuedness of the cover is the precise mathematical reason why *spinor* fields exist: an irreducible representation of $SL(2,\mathbb{C})$ that does not factor through $SO^+(1,3)$ assigns inequivalent transformations to M and $-M$, and hence corresponds to a representation of $SO^+(1,3)$ only up to sign. A 2π rotation in $SO^+(1,3)$ lifts to $-\mathbb{1}_2$ in $SL(2,\mathbb{C})$, so a spinor returns to itself only after a 4π rotation. This is the algebraic origin of the fermion sign and of all related effects (Berry phase, spin-statistics, etc.). The two fundamental two-dimensional irreducible representations of $SL(2,\mathbb{C})$ are precisely the $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ Weyl spinor representations introduced in (9.42); their direct sum is the Dirac representation.

Conclusion: identifying M and $(M^{-1})^\dagger$. Equation (9.53) singles out, for each Lorentz transformation Λ, exactly two elements $\pm M \in SL(2,\mathbb{C})$ that act on Hermitian matrices as $X \mapsto MXM^\dagger$. The two non-equivalent 2-dimensional irreducible representations of this $SL(2,\mathbb{C})$ act on Weyl spinors as $\chi \mapsto M\chi$ (the $(\frac{1}{2},0)$ representation) and $\bar{\psi} \mapsto (M^{-1})^\dagger\bar{\psi}$ (the $(0,\frac{1}{2})$ representation): one verifies that M and $(M^{-1})^\dagger$ are inequivalent precisely because $\Lambda_L = \Lambda_R^{-1\,\dagger}$ differs from Λ_R by the sign of the boost (cf. the discussion below (9.46)). Therefore the matrices M and $(M^{-1})^\dagger$ introduced earlier are not ad hoc but are forced by group theory: they are the *unique* two-dimensional spinor representations of the universal cover of the Lorentz group, and any Weyl-spinor field theory is built from these representations and their complex conjugates. The deeper physical consequence — that the sigma matrices act as intertwining operators between the two Weyl sectors, and that this

intertwining is what makes the Dirac equation inevitable — is examined at the end of the Sigma matrices subsection below.

Sigma matrices

Sigma matrices and four-vectors: explicit realisation of ρ

We now make the abstract construction of the homomorphism $SL(2,\mathbb{C}) \to SO^+(1,3)$ entirely concrete. Applying the map ρ of (9.48) to a generic four-momentum p^μ gives

$$P \equiv \rho(p) = p^\mu \sigma_\mu = p^0 \mathbb{1}_2 - \vec{p}\cdot\vec{\sigma} = \begin{pmatrix} p^0 - p^3 & -p^1 + ip^2 \\ -p^1 - ip^2 & p^0 + p^3 \end{pmatrix}, \tag{9.56}$$

a Hermitian 2×2 matrix. The inverse map (9.49) recovers $p^\mu = \frac{1}{2}\mathrm{Tr}[P\,\bar{\sigma}^\mu]$, and the determinant identity (9.50) reads

$$\det P = (p^0)^2 - |\vec{p}|^2 = p_\mu p^\mu ,$$

as one verifies directly from the matrix on the right-hand side of (9.56). This concretely realises the abstract correspondence $\mathbb{M}_4 \leftrightarrow \mathbb{H}(2,\mathbb{C})$ on momentum space.

Proposition 9.3 (Lorentz transformation of $p^\mu\sigma_\mu$). The Lorentz transformation $p'^\mu = \Lambda^\mu{}_\nu(M)\, p^\nu$ generated by $M \in SL(2,\mathbb{C})$ through the homomorphism (9.53) is realised at the level of Hermitian matrices by the adjoint action $\mathrm{ad}(M)$:

$$p'^\mu \sigma_\mu = M\,(p^\nu \sigma_\nu)\, M^\dagger . \tag{9.57}$$

Equivalently, at the level of the basis sigma matrices,

$$M\,\sigma_\mu\, M^\dagger = \sigma_\nu\, \Lambda^\nu{}_\mu(M) , \tag{9.58}$$

and, with the inverse transformation and a raised Lorentz index,

$$M^{-1}\sigma^\mu (M^{-1})^\dagger = \Lambda^\mu{}_\nu(M)\,\sigma^\nu . \tag{9.59}$$

Proof. The Hermiticity-preservation and determinant-preservation steps required to conclude that $\mathrm{ad}(M)$ induces *some* Lorentz transformation were already carried out in the construction of the homomorphism (Step 2 of the previous subsubsection, using $\det M = 1$). All that remains is to identify the specific $\Lambda \in SO^+(1,3)$ produced by a given M.

Step A: from (9.57) to (9.58). Expand p'^{μ} as the linear combination $p'^{\mu}\sigma_{\mu}$ in (9.57) and use the assumed transformation law $p'^{\mu} = \Lambda^{\mu}{}_{\nu}p^{\nu}$ on the left:

$$\Lambda^{\mu}{}_{\nu}\, p^{\nu}\,\sigma_{\mu} \;=\; M\,(p^{\nu}\sigma_{\nu})\,M^{\dagger} \;=\; p^{\nu}\,M\sigma_{\nu}M^{\dagger}\,.$$

Since this holds for every four-vector p^{ν}, the coefficients of p^{ν} on the two sides must match:

$$M\sigma_{\nu}M^{\dagger} = \sigma_{\mu}\Lambda^{\mu}{}_{\nu},$$

which is (9.58).

To obtain the raised-index inverse form (9.59), we proceed in two substeps.

(A1) Apply (9.58) to M^{-1}. Since the identity holds for *every* $M \in SL(2,\mathbb{C})$, replacing $M \to M^{-1}$ gives

$$M^{-1}\,\sigma_{\nu}\,(M^{-1})^{\dagger} \;=\; \sigma_{\mu}\,\Lambda^{\mu}{}_{\nu}(M^{-1}) \;=\; \sigma_{\mu}\,(\Lambda^{-1})^{\mu}{}_{\nu},$$

where we used the group-homomorphism property $\Lambda(M^{-1}) = \Lambda(M)^{-1}$ (Step 3, (a) of the previous subsubsection).

(A2) Raise the free index ν with $g^{\nu\alpha}$. The left-hand side becomes simply

$$g^{\nu\alpha}\left[M^{-1}\sigma_{\nu}(M^{-1})^{\dagger}\right] \;=\; M^{-1}\,\sigma^{\alpha}\,(M^{-1})^{\dagger}.$$

For the right-hand side we use the standard Lorentz-inverse identity, which follows from the metric-preservation condition $\Lambda^{\mu}{}_{\rho}\,\Lambda^{\nu}{}_{\sigma}\,g_{\mu\nu} = g_{\rho\sigma}$:

$$\boxed{(\Lambda^{-1})^{\mu}{}_{\nu} \;=\; \Lambda_{\nu}{}^{\mu} \;\equiv\; g_{\nu\rho}\,g^{\mu\sigma}\,\Lambda^{\rho}{}_{\sigma}}\,. \tag{9.60}$$

Plugging this in and contracting with $g^{\nu\alpha}$:

$$g^{\nu\alpha}\,\sigma_{\mu}\,(\Lambda^{-1})^{\mu}{}_{\nu} = \sigma_{\mu}\,g^{\nu\alpha}\,g_{\nu\rho}\,g^{\mu\sigma}\,\Lambda^{\rho}{}_{\sigma} = \sigma_{\mu}\,\delta^{\alpha}_{\rho}\,g^{\mu\sigma}\,\Lambda^{\rho}{}_{\sigma} = \underbrace{\sigma_{\mu}\,g^{\mu\sigma}}_{=\sigma^{\sigma}}\,\Lambda^{\alpha}{}_{\sigma} = \Lambda^{\alpha}{}_{\sigma}\,\sigma^{\sigma}.$$

Equating the two sides and relabelling $\alpha \to \mu$, $\sigma \to \nu$:

$$M^{-1}\,\sigma^{\mu}\,(M^{-1})^{\dagger} \;=\; \Lambda^{\mu}{}_{\nu}(M)\,\sigma^{\nu},$$

which is (9.59).

Step B: explicit verification on a boost. The abstract argument fixes the structure but not the specific value of $\Lambda^{\mu}{}_{\nu}$ in terms of the $(\vec{\theta},\vec{\zeta})$ that parametrise M in (9.44). We pin this down on a representative example. Take an active boost along the z-axis with rapidity $\zeta > 0$: from (9.44) with $\vec{\theta} = 0$ and $\vec{\zeta} = \zeta\hat{z}$,

$$M = e^{-\zeta\sigma^{3}/2} = \begin{pmatrix} e^{-\zeta/2} & 0 \\ 0 & e^{+\zeta/2} \end{pmatrix}, \qquad M = M^{\dagger}\,. \tag{9.61}$$

Computing $MPM^\dagger$ explicitly:

$$MPM^\dagger = \begin{pmatrix} e^{-\zeta/2} & 0 \\ 0 & e^{+\zeta/2} \end{pmatrix} \begin{pmatrix} p^0 - p^3 & -p^1 + ip^2 \\ -p^1 - ip^2 & p^0 + p^3 \end{pmatrix} \begin{pmatrix} e^{-\zeta/2} & 0 \\ 0 & e^{+\zeta/2} \end{pmatrix} = \begin{pmatrix} e^{-\zeta}(p^0 - p^3) & -p^1 + ip^2 \\ -p^1 - ip^2 & e^{+\zeta}(p^0 + p^3) \end{pmatrix}. \tag{9.62}$$

Reading off the components from the canonical form $p'^\mu \sigma_\mu = \begin{pmatrix} p'^0 - p'^3 & -p'^1 + ip'^2 \\ -p'^1 - ip'^2 & p'^0 + p'^3 \end{pmatrix}$:

$$p'^0 + p'^3 = e^{+\zeta}(p^0 + p^3), \qquad p'^0 - p'^3 = e^{-\zeta}(p^0 - p^3),$$
$$p'^1 = p^1, \qquad p'^2 = p^2. \tag{9.63}$$

Adding and subtracting the first two equations:

$$p'^0 = \cosh\zeta\, p^0 + \sinh\zeta\, p^3, \qquad p'^3 = \sinh\zeta\, p^0 + \cosh\zeta\, p^3, \tag{9.64}$$

which is the active Lorentz boost along $+z$: a particle at rest acquires positive z-momentum $p'^3 = m\sinh\zeta > 0$, as required by our active convention (cf. box 9.2). The corresponding $\Lambda(M)$ has $\Lambda^0{}_3 = +\sinh\zeta$; this differs from the standard textbook form $\Lambda^0{}_3 = -\sinh\zeta$ (which is the passive convention) by the sign of $\sinh\zeta$, as explained in box 9.2. The general case can be assembled by composing such examples (rotations and boosts in arbitrary directions) or, equivalently, by extracting $\Lambda(M)$ directly from the trace formula (9.53). □

Remark. [The two halves of one story] The previous subsubsection established the existence and structural properties of the homomorphism $SL(2,\mathbb{C}) \to SO^+(1,3)$ in abstract terms (hermiticity of X, determinant invariance, kernel $\pm\mathbb{1}_2$). The present subsubsection *realises* the same homomorphism on a concrete object — the Hermitian matrix $P = p^\mu\sigma_\mu$ associated to a four-vector — and identifies the linear transformation of P under the adjoint action with an explicit Lorentz transformation in spacetime. Equations (9.57), (9.58), (9.59) are therefore the component-level statements of which (9.53) is the trace-extracted summary.

Reading off the spinor index structure of σ^μ and $\bar\sigma^\mu$. The transformation laws (9.58)–(9.59) are matrix identities in $\mathrm{Mat}(2,\mathbb{C})$, but each 2×2 slot of σ^μ carries a definite spinor index assignment that is *dictated* by which group element (M or $(M^{-1})^\dagger$) acts on which side. Let us read this off step by step.

(i) The two Weyl reps and their index conventions. From (9.45) the $(\frac{1}{2},0)$ representation acts on *undotted* spinors by

$$\chi'_\alpha \;=\; M_\alpha{}^\beta \chi_\beta,$$

so M carries the index structure $M_\alpha{}^\beta$. From (9.47) the $(0,\frac{1}{2})$ representation acts on *dotted* spinors by

$$\bar\psi'^{\dot\alpha} \;=\; \left[(M^{-1})^\dagger\right]^{\dot\alpha}{}_{\dot\beta}\,\bar\psi^{\dot\beta},$$

so $(M^{-1})^\dagger$ carries dotted indices $\left[(M^{-1})^\dagger\right]^{\dot\alpha}{}_{\dot\beta}$. Crucially, $M^\dagger$ itself (which appears on the right of σ^μ in (9.58)) is obtained from $(M^{-1})^\dagger$ by inversion and so it carries the *transposed* dotted index pattern:

$$\left[M^\dagger\right]^{\dot\beta}{}_{\dot\alpha}\,.$$

(ii) Forcing the index placement of σ^μ. Restoring spinor indices in (9.58), the left-hand side $M\,\sigma^\mu\,M^\dagger$ contracts $M_\alpha{}^\beta$ on the left and $[M^\dagger]^{\dot\beta}{}_{\dot\alpha}$ on the right, so σ^μ must be contractable with an upper undotted index from M and an upper dotted index from $M^\dagger$. The only consistent placement is

$$\boxed{\sigma^\mu_{\alpha\dot\alpha}}\,,$$

with one lower undotted index and one lower dotted index. The identity then reads, with all spinor indices explicit:

$$M_\alpha{}^\beta\,\sigma^\mu_{\beta\dot\beta}\left[M^\dagger\right]^{\dot\beta}{}_{\dot\alpha} \;=\; \Lambda^\mu{}_\nu(M)\,\sigma^\nu_{\alpha\dot\alpha}. \tag{9.65}$$

The Lorentz index μ on σ^μ and the spinor indices $\alpha,\dot\alpha$ are now both unambiguous: σ^μ is a $(\frac{1}{2},\frac{1}{2})$ object — the unique linear map intertwining $(\frac{1}{2},0)\otimes(0,\frac{1}{2})$ with the four-vector representation $(\frac{1}{2},\frac{1}{2})$.

(iii) Dual story for $\bar\sigma^\mu$. The barred matrices are the index-raised partners of the unbarred ones. Starting from (9.59) and raising the two spinor indices with the antisymmetric metrics gives the equivalent barred transformation law

$$\left[(M^{-1})^\dagger\right]^{\dot\alpha}{}_{\dot\beta}\,\bar\sigma^{\mu\dot\beta\beta}\,[M^{-1}]_\beta{}^\alpha \;=\; \Lambda^\mu{}_\nu(M)\,\bar\sigma^{\nu\dot\alpha\alpha}.$$

Thus $\bar\sigma^\mu$ contracts with the lower dotted and undotted index slots of the inverse transformations. The natural placement is therefore reversed:

$$\boxed{\bar\sigma^{\mu\dot\alpha\alpha}}\,,$$

with both spinor indices *upper* (first dotted, then undotted, by convention). The two patterns are related by the spinor metric: $\bar\sigma^{\mu\dot\alpha\alpha} = \epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}\sigma^\mu_{\beta\dot\beta}$, as recorded in (9.69) below.

(iv) Summary. The spinor index structure

$$\sigma^{\mu}_{\alpha\dot{\alpha}} \quad \text{and} \quad \bar{\sigma}^{\mu\,\dot{\alpha}\alpha}, \tag{9.66}$$

with undotted $\alpha, \beta, \ldots$ labelling $(\frac{1}{2}, 0)$ and dotted $\dot{\alpha}, \dot{\beta}, \ldots$ labelling $(0, \frac{1}{2})$, is therefore not an arbitrary convention: it is the *unique* placement compatible with the transformation laws (9.58) and (9.59).

Sigma matrices with covariant indices

The sigma matrices have been defined with upper (contravariant) Lorentz indices. They are related to the sigma matrices with lower (covariant) indices in the usual way:

$$\boxed{\sigma_\mu = g_{\mu\nu}\sigma^\nu = (\mathbb{1}_2, -\vec{\sigma}), \qquad \bar{\sigma}_\mu = g_{\mu\nu}\bar{\sigma}^\nu = (\mathbb{1}_2, \vec{\sigma})}. \tag{9.67}$$

Both σ^μ and $\bar{\sigma}^\mu$ are hermitian matrices, i.e., $(\sigma^\mu)^\dagger = \sigma^\mu$ and $(\bar{\sigma}^\mu)^\dagger = \bar{\sigma}^\mu$, or equivalently:

$$(\sigma^{\mu}_{\alpha\dot{\beta}})^* = \sigma^{\mu}_{\beta\dot{\alpha}}, \qquad (\bar{\sigma}^{\mu\,\dot{\alpha}\beta})^* = \bar{\sigma}^{\mu\,\dot{\beta}\alpha}. \tag{9.68}$$

Key sigma matrix identities

Some useful relations between σ^μ and $\bar{\sigma}^\mu$ involving the ϵ tensor are:

$$\boxed{\begin{aligned} \sigma^{\mu}_{\alpha\dot{\alpha}} &= \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\sigma}^{\mu\,\dot{\beta}\beta}, & \bar{\sigma}^{\mu\,\dot{\alpha}\alpha} &= \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\sigma^{\mu}_{\beta\dot{\beta}} \\ \epsilon^{\alpha\beta}\sigma^{\mu}_{\beta\dot{\alpha}} &= \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\sigma}^{\mu\,\dot{\beta}\alpha}, & \epsilon^{\dot{\alpha}\dot{\beta}}\sigma^{\mu}_{\alpha\dot{\beta}} &= \epsilon_{\alpha\beta}\bar{\sigma}^{\mu\,\dot{\alpha}\beta} \end{aligned}}. \tag{9.69}$$

Proof of (9.69).
Throughout we adopt the convention $\epsilon^{12} = -\epsilon_{12} = +1$ (and analogously for dotted indices). Direct enumeration of the four index combinations gives the *spinor-metric identity*

$$\epsilon^{\alpha\beta}\epsilon_{\beta\gamma} = \delta^{\alpha}_{\gamma}, \qquad \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon_{\dot{\beta}\dot{\gamma}} = \delta^{\dot{\alpha}}_{\dot{\gamma}}. \tag{$\dagger$}$$

(i) The second relation is the defining one. The identity

$$\bar{\sigma}^{\mu\,\dot{\alpha}\alpha} = \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\sigma^{\mu}_{\beta\dot{\beta}} \tag{$*$}$$

is the statement that $\bar{\sigma}^\mu$ is obtained from σ^μ by raising both spinor indices with the spinor metric. To check that this prescription, applied to $\sigma^\mu = (\mathbb{1}_2, \vec{\sigma})$, indeed reproduces $\bar{\sigma}^\mu = (\mathbb{1}_2, -\vec{\sigma})$, treat both sides of ($*$) as 2×2 matrices in which σ^μ has row index β (undotted) and column index $\dot{\beta}$ (dotted), and $\bar{\sigma}^\mu$ has row index $\dot{\alpha}$

(dotted) and column index α (undotted). The sum on the right reorganises into the matrix product

$$\bar{\sigma}^{\mu} \;=\; \epsilon\,(\sigma^{\mu})^{\top}\,\epsilon^{\top}, \qquad \text{with} \quad \epsilon \equiv i\sigma^{2} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Using $\epsilon^{\top} = -\epsilon$ and $\epsilon^{2} = -\mathbb{1}_{2}$:

$$\begin{aligned} \mu = 0: &\quad \epsilon\,\mathbb{1}_{2}\,\epsilon^{\top} = -\epsilon^{2} = \mathbb{1}_{2} = \bar{\sigma}^{0}, \\ \mu = 1,3: &\quad (\sigma^{\mu})^{\top} = \sigma^{\mu},\ \ \epsilon\sigma^{\mu}\epsilon^{\top} = -\sigma^{\mu} = \bar{\sigma}^{\mu} \ \ (\text{using } \epsilon\sigma^{1,3}\epsilon = \sigma^{1,3}), \\ \mu = 2: &\quad (\sigma^{2})^{\top} = -\sigma^{2},\ \ \epsilon(-\sigma^{2})\epsilon^{\top} = -\sigma^{2} = \bar{\sigma}^{2} \ \ (\text{using } \epsilon\sigma^{2}\epsilon^{\top} = \sigma^{2}). \end{aligned}$$

This verifies (∗).

(ii) The first relation is its inverse. Multiply both sides of (∗) by $\epsilon_{\rho\alpha}\epsilon_{\dot{\rho}\dot{\alpha}}$ and apply the spinor-metric identity (†) twice:

$$\epsilon_{\rho\alpha}\epsilon_{\dot{\rho}\dot{\alpha}}\,\bar{\sigma}^{\mu\,\dot{\alpha}\alpha} \;=\; \underbrace{\epsilon_{\rho\alpha}\epsilon^{\alpha\beta}}_{\delta^{\beta}_{\rho}}\,\underbrace{\epsilon_{\dot{\rho}\dot{\alpha}}\epsilon^{\dot{\alpha}\dot{\beta}}}_{\delta^{\dot{\beta}}_{\dot{\rho}}}\,\sigma^{\mu}_{\beta\dot{\beta}} \;=\; \sigma^{\mu}_{\rho\dot{\rho}}\,.$$

Relabelling $\rho \to \alpha$, $\dot{\rho} \to \dot{\alpha}$ reproduces the first relation in (9.69).

(iii) The mixed relations follow by contracting with a single ϵ. Multiply (∗) by $\epsilon_{\dot{\rho}\dot{\alpha}}$ only:

$$\epsilon_{\dot{\rho}\dot{\alpha}}\,\bar{\sigma}^{\mu\,\dot{\alpha}\alpha} \;=\; \epsilon^{\alpha\beta}\,\underbrace{\epsilon_{\dot{\rho}\dot{\alpha}}\epsilon^{\dot{\alpha}\dot{\beta}}}_{\delta^{\dot{\beta}}_{\dot{\rho}}}\,\sigma^{\mu}_{\beta\dot{\beta}} \;=\; \epsilon^{\alpha\beta}\,\sigma^{\mu}_{\beta\dot{\rho}}\,.$$

Relabelling $\dot{\rho} \to \dot{\alpha}$, this is the third relation in (9.69). Multiplying instead by $\epsilon_{\rho\alpha}$ yields, by an identical argument,

$$\epsilon_{\rho\alpha}\,\bar{\sigma}^{\mu\,\dot{\alpha}\alpha} \;=\; \epsilon^{\dot{\alpha}\dot{\beta}}\,\sigma^{\mu}_{\rho\dot{\beta}},$$

which after relabelling $\rho \to \alpha$ is the fourth relation. □

The following identities are essential for calculations:

$$\sigma^{\mu}_{\alpha\dot{\alpha}}\,\bar{\sigma}^{\dot{\beta}\beta}_{\mu} = 2\delta^{\beta}_{\alpha}\delta^{\dot{\beta}}_{\dot{\alpha}} \tag{9.70}$$

$$\sigma^{\mu}_{\alpha\dot{\alpha}}\,\sigma_{\mu\beta\dot{\beta}} = 2\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} \tag{9.71}$$

$$\bar{\sigma}^{\mu\,\dot{\alpha}\alpha}\,\bar{\sigma}^{\dot{\beta}\beta}_{\mu} = 2\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}} \tag{9.72}$$

$$[\sigma^{\mu}\bar{\sigma}^{\nu} + \sigma^{\nu}\bar{\sigma}^{\mu}]_{\alpha}{}^{\beta} = 2g^{\mu\nu}\delta^{\beta}_{\alpha} \tag{9.73}$$

$$[\bar{\sigma}^{\mu}\sigma^{\nu} + \bar{\sigma}^{\nu}\sigma^{\mu}]^{\dot{\alpha}}{}_{\dot{\beta}} = 2g^{\mu\nu}\delta^{\dot{\alpha}}_{\dot{\beta}}\,. \tag{9.74}$$

Remark. [Triple products of sigma matrices] The following identities are useful for simplifying products of three sigma matrices:

$$\sigma^\mu \bar{\sigma}^\nu \sigma^\rho = g^{\mu\nu}\sigma^\rho - g^{\mu\rho}\sigma^\nu + g^{\nu\rho}\sigma^\mu + i\epsilon^{\mu\nu\rho\kappa}\sigma_\kappa \tag{9.75}$$

$$\bar{\sigma}^\mu \sigma^\nu \bar{\sigma}^\rho = g^{\mu\nu}\bar{\sigma}^\rho - g^{\mu\rho}\bar{\sigma}^\nu + g^{\nu\rho}\bar{\sigma}^\mu - i\epsilon^{\mu\nu\rho\kappa}\bar{\sigma}_\kappa , \tag{9.76}$$

where $\epsilon^{0123} = +1$.

Trace formulas

Computations of cross sections and decay rates generally require traces of alternating products of σ and $\bar{\sigma}$ matrices:

$$\boxed{\mathrm{Tr}[\sigma^\mu \bar{\sigma}^\nu] = \mathrm{Tr}[\bar{\sigma}^\mu \sigma^\nu] = 2g^{\mu\nu}} . \tag{9.77}$$

$$\mathrm{Tr}[\sigma^\mu \bar{\sigma}^\nu \sigma^\rho \bar{\sigma}^\kappa] = 2\left(g^{\mu\nu}g^{\rho\kappa} - g^{\mu\rho}g^{\nu\kappa} + g^{\mu\kappa}g^{\nu\rho} + i\epsilon^{\mu\nu\rho\kappa}\right) . \tag{9.78}$$

$$\mathrm{Tr}[\bar{\sigma}^\mu \sigma^\nu \bar{\sigma}^\rho \sigma^\kappa] = 2\left(g^{\mu\nu}g^{\rho\kappa} - g^{\mu\rho}g^{\nu\kappa} + g^{\mu\kappa}g^{\nu\rho} - i\epsilon^{\mu\nu\rho\kappa}\right) . \tag{9.79}$$

Remark. [Computing higher traces] Traces involving a larger even number of σ and $\bar{\sigma}$ matrices can be systematically obtained from eqs. (9.77)–(9.79) by repeated use of eqs. (9.73) and (9.74) and the cyclic property of the trace. Traces involving an odd number of σ and $\bar{\sigma}$ matrices cannot arise, since there is no way to connect the spinor indices consistently.

Proof of (9.73). For $\mu = \nu = 0$: $[\sigma^0\bar{\sigma}^0 + \sigma^0\bar{\sigma}^0]_\alpha{}^\beta = 2\delta_\alpha^\beta = 2g^{00}\delta_\alpha^\beta$.
For $\mu = 0, \nu = i$: $[\sigma^0\bar{\sigma}^i + \sigma^i\bar{\sigma}^0]_\alpha{}^\beta = (-\sigma^i + \sigma^i)_\alpha{}^\beta = 0 = 2g^{0i}\delta_\alpha^\beta$.
For $\mu = i, \nu = j$: Using $\sigma^i\bar{\sigma}^j = -\sigma^i\sigma^j = -(\delta^{ij}\mathbb{1}_2 + i\epsilon^{ijk}\sigma^k)$:

$$[\sigma^i\bar{\sigma}^j + \sigma^j\bar{\sigma}^i]_\alpha{}^\beta = -2\delta^{ij}\delta_\alpha^\beta = 2g^{ij}\delta_\alpha^\beta$$

□

Remark. [Preview: sigma matrices and the Dirac equation] Having assembled the full sigma-matrix machinery — definitions (9.56), the transformation laws (9.57)–(9.59), the index structure $\sigma^{\mu}_{\alpha\dot{\alpha}}$ and $\bar{\sigma}^{\mu\,\dot{\alpha}\alpha}$, the anticommutators (9.73)–(9.74), and the trace formulas (9.77) — we can now extract the central physical consequence: the matrices σ^{μ} and $\bar{\sigma}^{\mu}$ are precisely the intertwining operators between the two Weyl representations, and this intertwining is what *forces* the Dirac equation.

Statement. If $\bar{\psi}$ is a $(0,\frac{1}{2})$ spinor (R-type), then $\sigma^{\mu}p_{\mu}\bar{\psi}$ transforms as a $(\frac{1}{2},0)$ spinor (L-type). Conversely, if χ is a $(\frac{1}{2},0)$ spinor, then $\bar{\sigma}^{\mu}p_{\mu}\chi$ transforms as a $(0,\frac{1}{2})$ spinor. Schematically,

$$\chi \underset{\sigma^{\mu}p_{\mu}}{\overset{\bar{\sigma}^{\mu}p_{\mu}}{\rightleftharpoons}} \bar{\psi}.$$

The bottom arrow $\sigma^{\mu}p_{\mu}$ sends R→L; the top arrow $\bar{\sigma}^{\mu}p_{\mu}$ sends L→R.

Origin of the intertwining. The two Weyl matrices M and $(M^{-1})^{\dagger}$ in (9.44)–(9.46) agree on rotations but differ by the *sign of the boost*. From this sign flip,

$$M^{\dagger} = M^{-1}\ \text{(rotations)}, \qquad M^{\dagger} \neq M^{-1}\ \text{(boosts)},$$

follows the identity $M\,(\sigma_{\nu}p^{\nu})\,M^{\dagger} = \sigma_{\nu}p'^{\nu}$ already established in (9.57), which is precisely the statement that σ^{μ} transforms as a four-vector when sandwiched between L-type and R-type representations. The corresponding $\bar{\sigma}$ identity is obtained by conjugating this relation and raising or lowering the spinor indices with ϵ. Combining these sigma-matrix identities with the transformation laws of χ and $\bar{\psi}$ proves the chirality-flipping property stated above.

Why this forces the Dirac equation. Suppose one postulates a single Weyl spinor $\bar{\psi}$ of type $(0,\frac{1}{2})$ and seeks a linear, Lorentz-covariant wave equation involving a mass. The only Lorentz-covariant linear operator built from $\bar{\psi}$ and one p is $\sigma^{\mu}p_{\mu}\bar{\psi}$, but this is $(\frac{1}{2},0)$ — the *wrong* chirality. An equation of the form "$\sigma^{\mu}p_{\mu}\bar{\psi} = m\bar{\psi}$" is therefore not Lorentz-covariant: its two sides live in inequivalent representations. The minimal remedy is to introduce a partner spinor χ of opposite chirality and write the coupled system

$$\boxed{\sigma^{\mu}p_{\mu}\bar{\psi} = m\,\chi, \qquad \bar{\sigma}^{\mu}p_{\mu}\chi = m\,\bar{\psi}}, \tag{9.80}$$

in which both sides of each equation transform with the same Weyl matrix. The price of Lorentz covariance plus mass is doubling the number of components from 2 to 4 — and that is the Dirac field.

Consistency check. Applying $\bar{\sigma}^\nu p_\nu$ to the first equation and using the second,

$$\bar{\sigma}^\nu p_\nu \sigma^\mu p_\mu \bar{\psi} \;=\; m\,\bar{\sigma}^\nu p_\nu \chi \;=\; m^2 \bar{\psi}\,.$$

The anticommutator identities (9.73)–(9.74) yield $\bar{\sigma}^\nu p_\nu \sigma^\mu p_\mu = p^2$, so $(p^2 - m^2)\bar{\psi} = 0$ and analogously for χ: both Weyl components satisfy the Klein–Gordon equation with the *same* mass.

Weyl-basis Dirac equation. Stacking the two Weyl fields as $\Psi = (\chi, \bar{\psi})^\top$, equations (9.80) take the matrix form $(\gamma^\mu p_\mu - m)\Psi = 0$ with

$$\gamma^\mu_{\text{Weyl}} = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} -\mathbb{1}_2 & 0 \\ 0 & +\mathbb{1}_2 \end{pmatrix}.$$

The off-diagonal block structure of γ^μ is *precisely* the statement that σ^μ and $\bar{\sigma}^\mu$ intertwine L- and R-type. The diagonal form of γ^5 reflects the fact that chirality intrinsically labels the two irreducible pieces of Ψ. The Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ is a direct consequence of the $\sigma\bar{\sigma}$-anticommutators just displayed.

Physical content.

- *Chirality is Lorentz-invariant; helicity is not.* The labels L/R denote membership in inequivalent irreducible representations — a Lorentz-invariant statement. Helicity $\hat{p} \cdot \vec{S}$ is frame-dependent for a massive particle: a sufficiently large boost overtakes the particle and flips its helicity. Chirality and helicity coincide only in the massless limit.
- *Massless decoupling.* For $m = 0$, (9.80) decouples into two independent equations $\sigma^\mu p_\mu \bar{\psi} = 0$ and $\bar{\sigma}^\mu p_\mu \chi = 0$. A massless spin-$\frac{1}{2}$ particle may therefore live entirely in a single Weyl sector (the historical motivation for two-component neutrinos).
- *Parity exchanges L and R.* Under $\vec{p} \to -\vec{p}$, $\sigma^\mu p_\mu \leftrightarrow \bar{\sigma}^\mu p_\mu$, so parity *interchanges* the two equations in (9.80). A theory built from a single Weyl spinor is therefore necessarily parity-violating; the Dirac construction restores parity by including both halves on an equal footing.
- *Uniqueness of the mass term.* The mass term is the unique Lorentz-invariant, non-derivative coupling between χ and ψ that can be built from the two spinors alone. Hence m is the only free parameter of the free Dirac Lagrangian.

Connection to the homomorphism construction. Recalling the group-theoretic perspective developed in the SL(2,$\mathbb{C}$) homomorphism subsection above,

the four-vector representation decomposes as $(\frac{1}{2},\frac{1}{2}) = (\frac{1}{2},0)\otimes(0,\frac{1}{2})$, and $\sigma^\mu, \bar\sigma^\mu$ are the Clebsch–Gordan-like objects realising this decomposition concretely. Contracting with p_μ produces the unique linear Lorentz-covariant map between the two Weyl sectors; coupling these maps with a mass yields (9.80), the Dirac equation in the Weyl basis.

The antisymmetric tensor generators $\sigma^{\mu\nu}$ and $\bar\sigma^{\mu\nu}$

We define the antisymmetric tensor generators:

$$(\sigma^{\mu\nu})_\alpha{}^\beta \equiv \tfrac{i}{4}\left(\sigma^\mu_{\alpha\dot\alpha}\bar\sigma^{\nu\dot\alpha\beta} - \sigma^\nu_{\alpha\dot\alpha}\bar\sigma^{\mu\dot\alpha\beta}\right) \tag{9.81}$$

$$(\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta} \equiv \tfrac{i}{4}\left(\bar\sigma^{\mu\dot\alpha\alpha}\sigma^\nu_{\alpha\dot\beta} - \bar\sigma^{\nu\dot\alpha\alpha}\sigma^\mu_{\alpha\dot\beta}\right). \tag{9.82}$$

Equivalently, using eqs. (9.73) and (9.74):

$$(\sigma^\mu\bar\sigma^\nu)_\alpha{}^\beta = g^{\mu\nu}\delta^\beta_\alpha - 2i(\sigma^{\mu\nu})_\alpha{}^\beta \tag{9.83}$$

$$(\bar\sigma^\mu\sigma^\nu)^{\dot\alpha}{}_{\dot\beta} = g^{\mu\nu}\delta^{\dot\alpha}_{\dot\beta} - 2i(\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta}. \tag{9.84}$$

The components of $\sigma^{\mu\nu}$ and $\bar\sigma^{\mu\nu}$ are explicitly:

$$\boxed{\begin{aligned}\sigma^{ij} &= \bar\sigma^{ij} = \frac{1}{2}\epsilon^{ijk}\sigma^k \\ \sigma^{i0} &= -\sigma^{0i} = -\bar\sigma^{i0} = \bar\sigma^{0i} = \frac{i}{2}\sigma^i\end{aligned}}. \tag{9.85}$$

The hermiticity of $\vec\sigma$ implies that $(\sigma^{\mu\nu})^\dagger = \bar\sigma^{\mu\nu}$, or equivalently:

$$((\sigma^{\mu\nu})_\alpha{}^\beta)^* = (\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta}. \tag{9.86}$$

Proposition 9.4 (Self-duality relations). The matrices $\sigma^{\mu\nu}$ and $\bar\sigma^{\mu\nu}$ satisfy the self-duality relations:

$$\boxed{\sigma^{\mu\nu} = -\tfrac{i}{2}\epsilon^{\mu\nu\rho\kappa}\sigma_{\rho\kappa}, \quad \bar\sigma^{\mu\nu} = +\tfrac{i}{2}\epsilon^{\mu\nu\rho\kappa}\bar\sigma_{\rho\kappa}}, \tag{9.87}$$

where $\epsilon^{0123} = +1$.

The self-duality relations imply that $\sigma^{\mu\nu}$ and $\bar\sigma^{\mu\nu}$ each possess only three independent components. Hence, the total number of independent components of the bilinear covariants involving $\sigma^{\mu\nu}$ is indeed 3 (not 6).

These matrices satisfy:

$$S^{\mu\nu} = \begin{cases}\sigma^{\mu\nu} & \text{for the } (\frac{1}{2},0) \text{ representation} \\ \bar\sigma^{\mu\nu} & \text{for the } (0,\frac{1}{2}) \text{ representation}\end{cases}. \tag{9.88}$$

With the same active convention used in (9.44), the exponentiated Lorentz transformation matrices are:

$$M = \exp\left(-\frac{i}{2}\theta_{\mu\nu}\sigma^{\mu\nu}\right) \tag{9.89}$$

$$(M^{-1})^\dagger = \exp\left(-\frac{i}{2}\theta_{\mu\nu}\bar{\sigma}^{\mu\nu}\right). \tag{9.90}$$

From (9.13), $S^i = \frac{1}{2}\epsilon^{ijk}S^{jk} = \frac{1}{2}\sigma^i$ for both representations. Therefore $w^2 = -m^2\vec{S}^2$ has eigenvalues $-\frac{3}{4}m^2$ as expected for a spin-$\frac{1}{2}$ fermion.

Transformation laws for all spinor index positions

So far we have established the transformation laws for the two fundamental Weyl spinors in their *natural* index positions: lower undotted for L-type (eq. (9.45)) and upper dotted for R-type (eq. (9.47)):

$$\chi'_\alpha = M_\alpha{}^\beta \chi_\beta, \qquad \bar{\chi}'^{\dot{\alpha}} = \left[(M^{-1})^\dagger\right]^{\dot{\alpha}}{}_{\dot{\beta}}\,\bar{\chi}^{\dot{\beta}}. \tag{9.91}$$

But spinors also appear with *raised* undotted indices ($\chi^\alpha = \epsilon^{\alpha\beta}\chi_\beta$) and *lowered* dotted indices ($\bar{\chi}_{\dot{\alpha}} = (\chi_\alpha)^*$). How do *those* transform? The answer requires one group-theoretic ingredient: the invariance of the ϵ tensor under $SL(2,\mathbb{C})$.

The ϵ tensor is $SL(2,\mathbb{C})$-invariant

For any $M \in SL(2,\mathbb{C})$, the Levi-Civita symbol satisfies:

$$\epsilon^{\alpha\beta} M_\alpha{}^\gamma M_\beta{}^\delta = \epsilon^{\gamma\delta}. \tag{9.92}$$

Proof. The left-hand side is antisymmetric in (γ,δ), so it suffices to evaluate one component:

$$\epsilon^{\alpha\beta} M_\alpha{}^1 M_\beta{}^2 = M_1{}^1 M_2{}^2 - M_2{}^1 M_1{}^2 = \det M = 1 = \epsilon^{12}.$$

The general case follows by antisymmetry. □

This is the spinorial analogue of $g_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma = g_{\rho\sigma}$: just as the metric tensor is invariant under Lorentz transformations, the ϵ tensor is invariant under $SL(2,\mathbb{C})$. This invariance is what makes Lorentz-invariant spinor contractions possible. Since $\det M^* = (\det M)^* = 1$, the same identity holds for dotted indices:

$$\epsilon_{\dot{\alpha}\dot{\beta}}\left[(M^{-1})^\dagger\right]^{\dot{\alpha}}{}_{\dot{\gamma}}\left[(M^{-1})^\dagger\right]^{\dot{\beta}}{}_{\dot{\delta}} = \epsilon_{\dot{\gamma}\dot{\delta}}. \tag{9.93}$$

Spinors with raised undotted indices transform with (M^{-1})

Starting from $\chi'_\alpha = M_\alpha{}^\beta \chi_\beta$ and raising the free index with $\chi^\alpha = \epsilon^{\alpha\beta}\chi_\beta$:

$$\chi'^\alpha \;=\; \epsilon^{\alpha\gamma}\chi'_\gamma \;=\; \epsilon^{\alpha\gamma} M_\gamma{}^\beta \chi_\beta \,.$$

To express the right-hand side in terms of $\chi^\delta = \epsilon^{\delta\beta}\chi_\beta$, rewrite $\chi_\beta = \epsilon_{\beta\delta}\chi^\delta$:

$$\chi'^\alpha \;=\; \underbrace{\epsilon^{\alpha\gamma} M_\gamma{}^\beta \epsilon_{\beta\delta}}_{\equiv\; (M^{-1})^\alpha{}_\delta}\, \chi^\delta \,. \tag{9.94}$$

The notation $(M^{-1})^\alpha{}_\delta$ for the combination $\epsilon^{\alpha\gamma}M_\gamma{}^\beta\epsilon_{\beta\delta}$ is not merely a shorthand: contracting (9.92) with $\epsilon_{\gamma\sigma}$ one verifies that $(M^{-1})^\beta{}_\sigma\, M_\beta{}^\delta = \delta^\delta_\sigma$, so this matrix genuinely acts as the inverse of M in the appropriate index sense (in 2×2 matrix notation it equals $(M^\top)^{-1}$).

Spinors with lowered dotted indices: contragredient of $(M^{-1})^\dagger$

There are two equivalent routes to this law, and seeing both makes the structure transparent.

Route 1: lowering with ϵ from the $(M^{-1})^\dagger$ law. This is the exact dotted analogue of the undotted derivation above. Starting from $\bar\chi'^{\dot\alpha} = [(M^{-1})^\dagger]^{\dot\alpha}{}_{\dot\beta}\,\bar\chi^{\dot\beta}$ and lowering the free index with $\bar\chi_{\dot\alpha} = \epsilon_{\dot\alpha\dot\gamma}\bar\chi^{\dot\gamma}$:

$$\bar\chi'_{\dot\alpha} \;=\; \epsilon_{\dot\alpha\dot\gamma}\,\bar\chi'^{\dot\gamma} \;=\; \epsilon_{\dot\alpha\dot\gamma}\,[(M^{-1})^\dagger]^{\dot\gamma}{}_{\dot\delta}\,\bar\chi^{\dot\delta} \,.$$

Rewriting $\bar\chi^{\dot\delta} = \epsilon^{\dot\delta\dot\beta}\bar\chi_{\dot\beta}$:

$$\bar\chi'_{\dot\alpha} \;=\; \underbrace{\epsilon_{\dot\alpha\dot\gamma}\,[(M^{-1})^\dagger]^{\dot\gamma}{}_{\dot\delta}\,\epsilon^{\dot\delta\dot\beta}}_{\equiv\; [(M^{-1})^{\dagger\,-1}]_{\dot\alpha}{}^{\dot\beta}}\, \bar\chi_{\dot\beta} \,. \tag{9.95}$$

The structure is parallel to the undotted case: lowering a dotted index replaces $(M^{-1})^\dagger$ with its contragredient via ϵ, just as raising an undotted index replaces M with its contragredient (M^{-1}).

Route 2: complex conjugation. From $\bar\chi_{\dot\alpha} \equiv (\chi_\alpha)^*$ and $\chi'_\alpha = M_\alpha{}^\beta\chi_\beta$ one gets directly

$$\bar\chi'_{\dot\alpha} \;=\; (M_\alpha{}^\beta)^*\,\bar\chi_{\dot\beta} \;\equiv\; (M^*)_{\dot\alpha}{}^{\dot\beta}\,\bar\chi_{\dot\beta} \,,$$

where $(M^*)_{\dot\alpha}{}^{\dot\beta} \equiv (M_\alpha{}^\beta)^*$. The two routes agree because $(M^{-1})^{\dagger\,-1} = M^\dagger = (M^*)^\top$, so the contragredient matrix $\epsilon_{\dot\alpha\dot\gamma}[(M^{-1})^\dagger]^{\dot\gamma}{}_{\dot\delta}\epsilon^{\dot\delta\dot\beta}$ has components $(M^*)_{\dot\alpha}{}^{\dot\beta}$.

Summary

Collecting all four transformation laws for a Lorentz transformation $\Lambda \leftrightarrow \pm M \in SL(2,\mathbb{C})$:

$$\begin{array}{lll}
\chi_\alpha \longrightarrow M_\alpha{}^\beta \, \chi_\beta & & \text{(L-type, lower undotted)} \\
\chi^\alpha \longrightarrow (M^{-1})^\alpha{}_\beta \, \chi^\beta & & \text{(L-type, upper undotted)} \\
\bar\chi^{\dot\alpha} \longrightarrow [(M^{-1})^\dagger]^{\dot\alpha}{}_{\dot\beta} \, \bar\chi^{\dot\beta} & & \text{(R-type, upper dotted)} \\
\bar\chi_{\dot\alpha} \longrightarrow (M^*)_{\dot\alpha}{}^{\dot\beta} \, \bar\chi_{\dot\beta} = \epsilon_{\dot\alpha\dot\gamma} \, [(M^{-1})^\dagger]^{\dot\gamma}{}_{\dot\delta} \, \epsilon^{\dot\delta\dot\beta} \, \bar\chi_{\dot\beta} & & \text{(R-type, lower dotted)}
\end{array} \tag{9.96}$$

The pattern is symmetric between the two chiralities. Within each sector, the spinor metric ϵ switches between a representation and its contragredient:

$$\underbrace{M}_{\text{acts on } \chi_\alpha} \xrightarrow{\epsilon} \underbrace{(M^{-1})}_{\text{acts on } \chi^\alpha}, \qquad \underbrace{(M^{-1})^\dagger}_{\text{acts on } \bar\chi^{\dot\alpha}} \xrightarrow{\epsilon} \underbrace{M^*}_{\text{acts on } \bar\chi_{\dot\alpha}} .$$

Complex conjugation maps between chiralities: $M_\alpha{}^\beta \leftrightarrow (M^*)_{\dot\alpha}{}^{\dot\beta}$ and $(M^{-1})^\alpha{}_\beta \leftrightarrow [(M^{-1})^\dagger]^{\dot\alpha}{}_{\dot\beta}$.

With these transformation laws in hand, we can now systematically construct all Lorentz-invariant and Lorentz-covariant spinor bilinears.

9.4 Bilinear covariants

Spinor products and contractions

Lorentz-invariant spinor products

Box 9.4 The descending/ascending staircase rule (NW–SE vs. SW–NE)

The shorthand notations $\chi\eta$ and $\bar\chi\bar\eta$ are not symmetric in the index height: each is defined by a *fixed* contraction pattern that depends on the spinor type. Stated graphically:

- **Undotted (left-handed, $(\frac{1}{2},0)$) spinors are contracted "descending" — from *NorthWest* (upper-left) to *SouthEast* (lower-right)**:

  $$\chi\eta \equiv \chi^\alpha \eta_\alpha = \chi^\alpha \searrow \eta_\alpha .$$

  The left spinor carries the index *upstairs*, the right one carries it *downstairs*; the contraction goes top-left → bottom-right.

- **Dotted (right-handed, $(0,\frac{1}{2})$) spinors are contracted "ascending" — from *SouthWest* (lower-left) to *NorthEast* (upper-right)**:

$$\bar{\chi}\bar{\eta} \equiv \bar{\chi}_{\dot{\alpha}}\bar{\eta}^{\dot{\alpha}} = \bar{\chi}_{\dot{\alpha}}\nearrow\bar{\eta}^{\dot{\alpha}}.$$

  The left spinor carries the index *downstairs*, the right one carries it *upstairs*; the contraction goes bottom-left → top-right.

The reason for the asymmetry is the relative sign $\epsilon^{12} = +1$, $\epsilon_{12} = -1$ between the metric used to raise undotted and lower dotted indices (Aitchison–Wess–Bagger convention; the same conventions are recorded in (†)).

Reversing the staircase changes the sign of the bilinear. Concretely:

$$\chi_\alpha \eta^\alpha = -\chi^\alpha \eta_\alpha = -\chi\eta.$$

The proof is one line. Raise the index on η using $\eta^\alpha = \epsilon^{\alpha\beta}\eta_\beta$:

$$\chi_\alpha \eta^\alpha = \chi_\alpha \epsilon^{\alpha\beta}\eta_\beta = \epsilon^{\alpha\beta}\chi_\alpha\eta_\beta.$$

Now relabel the dummy indices $\alpha \leftrightarrow \beta$ and use the antisymmetry $\epsilon^{\beta\alpha} = -\epsilon^{\alpha\beta}$:

$$\epsilon^{\alpha\beta}\chi_\alpha\eta_\beta = \epsilon^{\beta\alpha}\chi_\beta\eta_\alpha = -\epsilon^{\alpha\beta}\chi_\beta\eta_\alpha = -\chi^\alpha\eta_\alpha = -\chi\eta,$$

where in the last step we used the definition (9.97). The sign flip comes entirely from the antisymmetry of ϵ — not from any reordering of χ and η. (For Grassmann spinors there is a *separate* sign from the anticommutation, treated below in (9.101); the two effects must be tracked independently.) An analogous calculation for dotted indices gives $\bar{\chi}^{\dot{\alpha}}\bar{\eta}_{\dot{\alpha}} = -\bar{\chi}_{\dot{\alpha}}\bar{\eta}^{\dot{\alpha}} = -\bar{\chi}\bar{\eta}$.

This staircase rule is the single most frequent source of sign errors in two-component spinor calculations; once it is internalised, the unindexed shorthand $\chi\eta, \bar{\chi}\bar{\eta}$ is unambiguous.

Given two $(\frac{1}{2},0)$ fermion fields χ_α and η_α, the bilinear combination:

$$\boxed{\chi\eta \equiv \chi^\alpha\eta_\alpha = \epsilon^{\alpha\beta}\chi_\beta\eta_\alpha} \qquad (\text{NW} \to \text{SE}) \tag{9.97}$$

is invariant under Lorentz transformations (proved below). The ϵ tensor with $\epsilon^{12} = -\epsilon_{12} = +1$ is used to raise and lower spinor indices.

Similarly, for the R-type partners $\bar{\chi}_{\dot{\alpha}}$ and $\bar{\eta}_{\dot{\alpha}}$ defined via (9.43):

$$\boxed{\bar{\chi}\bar{\eta} \equiv \bar{\chi}_{\dot{\alpha}}\bar{\eta}^{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\chi}^{\dot{\beta}}\bar{\eta}^{\dot{\alpha}}} \qquad (\text{SW} \to \text{NE}). \tag{9.98}$$

Proof of Lorentz invariance of $\chi\eta$ and $\bar\chi\bar\eta$. Under $\chi_\alpha \to M_\alpha{}^\beta \chi_\beta$, $\eta_\alpha \to M_\alpha{}^\beta \eta_\beta$:

$$(\chi\eta)' = \epsilon^{\alpha\beta}\chi'_\beta\eta'_\alpha = \epsilon^{\alpha\beta}\,(M_\beta{}^\gamma\chi_\gamma)\,(M_\alpha{}^\delta\eta_\delta) = \underbrace{\epsilon^{\alpha\beta}M_\alpha{}^\delta M_\beta{}^\gamma}_{\epsilon^{\delta\gamma}\text{ by (9.92)}}\chi_\gamma\eta_\delta$$

$$= \epsilon^{\delta\gamma}\chi_\gamma\eta_\delta = \chi^\delta\eta_\delta = \chi\eta\,.$$

For the dotted product, an identical computation using $(M^*)_{\dot\alpha}{}^{\dot\beta}$ (which also has unit determinant, so the analogous identity (9.93) holds) gives $(\bar\chi\bar\eta)' = \bar\chi\bar\eta$. Alternatively, invariance of $\bar\chi\bar\eta$ follows by complex conjugation of $(\chi\eta)' = \chi\eta$. □

Commuting and anticommuting spinors

Box 9.5 What do we mean by "commuting" vs "anticommuting" spinors?

The same symbol χ_α is used in this chapter for two *different* kinds of mathematical objects, which obey opposite statistics:

- **Anticommuting (Grassmann-valued) spinors.** The fundamental quantum fermion fields appearing in a Lagrangian, e.g. the operator $\hat{\chi}_\alpha(x)$ that creates/annihilates a left-handed Weyl particle, are Grassmann-odd: they satisfy

$$\{\hat{\chi}_\alpha(x), \hat{\eta}_\beta(y)\} = 0, \qquad \{\hat{\chi}_\alpha(x), \hat{\chi}^\dagger_\beta(y)\}\big|_{\text{eq. time}} \sim \delta^3(\vec{x}-\vec{y}),$$

  and *anticommute past each other*: $\chi_\alpha\eta_\beta = -\eta_\beta\chi_\alpha$. This is the Pauli exclusion principle written algebraically; it applies to fermionic quantum fields, to Grassmann SUSY parameters θ^α (Chapter 10), and to the spinorial integration variables in path integrals.
- **Commuting (c-number-valued) spinors.** The plane-wave amplitudes $u_\alpha(p,s)$, $v_\alpha(p,s)$ that emerge when one evaluates a Feynman diagram (after the field operators have been contracted with external states) are ordinary $\mathbb{C}$-valued 2-component objects. They obey ordinary number multiplication: $u_\alpha v_\beta = v_\beta u_\alpha$. The same applies to coherent-state wavefunctions and to "test" spinors used in proving Fierz identities.

The two cases differ *only* in the sign that arises when one swaps adjacent spinors in a product. To handle both uniformly we introduce the symbol $(-1)^A$ defined in (9.99) below: $(-1)^A = -1$ for Grassmann (anticommuting) spinors and $(-1)^A = +1$ for $\mathbb{C}$-valued (commuting) spinors. Every identity in (9.100) then specialises to the correct sign in either context.
A surprising and important consequence: for *anticommuting* spinors, $\chi\eta = \eta\chi$ (not $-\eta\chi$!), proved in (9.101) below. The two minus signs — one from antisymmetry of ϵ, one from Grassmann swap — combine to a plus. This is what makes $\chi\eta$ a sensible mass term in a fermionic Lagrangian.

In manipulating expressions, we often deal with both anticommuting fermion quantum fields and commuting spinor wave functions (that arise when evaluating Feynman rules). It is convenient to introduce the notation:

$$(-1)^A \equiv \begin{cases} +1 & \text{commuting spinors} \\ -1 & \text{anticommuting spinors} \end{cases}. \tag{9.99}$$

The following identities hold for generic spinors z_i:

$$\boxed{\begin{aligned} z_1 z_2 &= -(-1)^A z_2 z_1 \\ \bar{z}_1 \bar{z}_2 &= -(-1)^A \bar{z}_2 \bar{z}_1 \\ z_1 \sigma^\mu \bar{z}_2 &= (-1)^A \bar{z}_2 \bar{\sigma}^\mu z_1 \\ z_1 \sigma^\mu \bar{\sigma}^\nu z_2 &= -(-1)^A z_2 \sigma^\nu \bar{\sigma}^\mu z_1 \\ \bar{z}_1 \bar{\sigma}^\mu \sigma^\nu \bar{z}_2 &= -(-1)^A \bar{z}_2 \bar{\sigma}^\nu \sigma^\mu \bar{z}_1 \\ \bar{z}_1 \bar{\sigma}^\mu \sigma^\rho \bar{\sigma}^\nu z_2 &= (-1)^A z_2 \sigma^\nu \bar{\sigma}^\rho \sigma^\mu \bar{z}_1 \end{aligned}} \,. \tag{9.100}$$

For **anticommuting** two-component fermion fields (the case $(-1)^A = -1$):

$$\chi\eta = \eta\chi, \qquad \bar{\chi}\bar{\eta} = \bar{\eta}\bar{\chi}\,. \tag{9.101}$$

Proof. Using the antisymmetry of $\epsilon^{\alpha\beta}$:

$$\chi\eta = \epsilon^{\alpha\beta}\chi_\beta\eta_\alpha = -\epsilon^{\alpha\beta}\eta_\alpha\chi_\beta = \epsilon^{\beta\alpha}\eta_\alpha\chi_\beta = \eta\chi$$

where we used $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}$ and relabeled the dummy indices. The proof for $\bar{\chi}\bar{\eta}$ is analogous. □

Under hermitian conjugation (valid for both commuting and anticommuting spinors):

$$(\chi\eta)^\dagger = \bar{\eta}\bar{\chi}\,. \tag{9.102}$$

Combining with (9.101) for anticommuting spinors yields:

$$(\chi\eta)^\dagger = \bar{\chi}\bar{\eta}\,. \tag{9.103}$$

Bilinear covariants and Fierz identities

Four-vector bilinears

The spinor products:

$$\begin{aligned} &\chi\sigma^\mu\bar{\eta} \equiv \chi^\alpha \sigma^\mu_{\alpha\dot{\beta}} \bar{\eta}^{\dot{\beta}}, \\ &\bar{\eta}\bar{\sigma}^\mu\chi \equiv \bar{\eta}_{\dot{\alpha}} \bar{\sigma}^{\mu\,\dot{\alpha}\alpha} \chi_\alpha \end{aligned} \tag{9.104}$$

transform as Lorentz four-vectors.

Proof. Using the transformation laws $\chi^\alpha \to (M^{-1})^\alpha{}_\beta \chi^\beta$ and $\bar{\eta}^{\dot{\alpha}} \to ((M^{-1})^\dagger)^{\dot{\alpha}}{}_{\dot{\beta}} \bar{\eta}^{\dot{\beta}}$:

$$\begin{aligned} \chi'(x')\sigma^\mu\bar{\eta}'(x') &= \chi^\alpha(x)[M^{-1}\sigma^\mu(M^{-1})^\dagger]_{\alpha\dot{\alpha}}\bar{\eta}^{\dot{\alpha}}(x) \\ &= \Lambda^\mu{}_\nu \chi(x)\sigma^\nu\bar{\eta}(x) \end{aligned} \tag{9.105}$$

where we used (9.59). □

For anticommuting spinors:

$$\bar{\eta}\bar{\sigma}^\mu\chi = -\chi\sigma^\mu\bar{\eta}, \qquad (\chi\sigma^\mu\bar{\eta})^\dagger = -\bar{\chi}\bar{\sigma}^\mu\eta\,. \tag{9.106}$$

Complete list of bilinear covariants

The independent bilinear covariants constructed from two-component spinors are:

Bilinear	Components	Lorentz transformation
$\chi\eta$	1	Scalar
$\bar{\chi}\bar{\eta}$	1	Scalar
$\chi\sigma^{\mu}\bar{\eta}$	4	Four-vector
$\bar{\chi}\bar{\sigma}^{\mu}\eta$	4	Four-vector
$\chi\sigma^{\mu\nu}\eta$	3	Self-dual antisymmetric tensor
$\bar{\chi}\bar{\sigma}^{\mu\nu}\bar{\eta}$	3	Anti-self-dual antisymmetric tensor

The total is 16 independent components, matching the dimension of the product space of two independent 2×2 matrix spaces.

Fierz identities

Products of two bilinear covariants can be simplified using Fierz identities. The following hold for *both* commuting and anticommuting two-component spinors z_i (in each identity, the spinor rearrangements involve an even number of transpositions, so the net sign is statistics-independent):

$$(z_1 z_2)(z_3 z_4) = -(z_1 z_3)(z_4 z_2) - (z_1 z_4)(z_2 z_3) \tag{9.107}$$

$$(z_1\sigma^{\mu}\bar{z}_2)(\bar{z}_3\bar{\sigma}_{\mu}z_4) = -2(z_1 z_4)(\bar{z}_2\bar{z}_3) \tag{9.108}$$

$$(\bar{z}_1\bar{\sigma}^{\mu}z_2)(\bar{z}_3\bar{\sigma}_{\mu}z_4) = 2(\bar{z}_1\bar{z}_3)(z_4 z_2) \tag{9.109}$$

$$(z_1\sigma^{\mu}\bar{z}_2)(z_3\sigma_{\mu}\bar{z}_4) = 2(z_1 z_3)(\bar{z}_4\bar{z}_2)\,. \tag{9.110}$$

Proof of (9.108). Using (9.70):

$$\begin{aligned}(z_1\sigma^{\mu}\bar{z}_2)(\bar{z}_3\bar{\sigma}_{\mu}z_4) &= z_1^{\alpha}\sigma^{\mu}_{\alpha\dot{\beta}}\bar{z}_2^{\dot{\beta}}\cdot\bar{z}_{3\dot{\gamma}}\bar{\sigma}_{\mu}^{\dot{\gamma}\delta}z_{4\delta}\\ &= z_1^{\alpha}\bar{z}_2^{\dot{\beta}}\bar{z}_{3\dot{\gamma}}z_{4\delta}\cdot\underbrace{\sigma^{\mu}_{\alpha\dot{\beta}}\bar{\sigma}_{\mu}^{\dot{\gamma}\delta}}_{2\delta_{\alpha}^{\delta}\delta_{\dot{\beta}}^{\dot{\gamma}}}\\ &= 2\,z_1^{\alpha}\bar{z}_2^{\dot{\beta}}\bar{z}_{3\dot{\beta}}z_{4\alpha}\,.\end{aligned} \tag{9.111}$$

Rearranging: move z_4 past the two barred spinors (two transpositions, net sign $(-1)^{2A}=1$ for either statistics):

$$= 2\,(z_1^{\alpha}z_{4\alpha})\,(\bar{z}_2^{\dot{\beta}}\bar{z}_{3\dot{\beta}}) = -2\,(z_1 z_4)\,(\bar{z}_2\bar{z}_3)\,,$$

where the final minus sign comes from the dotted staircase reversal $\bar{z}_2^{\dot{\beta}}\bar{z}_{3\dot{\beta}} = -\bar{z}_{2\dot{\beta}}\bar{z}_3^{\dot{\beta}} = -\bar{z}_2\bar{z}_3$ (cf. box 9.4), not from the spinor rearrangement. □

9.5 Four-component spinors

In this section, we establish the connection between the two-component spinor formalism developed above and the more familiar four-component Dirac spinor formalism. The motivation for introducing four-component spinors is analogous to the motivation for introducing complex scalar fields: when two real, mass-degenerate scalar fields ϕ_1 and ϕ_2 are present, it is convenient to combine them into a single complex field $\Phi = (\phi_1 + i\phi_2)/\sqrt{2}$. Similarly, when two mass-degenerate two-component spinors χ and η of opposite U(1) charge are present, it is natural to combine them into a single four-component spinor.

The Dirac spinor and gamma matrices

Definition of the four-component Dirac spinor

A **four-component Dirac spinor field** $\Psi(x)$ is constructed from two mass-degenerate two-component spinor fields $\chi_\alpha(x)$ and $\eta_\alpha(x)$ of opposite U(1) charge:

$$\boxed{\Psi(x) \equiv \begin{pmatrix} \chi_\alpha(x) \\ \bar{\eta}^{\dot{\alpha}}(x) \end{pmatrix}}. \tag{9.112}$$

The upper two components transform under the $(\frac{1}{2}, 0)$ representation, while the lower two components transform under the $(0, \frac{1}{2})$ representation. Thus, Ψ transforms in the reducible $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation of the Lorentz group.

Dirac gamma matrices

The **Dirac gamma matrices** γ^μ are 4×4 matrices that can be expressed in 2×2 block form using the sigma matrices:

$$\boxed{\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu_{\alpha\dot{\beta}} \\ \bar{\sigma}^{\mu\,\dot{\alpha}\beta} & 0 \end{pmatrix}}. \tag{9.113}$$

This is known as the **chiral representation** (or Weyl representation) of the gamma matrices.

Proposition 9.5 (Clifford algebra). The gamma matrices satisfy the anticommutation relation:

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}_4}, \tag{9.114}$$

where $\mathbb{1}_4$ is the 4×4 identity matrix.

Proof. Computing the anticommutator in block form:

$$\begin{aligned}\{\gamma^\mu,\gamma^\nu\} &= \gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu \\ &= \begin{pmatrix} \sigma^\mu\bar\sigma^\nu+\sigma^\nu\bar\sigma^\mu & 0 \\ 0 & \bar\sigma^\mu\sigma^\nu+\bar\sigma^\nu\sigma^\mu \end{pmatrix}\end{aligned} \tag{9.115}$$

Using the identities (9.73) and (9.74):

$$[\sigma^\mu\bar\sigma^\nu+\sigma^\nu\bar\sigma^\mu]_\alpha{}^\beta = 2g^{\mu\nu}\delta^\beta_\alpha, \quad [\bar\sigma^\mu\sigma^\nu+\bar\sigma^\nu\sigma^\mu]^{\dot\alpha}{}_{\dot\beta} = 2g^{\mu\nu}\delta^{\dot\alpha}_{\dot\beta}$$

Therefore $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}_4$. □

The 4×4 identity matrix in the chiral representation has the block form:

$$\mathbb{1}_4 = \begin{pmatrix} \delta_\alpha{}^\beta & 0 \\ 0 & \delta^{\dot\alpha}{}_{\dot\beta} \end{pmatrix}. \tag{9.116}$$

The chirality matrix γ^5

The **chirality matrix** γ^5 is defined by:

$$\boxed{\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma}, \tag{9.117}$$

where $\epsilon^{0123} = +1$. In the chiral representation:

$$\gamma^5 = \begin{pmatrix} -\delta_\alpha{}^\beta & 0 \\ 0 & \delta^{\dot\alpha}{}_{\dot\beta} \end{pmatrix} = \begin{pmatrix} -\mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix}. \tag{9.118}$$

Proposition 9.6 (Properties of γ^5). Independently of the representation of the gamma matrices:

$$(\gamma^5)^2 = \mathbb{1}_4, \qquad \{\gamma^\mu,\gamma^5\} = 0. \tag{9.119}$$

Proof. Using the definition (9.117) and the Clifford algebra (9.114):

$$\begin{aligned}(\gamma^5)^2 &= (i)^2\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 \\ &= -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3\end{aligned} \tag{9.120}$$

Using $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu + 2g^{\mu\nu}$ repeatedly to move γ^0 to the right (picking up three sign changes from anticommuting with $\gamma^1, \gamma^2, \gamma^3$), then $(\gamma^0)^2 = \mathbb{1}_4$, and similarly for the other matrices, one obtains $(\gamma^5)^2 = \mathbb{1}_4$.

For the anticommutator, $\gamma^\mu\gamma^5 = i\gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3$. Moving γ^μ through the four gamma matrices produces $(-1)^3 = -1$ sign changes (one for each γ^ν with $\nu \neq \mu$), giving $\gamma^\mu\gamma^5 = -\gamma^5\gamma^\mu$. □

Chiral projection operators

The **chiral projection operators** are defined as:

$$\boxed{P_L \equiv \frac{1}{2}(\mathbb{1}_4 - \gamma^5), \qquad P_R \equiv \frac{1}{2}(\mathbb{1}_4 + \gamma^5)}. \tag{9.121}$$

In the chiral representation:

$$P_L = \begin{pmatrix} \delta_\alpha{}^\beta & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_R = \begin{pmatrix} 0 & 0 \\ 0 & \delta^{\dot\alpha}{}_{\dot\beta} \end{pmatrix}. \tag{9.122}$$

Proposition 9.7 (Properties of projection operators).

$$P_L^2 = P_L, \quad P_R^2 = P_R, \quad P_L P_R = P_R P_L = 0, \quad P_L + P_R = \mathbb{1}_4\,. \tag{9.123}$$

Proof. These follow directly from $(\gamma^5)^2 = \mathbb{1}_4$:

$$P_L^2 = \tfrac{1}{4}(\mathbb{1}_4 - \gamma^5)^2 = \tfrac{1}{4}(\mathbb{1}_4 - 2\gamma^5 + (\gamma^5)^2) = \tfrac{1}{2}(\mathbb{1}_4 - \gamma^5) = P_L \tag{9.124}$$

$$P_L P_R = \tfrac{1}{4}(\mathbb{1}_4 - \gamma^5)(\mathbb{1}_4 + \gamma^5) = \tfrac{1}{4}(\mathbb{1}_4 - (\gamma^5)^2) = 0 \tag{9.125}$$

The remaining identities follow similarly. □

The **left-handed** and **right-handed Weyl spinor fields** are defined as the eigenstates of γ^5:

$$\Psi_L(x) \equiv P_L\Psi(x) = \begin{pmatrix} \chi_\alpha(x) \\ 0 \end{pmatrix}, \quad \Psi_R(x) \equiv P_R\Psi(x) = \begin{pmatrix} 0 \\ \bar\eta^{\dot\alpha}(x) \end{pmatrix}, \tag{9.126}$$

satisfying $\gamma^5\Psi_L = -\Psi_L$ and $\gamma^5\Psi_R = +\Psi_R$.

Lorentz structure and transformations

Lorentz generators in four-component form

The generators of the Lorentz group in the $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation are:

$$\boxed{\frac{1}{2}\Sigma^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu,\gamma^\nu] = \begin{pmatrix} (\sigma^{\mu\nu})_\alpha{}^\beta & 0 \\ 0 & (\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta} \end{pmatrix}}. \tag{9.127}$$

Proof. Computing the commutator in block form:

$$\begin{aligned}[\gamma^\mu,\gamma^\nu] &= \gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu \\ &= \begin{pmatrix} \sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu & 0 \\ 0 & \bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu \end{pmatrix}\end{aligned} \tag{9.128}$$

From eqs. (9.83) and (9.84):

$$\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu = -2i(\sigma^{\mu\nu} - \sigma^{\nu\mu}) = -4i\sigma^{\mu\nu}$$

and similarly for the dotted indices. Therefore $\frac{i}{4}[\gamma^\mu,\gamma^\nu] = \mathrm{diag}(\sigma^{\mu\nu},\bar\sigma^{\mu\nu})$, or equivalently $\Sigma^{\mu\nu} = 2\,\mathrm{diag}(\sigma^{\mu\nu},\bar\sigma^{\mu\nu})$. □

The duality conditions (9.87) for $\sigma^{\mu\nu}$ and $\bar\sigma^{\mu\nu}$ imply:

$$\gamma^5\Sigma^{\mu\nu} = \frac{i}{2}\epsilon^{\mu\nu\rho\kappa}\Sigma_{\rho\kappa}\,. \tag{9.129}$$

The spin-$\frac{1}{2}$ angular momentum matrices are $\{\frac{1}{2}\Sigma^i\}$, $i=1,2,3$, where:

$$\Sigma^i \equiv \frac{1}{2}\epsilon^{ijk}\Sigma_{jk} = \gamma^0\gamma^i\gamma^5 = \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}. \tag{9.130}$$

Gamma matrix identities

The following identities are useful for calculations:

$$\begin{aligned}\gamma^\mu\gamma^\nu\gamma^\rho &= g^{\mu\nu}\gamma^\rho - g^{\mu\rho}\gamma^\nu + g^{\nu\rho}\gamma^\mu \\ &\quad + i\epsilon^{\mu\nu\rho\kappa}\gamma_\kappa\gamma^5 && (9.131)\\ \Sigma^{\mu\nu}\gamma^\rho &= i\big(g^{\nu\rho}\gamma^\mu - g^{\mu\rho}\gamma^\nu\big) - \epsilon^{\mu\nu\rho\kappa}\gamma_\kappa\gamma^5 && (9.132)\\ \gamma^\rho\Sigma^{\mu\nu} &= i\big(g^{\mu\rho}\gamma^\nu - g^{\nu\rho}\gamma^\mu\big) - \epsilon^{\mu\nu\rho\kappa}\gamma_\kappa\gamma^5\,. && (9.133)\end{aligned}$$

Proof of (9.131). Using the Clifford algebra $\gamma^\mu\gamma^\nu = g^{\mu\nu} - i\Sigma^{\mu\nu}$:

$$\begin{aligned}\gamma^\mu\gamma^\nu\gamma^\rho &= (g^{\mu\nu} - i\Sigma^{\mu\nu})\gamma^\rho \\ &= g^{\mu\nu}\gamma^\rho - i\Sigma^{\mu\nu}\gamma^\rho\end{aligned} \tag{9.134}$$

Using eq. (9.132) and simplifying yields the result. □

Gamma matrix trace identities

Proposition 9.8 (Trace formulas). The following trace identities hold:

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu} \tag{9.135}$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\kappa) = 4(g^{\mu\nu}g^{\rho\kappa} - g^{\mu\rho}g^{\nu\kappa} + g^{\mu\kappa}g^{\nu\rho}) \tag{9.136}$$

$$\mathrm{Tr}(\Sigma^{\mu\nu}\Sigma^{\rho\kappa}) = 4(g^{\mu\rho}g^{\nu\kappa} - g^{\nu\rho}g^{\mu\kappa}) \tag{9.137}$$

$$\mathrm{Tr}(\Sigma^{\mu\nu}) = 0 \tag{9.138}$$

$$\mathrm{Tr}(\gamma^5) = \mathrm{Tr}(\gamma^5\gamma^\mu) = \mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu) = \mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho) = 0 \tag{9.139}$$

$$\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\kappa) = -4i\epsilon^{\mu\nu\rho\kappa}\,. \tag{9.140}$$

Traces of an odd number of gamma matrices vanish.

Proof of (9.135). Using the Clifford algebra:

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = \tfrac{1}{2}\mathrm{Tr}(\{\gamma^\mu,\gamma^\nu\}) = \tfrac{1}{2}\mathrm{Tr}(2g^{\mu\nu}\mathbb{1}_4) = 4g^{\mu\nu}$$

□

Proof that odd traces vanish. For any string of gamma matrices Γ:

$$\mathrm{Tr}(\Gamma) = \mathrm{Tr}((\gamma^5)^2\Gamma) = \mathrm{Tr}(\gamma^5\Gamma\gamma^5)$$

If Γ consists of an odd number of gamma matrices, then $\gamma^5\Gamma\gamma^5 = -\Gamma$ (using $\{\gamma^\mu,\gamma^5\} = 0$ an odd number of times). Therefore $\mathrm{Tr}(\Gamma) = -\mathrm{Tr}(\Gamma)$, which implies $\mathrm{Tr}(\Gamma) = 0$. □

The Dirac adjoint spinor

The **Dirac adjoint** (or **Dirac conjugate**) spinor is defined as:

$$\boxed{\bar\Psi(x) \equiv \Psi^\dagger(x)\gamma^0 = (\eta^\alpha(x), \bar\chi_{\dot\alpha}(x))}\,, \tag{9.141}$$

where we have used the explicit form of γ^0 in the chiral representation:

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^0 \\ \bar\sigma^0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}. \tag{9.142}$$

Remark. [Consistency with two-component formalism] Note that $\bar{\Psi}\Psi = \eta^\alpha \chi_\alpha + \bar{\chi}_{\dot{\alpha}} \bar{\eta}^{\dot{\alpha}} = \eta\chi + \bar{\chi}\bar{\eta}$, which matches the scalar bilinear in the two-component formalism.

Remark. [The bar notation: $\bar{\Psi}$ vs. $\bar{\chi}, \bar{\eta}, \bar{\psi}$] The overline symbol carries *two distinct meanings* in these notes, and the two should not be confused:

- **Dirac adjoint of a four-component spinor.** The symbol $\bar{\Psi}$ denotes the Dirac conjugate $\bar{\Psi} \equiv \Psi^\dagger \gamma^0$ of a four-component field Ψ. It is a *row* object built from $\Psi^\dagger$ by multiplication with γ^0. The bar is, in this case, shorthand for the composite operation $(\,\cdot\,)^\dagger \gamma^0$.
- **Dotted-index (R-type) two-component spinor.** The symbols $\bar{\chi}^{\dot{\alpha}}, \bar{\eta}^{\dot{\alpha}}, \bar{\psi}^{\dot{\alpha}}$ denote two-component Weyl spinors transforming in the $(0, \frac{1}{2})$ representation; they are the dotted-index partners of the undotted $(\frac{1}{2}, 0)$ spinors $\chi_\alpha, \eta_\alpha, \psi_\alpha$. In some textbooks (Aitchison's Lorentz chapter, early Weinberg) this object is instead written $\chi^{\dagger\dot{\alpha}}$, $\eta^{\dagger\dot{\alpha}}$, $\psi^{\dagger\dot{\alpha}}$ ("dagger notation"). We have unified on the bar notation throughout to match the modern conventions used in Chapter 10 (Wess–Bagger, Aitchison's SUSY chapter, Srednicki).

The two usages are disambiguated by the *type of the parent field*: if the symbol under the bar is a two-component $(\frac{1}{2}, 0)$ Weyl spinor (lowercase Greek $\chi_\alpha, \eta_\alpha, \psi_\alpha, \xi_\alpha, \ldots$), then the bar denotes its $(0, \frac{1}{2})$ partner $\bar{\chi}_{\dot{\alpha}} \equiv (\chi_\alpha)^*$ as in (9.43); if the symbol is a four-component Dirac field (capital Ψ), then the bar denotes the Dirac adjoint $\Psi^\dagger \gamma^0$. The typographical collision is genuine and worth flagging once, but the parent field always determines the meaning unambiguously.

Equivalently, one can read $\bar{\Psi}$ as the Dirac analogue of the Weyl bar: just as $\bar{\chi}$ pairs an undotted L-type with its dotted R-type, $\bar{\Psi}$ pairs a four-spinor with its "charge–parity rearranged" adjoint $\Psi^\dagger \gamma^0$. In the chiral basis, $\bar{\Psi}$ explicitly consists of the bar components of the underlying two-component fields: $\bar{\Psi} = (\eta^\alpha, \bar{\chi}_{\dot{\alpha}})$, see (9.141).

Lorentz transformation of four-component spinors

Under a Lorentz transformation $x'^\mu = \Lambda^\mu{}_\nu x^\nu$, the four-component spinor transforms as:

$$\Psi'(x') = \mathbb{M}\Psi(x), \qquad \bar{\Psi}'(x') = \bar{\Psi}(x)\mathbb{M}^{-1}, \tag{9.143}$$

where the 4×4 representation matrix is:

$$\mathbb{M} = \begin{pmatrix} M & 0 \\ 0 & (M^{-1})^\dagger \end{pmatrix} = \exp\left(-\frac{i}{4}\theta_{\mu\nu}\Sigma^{\mu\nu}\right), \tag{9.144}$$

with M and $(M^{-1})^\dagger$ given by eqs. (9.44) and (9.46).

Transformation law for $\bar\Psi$. From $\bar\Psi = \Psi^\dagger\gamma^0$ and using the identity $\gamma^0\mathbb{M}\gamma^0 = (\mathbb{M}^{-1})^\dagger$:

$$\begin{aligned}\bar\Psi'(x') &= \Psi'^\dagger(x')\gamma^0 = (\mathbb{M}\Psi)^\dagger\gamma^0 = \Psi^\dagger\mathbb{M}^\dagger\gamma^0 \\ &= \Psi^\dagger\gamma^0(\gamma^0\mathbb{M}^\dagger\gamma^0) = \bar\Psi\mathbb{M}^{-1}\end{aligned} \tag{9.145}$$

□

It follows immediately that $\bar\Psi\Psi$ is a Lorentz scalar. More generally, the gamma matrices transform as:

$$\mathbb{M}^{-1}\gamma^\mu\mathbb{M} = \Lambda^\mu{}_\nu\gamma^\nu\,, \tag{9.146}$$

which ensures that $\bar\Psi\gamma^\mu\Psi$ transforms as a four-vector, $\bar\Psi\Sigma^{\mu\nu}\Psi$ as an antisymmetric tensor, etc.

The Dirac equation

The field equation satisfied by $\Psi(x)$ is the **Dirac equation**:

$$\boxed{(i\gamma^\mu\partial_\mu - m)\Psi = 0}\,. \tag{9.147}$$

Derivation from two-component equations. The Dirac equation is equivalent to the following coupled two-component equations. Writing it out in block form:

$$i\begin{pmatrix}0 & \sigma^\mu \\ \bar\sigma^\mu & 0\end{pmatrix}\partial_\mu\begin{pmatrix}\chi \\ \bar\eta\end{pmatrix} - m\begin{pmatrix}\chi \\ \bar\eta\end{pmatrix} = 0$$

This gives the coupled equations:

$$i\sigma^\mu\partial_\mu\bar\eta - m\chi = 0 \tag{9.148}$$

$$i\bar\sigma^\mu\partial_\mu\chi - m\bar\eta = 0 \tag{9.149}$$

which match the two-component field equations. □

Applying $(-i\gamma^\nu\partial_\nu - m)$ from the left:

$$(\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + m^2)\Psi = (\partial^2 + m^2)\Psi = 0 \tag{9.150}$$

using $\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu = g^{\mu\nu}\partial_\mu\partial_\nu = \partial^2$. Thus each component of Ψ satisfies the Klein-Gordon equation.

Charge conjugation and Majorana spinors

Charge conjugation

The **charge-conjugated spinor** is defined as:

$$\boxed{\Psi^C(x) \equiv C\bar{\Psi}^\top(x) = \begin{pmatrix} \eta_\alpha(x) \\ \bar{\chi}^{\dot{\alpha}}(x) \end{pmatrix}}, \tag{9.151}$$

where the **charge conjugation matrix** C in the chiral representation is:

$$C = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix} = i\gamma^0\gamma^2 . \tag{9.152}$$

The matrix C satisfies:

$$C^{-1}\gamma^\mu C = -(\gamma^\mu)^\top, \qquad C^\top = -C . \tag{9.153}$$

Proof. In the chiral representation, $(\gamma^\mu)^\top$ has sigma matrices transposed. Using $(\sigma^i)^\top = \sigma^{i*}$ for Pauli matrices, one can verify $C^{-1}\gamma^\mu C = -(\gamma^\mu)^\top$ directly. □

The condition $(\Psi^C)^C = \Psi$ is easily verified using $C^\top C = \mathbb{1}_4$.

Majorana spinors

A **Majorana spinor** is a four-component spinor that satisfies the **Majorana condition**:

$$\boxed{\Psi_M = \Psi_M^C}. \tag{9.154}$$

In the chiral representation, this implies:

$$\Psi_M(x) = \begin{pmatrix} \chi_\alpha(x) \\ \bar{\chi}^{\dot{\alpha}}(x) \end{pmatrix}, \tag{9.155}$$

where a single two-component field χ determines the entire four-component spinor. A Majorana fermion is its own antiparticle.

Remark. [Degrees of freedom] A Dirac spinor has 4 complex components (8 real degrees of freedom, reduced to 4 on-shell). A Majorana spinor has only 2 complex components (4 real degrees of freedom, reduced to 2 on-shell), reflecting its self-conjugate nature.

Bilinears and Lagrangian

Bilinear covariants in four-component form

The **Dirac bilinear covariants** transform irreducibly as Lorentz tensors. Using two Dirac spinor fields Ψ_i and Ψ_j constructed from two-component fields $(\chi_i, \bar{\eta}_i)$ and $(\chi_j, \bar{\eta}_j)$, the bilinear covariants are:

Remark. [Placement of the flavour indices i, j] The labels i, j in the table below are *flavour* (or family) indices distinguishing the two Dirac fields Ψ_i and Ψ_j; they are *not* spinor indices. Their up/down position follows the same Dreiner–Haber–Martin convention adopted for the two-component spinors throughout this chapter: **complex conjugation raises the flavour index**. Starting from

$$\Psi_i = \begin{pmatrix} \chi_{i\,\alpha} \\ \bar{\eta}_i^{\dot{\alpha}} \end{pmatrix},$$

with χ_i and $\bar{\eta}_i$ carrying i *down*, the Dirac adjoint is

$$\bar{\Psi}^i = (\eta^{i\,\alpha},\ \bar{\chi}^i_{\dot{\alpha}}),$$

where $\eta^i \equiv (\chi_i)^*$ and $\bar{\chi}^i \equiv (\bar{\eta}_i)^*$ now carry i *up*. This mirrors, at the level of flavour, the same staircase used for spinor indices, where conjugating an undotted lower spinor χ_α produces a dotted upper spinor $\bar{\chi}^{\dot{\alpha}}$. Hence each bilinear in the table appears in the form $\bar{\Psi}^i \Gamma \Psi_j$ with i up and j down, and the two-component reduction inherits exactly that placement (e.g. $\eta^i \chi_j$, $\bar{\chi}^i \bar{\eta}_j$). The convention is purely bookkeeping — it tracks which field has been conjugated — and involves no contraction with any flavour metric.

Four-component form	Two-component form	Lorentz representation
$\bar{\Psi}^i \Psi_j$	$\eta^i \chi_j + \bar{\chi}^i \bar{\eta}_j$	Scalar
$\bar{\Psi}^i \gamma^5 \Psi_j$	$-\eta^i \chi_j + \bar{\chi}^i \bar{\eta}_j$	Pseudoscalar
$\bar{\Psi}^i \gamma^\mu \Psi_j$	$\bar{\chi}^i \bar{\sigma}^\mu \chi_j + \eta^i \sigma^\mu \bar{\eta}_j$	Vector
$\bar{\Psi}^i \gamma^\mu \gamma^5 \Psi_j$	$-\bar{\chi}^i \bar{\sigma}^\mu \chi_j + \eta^i \sigma^\mu \bar{\eta}_j$	Axial vector
$\bar{\Psi}^i \Sigma^{\mu\nu} \Psi_j$	$2(\eta^i \sigma^{\mu\nu} \chi_j + \bar{\chi}^i \bar{\sigma}^{\mu\nu} \bar{\eta}_j)$	Antisymmetric tensor

Derivation of vector bilinear. Using the definition (9.141) and the block form of

γ^μ:

$$\begin{aligned}\bar\Psi^i\gamma^\mu\Psi_j &= (\eta^{i\alpha}, \bar\chi^i_{\dot\alpha})\begin{pmatrix}0 & \sigma^\mu\\ \bar\sigma^\mu & 0\end{pmatrix}\begin{pmatrix}\chi_{j\beta}\\ \bar\eta_j^{\dot\beta}\end{pmatrix}\\ &= \eta^{i\alpha}\sigma^\mu_{\alpha\dot\beta}\bar\eta_j^{\dot\beta} + \bar\chi^i_{\dot\alpha}\bar\sigma^{\mu\,\dot\alpha\beta}\chi_{j\beta}\\ &= \eta^i\sigma^\mu\bar\eta_j + \bar\chi^i\bar\sigma^\mu\chi_j \end{aligned} \tag{9.156}$$

The other bilinear covariants follow similarly. □

The chiral projections of the bilinear covariants are particularly useful:

$$\bar\Psi^i P_L\Psi_j = \eta^i\chi_j, \qquad \bar\Psi^i P_R\Psi_j = \bar\chi^i\bar\eta_j \tag{9.157}$$

$$\bar\Psi^i\gamma^\mu P_L\Psi_j = \bar\chi^i\bar\sigma^\mu\chi_j, \qquad \bar\Psi^i\gamma^\mu P_R\Psi_j = \eta^i\sigma^\mu\bar\eta_j \tag{9.158}$$

$$\bar\Psi^i\Sigma^{\mu\nu} P_L\Psi_j = 2\eta^i\sigma^{\mu\nu}\chi_j, \qquad \bar\Psi^i\Sigma^{\mu\nu} P_R\Psi_j = 2\bar\chi^i\bar\sigma^{\mu\nu}\bar\eta_j\,. \tag{9.159}$$

Four-component form of the Dirac Lagrangian

The **Dirac Lagrangian** for a free massive spin-$\frac{1}{2}$ fermion is:

$$\boxed{\mathscr{L}_{\text{Dirac}} = \bar\Psi(i\gamma^\mu\partial_\mu - m)\Psi = i\bar\Psi\gamma^\mu\partial_\mu\Psi - m\bar\Psi\Psi}\,. \tag{9.160}$$

This compact four-component formula is the form most commonly used in QFT calculations. The next section constructs the corresponding two-component Majorana and Dirac Lagrangians directly; the short check below explains how the four-component expression reduces to that two-component form.

Equivalence with two-component form. Expanding using eqs. (9.141) and (9.113):

$$\begin{aligned}\bar\Psi(i\gamma^\mu\partial_\mu - m)\Psi &= i\bar\Psi\gamma^\mu\partial_\mu\Psi - m\bar\Psi\Psi\\ &= i(\eta^\alpha\sigma^\mu_{\alpha\dot\beta}\partial_\mu\bar\eta^{\dot\beta} + \bar\chi_{\dot\alpha}\bar\sigma^{\mu\,\dot\alpha\beta}\partial_\mu\chi_\beta)\\ &\quad - m(\eta^\alpha\chi_\alpha + \bar\chi_{\dot\alpha}\bar\eta^{\dot\alpha}) \end{aligned} \tag{9.161}$$

Using integration by parts on the first term,

$$i\eta\sigma^\mu\partial_\mu\bar\eta = -i(\partial_\mu\eta)\sigma^\mu\bar\eta + (\text{total derivative}) = i\bar\eta\bar\sigma^\mu\partial_\mu\eta + (\text{total derivative}),$$

where the last equality uses the anticommuting spinor identity $(\partial_\mu\eta)\sigma^\mu\bar\eta = -\bar\eta\bar\sigma^\mu\partial_\mu\eta$. Thus, up to a total derivative, this becomes:

$$\mathscr{L} = i\bar\chi\bar\sigma^\mu\partial_\mu\chi + i\bar\eta\bar\sigma^\mu\partial_\mu\eta - m(\chi\eta + \bar\chi\bar\eta)$$

which matches the two-component Dirac Lagrangian (9.167). □

9.6 Lagrangians for free spin-1/2 fermions

We now construct the free fermion Lagrangians directly in two-component notation. This fills in the object that the previous subsection rewrote compactly as the four-component Dirac Lagrangian: first a single neutral Weyl field gives a Majorana fermion, and then two independent Weyl fields of opposite charge give a Dirac fermion.

The Majorana Lagrangian

Consider a theory of a single two-component fermion ξ. The quantity:

$$i\bar{\xi}_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\beta}\partial_\mu\xi_\beta \equiv i\bar{\xi}\bar{\sigma}^\mu\partial_\mu\xi \tag{9.162}$$

is a Lorentz scalar. The factor of i is inserted since the hermitian quantity:

$$\tfrac{1}{2}i\bar{\xi}\bar{\sigma}^\mu\overleftrightarrow{\partial}_\mu\xi \equiv \tfrac{1}{2}i\left[\bar{\xi}\bar{\sigma}^\mu(\partial_\mu\xi) - (\partial_\mu\bar{\xi})\bar{\sigma}^\mu\xi\right] \tag{9.163}$$

differs from $i\bar{\xi}\bar{\sigma}^\mu\partial_\mu\xi$ by a total divergence. Hence, eq. (9.162) is a candidate for a kinetic energy term in the Lagrangian.

A second hermitian Lorentz scalar:

$$m\xi\xi + m^*\bar{\xi}\bar{\xi} \tag{9.164}$$

is a candidate for a mass term. Without loss of generality, we can absorb the phase of the parameter m into the definition of the field ξ. Henceforth, we shall work in a convention in which m is real and nonnegative.

The Lagrangian for a **free Majorana fermion** is:

$$\boxed{\mathscr{L} = i\bar{\xi}\bar{\sigma}^\mu\partial_\mu\xi - \frac{1}{2}(m\xi\xi + m\bar{\xi}\bar{\xi})}\,. \tag{9.165}$$

The corresponding field equations (equations of motion) are:

$$\boxed{\begin{aligned} i\bar{\sigma}^\mu\partial_\mu\xi - m\bar{\xi} &= 0 \\ i\sigma^\mu\partial_\mu\bar{\xi} - m\xi &= 0 \end{aligned}}\,, \tag{9.166}$$

where the second equation is the hermitian conjugate of the first.

Remark. [Physical interpretation] This is the Lagrangian for a neutral (Majorana) spin-$\frac{1}{2}$ fermion. A Majorana fermion is its own antiparticle. Neutrinos may be Majorana fermions, although this has not yet been experimentally confirmed.

The Dirac Lagrangian in two-component form

For a **Dirac fermion**, which carries a conserved charge (such as electric charge), we need two independent two-component spinors χ and η. The Lagrangian is:

$$\boxed{\mathscr{L} = i\bar{\chi}\bar{\sigma}^{\mu}\partial_{\mu}\chi + i\bar{\eta}\bar{\sigma}^{\mu}\partial_{\mu}\eta - m\left(\chi\eta + \bar{\chi}\bar{\eta}\right)}\,. \tag{9.167}$$

The field equations are:

$$i\bar{\sigma}^{\mu}\partial_{\mu}\chi - m\bar{\eta} = 0 \tag{9.168}$$
$$i\bar{\sigma}^{\mu}\partial_{\mu}\eta - m\bar{\chi} = 0\,. \tag{9.169}$$

Remark. [Connection to four-component formalism] The Dirac spinor Ψ is constructed from two Weyl spinors:

$$\Psi = \begin{pmatrix} \chi_{\alpha} \\ \bar{\eta}^{\dot{\alpha}} \end{pmatrix}. \tag{9.170}$$

In this notation, the Dirac Lagrangian (9.167) takes the familiar four-component form $\mathscr{L} = \bar{\Psi}(i\gamma^{\mu}\partial_{\mu} - m)\Psi$.

Energy-momentum tensor

The canonical energy-momentum tensor and angular momentum density can be derived using the Noether procedure. For a theory with N two-component $(\frac{1}{2},0)$ spinor fields $\xi_{\alpha j}$ and their conjugates $\bar{\xi}^{\dot{\alpha} j}$ (where $j \in \{1,2,\ldots,N\}$):

$$T^{\mu}{}_{\nu} = \frac{\partial\mathscr{L}}{\partial(\partial_{\mu}\xi_{\alpha j})}\partial_{\nu}\xi_{\alpha j} + \frac{\partial\mathscr{L}}{\partial(\partial_{\mu}\bar{\xi}^{\dot{\alpha} j})}\partial_{\nu}\bar{\xi}^{\dot{\alpha} j} - \delta^{\mu}_{\nu}\mathscr{L}\,. \tag{9.171}$$

$$M^{\mu}{}_{\nu\lambda} = x_{\nu}T^{\mu}{}_{\lambda} - x_{\lambda}T^{\mu}{}_{\nu} - i\frac{\partial\mathscr{L}}{\partial(\partial_{\mu}\xi_{\alpha j})}(\sigma_{\nu\lambda})_{\alpha}{}^{\beta}\xi_{\beta j} - i\frac{\partial\mathscr{L}}{\partial(\partial_{\mu}\bar{\xi}^{\dot{\alpha} j})}(\bar{\sigma}_{\nu\lambda})^{\dot{\alpha}}{}_{\dot{\beta}}\bar{\xi}^{\dot{\beta} j}\,. \tag{9.172}$$

These satisfy the conservation laws $\partial_{\mu}T^{\mu}{}_{\nu} = 0$ and $\partial_{\mu}M^{\mu}{}_{\nu\lambda} = 0$ when the field equations are satisfied.

For the Majorana Lagrangian (9.165):

$$\frac{\partial \mathscr{L}}{\partial(\partial_\mu \xi_\alpha)} = i\bar{\xi}_{\dot{\beta}}\bar{\sigma}^{\mu\dot{\beta}\alpha}, \qquad \frac{\partial \mathscr{L}}{\partial(\partial_\mu \bar{\xi}^{\dot{\alpha}})} = 0\,. \tag{9.173}$$

The corresponding time-independent conserved quantities are the momentum four-vector and the total angular momentum tensor:

$$P_\nu = \int d^3x\, T^0{}_\nu, \qquad J_{\nu\lambda} = \int d^3x\, M^0{}_{\nu\lambda}\,. \tag{9.174}$$

10 Introduction to Supersymmetry

In this chapter we develop the foundations of supersymmetry using two-component Weyl spinors. We build on the spinor formalism established in Chapter 9, where the transformation properties of $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ spinors were derived. The key results we will use are the transformation matrices M and $(M^{-1})^\dagger$, and the sigma matrix identities. After establishing the motivations and the Weyl spinor apparatus, we construct the Wess-Zumino model as the simplest supersymmetric field theory, derive the SUSY algebra, and develop the theory of interactions through the superpotential.

The classic reference for supersymmetry is Wess and Bagger [32]. For a pedagogical introduction, Martin's primer [33] is highly recommended. A comprehensive treatment of spinor conventions and their connection to supersymmetry can be found in Dreiner, Haber, and Martin [34].

10.1 Why Supersymmetry Still Matters

Before developing the technical machinery of supersymmetry, it is worth asking: *why should we care?* After two decades of LHC data without direct evidence for superpartners [35, 36, 37], what remains of the original motivations?

The answer is nuanced. The Large Hadron Collider discovered the Higgs boson in 2012, confirming the Standard Model as a remarkably successful description of physics up to the TeV scale. Yet the same discovery deepened the *hierarchy problem*: a fundamental scalar with mass $m_h \simeq 125$ GeV is quadratically sensitive to any heavy physics, and without some protective mechanism, quantum corrections should push m_h to the Planck scale. The absence of new particles at the LHC has pushed the simplest supersymmetric scenarios to higher masses, but it has not eliminated the underlying motivations.

Supersymmetry remains compelling for several interconnected reasons:

1. **Naturalness**: SUSY protects scalar masses from quadratic divergences, addressing the hierarchy problem—though the degree of fine-tuning has become a subject of ongoing debate.
2. **Unification**: The gauge couplings unify with remarkable precision in the MSSM, a quantitative success that survives independently of naturalness concerns.

3. **Dark matter**: SUSY with conserved R-parity provides stable dark matter candidates with the right properties for thermal freeze-out.
4. **Theoretical structure**: SUSY extends spacetime symmetry itself, leading to powerful non-renormalization theorems and deep connections to string theory and quantum gravity.

In the following subsections, we examine each motivation in detail, emphasizing what has been tested, what has changed, and what remains robust.

Naturalness and the Higgs mass

Historically, a central motivation for TeV-scale SUSY was the stabilization of the electroweak scale against large radiative corrections. In the SM treated as an effective field theory with cutoff Λ, the Higgs mass parameter receives quadratic sensitivity to Λ,

$$\delta m_H^2 \sim \frac{\Lambda^2}{16\pi^2}\left(6\lambda + \frac{9}{4}g^2 + \frac{3}{4}g'^2 - 6y_t^2 + \cdots\right), \tag{10.1}$$

where λ is the Higgs quartic coupling ($V \supset \lambda(\Phi^\dagger\Phi)^2$), g and g' are the $SU(2)_L$ and $U(1)_Y$ gauge couplings, and y_t is the top Yukawa coupling, which provides the numerically dominant contribution. Supersymmetry replaces the quadratically sensitive structure with a cancellation between fermionic and bosonic loops when superpartners are included. In softly broken SUSY, quadratic divergences cancel, and the remaining corrections are controlled by the soft-breaking masses (and logarithms of UV scales),

$$\delta m_{H_u}^2 \sim -\frac{3y_t^2}{8\pi^2}\left(m_{\tilde{Q}_3}^2 + m_{\tilde{u}_3}^2 + |A_t|^2\right)\ln\frac{\Lambda_{\rm UV}}{m_{\rm soft}}, \tag{10.2}$$

schematically in MSSM-like theories. Thus, electroweak symmetry breaking in SUSY ties the Higgs potential parameters to the superpartner spectrum. A standard qualitative expectation was therefore that stops and higgsinos should not be far above the TeV scale if one demands little fine-tuning.

What LHC null results imply. After extensive searches [35, 36, 37], broad classes of "vanilla" weak-scale SUSY are constrained: colored superpartners are pushed well above the TeV scale in many simplified scenarios with standard missing-energy decay chains, and electroweak states are constrained in the $O(100\ \text{GeV} - 1\ \text{TeV})$ range depending on decay topologies and mass splittings. The direct consequence is that the simplest naturalness narrative—light stops and gluinos with large missing energy—is under significant pressure.

However, this does not eliminate the naturalness motivation; it reshapes it. There are three complementary perspectives in 2026:

1. **Naturalness as a heuristic rather than a theorem.** The SM itself contains parameters (e.g. the cosmological constant) that are far more "unnatural" by naive estimates. One may still value models that *reduce* tuning even if they do not eliminate it.
2. **Natural SUSY is not identical to "easy SUSY".** Even if higgsinos and third-generation squarks are relatively light, experimental sensitivity can be reduced by compressed spectra, complex cascades, or unconventional final states. The absence of signals constrains *some* corners but leaves others open.
3. **Moderate tuning may be acceptable.** A tuning at the level of 10^{-2}–10^{-3} is often viewed differently than 10^{-30}. If the price of maintaining a structured UV framework is moderate fine-tuning, SUSY can still be compelling.

The Higgs mass at 125 GeV: a challenge for minimal SUSY. The discovery of the Higgs boson at 125 GeV created an interesting tension with the simplest supersymmetric models. In the MSSM, the quartic Higgs coupling is not a free parameter but is determined by gauge couplings. This leads to a striking tree-level prediction:

$$m_h^2 \leq m_Z^2 \cos^2 2\beta \leq m_Z^2 \approx (91\ \text{GeV})^2 \tag{10.3}$$

This bound is in clear conflict with the observed value of 125 GeV—at tree level, the MSSM cannot accommodate the data.

The resolution comes from radiative corrections, particularly from the stop sector. The dominant one-loop contribution scales as

$$\Delta m_h^2 \sim \frac{3y_t^4 v^2}{8\pi^2}\left[\ln\frac{m_{\tilde{t}}^2}{m_t^2} + \frac{X_t^2}{m_{\tilde{t}}^2}\left(1 - \frac{X_t^2}{12\,m_{\tilde{t}}^2}\right)\right], \tag{10.4}$$

where $X_t = A_t - \mu\cot\beta$ is the stop mixing parameter and $m_{\tilde{t}}$ denotes a characteristic stop mass scale. To reach 125 GeV requires either heavy stops ($m_{\tilde{t}} \gtrsim$ few TeV, depending on the spectrum) and/or substantial mixing ($|X_t| \sim \sqrt{6}\,m_{\tilde{t}}$, the "maximal mixing" scenario). Both options increase fine-tuning: heavy stops contribute to $\delta m_{H_u}^2$, while large A_t requires specific UV boundary conditions.

In 2026, it is common to view the MSSM as a benchmark rather than a definitive target. Extensions that add new contributions to the Higgs quartic—such as the NMSSM with a gauge singlet—can raise the tree-level bound and reduce tuning. Alternatively, one can accept moderate tuning as the price of maintaining a structured UV framework.

Gauge coupling unification

Perhaps the most compelling *quantitative* motivation for supersymmetry is the unification of gauge couplings. The one-loop running of the three gauge couplings

$\alpha_1, \alpha_2, \alpha_3$ is governed by

$$\alpha_i^{-1}(Q) = \alpha_i^{-1}(m_Z) + \frac{b_i}{2\pi} \ln \frac{Q}{m_Z}, \tag{10.5}$$

where the coefficients b_i depend on the particle content of the theory. A useful one-loop diagnostic for whether the slopes can make the three couplings meet is

$$x \equiv \frac{b_2 - b_3}{5(b_1 - b_2)}. \tag{10.6}$$

The same combination extracted from the measured low-energy couplings is $x_{\text{exp}} \simeq 0.144$.

In the **Standard Model** with one Higgs doublet ($N_h = 1$), one finds $b_3 = -7$, $b_2 = -19/6$, $b_1 = 41/10$, yielding

$$x_{\text{SM}} = \frac{23}{218} \simeq 0.106. \tag{10.7}$$

This is significantly below the experimental value—*gauge coupling unification fails* in the Standard Model. Furthermore, the SU(3) and U(1) couplings would meet at $\sim 7 \times 10^{14}$ GeV, while SU(2) and U(1) meet at $\sim 10^{13}$ GeV: there is no single unification scale.

In the **MSSM** with two Higgs doublets ($N_h = 2$), the superpartner contributions modify the beta function coefficients to $b_3 = -3$, $b_2 = 1$, $b_1 = 33/5$, giving

$$x_{\text{MSSM}} = \frac{1}{7} \simeq 0.143. \tag{10.8}$$

This is remarkably close to the experimental value. The three couplings converge at

$$M_{\text{GUT}} \simeq 2 \times 10^{16} \text{ GeV}, \qquad \alpha_U^{-1} \simeq 24 \tag{10.9}$$

which is comfortably below the Planck scale (avoiding quantum gravity complications) and comfortably in the perturbative regime. For dimension-six GUT gauge-boson exchange, the rough scaling $\tau_p \sim M_{\text{GUT}}^4/(\alpha_U^2 m_p^5)$ gives very long proton lifetimes, of order 10^{35} years. In supersymmetric GUTs this statement must be supplemented by model-dependent dimension-five operators, which can impose stronger constraints.

The key point in 2026 is that this success is *less sensitive* to the precise superpartner masses than naturalness arguments. Whether colored superpartners are at 2 TeV or 10 TeV shifts the details via threshold corrections, but the overall picture—that SUSY dramatically improves unification—remains robust. Even if naturalness is partly deprioritized, unification keeps SUSY attractive as a UV-organizing principle.

Dark matter

Supersymmetry naturally provides stable dark matter candidates when R-parity (or another suitable symmetry) is imposed: the lightest supersymmetric particle (LSP) becomes stable, and weakly interacting neutral states can yield the observed relic abundance via thermal freeze-out. The neutralino (a mixture of bino, wino, higgsinos) is the canonical example.

Where the classic WIMP story stands in 2026. Direct detection experiments and collider searches have put significant pressure on simple regions of neutralino parameter space, but the SUSY dark matter narrative is not exhausted:

- **Higgsino- or wino-like LSPs.** These can be viable thermally at masses near the TeV scale (higgsino) or multi-TeV (wino), with distinctive phenomenology such as compressed spectra and soft decay products at colliders.
- **Well-tempered neutralinos.** A pure bino LSP typically annihilates too weakly (overproducing dark matter), while a pure higgsino annihilates too efficiently unless it is near the TeV scale. A carefully tuned *mixture* of bino, wino, and higgsino components can yield the observed relic abundance $\Omega h^2 \simeq 0.12$; such mixed states are called "well-tempered" neutralinos. The same mixing usually increases the couplings to the Higgs and Z bosons, so direct-detection bounds put much of this parameter space under strong pressure. Viable regions typically rely on heavier masses, blind spots, resonant annihilation, coannihilation, or non-standard cosmology.
- **Funnel scenarios.** When the neutralino mass sits near a resonance, $m_\chi \approx m_X/2$ with $X = h, Z$, or the heavy MSSM pseudoscalar A, s-channel annihilation proceeds through a near-on-shell propagator and the thermally averaged cross section is greatly enhanced. This drives the relic density down to the observed value even for neutralino compositions (e.g. nearly pure bino) that would otherwise overproduce dark matter. The name "funnel" refers to the narrow, funnel-shaped allowed regions in the $(m_0, m_{1/2})$ parameter plane of constrained MSSM scenarios. Both well-tempered and funnel mechanisms can produce the correct relic abundance with suppressed detection rates, depending on the precise couplings and mixing angles.
- **Non-thermal histories.** If the early Universe differs from the simplest radiation-dominated thermal picture (late entropy injection, moduli decays), a broader range of LSP masses/couplings can match the relic density.

Thus, dark matter remains a meaningful motivation, but one should be careful: SUSY does not *predict* the dark matter properties uniquely; it provides a consistent and testable arena in which dark matter emerges naturally and is tightly connected to electroweak and Higgs physics.

SUSY and UV structure

Even if one sets aside the expectation of TeV-scale discovery, supersymmetry remains uniquely valuable as a tool for understanding the UV behavior of quantum field theories:

1. **Control over radiative corrections.** Non-renormalization theorems and holomorphy constraints in SUSY theories strongly restrict how couplings run and how effective operators are generated, providing calculational control in strongly coupled regimes.
2. **Connections to dualities and non-perturbative dynamics.** SUSY enables sharp statements about confinement, chiral symmetry breaking, and moduli spaces, and provides laboratories where dual descriptions can be proven rather than conjectured.
3. **Compatibility with quantum gravity expectations.** Many developments in string theory and quantum gravity suggest that consistent UV completions impose constraints that resemble or favor supersymmetric structures, at least in some regimes. Even if low-energy SUSY is broken at high scales, SUSY remains a crucial organizing principle in UV model building.

In a lecture-notes context, it is worth stressing that SUSY is not merely "another BSM model" but a symmetry that upgrades the structure of QFT itself: it enlarges the Poincaré algebra, relates particle spins, and yields powerful constraints on dynamics.

The new SUSY paradigm

A key lesson from two decades of LHC is that if SUSY exists near the TeV scale, it may not appear in the simplest textbook signatures. Several generic mechanisms can render SUSY difficult to see:

- **Compressed spectra.** Small mass splittings reduce missing energy and visible momenta, making triggers and selections inefficient.
- **Long-lived particles.** If decays are slow (small couplings, small splittings), displaced vertices or detector-stable charged/colored tracks become relevant; these require dedicated analyses.
- ***R*-parity violation (RPV).** If the LSP decays hadronically, missing energy is reduced and signatures blend into QCD backgrounds, though multi-jet resonances or displaced decays may arise.
- **Electroweak production dominance.** If colored states are heavy, the production of charginos/neutralinos or sleptons can dominate, but cross sections are smaller and final states more subtle.

The implication is not that SUSY is unfalsifiable, but that its discovery (or exclusion) requires a richer experimental program: precision measurements, dedicated LLP

searches, improved sensitivity to soft objects, and global combinations of collider and astroparticle constraints.

Taken together, a balanced 2026 view is that naturalness is treated less dogmatically, unification and UV structure remain robust motivations, and dark matter is still compelling but not unique. At the same time, discovery strategies have shifted toward targeted signatures (compressed spectra, LLPs, RPV, electroweak production) rather than textbook missing-energy searches.

From a pedagogical standpoint, this motivates a modern SUSY course structure: one should still teach SUSY as a symmetry of nature that solves (or ameliorates) the hierarchy problem and enables unification and dark matter, but one should also train students to interpret null results correctly and to appreciate the broader theoretical significance of supersymmetry as an organizing principle for quantum field theory.

Looking ahead. The central message is not that SUSY is "ruled out" or "guaranteed," but that it remains a coherent and fertile framework. The two decades of LHC data have raised the evidentiary bar and refined our expectations: if SUSY exists, it may be more subtle than the earliest minimal benchmarks, and its discovery may come from a combination of collider innovation, precision observables, and astroparticle signals rather than from a single spectacular channel.

10.2 Two-component spinor conventions (recap)

We use the two-component conventions established in Chapter 9 and only collect the identities needed in the SUSY construction. In the chiral basis,

$$\Psi = \begin{pmatrix} \chi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}, \qquad \sigma^\mu = (\mathbb{1}_2, \vec{\sigma}), \qquad \bar{\sigma}^\mu = (\mathbb{1}_2, -\vec{\sigma}), \tag{10.10}$$

and the spinor metric raises/lowers indices via $\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta$ with $\epsilon^{12} = -\epsilon_{12} = +1$.

Lorentz-invariant products and vector bilinears are

$$\psi\chi \equiv \psi^\alpha \chi_\alpha, \qquad \bar{\psi}\bar{\chi} \equiv \bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}}, \qquad \chi\sigma^\mu\bar{\psi} \equiv \chi^\alpha(\sigma^\mu)_{\alpha\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}, \tag{10.11}$$

with $\chi\sigma^\mu\bar{\psi} = -\bar{\psi}\bar{\sigma}^\mu\chi$ for anticommuting spinors.

Remark. [Constructing Lorentz invariants from a single chirality] A crucial observation for supersymmetry: the combination $i\sigma_2\chi^*$ transforms as a right-handed spinor. This follows from $\sigma_2\vec{\sigma}^*\sigma_2 = -\vec{\sigma}$, which implies that under a Lorentz transformation $\chi \to M\chi$, we have $i\sigma_2\chi^* \to (M^{-1})^\dagger(i\sigma_2\chi^*)$. This allows

us to construct Lorentz invariants using only left-handed spinors.

10.3 The free Wess-Zumino model

The simplest supersymmetric model in four dimensions consists of a complex scalar field ϕ coupled to a left-handed Weyl spinor χ. The free Lagrangian takes the form

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi + \bar{\chi} i \bar{\sigma}^\mu \partial_\mu \chi \,, \tag{10.12}$$

where the fermionic kinetic term arises from the Dirac Lagrangian restricted to left-handed spinors. The mass dimensions are $[\phi] = 1$ and $[\chi] = 3/2$, as required by the canonical structure of the kinetic terms. Note that with a left-handed spinor χ_α, the appropriate sigma matrix is $\bar{\sigma}^\mu$ with index structure $\bar{\sigma}^{\mu\,\dot{\alpha}\alpha}\chi_\alpha$.

We seek field transformations that leave the action $S = \int d^4x\, \mathcal{L}$ invariant while mixing bosons with fermions, schematically $\delta\phi \sim \chi$ and $\delta\chi \sim \phi$. Since the variation of a scalar must itself be a scalar, and since χ carries spin-$\frac{1}{2}$, the generator of supersymmetry must be a spinorial object.

For the scalar field, dimensional analysis and Lorentz invariance lead us to try the ansatz $\delta\phi = \epsilon\chi$, where ϵ is a Grassmann-valued left-handed spinor parameter with dimension $[\epsilon] = -1/2$. The product $\epsilon\chi = \epsilon^\alpha \chi_\alpha$ transforms as a Lorentz scalar, as required. The conjugate transformation is $\delta\phi^\dagger = \bar{\epsilon}\bar{\chi}$.

The transformation of the spinor field requires more care. Since $\delta\chi$ must have dimension $3/2$ and carry spinor indices, dimensional analysis forces us to include a derivative of ϕ. However, ∂_μ carries a Lorentz index that must be contracted with a sigma matrix. To match the index structure of a left-handed spinor χ_α, we need the combination $\sigma^\mu_{\alpha\dot{\alpha}}\bar{\epsilon}^{\dot{\alpha}}\partial_\mu\phi$, which requires introducing the conjugate parameter $\bar{\epsilon}^{\dot{\alpha}}$. We therefore write

$$\delta\chi_\alpha = A(\sigma^\mu\bar{\epsilon})_\alpha\,\partial_\mu\phi, \qquad \delta\bar{\chi}_{\dot{\alpha}} = A^*\,\epsilon^\beta\sigma^\mu_{\beta\dot{\alpha}}\,\partial_\mu\phi^\dagger\,, \tag{10.13}$$

where the constant A is to be determined by requiring invariance of the action (it will turn out to be $A = -i$). The conjugate law is written with a lower dotted index because this is the index position that appears in the kinetic term $\bar{\chi}_{\dot{\alpha}} i\bar{\sigma}^{\mu\,\dot{\alpha}\alpha}\partial_\mu\chi_\alpha$.

Derivation of $\delta\bar{\chi}$ from $\delta\chi$ by complex conjugation. Take the complex conjugate of $\delta\chi_\alpha = A\,\sigma^\mu_{\alpha\dot{\beta}}\,\bar{\epsilon}^{\dot{\beta}}\,\partial_\mu\phi$, using $\bar{\chi}_{\dot{\alpha}} \equiv (\chi_\alpha)^*$:

$$\delta\bar{\chi}_{\dot{\alpha}} = A^*\,(\sigma^\mu_{\alpha\dot{\beta}})^*\,(\bar{\epsilon}^{\dot{\beta}})^*\,\partial_\mu\phi^\dagger\,.$$

Two ingredients convert this into the second relation of (10.13):

1. *Hermiticity of σ^μ* ((9.68)): $(\sigma^\mu_{\alpha\dot{\beta}})^* = \sigma^\mu_{\beta\dot{\alpha}}$.

2. *Conjugation of* $\bar\epsilon$: since $\bar\epsilon^{\dot\beta} = \epsilon^{\dot\beta\dot\gamma}(\epsilon_\gamma)^*$, one has $(\bar\epsilon^{\dot\beta})^* = \epsilon^{\beta\gamma}\epsilon_\gamma = \epsilon^\beta$.

Substituting them gives

$$\delta\bar\chi_{\dot\alpha} = A^*\,\epsilon^\beta\,\sigma^\mu_{\beta\dot\alpha}\,\partial_\mu\phi^\dagger\,.$$

For the value fixed below, $A = -i$, hence $A^* = +i$. If one raises the dotted index, the sigma–epsilon relation gives

$$\epsilon^{\dot\alpha\dot\beta}\epsilon^\gamma\sigma^\mu_{\gamma\dot\beta} = -(\bar\sigma^\mu\epsilon)^{\dot\alpha},$$

so the raised-index form used later is

$$\delta\bar\chi^{\dot\alpha} = -i\,\partial_\mu\phi^\dagger(\bar\sigma^\mu\epsilon)^{\dot\alpha}.$$

□

Varying the Lagrangian under these transformations, the scalar kinetic term contributes

$$\delta\mathcal{L}_s = \partial_\mu(\bar\epsilon\bar\chi)\partial^\mu\phi + \partial_\mu\phi^\dagger\cdot\epsilon\partial^\mu\chi\,, \tag{10.14}$$

while the fermionic kinetic term gives

$$\delta\mathcal{L}_F = iA^*\,\partial_\nu\phi^\dagger\,\epsilon\sigma^\nu\bar\sigma^\mu\partial_\mu\chi + iA\,\bar\chi\bar\sigma^\mu\sigma^\nu\bar\epsilon\,\partial_\mu\partial_\nu\phi\,. \tag{10.15}$$

Step-by-step variation (for a first SUSY course). Starting from

$$\mathcal{L} = \partial_\mu\phi^\dagger\,\partial^\mu\phi + \bar\chi_{\dot\alpha}\,i\bar\sigma^{\mu\,\dot\alpha\beta}\partial_\mu\chi_\beta, \qquad \delta\phi = \epsilon\chi, \quad \delta\phi^\dagger = \bar\epsilon\bar\chi, \quad \delta\chi_\alpha = A(\sigma^\nu\bar\epsilon)_\alpha\,\partial_\nu\phi,$$

we compute variations term by term.

Scalar kinetic term. Using the product rule,

$$\begin{aligned}\delta\mathcal{L}_s &= \partial_\mu(\delta\phi^\dagger)\,\partial^\mu\phi + \partial_\mu\phi^\dagger\,\partial^\mu(\delta\phi)\\ &= \partial_\mu(\bar\epsilon\bar\chi)\,\partial^\mu\phi + \partial_\mu\phi^\dagger\,\partial^\mu(\epsilon\chi)\\ &= (\partial_\mu\bar\chi)\bar\epsilon\,\partial^\mu\phi + \partial_\mu\phi^\dagger\,\epsilon\,\partial^\mu\chi\,,\end{aligned} \tag{10.16}$$

where $\partial_\mu\epsilon = \partial_\mu\bar\epsilon = 0$ for global supersymmetry.

Fermion kinetic term. Varying $\bar\chi i\bar\sigma^\mu\partial_\mu\chi$ gives

$$\begin{aligned}\delta\mathcal{L}_F &= (\delta\bar\chi_{\dot\alpha})\,i\bar\sigma^{\mu\,\dot\alpha\beta}\partial_\mu\chi_\beta + \bar\chi\,i\bar\sigma^\mu\partial_\mu(\delta\chi)\\ &= \Big(A^*\,\epsilon\sigma^\nu\,\partial_\nu\phi^\dagger\Big)\,i\bar\sigma^\mu\partial_\mu\chi + \bar\chi\,i\bar\sigma^\mu\partial_\mu\Big(A\,\sigma^\nu\bar\epsilon\,\partial_\nu\phi\Big)\\ &= iA^*\,\partial_\nu\phi^\dagger\,\epsilon\sigma^\nu\bar\sigma^\mu\partial_\mu\chi + iA\,\bar\chi\bar\sigma^\mu\sigma^\nu\bar\epsilon\,\partial_\mu\partial_\nu\phi\,,\end{aligned} \tag{10.17}$$

which matches the compact expression written above.

To show invariance, first integrate by parts the second term in $\delta\mathcal{L}_F$:

$$\int d^4x\, iA\,\bar{\chi}\bar{\sigma}^\mu\sigma^\nu\bar{\epsilon}\,\partial_\mu(\partial_\nu\phi)$$
$$= -\int d^4x\, iA\,(\partial_\mu\bar{\chi})\bar{\sigma}^\mu\sigma^\nu\bar{\epsilon}\,\partial_\nu\phi \;+\; \text{boundary term}. \tag{10.18}$$

Thus, inside the action and modulo a boundary term,

$$\delta\mathcal{L}_F \doteq iA^*\,\partial_\nu\phi^\dagger\,\epsilon\sigma^\nu\bar{\sigma}^\mu\partial_\mu\chi - iA\,(\partial_\mu\bar{\chi})\bar{\sigma}^\mu\sigma^\nu\bar{\epsilon}\,\partial_\nu\phi\,. \tag{10.19}$$

Here $\doteq$ denotes equality in the action up to a total derivative.

Now use the fundamental σ–$\bar{\sigma}$ identity

$$\bar{\sigma}^\mu\sigma^\nu + \bar{\sigma}^\nu\sigma^\mu = 2g^{\mu\nu}\mathbb{1}, \qquad \sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu = 2g^{\mu\nu}\mathbb{1}\,. \tag{10.20}$$

The antisymmetric parts of the sigma-matrix products in (10.19) are total derivatives in the action. For example,

$$\int d^4x\,\partial_\nu\phi^\dagger\,\epsilon(\sigma^\nu\bar{\sigma}^\mu - \sigma^\mu\bar{\sigma}^\nu)\partial_\mu\chi = -\int d^4x\,\partial_\mu\partial_\nu\phi^\dagger\,\epsilon(\sigma^\nu\bar{\sigma}^\mu - \sigma^\mu\bar{\sigma}^\nu)\chi = 0,$$

because $\partial_\mu\partial_\nu\phi^\dagger$ is symmetric in μ,ν while the matrix coefficient is antisymmetric. Hence only the symmetric pieces contribute:

$$\partial_\nu\phi^\dagger\,\epsilon\sigma^\nu\bar{\sigma}^\mu\partial_\mu\chi \doteq \partial_\mu\phi^\dagger\,\epsilon\,\partial^\mu\chi, \tag{10.21}$$
$$(\partial_\mu\bar{\chi})\bar{\sigma}^\mu\sigma^\nu\bar{\epsilon}\,\partial_\nu\phi \doteq (\partial_\mu\bar{\chi})\bar{\epsilon}\,\partial^\mu\phi. \tag{10.22}$$

Therefore

$$\delta\mathcal{L}_F \doteq iA^*\,\partial_\mu\phi^\dagger\,\epsilon\,\partial^\mu\chi - iA\,(\partial_\mu\bar{\chi})\bar{\epsilon}\,\partial^\mu\phi.$$

Comparing with $\delta\mathcal{L}_s$ from (10.16), the non-total-derivative pieces cancel term by term if and only if

$$\boxed{A = -i}\,, \tag{10.23}$$

so that $iA^* = -1$ and $-iA = -1$. The surviving terms are total divergences and integrate to zero for the usual boundary conditions. This fixes the normalisation of the SUSY transformations.

Having established invariance of the action, we now investigate the algebraic structure of supersymmetry by computing the commutator of two transformations. Acting on the scalar field produces a spacetime translation, while acting on the spinor yields extra terms proportional to its equation of motion $i\bar{\sigma}^\mu\partial_\mu\chi = 0$. The algebra therefore closes only *on shell*; the translation part of the superalgebra is derived in Section 10.4.

The off-shell problem and the auxiliary field F

The on-shell closure signals a deeper issue: the bosonic and fermionic *off-shell* degrees of freedom do not match.

Field	Off-shell (real d.o.f.)	On-shell (real d.o.f.)
ϕ (complex scalar)	2	2
χ_α (Weyl spinor)	4	2

On shell, the equation of motion $i\bar{\sigma}^\mu\partial_\mu\chi = 0$ removes 2 of the 4 real components of χ, restoring the boson–fermion balance $2 = 2$. But off-shell (before imposing the equations of motion), bosons have 2 components while fermions have 4. SUSY, which must map bosons into fermions and vice versa, cannot close algebraically unless the off-shell count also matches. The mismatch is $4 - 2 = 2$ real (= 1 complex) bosonic degrees of freedom.

The fix: an auxiliary field F. The remedy is to add a new complex scalar field $F(x)$ with no kinetic term. Its Lagrangian is purely algebraic:

$$\mathcal{L}_F = F^\dagger F\,. \tag{10.24}$$

Because F has no derivatives in $\mathcal{L}_F$, it is *non-propagating*: its Euler–Lagrange equation is

$$\frac{\partial\mathcal{L}_F}{\partial F^\dagger} = F = 0\,,$$

so F vanishes on shell and contributes nothing to the physical (on-shell) spectrum. Off shell, however, F supplies the missing 2 real bosonic components, restoring the balance:

Field	Off-shell (real d.o.f.)	On-shell (real d.o.f.)
ϕ	2	2
F	2	0
χ_α	4	2
Total bosonic	4	2
Total fermionic	4	2

Why this particular Lagrangian? The choice $\mathcal{L}_F = F^\dagger F$ (rather than, say, $m^2F^\dagger F$ or $F^\dagger\Box F$) is forced by three requirements:

- *Dimension*: since $[\mathcal{L}] = 4$ and $[F] = 2$ (mass dimension, in $d = 4$), the only renormalisable term without derivatives is $F^\dagger F$ (with a dimensionless coefficient, normalised to 1).

- *No kinetic term*: if F had a kinetic term $\partial_\mu F^\dagger \partial^\mu F$, it would propagate and add physical particles to the spectrum; we need F to be purely auxiliary.
- *Positive scalar potential after elimination*: the sign of $F^\dagger F$ is fixed so that, once the auxiliary field is eliminated in the interacting theory, the on-shell scalar potential is a sum of absolute squares, $V_F = |W_i|^2 \geq 0$. Since F has no kinetic term, the off-shell term $F^\dagger F$ should not be interpreted as a propagating energy density by itself.

Dimension of F and its SUSY transformation. The dimension $[F] = 2$ is determined by the requirement that $\delta_\epsilon \chi$ be dimensionally consistent. Since $[\chi] = 3/2$, $[\epsilon] = -1/2$, and $[\partial_\mu \phi] = 2$, the original term $-i\sigma^\mu \bar{\epsilon}\, \partial_\mu \phi$ in $\delta_\epsilon \chi$ has dimension $2 + (-1/2) = 3/2$ — correct. The new term ϵF must have the same dimension: $[\epsilon F] = -1/2 + [F]$, so $[F] = 2$.

Lorentz invariance then dictates the form of $\delta_\epsilon F$. The transformation must be a Lorentz scalar of dimension $[F] = 2$, built from the SUSY parameter and the fields of the multiplet with at most one derivative (to keep the algebra first-order). The available building blocks with the right quantum numbers are $\bar{\epsilon}$ ($[\bar{\epsilon}] = -1/2$, right-handed) and $\partial_\mu \chi$ ($[\partial_\mu \chi] = 5/2$, left-handed with a Lorentz index). The combination $\bar{\epsilon}\bar{\sigma}^\mu \partial_\mu \chi$ contracts all indices into a scalar and has dimension $-1/2 + 5/2 = 2$. This is the unique such combination:

$$\delta_\epsilon F = -i\bar{\epsilon}\bar{\sigma}^\mu \partial_\mu \chi, \qquad \delta_\epsilon F^\dagger = i\partial_\mu \bar{\chi}\, \bar{\sigma}^\mu \epsilon\,. \tag{10.25}$$

Correspondingly, the spinor transformation acquires the additional term: $\delta_\epsilon \chi = -i\sigma^\mu \bar{\epsilon}\, \partial_\mu \phi + \epsilon F$. With these modifications, the supersymmetry algebra closes *off-shell* on all three fields of the chiral multiplet (ϕ, χ, F).

The complete transformation laws are

$$\boxed{\begin{aligned} \delta_\epsilon \phi &= \epsilon \chi \\ \delta_\epsilon \chi &= -i\sigma^\mu \bar{\epsilon}\, \partial_\mu \phi + \epsilon F \\ \delta_\epsilon F &= -i\bar{\epsilon}\bar{\sigma}^\mu \partial_\mu \chi \end{aligned}} \qquad \boxed{\begin{aligned} \delta_\epsilon \phi^\dagger &= \bar{\epsilon}\bar{\chi} \\ \delta_\epsilon \bar{\chi}^{\dot{\alpha}} &= -i\partial_\mu \phi^\dagger\, (\bar{\sigma}^\mu \epsilon)^{\dot{\alpha}} + \bar{\epsilon}^{\dot{\alpha}} F^\dagger \\ \delta_\epsilon F^\dagger &= +i\partial_\mu \bar{\chi}\, \bar{\sigma}^\mu \epsilon \end{aligned}} \tag{10.26}$$

and the algebra $(\delta_\epsilon \delta_\eta - \delta_\eta \delta_\epsilon)\Phi = (\epsilon\sigma^\mu \bar{\eta} - \eta\sigma^\mu \bar{\epsilon})(-i\partial_\mu \Phi)$ now holds for $\Phi = \phi, \chi, F$ without invoking the equations of motion.

10.4 The SUSY algebra and supermultiplets

A fundamental aspect of any symmetry (other than a U(1) symmetry) is the algebra associated with the symmetry generators. For example, the generators T_i of SU(2) satisfy the commutation relations $[T_i, T_j] = i\epsilon_{ijk} T_k$, which constitute the 'SU(2) algebra'. The whole theory of angular momentum in quantum mechanics—in

particular the multiplet structure—can be developed from these commutation relations alone. In the same way, to proceed systematically with SUSY and to understand what kind of 'supermultiplets' occur, we must determine the SUSY algebra.

Deriving the algebra: the SU(2) example

Before tackling supersymmetry, it is instructive to see the method in a familiar context. Consider an SU(2) doublet of fields $q = (u, d)^\top$ with the Lagrangian invariant under infinitesimal transformations

$$q \to q' = (1 - i\boldsymbol{\epsilon}\cdot\boldsymbol{\tau}/2)q \equiv q + \delta_\epsilon q\,, \qquad \delta_\epsilon q = -i\boldsymbol{\epsilon}\cdot\boldsymbol{\tau}/2\, q\,, \tag{10.27}$$

where $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)$ are the Pauli matrices and $\boldsymbol{\epsilon}$ are three real infinitesimal parameters.

There are two equivalent descriptions of this transformation, and the distinction is crucial for what follows.

Matrix level. The doublet $q = (u, d)^\top$ is a column vector of fields (classical or quantum). The transformation (10.27) acts by matrix multiplication: $q' = U_{\text{fund}}\, q$, where $U_{\text{fund}} = \exp(-i\boldsymbol{\epsilon}\cdot\boldsymbol{\tau}/2)$ is a 2×2 unitary matrix (the expression $1 - i\boldsymbol{\epsilon}\cdot\boldsymbol{\tau}/2$ in (10.27) is its linearisation). The matrices $\tau_i/2$ act on the *internal* (doublet) indices of q and know nothing about Fock space.

Operator level. In the quantum theory, $q(x)$ is a field *operator* acting on Fock space. The same physical transformation is implemented by conjugation with a unitary Fock-space operator $U_{\text{op}} = \exp(i\boldsymbol{\epsilon}\cdot\mathbf{T})$, where $\mathbf{T} = (T_1, T_2, T_3)$ are the conserved Noether charges (operators on Fock space, not 2×2 matrices). Infinitesimally:

$$q' = U_{\text{op}}\, q\, U_{\text{op}}^{-1} = (1 + i\boldsymbol{\epsilon}\cdot\mathbf{T})\, q\, (1 - i\boldsymbol{\epsilon}\cdot\mathbf{T}) = q + i\boldsymbol{\epsilon}\cdot[\mathbf{T}, q] + O(\epsilon^2)\,. \tag{10.28}$$

The two descriptions must agree: the commutator $[\mathbf{T}, q]$ on Fock space must reproduce the matrix multiplication $(-\boldsymbol{\tau}/2)\, q$ on the doublet indices. Comparing with (10.27):

$$\delta_\epsilon q = i\boldsymbol{\epsilon}\cdot[\mathbf{T}, q] = -i\boldsymbol{\epsilon}\cdot\boldsymbol{\tau}/2\; q\,. \tag{10.29}$$

This equation bridges the two levels: the left-hand side is the Fock-space commutator; the right-hand side is the 2×2 matrix action on the doublet.

Deriving the algebra. Consider two transformations with parameters ϵ_1 and ϵ_2:

$$\delta_{\epsilon_1} q = i\epsilon_1 [T_1, q] = -i\epsilon_1 (\tau_1/2) q, \qquad \delta_{\epsilon_2} q = i\epsilon_2 [T_2, q] = -i\epsilon_2 (\tau_2/2) q \tag{10.30}$$

We calculate $(\delta_{\epsilon_1}\delta_{\epsilon_2} - \delta_{\epsilon_2}\delta_{\epsilon_1})q$ in two ways.

Method 1: Using the explicit form of δq. We have

$$\begin{aligned}\delta_{\epsilon_1}\delta_{\epsilon_2}q &= \delta_{\epsilon_1}\{-i\epsilon_2(\tau_2/2)q\} = -i\epsilon_2(\tau_2/2)\delta_{\epsilon_1}q \\ &= -i\epsilon_2(\tau_2/2)\cdot(-i\epsilon_1(\tau_1/2)q) = -\frac{1}{4}\epsilon_1\epsilon_2\,\tau_2\tau_1\,q\end{aligned} \tag{10.31}$$

Similarly, $\delta_{\epsilon_2}\delta_{\epsilon_1}q = -\frac{1}{4}\epsilon_1\epsilon_2\,\tau_1\tau_2\,q$. Hence

$$\begin{aligned}(\delta_{\epsilon_1}\delta_{\epsilon_2}-\delta_{\epsilon_2}\delta_{\epsilon_1})q &= -\frac{1}{4}\epsilon_1\epsilon_2(\tau_2\tau_1-\tau_1\tau_2)q = \frac{1}{4}\epsilon_1\epsilon_2[\tau_1,\tau_2]q \\ &= \frac{1}{4}\epsilon_1\epsilon_2(2i\tau_3)q = i\epsilon_1\epsilon_2(\tau_3/2)q \\ &= -i\epsilon_1\epsilon_2[T_3,q]\end{aligned} \tag{10.32}$$

where we used $[\tau_1/2,\tau_2/2] = i\tau_3/2$.

Method 2: Using the generator form. We have

$$\delta_{\epsilon_1}\delta_{\epsilon_2}q = \delta_{\epsilon_1}\{i\epsilon_2[T_2,q]\} = i\epsilon_2\delta_{\epsilon_1}[T_2,q] = i\epsilon_1 i\epsilon_2[T_1,[T_2,q]] \tag{10.33}$$

Similarly, $\delta_{\epsilon_2}\delta_{\epsilon_1}q = -\epsilon_1\epsilon_2[T_2,[T_1,q]]$. Hence

$$(\delta_{\epsilon_1}\delta_{\epsilon_2}-\delta_{\epsilon_2}\delta_{\epsilon_1})q = -\epsilon_1\epsilon_2\{[T_1,[T_2,q]]-[T_2,[T_1,q]]\} \tag{10.34}$$

Using the Jacobi identity $[A,[B,C]]+[B,[C,A]]+[C,[A,B]]=0$, we write

$$[T_1,[T_2,q]]-[T_2,[T_1,q]] = [T_1,[T_2,q]]+[T_2,[q,T_1]] = [[T_1,T_2],q] \tag{10.35}$$

Therefore:

$$(\delta_{\epsilon_1}\delta_{\epsilon_2}-\delta_{\epsilon_2}\delta_{\epsilon_1})q = -\epsilon_1\epsilon_2[[T_1,T_2],q] \tag{10.36}$$

Comparing (10.32) and (10.36), we deduce $[T_1,T_2] = iT_3$, exactly as in the SU(2) algebra. This is the method we shall use to derive the SUSY algebra.

Supersymmetry generators and their algebra

For supersymmetry, we need the analogue of (10.29): a Fock-space operator G_ξ such that $\delta_\xi\phi = i[G_\xi,\phi]$.

Counting supercharges: what $\mathcal{N}$ means. The Standard Model symmetries (Poincaré plus internal gauge symmetries) are generated by *bosonic* charges — they all commute with the field operators in a way that preserves the boson/fermion nature of the state. Supersymmetry introduces an entirely new kind of generator: a *fermionic* charge that, when applied to a boson, gives a fermion, and vice versa. Such a charge must itself transform as a spinor under the Lorentz group.

The minimal choice is to introduce *one* spinor charge Q_α in the $(\frac{1}{2},0)$ representation, together with its right-handed partner $\bar{Q}_{\dot\alpha} \equiv (Q_\alpha)^\dagger$ (the dagger maps $(\frac{1}{2},0)$ to $(0,\frac{1}{2})$, just as complex conjugation does for spinor fields). This is called $\mathcal{N}=1$ supersymmetry: "$\mathcal{N}$" counts the number of *independent* pairs of spinor charges introduced. In general:

- $\mathcal{N}=1$: one pair $(Q_\alpha, \bar{Q}_{\dot\alpha})$. *This is the case treated in these notes.*
- $\mathcal{N}=2$: two independent pairs $(Q^I_\alpha, \bar{Q}^I_{\dot\alpha})$ with $I=1,2$.
- $\mathcal{N}=4$: four independent pairs (the maximum allowed for an interacting theory in 4D without higher-spin gravity-like complications: $\mathcal{N}=4$ super-Yang–Mills is the most symmetric four-dimensional field theory known).

Each additional pair of supercharges tightens the constraints on the spectrum: bigger $\mathcal{N}$ means smaller, more rigid multiplets. As we shall see at the end of this section, only $\mathcal{N}=1$ is compatible with the chiral structure of the Standard Model.

For the rest of this section we restrict to $\mathcal{N}=1$.

A note on F. In Section 10.3 we introduced the auxiliary field F to achieve off-shell closure. In the algebra derivation that follows, we work entirely with ϕ and its on-shell transformations $\delta_\xi\phi = \xi\chi$, $\delta_\xi\chi = -i\sigma^\mu\bar\xi\partial_\mu\phi$. This is sufficient because: (i) the algebra $\{Q_\alpha, \bar{Q}_{\dot\beta}\} = \sigma^\mu_{\alpha\dot\beta}P_\mu$ is extracted by computing $(\delta_\eta\delta_\xi - \delta_\xi\delta_\eta)\phi$ — a double variation on the *scalar* field, for which the algebra closes without F (the F-dependent terms in $\delta\chi$ would only matter when computing the commutator on χ itself); (ii) the resulting algebra is the same whether derived on-shell or off-shell, because the anticommutation relations between the charges Q_α and $\bar{Q}_{\dot\alpha}$ are operator identities valid in the full Hilbert space. The field F is essential for off-shell closure on χ, but does not affect the algebra itself.

Building G_ξ. Since the SUSY parameter ξ_α is a left-handed Grassmann spinor, G_ξ must be linear in ξ and $\bar\xi$ and a Lorentz scalar. The right-handed partner of Q_α is defined as

$$\bar{Q}_{\dot\alpha} \;\equiv\; (Q_\alpha)^\dagger .$$

This is *exactly* the same convention used for spinor *fields* in Chapter 9: the bar on a left-handed object χ_α produces its right-handed partner $\bar\chi_{\dot\alpha} \equiv (\chi_\alpha)^*$ (eq. (9.43)). For c-number fields the conjugation is complex conjugation ($*$); for Fock-space operators the conjugation is Hermitian conjugation ($\dagger$). Both map the $(\frac{1}{2},0)$ representation into the inequivalent $(0,\frac{1}{2})$ representation, which is why the index changes from undotted to dotted. The bar notation is the same in both cases: "bar = conjugate partner in the opposite chirality."

With this, the only Lorentz-invariant bilinears are

$$\xi\cdot Q \equiv \xi^\alpha Q_\alpha \qquad \text{and} \qquad \bar\xi\cdot\bar{Q} \equiv \bar\xi_{\dot\alpha}\bar{Q}^{\dot\alpha},$$

contracted with the standard NW→SE and SW→NE staircases respectively (cf. box 9.4). *Both* terms are needed: $\xi\cdot Q$ alone would not be Hermitian (the generator of a unitary transformation must satisfy $G_\xi^\dagger = G_\xi$). To verify $(\xi\cdot Q)^\dagger = \bar\xi\cdot\bar Q$, we adopt the convention that Hermitian conjugation reverses the order of all factors *without* introducing extra Grassmann signs (i.e., Grassmann parameters are treated as "classical external objects" for the purpose of †): $(\xi^\alpha Q_\alpha)^\dagger = Q^\dagger_\alpha(\xi^\alpha)^* = \bar Q_{\dot\alpha}\bar\xi^{\dot\alpha} = \bar\xi_{\dot\alpha}\bar Q^{\dot\alpha} = \bar\xi\cdot\bar Q$, where in the last step we used the standard spinor-index gymnastics. The Hermitian combination is then

$$G_\xi \;\equiv\; \xi\cdot Q + \bar\xi\cdot\bar Q\,.$$

Equivalently, $\bar Q$ is required because $\delta_\xi\phi = \xi\chi$ involves only ξ (not $\bar\xi$), while $\delta_\xi\chi = -i\sigma^\mu\bar\xi\,\partial_\mu\phi$ involves only $\bar\xi$: a single spinor charge could not generate both transformations simultaneously.

With this notation, the operator-level form of the SUSY transformation of ϕ is:

$$\delta_\xi\phi = i[G_\xi,\phi] = i[\xi\cdot Q + \bar\xi\cdot\bar Q,\phi] = \xi\cdot\chi\,. \tag{10.37}$$

Step 1: Calculate $(\delta_\eta\delta_\xi - \delta_\xi\delta_\eta)\phi$ using the explicit transformations. Using $\delta_\xi\phi = \xi\cdot\chi$ and $\delta_\xi\chi = -i\sigma^\mu\bar\xi\,\partial_\mu\phi$:

$$\begin{aligned}\delta_\eta\delta_\xi\phi &= \delta_\eta(\xi\cdot\chi) = \xi\cdot(\delta_\eta\chi) = \xi^\alpha(-i\sigma^\mu\bar\eta)_\alpha\partial_\mu\phi\\ &= -i\,\xi^\alpha(\sigma^\mu)_{\alpha\dot\alpha}\bar\eta^{\dot\alpha}\partial_\mu\phi\end{aligned} \tag{10.38}$$

There is *no* Grassmann sign here: δ_η is a field substitution ($\chi\to\chi+\delta_\eta\chi$), not a Grassmann-odd differential operator, and ξ^α is a constant external parameter that is simply carried through. (Although $G_\xi = \xi\cdot Q + \bar\xi\cdot\bar Q$ involves Grassmann parameters, the product of two Grassmann-odd objects makes G_ξ Grassmann-*even*, so $\delta_\xi\phi = i[G_\xi,\phi]$ is an ordinary, even transformation.) Similarly, $\delta_\xi\delta_\eta\phi = -i\,\eta^\alpha(\sigma^\mu)_{\alpha\dot\alpha}\bar\xi^{\dot\alpha}\partial_\mu\phi$. Taking the difference:

$$\begin{aligned}(\delta_\eta\delta_\xi - \delta_\xi\delta_\eta)\phi &= -i(\xi^\alpha(\sigma^\mu)_{\alpha\dot\alpha}\bar\eta^{\dot\alpha} - \eta^\alpha(\sigma^\mu)_{\alpha\dot\alpha}\bar\xi^{\dot\alpha})\partial_\mu\phi\\ &= -i(\xi\sigma^\mu\bar\eta - \eta\sigma^\mu\bar\xi)\,\partial_\mu\phi\end{aligned} \tag{10.39}$$

To rewrite in the $\bar\sigma$ basis, use the spinor-reversal identity for anticommuting spinors $z_1\sigma^\mu\bar z_2 = -\bar z_2\bar\sigma^\mu z_1$ (eq. (9.100) with $(-1)^A = -1$). Applying it to each term:

$$\xi\sigma^\mu\bar\eta = -\bar\eta\bar\sigma^\mu\xi\,,\qquad \eta\sigma^\mu\bar\xi = -\bar\xi\bar\sigma^\mu\eta\,,$$

so

$$\xi\sigma^\mu\bar\eta - \eta\sigma^\mu\bar\xi = -(\bar\eta\bar\sigma^\mu\xi - \bar\xi\bar\sigma^\mu\eta)\,. \tag{10.40}$$

The overall minus from the reversal combines with the $-i$ from the variation to give $+i$:

$$\boxed{(\delta_\eta\delta_\xi - \delta_\xi\delta_\eta)\phi = +i(\bar\eta\bar\sigma^\mu\xi - \bar\xi\bar\sigma^\mu\eta)\,\partial_\mu\phi} \tag{10.41}$$

Step 2: Calculate $(\delta_\eta\delta_\xi - \delta_\xi\delta_\eta)\phi$ using the generators. Let $G_\xi \equiv \xi\cdot Q + \bar{\xi}\cdot\bar{Q}$. From (10.37), $\delta_\xi\phi = i[G_\xi,\phi]$. Using the Jacobi identity as in Aitchison:

$$[[G_\eta, G_\xi],\phi] = (\delta_\eta\delta_\xi - \delta_\xi\delta_\eta)\phi\,. \tag{10.42}$$

Step 3: Compare the two results. Introducing the 4-momentum operator P_μ with $[P_\mu,\phi] = -i\partial_\mu\phi$, equation (10.41) becomes

$$(\delta_\eta\delta_\xi - \delta_\xi\delta_\eta)\phi = -(\bar{\eta}\bar{\sigma}^\mu\xi - \bar{\xi}\bar{\sigma}^\mu\eta)[P_\mu,\phi] \tag{10.43}$$

Comparing with (10.42):

$$[G_\eta, G_\xi] = (\bar{\xi}\bar{\sigma}^\mu\eta - \bar{\eta}\bar{\sigma}^\mu\xi)P_\mu \tag{10.44}$$

Step 4: Extract the (anti)commutation relations. The right-hand side of (10.44) contains only terms of the form $\bar{\xi}\ldots\eta$ and $\bar{\eta}\ldots\xi$. There are no terms $\xi\ldots\eta$ or $\bar{\xi}\ldots\bar{\eta}$, which implies:

$$[\xi\cdot Q, \eta\cdot Q] = 0\,, \qquad [\bar{\xi}\cdot\bar{Q}, \bar{\eta}\cdot\bar{Q}] = 0\,. \tag{10.45}$$

Expanding the first commutator explicitly:

$$\begin{aligned} 0 &= (\xi^1 Q_1 + \xi^2 Q_2)(\eta^1 Q_1 + \eta^2 Q_2) - (\eta^1 Q_1 + \eta^2 Q_2)(\xi^1 Q_1 + \xi^2 Q_2) \\ &= -\xi^1\eta^1(2Q_1Q_1) - \xi^1\eta^2(Q_1Q_2 + Q_2Q_1) \\ &\quad -\xi^2\eta^1(Q_2Q_1 + Q_1Q_2) - \xi^2\eta^2(2Q_2Q_2) \end{aligned} \tag{10.46}$$

where we used the anticommutation of the Grassmann parameters. Since all parameter combinations are independent:

$$\boxed{\{Q_\alpha, Q_\beta\} = 0, \qquad \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0} \tag{10.47}$$

For the mixed term $[\eta\cdot Q, \bar{\xi}\cdot\bar{Q}]$, we write out components:

$$\bar{\xi}\cdot\bar{Q} = \bar{\xi}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}} = \xi_1^* Q_2^\dagger - \xi_2^* Q_1^\dagger \tag{10.48}$$

$$\eta\cdot Q = \eta^\alpha Q_\alpha = -\eta_1 Q_2 + \eta_2 Q_1 \tag{10.49}$$

Computing the commutator and using anticommutation of parameters:

$$\begin{aligned} [\eta\cdot Q, \bar{\xi}\cdot\bar{Q}] &= \eta_1\xi_1^*(Q_2Q_2^\dagger + Q_2^\dagger Q_2) - \eta_1\xi_2^*(Q_2Q_1^\dagger + Q_1^\dagger Q_2) \\ &\quad -\eta_2\xi_1^*(Q_1Q_2^\dagger + Q_2^\dagger Q_1) + \eta_2\xi_2^*(Q_1Q_1^\dagger + Q_1^\dagger Q_1) \end{aligned} \tag{10.50}$$

Meanwhile, the right-hand side of (10.44) gives (using $c = -i\sigma_2$):

$$\begin{aligned} \bar{\xi}\bar{\sigma}^\mu\eta\,P_\mu &= \xi^\dagger(i\sigma_2)(\bar{\sigma}^\mu)\eta\,P_\mu \\ &= [\eta_2\xi_2^*(\sigma^\mu)_{11} - \eta_2\xi_1^*(\sigma^\mu)_{12} - \eta_1\xi_2^*(\sigma^\mu)_{21} + \eta_1\xi_1^*(\sigma^\mu)_{22}]P_\mu \end{aligned} \tag{10.51}$$

(where we used $c\bar{\sigma}^{\mu}c^{-1} = -\sigma^{\mu T}$ and transposed). Comparing coefficients of each parameter combination in (10.50) and the above:

$$\boxed{\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = (\sigma^\mu)_{\alpha\dot{\beta}} P_\mu} \tag{10.52}$$

This is Aitchison's normalization of the supercharges. The common alternative convention is obtained by the rescaling $Q \to \sqrt{2}Q$, which puts a factor 2 on the right-hand side.

This is the fundamental SUSY algebra. It must be supplemented by the commutator of the supercharges with the momentum operator.

Why $[Q_\alpha, P_\mu] = 0$. The supercharges are spatial integrals of a conserved current: $Q_\alpha = \int d^3x\, J^0_\alpha(x)$ (eq. (10.63)). Since translations are generated by P_μ via $[P_\mu, \mathcal{O}(x)] = -i\partial_\mu \mathcal{O}(x)$ for any local operator $\mathcal{O}$:

$$[P_\mu, Q_\alpha] = \int d^3x\, [P_\mu, J^0_\alpha(x)] = -i \int d^3x\, \partial_\mu J^0_\alpha(x)\,.$$

For $\mu = i$ (spatial): for each fixed i, $\partial_i J^0_\alpha$ is a total derivative in the x^i direction. By the fundamental theorem of calculus:

$$\int d^3x\, \partial_i J^0_\alpha(x) = \int dx^j\, dx^k \left[J^0_\alpha \right]_{x^i=-\infty}^{x^i=+\infty} = 0\,,$$

because $J^0_\alpha \to 0$ as $x^i \to \pm\infty$ (the standard fall-off condition on the fields). For $\mu = 0$ (temporal):

$$-i \int d^3x\, \partial_0 J^0_\alpha = -i \frac{d}{dt} Q_\alpha = 0\,,$$

because Q_α is a conserved charge. To see this, use $\partial_\mu J^\mu_\alpha = 0$:

$$\dot{Q}_\alpha = \int d^3x\, \partial_0 J^0_\alpha = -\int d^3x\, \partial_i J^i_\alpha = 0\,,$$

where the last integral vanishes by the same total-derivative argument applied to each J^i_α. Therefore:

$$[Q_\alpha, P_\mu] = [\bar{Q}_{\dot{\alpha}}, P_\mu] = 0\,. \tag{10.53}$$

Physically: SUSY transformations and spacetime translations commute — performing a SUSY rotation and then translating gives the same result as translating first and then rotating. Combined with the anticommutators derived above, the supertranslation algebra is closed.

Box 10.1 The $\mathcal{N} = 1$ Supersymmetry Algebra

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = (\sigma^\mu)_{\alpha\dot\beta} P_\mu \quad \text{(Aitchison normalization)} \tag{10.54}$$

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0 \tag{10.55}$$

$$[Q_\alpha, P_\mu] = [\bar{Q}_{\dot\alpha}, P_\mu] = 0 \tag{10.56}$$

The SUSY generators are directly connected to the energy-momentum operator, justifying the view of SUSY as an extension of spacetime symmetry.

The Hamiltonian as a sum of squares

A striking consequence of the algebra is that the Hamiltonian can be expressed as a sum of squares of supercharges. From $\{Q_\alpha, \bar{Q}_{\dot\beta}\} = (\sigma^\mu)_{\alpha\dot\beta} P_\mu$ and recalling that $(\sigma^0)_{11} = (\sigma^0)_{22} = 1$, $(\sigma^3)_{11} = -(\sigma^3)_{22} = 1$, we have

$$\{Q_1, \bar{Q}_{\dot 1}\} = P_0 + P_3\,, \qquad \{Q_2, \bar{Q}_{\dot 2}\} = P_0 - P_3\,. \tag{10.57}$$

Following Aitchison, P_3 in these component formulas is the covariant component appearing in $\sigma^\mu P_\mu$; with the $(+---)$ metric, $P_3 = -P^3$. Adding these relations, the P_3 contributions cancel:

$$\boxed{H = P^0 = \frac{1}{2}(Q_1\bar{Q}_{\dot 1} + \bar{Q}_{\dot 1}Q_1 + Q_2\bar{Q}_{\dot 2} + \bar{Q}_{\dot 2}Q_2)} \tag{10.58}$$

This has a profound consequence: in any state $|\psi\rangle$,

$$\langle\psi|H|\psi\rangle = \frac{1}{2}\left(\|Q_1|\psi\rangle\|^2 + \|\bar{Q}_{\dot 1}|\psi\rangle\|^2 + \|Q_2|\psi\rangle\|^2 + \|\bar{Q}_{\dot 2}|\psi\rangle\|^2\right) \geq 0\,. \tag{10.59}$$

The energy is non-negative, and $E = 0$ if and only if all supercharges annihilate the state: $Q_\alpha|0\rangle = \bar{Q}_{\dot\alpha}|0\rangle = 0$. This is the condition for an *unbroken supersymmetric vacuum*.

The supersymmetry current

Review: Noether's theorem for exact and quasi-symmetries. Recall the logic of Noether's theorem. If a Lagrangian $\mathcal{L}(\phi_r, \partial_\mu\phi_r)$ is *exactly* invariant under $\phi_r \to \phi_r + \delta\phi_r$, then $\delta\mathcal{L} = 0$. On the other hand, the chain rule gives

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi_r}\delta\phi_r + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_r)}\partial_\mu(\delta\phi_r) = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_r)}\delta\phi_r\right) + \left(\frac{\partial\mathcal{L}}{\partial\phi_r} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_r)}\right)\delta\phi_r\,.$$

The second bracket vanishes by the Euler–Lagrange equations (on shell). Combining with $\delta\mathcal{L} = 0$:

$$\partial_\mu j^\mu = 0\,, \qquad j^\mu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)}\,\delta\phi_r\,. \tag{10.60}$$

If the symmetry is only a *quasi-symmetry* — the Lagrangian is not invariant but changes by a total derivative, $\delta\mathcal{L} = \partial_\mu K^\mu$ for some K^μ — then the same argument gives $\partial_\mu(j^\mu - K^\mu) = 0$, so the conserved current is $j^\mu - K^\mu$. This is precisely the situation for supersymmetry, where the action is invariant but $\mathcal{L}$ shifts by a total derivative (as shown in Section 10.3).

Application to the Wess–Zumino model. For the free Wess–Zumino Lagrangian $\mathcal{L} = \partial_\mu\phi^\dagger\partial^\mu\phi + i\bar\chi\bar\sigma^\mu\partial_\mu\chi$, the fields are ϕ, $\phi^\dagger$, and χ_α (with $\bar\chi$ determined by χ). Since the SUSY current carries a spinor index (it is a spinor-valued vector), we write $j^\mu = \xi\cdot J^\mu + \bar\xi\cdot\bar J^\mu$, where J^μ_α and $\bar J^{\mu\,\dot\alpha}$ are the supercurrent and its conjugate. The Noether construction gives:

$$\xi\cdot J^\mu + \bar\xi\cdot\bar J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\delta\phi + \delta\phi^\dagger\,\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^\dagger)} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\chi_\alpha)}\,\delta\chi_\alpha - K^\mu\,. \tag{10.61}$$

Important sign convention for fermions. For the fermionic field χ, the derivative $\partial\mathcal{L}/\partial(\partial_\mu\chi_\alpha)$ must be taken as a *left* derivative. This means: identify $\partial_\mu\chi_\alpha$ in each term, and read off the coefficient that multiplies it *from the left*. Concretely, the fermionic kinetic term is

$$\mathcal{L}_F = i\,\underbrace{\bar\chi_{\dot\beta}\bar\sigma^{\mu\dot\beta\alpha}}_{\text{left coefficient}}\,\partial_\mu\chi_\alpha\,.$$

The object $\partial_\mu\chi_\alpha$ already sits at the *rightmost* position of the product, so its left coefficient is read off directly without moving any Grassmann field past another:

$$\frac{\partial^L\mathcal{L}_F}{\partial(\partial_\mu\chi_\alpha)} = i\bar\chi_{\dot\beta}\bar\sigma^{\mu\dot\beta\alpha}\,.$$

No sign flip occurs precisely because $\partial_\mu\chi_\alpha$ does not need to be commuted past $\bar\chi$ — it is already in the correct position. Had the Lagrangian been written in the opposite order, $\mathcal{L}_F = i(\partial_\mu\chi_\alpha)\bar\chi_{\dot\beta}\bar\sigma^{\mu\dot\beta\alpha}$, then $\partial_\mu\chi_\alpha$ would be at the *left* and extracting it would again give no sign; but the remaining coefficient $\bar\chi\bar\sigma^\mu$ would now be a *right* coefficient, and the distinction between left and right derivatives matters for the overall sign of the Noether current. The convention used here (and in Aitchison) is: always write $\mathcal{L}_F$ with $\partial_\mu\chi$ to the right, and take the left derivative. (See Weinberg I, §7.3 for a systematic treatment of fermionic functional derivatives.)

Computing each piece. The partial derivatives of $\mathcal{L}$ are:

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \partial^\mu\phi^\dagger\,, \qquad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^\dagger)} = \partial^\mu\phi\,, \qquad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\chi_\alpha)} = i\bar\chi_{\dot\beta}\bar\sigma^{\mu\dot\beta\alpha}\,.$$

The SUSY variations (with $A=-i$):

$$\delta\phi = \xi\chi\,, \qquad \delta\phi^\dagger = \bar\xi\bar\chi\,, \qquad \delta\chi_\alpha = -i(\sigma^\nu\bar\xi)_\alpha\partial_\nu\phi\,.$$

Substituting into (10.61):

$$\begin{aligned}\xi\cdot J^\mu + \bar\xi\cdot\bar J^\mu &= \partial^\mu\phi^\dagger\cdot\xi\chi + \bar\xi\bar\chi\cdot\partial^\mu\phi + i\bar\chi\bar\sigma^\mu\cdot(-i\sigma^\nu\bar\xi\partial_\nu\phi) - K^\mu \\ &= \partial^\mu\phi^\dagger\,\xi\chi + \bar\xi\bar\chi\,\partial^\mu\phi + \bar\chi\bar\sigma^\mu\sigma^\nu\bar\xi\,\partial_\nu\phi - K^\mu\,. \end{aligned} \tag{10.62}$$

The total-derivative piece K^μ was determined during the invariance proof (Section 10.3); its role is to absorb the antisymmetric $\sigma^{\mu\nu}$ terms that arose there. After subtracting K^μ and simplifying using the equations of motion, the supercurrent takes the form:

$$\boxed{J^\mu_\alpha = (\sigma^\nu\bar\sigma^\mu\chi)_\alpha\,\partial_\nu\phi^\dagger} \tag{10.63}$$

with conjugate $\bar J^{\mu\dot\alpha} = (\bar\sigma^\nu\sigma^\mu\bar\chi)^{\dot\alpha}\partial_\nu\phi$. Conservation $\partial_\mu J^\mu_\alpha = 0$ holds on shell (using the free equations of motion $\Box\phi = 0$ and $i\bar\sigma^\mu\partial_\mu\chi = 0$).

Improvement terms. This result is not unique: the Noether current is defined only up to *improvement terms*. An improvement is any tensor $\Sigma^{\mu\nu} = -\Sigma^{\nu\mu}$ (antisymmetric in $\mu\nu$), since

$$J'^\mu \;\equiv\; J^\mu + \partial_\nu\Sigma^{\mu\nu}$$

is automatically conserved whenever J^μ is: $\partial_\mu J'^\mu = \partial_\mu J^\mu + \partial_\mu\partial_\nu\Sigma^{\mu\nu} = 0$ (the second term vanishes because $\partial_\mu\partial_\nu$ is symmetric while $\Sigma^{\mu\nu}$ is antisymmetric). The integrated charge is also unaffected: $Q' = \int d^3x\,J'^0 = \int d^3x\,J^0 + \int d^3x\,\partial_i\Sigma^{0i} = Q + (\text{boundary term at spatial infinity}) = Q$. Therefore J^μ and J'^μ generate the same symmetry and are physically equivalent. Different textbooks may present supercurrents that differ by such improvements; the algebra and the physics are identical.

The supercharges. Integrating the time component:

$$Q_\alpha = \int d^3x\,J^0_\alpha\,. \tag{10.64}$$

Using $\bar\sigma^0 = \mathbb{1}_2$, so $(\sigma^\nu\bar\sigma^0\chi)_\alpha = (\sigma^\nu\chi)_\alpha$:

$$Q_\alpha = \int d^3x\,(\sigma^\nu\chi)_\alpha\,\partial_\nu\phi^\dagger\,. \tag{10.65}$$

These charges generate the correct transformations via $\delta_\xi\phi = i[\xi\cdot Q + \bar\xi\cdot\bar Q, \phi] = \xi\cdot\chi$, as can be verified using the canonical (anti)commutation relations $[\phi(\vec x), \dot\phi^\dagger(\vec y)] = i\delta^3(\vec x-\vec y)$ and $\{\chi_\alpha(\vec x), \chi^\dagger_\beta(\vec y)\} = \delta_{\alpha\beta}\delta^3(\vec x-\vec y)$.

Supermultiplets

The SUSY algebra has profound consequences for the structure of particle multiplets.

Mass degeneracy. From $[Q_\alpha, P_\mu] = 0$, the operator $P^2 = P_\mu P^\mu$ commutes with all generators. States in a supermultiplet, connected by the action of Q and $\bar{Q}$, must all have the *same mass*.

Helicity content. Since Q_α transforms as a left-handed spinor under Lorentz transformations, its action on a one-particle state changes the angular momentum quantum number by $\pm\frac{1}{2}$. For the massless supermultiplets that follow, the relevant quantum number is the *helicity* $\lambda = \vec{J}\cdot\hat{p}$ (projection of total angular momentum along the direction of motion), not the spin s (the Casimir label of massive Poincaré representations). The reason is physical: a massless particle has no rest frame, so spin-as-"angular momentum at rest" is not defined; the only Lorentz-invariant label is the helicity. For our standard massless momentum $p^\mu = (E, 0, 0, E)$ along z, the helicity operator is simply J_3. We therefore need $[J_3, Q_\alpha]$.

To determine the precise sign for each component, we use the Lorentz algebra.

The supercharges Q_α belong to the $(\frac{1}{2}, 0)$ representation of the Lorentz group. We now derive the commutator $[J^i, Q_\alpha]$.

In quantum field theory, a Lorentz transformation Λ acts on an operator $\mathcal{O}_a(x)$ that transforms in a representation D as:

$$U(\Lambda)\,\mathcal{O}_a(x)\,U(\Lambda)^{-1} = D(\Lambda^{-1})_a{}^b\,\mathcal{O}_b(\Lambda x)\,.$$

For an infinitesimal rotation by $\vec{\theta}$, expanding both sides to first order gives:

$$[J^i, \mathcal{O}_a(x)] = \underbrace{-i(x^j\partial^k - x^k\partial^j)\mathcal{O}_a(x)}_{\text{orbital part}} + \underbrace{(S^i)_a{}^b\,\mathcal{O}_b(x)}_{\text{spin part}},$$

where S^i are the spin generators in the representation D.

For the supercharge $Q_\alpha = \int d^3x\, J^0_\alpha(x)$, the spatial integration kills the orbital part: terms like $\int d^3x\, x^j \partial^k J^0_\alpha$ can be integrated by parts in x^k (fundamental theorem of calculus), giving boundary contributions that vanish because $J^0_\alpha \to 0$ at spatial infinity. Only the spin part survives:

$$[J^i, Q_\alpha] = (S^i)_\alpha{}^\beta\, Q_\beta\,.$$

Since Q_α is in the $(\frac{1}{2}, 0)$ representation, $S^i = \frac{1}{2}\sigma^i$ (eq. (9.38)):

$$[J^i, Q_\alpha] = \frac{1}{2}(\sigma^i)_\alpha{}^\beta\, Q_\beta\,. \tag{10.66}$$

For $i = 3$: $(\sigma^3)_1{}^1 = +1$, $(\sigma^3)_2{}^2 = -1$, off-diagonal $= 0$. Therefore:

$$[J_3, Q_1] = +\frac{1}{2}Q_1\,, \qquad [J_3, Q_2] = -\frac{1}{2}Q_2\,. \tag{10.67}$$

The physical meaning: if $|\lambda\rangle$ is a helicity eigenstate with $J_3|\lambda\rangle = \lambda|\lambda\rangle$, then $Q_1|\lambda\rangle$ has helicity $\lambda + \frac{1}{2}$ (since $J_3(Q_1|\lambda\rangle) = ([J_3, Q_1] + Q_1 J_3)|\lambda\rangle = (\frac{1}{2}Q_1 + \lambda Q_1)|\lambda\rangle = (\lambda + \frac{1}{2})Q_1|\lambda\rangle$), and $Q_2|\lambda\rangle$ has helicity $\lambda - \frac{1}{2}$.

For the conjugate charges $\bar{Q}_{\dot\alpha} = Q_\alpha^\dagger$, which belong to the $(0, \frac{1}{2})$ representation: $Q_1^\dagger$ lowers helicity by $\frac{1}{2}$, and $Q_2^\dagger$ raises it by $\frac{1}{2}$.

In summary: Q_1 and $Q_2^\dagger$ raise the helicity by $\frac{1}{2}$; Q_2 and $Q_1^\dagger$ lower it by $\frac{1}{2}$.

Constructing massless supermultiplets. The strategy is to start from a lowest-helicity state and act on it with all possible supercharges, keeping only the non-zero states. The result is a short multiplet with very few states — far fewer than for massive particles.

Step 1: choose the reference momentum. For a massless particle, pick a standard momentum along the z-axis: $p^\mu = (E, 0, 0, E)$, so that $p_\mu = (E, 0, 0, -E)$ in the $(+---)$ convention. On this state:

$$P^0 = E\,, \quad P^3 = E\,, \quad P_0 = E\,, \quad P_3 = -E\,.$$

Step 2: evaluate the anticommutators on this state. The algebra $\{Q_\alpha, \bar{Q}_{\dot\beta}\} = (\sigma^\mu)_{\alpha\dot\beta} P_\mu$ becomes, on a state with momentum p:

$$\{Q_1, Q_1^\dagger\} = (\sigma^\mu)_{11} P_\mu = P_0 + P_3 = E + (-E) = 0\,, \tag{10.68}$$

$$\{Q_2, Q_2^\dagger\} = (\sigma^\mu)_{22} P_\mu = P_0 - P_3 = E - (-E) = 2E\,. \tag{10.69}$$

(Here we used $(\sigma^0)_{11} = (\sigma^0)_{22} = 1$, $(\sigma^3)_{11} = +1$, $(\sigma^3)_{22} = -1$, and $(\sigma^{1,2})_{11} = (\sigma^{1,2})_{22} = 0$.)

Equation (10.68) is remarkable: it says $\{Q_1, Q_1^\dagger\} = 0$. Since $\{Q_1, Q_1^\dagger\} = Q_1 Q_1^\dagger + Q_1^\dagger Q_1$, taking the expectation value in *any* state $|\psi\rangle$:

$$\|Q_1|\psi\rangle\|^2 + \|Q_1^\dagger|\psi\rangle\|^2 = 0\,.$$

Both terms are non-negative (norms squared), so both must vanish: $Q_1|\psi\rangle = Q_1^\dagger|\psi\rangle = 0$ for *every* state with this momentum. Thus Q_1 and $Q_1^\dagger$ act as zero on the entire massless representation. The only non-trivial generators are Q_2 and $Q_2^\dagger$.

Step 3: build the multiplet from the lowest-helicity state. Let $|p, \lambda_{\min}\rangle$ be a state with the lowest helicity $\lambda_{\min}$ in the multiplet. Since Q_2 lowers helicity by $\frac{1}{2}$ (eq. (10.67)), acting with it would produce a state of helicity $\lambda_{\min} - \frac{1}{2}$, which cannot exist if $\lambda_{\min}$ is the minimum. Therefore:

$$Q_2|p, \lambda_{\min}\rangle = 0\,.$$

Similarly, $Q_1^\dagger$ lowers helicity by $\frac{1}{2}$, but we already showed $Q_1^\dagger = 0$ on all massless states.

The only non-trivial action is $Q_2^\dagger$, which *raises* helicity by $\frac{1}{2}$:

$$|p, \lambda_{\min} + \tfrac{1}{2}\rangle \;\propto\; Q_2^\dagger|p, \lambda_{\min}\rangle\,.$$

This state is non-zero: from (10.69) with $Q_2|p, \lambda_{\min}\rangle = 0$,

$$\langle p, \lambda_{\min}|Q_2 Q_2^\dagger|p, \lambda_{\min}\rangle = 2E > 0\,,$$

confirming $Q_2^\dagger|p, \lambda_{\min}\rangle \neq 0$.

Step 4: the chain terminates. Acting with $Q_2^\dagger$ a second time: $(Q_2^\dagger)^2 = 0$ because $\{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0$ implies $(Q_2^\dagger)^2 = \frac{1}{2}\{Q_2^\dagger, Q_2^\dagger\} = 0$. So no further states are generated.

Result. Each massless $\mathcal{N} = 1$ supermultiplet consists of exactly **two** helicity states:

$$\boxed{\text{Massless supermultiplet: } |p, \lambda\rangle \text{ and } |p, \lambda + \tfrac{1}{2}\rangle} \tag{10.70}$$

where λ is the lower helicity. CPT invariance requires the inclusion of the conjugate states with helicities $-\lambda$ and $-\lambda - \frac{1}{2}$, so a CPT-complete massless supermultiplet has four states in total (or two, if the multiplet is self-conjugate under CPT).

Physical examples.

Chiral (matter) supermultiplet: $\lambda = 0$. The two states have helicities 0 and $+\frac{1}{2}$. CPT adds 0 and $-\frac{1}{2}$. The fields describing these states are:

- a complex scalar ϕ (helicity 0, two real degrees of freedom on shell),
- a Weyl fermion χ_α (helicities $\pm\frac{1}{2}$, two real degrees of freedom on shell).

This is exactly the Wess–Zumino multiplet (ϕ, χ, F) studied in Section 10.3 (with F the off-shell auxiliary field that vanishes on shell). Every quark and lepton of the Standard Model sits in a chiral supermultiplet, paired with a scalar "superpartner" (squark, slepton).

Vector (gauge) supermultiplet: $\lambda = -1$. The two states have helicities -1 and $-\frac{1}{2}$. CPT adds $+1$ and $+\frac{1}{2}$. The fields are:

- a massless gauge boson A_μ (helicities ± 1, as for the photon or gluon),
- a Weyl fermion λ_α called the *gaugino* (helicities $\pm\frac{1}{2}$).

Every gauge boson of the SM (photon, $W^\pm$, Z, gluons) sits in a vector supermultiplet, paired with a fermionic gaugino (photino, wino, zino, gluino).

Graviton supermultiplet: $\lambda = -2$. The two states have helicities -2 and $-\frac{3}{2}$. CPT adds $+2$ and $+\frac{3}{2}$. The fields are:

- the graviton $g_{\mu\nu}$ (helicities ± 2, the quantum of gravity),
- the *gravitino* ψ_μ (helicities $\pm\frac{3}{2}$, a spin-$\frac{3}{2}$ Rarita–Schwinger field).

This multiplet requires local (gauged) supersymmetry, i.e. *supergravity*, which is beyond the scope of these notes.

The boson–fermion degeneracy. In every supermultiplet, the number of bosonic and fermionic on-shell degrees of freedom is equal: 2 bosonic = 2 fermionic. This is a general theorem: SUSY maps bosons to fermions and vice versa, and since $[Q, P^2] = 0$, all states in a supermultiplet have the same mass. If SUSY were unbroken, every particle would have a superpartner of equal mass but spin differing by $\frac{1}{2}$. The absence of such partners at current collider energies means SUSY must be broken, with the superpartner masses pushed above the experimental reach.

Remark. [What does $\mathcal{N}$ mean, and why only $\mathcal{N} = 1$?] The integer $\mathcal{N}$ counts the number of *independent* sets of supercharges. In $\mathcal{N} = 1$ SUSY (the case treated in these notes), there is a single pair $(Q_\alpha, \bar{Q}_{\dot\alpha})$ — one left-handed and one right-handed spinor charge. In $\mathcal{N} = 2$ SUSY, there are two such pairs $(Q^1_\alpha, \bar{Q}^1_{\dot\alpha})$ and $(Q^2_\alpha, \bar{Q}^2_{\dot\alpha})$; more generally, $\mathcal{N}$ copies. Each additional pair of supercharges further constrains the multiplet structure and enlarges the minimal representations.

For example, in $\mathcal{N} = 2$ the smallest matter representation (the *hypermultiplet*) contains states with $\lambda = +\frac{1}{2}, -\frac{1}{2}$, and two $\lambda = 0$ states. The two fermionic components are tied together by the extended supersymmetry and therefore *cannot* be assigned different gauge quantum numbers. But in the Standard Model, $\mathrm{SU}(2)_L$ treats left- and right-handed fermions differently (only left-handed fermions form doublets). This chiral structure is incompatible with $\mathcal{N} \geq 2$ matter multiplets: in such multiplets, the left- and right-handed components are locked into the same representation and cannot be separated.

Realistic four-dimensional weak-scale model building therefore starts from $\mathcal{N} = 1$, where left and right chiral multiplets can be introduced independently with different gauge charges. Extended SUSY ($\mathcal{N} \geq 2$) can still appear in gauge sectors ($\mathcal{N} = 2$ gauge theories have remarkable properties), in higher-dimensional

constructions, or in effective descriptions after suitable breaking to $\mathcal{N} = 1$ at low energies.

10.5 Interactions and the superpotential

Having established the free Wess-Zumino model, we now turn to the crucial question of interactions. The requirement of supersymmetry invariance places severe constraints on the allowed interaction terms, leading to a remarkably elegant structure encoded in a single holomorphic function called the *superpotential*. We follow closely the treatment of Aitchison.

The free multi-field Lagrangian

Consider a collection of chiral supermultiplets (ϕ_i, χ_i, F_i) labelled by an index i running over all gauge and flavour degrees of freedom. The free Lagrangian generalises to

$$\mathcal{L}_{\text{free}} = \partial^\mu \phi^{\dagger i} \partial_\mu \phi_i + \bar{\chi}^i i\bar{\sigma}^\mu \partial_\mu \chi_i + F^{\dagger i} F_i \,, \tag{10.71}$$

where we sum over repeated indices with the convention that fields ϕ_i and χ_i carry lowered indices while their conjugates carry raised indices. This Lagrangian is invariant under the SUSY transformations

$$\delta_\epsilon \phi_i = \epsilon \chi_i, \qquad \delta_\epsilon \chi_i = -i\sigma^\mu \bar{\epsilon}\, \partial_\mu \phi_i + \epsilon F_i, \qquad \delta_\epsilon F_i = -i\bar{\epsilon}\bar{\sigma}^\mu \partial_\mu \chi_i \,, \tag{10.72}$$

together with their hermitian conjugates.

The most general interaction ansatz

The next step is to introduce interactions in such a way as to preserve SUSY. We impose that all interaction terms be *renormalizable*, i.e. have mass dimension ≤ 4. Since we already have $\mathcal{L}_{\text{free}}$, the interaction Lagrangian $\mathcal{L}_{\text{int}}$ must contain only fields *without* kinetic terms (no additional $\partial_\mu\phi\,\partial^\mu\phi$ or $\bar{\chi} i\bar{\sigma}^\mu \partial_\mu \chi$).

What building blocks are available? The fields have mass dimensions $[\phi] = 1$, $[\chi] = 3/2$, $[F] = 2$. Renormalizability restricts us to terms of total dimension ≤ 4. The possible Lorentz-scalar combinations are:

- **Terms linear in F_i:** $W^i(\phi, \phi^\dagger)\, F_i$, where $[W^i F_i] \leq 4$ requires $[W^i] \leq 2$, so W^i is at most quadratic in ϕ.
- **Bilinears in χ:** $W^{ij}(\phi, \phi^\dagger)\, \chi_i \chi_j$, where $[\chi\chi] = 3$ requires $[W^{ij}] \leq 1$, so W^{ij} is at most linear in ϕ.
- **Terms involving $\bar{\chi}^i$ and χ_j:** such as $\bar{\chi}^i \bar{\sigma}^\mu \chi_j\, \partial_\mu(\ldots)$; these would introduce derivatives and are already accounted for in $\mathcal{L}_{\text{free}}$.

- **Pure scalar terms $U(\phi,\phi^\dagger)$:** ruled out below.

The spinor product $\chi_i\chi_j \equiv \chi_i^\alpha \chi_{j\alpha} = \epsilon^{\alpha\beta}\chi_{i\beta}\chi_{j\alpha}$ is *symmetric* in $i \leftrightarrow j$ (the antisymmetry of $\epsilon^{\alpha\beta}$ compensates the sign from exchanging two Grassmann quantities), so W^{ij} must be symmetric too: $W^{ij} = W^{ji}$.

Therefore the most general renormalizable interaction Lagrangian is

$$\mathcal{L}_{\text{int}} = W^i(\phi,\phi^\dagger)F_i - \frac{1}{2}W^{ij}(\phi,\phi^\dagger)\chi_i\chi_j + \text{h.c.}, \tag{10.73}$$

where W^i and W^{ij} are, for the moment, arbitrary functions of the bosonic fields (not of χ or F, since including those would either violate Lorentz invariance or exceed dimension 4). The factor $1/2$ avoids double-counting the symmetric sum over i, j.

Why no pure scalar potential? One might ask: why not include a term $U(\phi,\phi^\dagger)$ involving only scalars? Under a SUSY transformation, such a term would vary as

$$\delta_\epsilon U = \frac{\partial U}{\partial \phi_k}(\epsilon\chi_k) + \frac{\partial U}{\partial \phi^{\dagger k}}(\bar\epsilon\bar\chi^k). \tag{10.74}$$

This contains no derivatives ∂_μ, no auxiliary fields F, and no conjugate spinors $\bar\chi$ in the first term. Inspecting the transformations (10.72), no other term in the Lagrangian can produce such a variation to cancel it. Therefore $U = 0$.

The holomorphicity constraint

Since we know $\mathcal{L}_{\text{free}}$ is invariant by itself, we need only require $\delta_\epsilon\mathcal{L}_{\text{int}} = 0$ (up to a total derivative). We analyse the variation piece by piece.

Step 1: The four-spinor terms. Since W^{ij} depends on the bosonic fields ϕ_k and $\phi^{\dagger k}$, the SUSY variation $\delta_\epsilon\phi_k = \epsilon\chi_k$ and $\delta_\epsilon\phi^{\dagger k} = \bar\epsilon\bar\chi^k$ acting on $-\frac{1}{2}W^{ij}\chi_i\chi_j$ (where $\chi_i\chi_j$ is *not* varied, since this produces a separate class of terms analysed in Steps 3–4) gives

$$\delta_\epsilon\mathcal{L}\big|_{\text{4-spinor}} = -\frac{1}{2}\frac{\partial W^{ij}}{\partial\phi_k}\underbrace{(\epsilon\chi_k)(\chi_i\chi_j)}_{\text{3 undotted spinors}} - \frac{1}{2}\frac{\partial W^{ij}}{\partial\phi^{\dagger k}}\underbrace{(\bar\epsilon\bar\chi^k)(\chi_i\chi_j)}_{\text{1 dotted + 2 undotted}} + \text{h.c.}. \tag{10.75}$$

These are "four-spinor" terms (four Grassmann fields, no derivatives, no F). No other term in $\mathcal{L}_{\text{int}}$ or $\mathcal{L}_{\text{free}}$ produces four-spinor terms under a SUSY variation: $\delta_\epsilon(W^iF_i)$ produces terms with F or ∂_μ, and $\delta_\epsilon\mathcal{L}_{\text{free}}$ produces terms with ∂_μ. Therefore the two terms in (10.75) must each vanish separately.

The first term involves four undotted (left-handed) spinors ϵ, χ_k, χ_i, χ_j—but two of them are already contracted in $(\chi_i\chi_j) = \chi_i^\alpha\chi_{j\alpha}$. The key tool is the following Schouten identity, valid for any four left-handed Weyl spinors α, β, γ, δ:

$$(\alpha\beta)(\gamma\delta) + (\alpha\gamma)(\delta\beta) + (\alpha\delta)(\beta\gamma) = 0. \tag{10.76}$$

Concretely, for $\alpha \to \epsilon$, $\beta \to \chi_i$, $\gamma \to \chi_j$, $\delta \to \chi_k$:

$$(\epsilon\chi_i)(\chi_j\chi_k) + (\epsilon\chi_j)(\chi_k\chi_i) + (\epsilon\chi_k)(\chi_i\chi_j) = 0\,. \tag{10.77}$$

Box 10.2 Proof of the Fierz identity ( )

For two-dimensional spinor indices the Schouten identity is

$$\epsilon_{\alpha\beta}\epsilon_{\gamma\delta} + \epsilon_{\alpha\gamma}\epsilon_{\delta\beta} + \epsilon_{\alpha\delta}\epsilon_{\beta\gamma} = 0\,. \tag{10.78}$$

Multiplying by the four anticommuting spinors $\alpha^\alpha\beta^\beta\gamma^\gamma\delta^\delta$ and keeping the NW-SE order of the contractions gives (10.76).

Now, the first term in (10.75), summed over repeated indices i, j, k, reads

$$-\frac{1}{2}\frac{\partial W^{ij}}{\partial\phi_k}(\epsilon\chi_k)(\chi_i\chi_j)\,. \tag{10.79}$$

By the identity (10.77), the factor $(\epsilon\chi_k)(\chi_i\chi_j)$ can be rewritten as $-(\epsilon\chi_i)(\chi_j\chi_k) - (\epsilon\chi_j)(\chi_k\chi_i)$. Since all three index labels i, j, k are summed over, these three terms are identical if $\partial W^{ij}/\partial\phi_k$ is *totally symmetric* in i, j, k. In that case

$$\frac{\partial W^{ij}}{\partial\phi_k}\Big[(\epsilon\chi_k)(\chi_i\chi_j) + (\epsilon\chi_i)(\chi_j\chi_k) + (\epsilon\chi_j)(\chi_k\chi_i)\Big] = 0\,, \tag{10.80}$$

and the first term vanishes identically. This is a *constraint on* W^{ij}: the quantity $\partial W^{ij}/\partial\phi_k$ must be symmetric in all three indices.

The second term involves the product $(\bar\epsilon\bar\chi^k)(\chi_i\chi_j)$ of one dotted spinor bilinear and one undotted spinor bilinear. There is *no* analogue of the Fierz identity for this mixed product: the identity (10.77) relies on contracting spinors of the *same* chirality with $\epsilon_{\alpha\beta}$, and dotted/undotted indices cannot be contracted against each other. Therefore, no symmetry or identity can make this term vanish, and the only way to eliminate it is to demand

$$\boxed{\frac{\partial W^{ij}}{\partial\phi^{\dagger k}} = 0}\,. \tag{10.81}$$

This is the **holomorphicity constraint**: W^{ij} must be a function of ϕ_k *only*, not of $\phi^{\dagger k}$. This is the single most important structural result in the construction of SUSY Lagrangians.

Step 2: Structure of W^{ij}. We have established that W^{ij} must be:

1. holomorphic: $W^{ij} = W^{ij}(\phi_k)$ (no dependence on $\phi^{\dagger k}$);

2. symmetric: $W^{ij} = W^{ji}$ (from $\chi_i\chi_j = \chi_j\chi_i$);
3. at most linear in ϕ_k: from $[W^{ij}] \leq 1$;
4. such that $\partial W^{ij}/\partial\phi_k$ is totally symmetric in i, j, k.

The most general function satisfying (1)–(3) is

$$W^{ij} = M^{ij} + y^{ijk}\phi_k\,, \tag{10.82}$$

where $M^{ij} = M^{ji}$ is a symmetric constant matrix (with dimension of mass) and y^{ijk} are dimensionless constants symmetric in $i \leftrightarrow j$.

Physical meaning of M^{ij}: the fermion mass matrix. The constant M^{ij} in W^{ij} is *not* an inert mathematical artefact — it is the **fermion mass matrix** of the theory, and it cannot be omitted without making the chiral multiplets massless. Indeed, substituting (10.82) into the interaction Lagrangian $\mathcal{L}_{\text{int}} \supset -\frac{1}{2}W^{ij}\chi_i\chi_j$:

$$-\tfrac{1}{2}W^{ij}\chi_i\chi_j = \underbrace{-\tfrac{1}{2}M^{ij}\chi_i\chi_j}_{\text{Dirac/Majorana fermion mass}} \underbrace{-\tfrac{1}{2}y^{ijk}\phi_k\chi_i\chi_j}_{\text{Yukawa coupling}}\,.$$

Each chiral multiplet contains a Weyl fermion χ_i; the matrix M^{ij} pairs these fermions and gives them mass. Supersymmetry then enforces the *same* mass on the bosonic partners ϕ_i: the F-term scalar potential $|W^i|^2$ contains

$$\left|M^{ij}\phi_j + \tfrac{1}{2}y^{ijk}\phi_j\phi_k\right|^2 \supset (M^\dagger M)^j{}_k\,\phi^{\dagger k}\phi_j\,,$$

which is a scalar mass term with the same matrix $M^\dagger M$ that controls fermion masses. The Wess–Zumino model with non-zero M describes a massive supermultiplet.

When can M^{ij} be set to zero? Only when one wants a *massless* (often conformal) theory: e.g. $\mathcal{N} = 4$ super-Yang–Mills, or chiral matter in the unbroken phase of the MSSM, where quark and lepton masses arise solely from Yukawa couplings $y^{ijk}\phi_k$ with ϕ_k the Higgs field, after electroweak symmetry breaking. Even in the MSSM, however, the Higgs supermultiplets themselves carry a non-zero superpotential mass: the famous μ-term $\mu\, H_u \cdot H_d$ is exactly of the form $M^{ij}\phi_i\phi_j$ and provides masses to the higgsinos. The "μ-problem" — why this mass is naturally at the TeV scale — is one of the central open puzzles of MSSM phenomenology.

Condition (4) then requires

$$\frac{\partial W^{ij}}{\partial\phi_k} = y^{ijk} \quad \text{to be totally symmetric in } i, j, k\,, \tag{10.83}$$

which forces y^{ijk} to be totally symmetric under *any* permutation of its three indices.

It is now convenient to observe that W^{ij} can be written as the second derivative of a single function. Define the **superpotential**

$$\boxed{W(\phi) = \frac{1}{2}M^{ij}\phi_i\phi_j + \frac{1}{6}y^{ijk}\phi_i\phi_j\phi_k}\,. \tag{10.84}$$

Then (using $M^{ij} = M^{ji}$ and y^{ijk} totally symmetric):

$$\frac{\partial W}{\partial \phi_i} = M^{ij}\phi_j + \frac{1}{2}y^{ijk}\phi_j\phi_k\,, \qquad \frac{\partial^2 W}{\partial\phi_i\partial\phi_j} = M^{ij} + y^{ijk}\phi_k = W^{ij}\,. \tag{10.85}$$

(In the first equation, the factor $1/2$ in W becomes 1 because $M^{ij}\phi_i\phi_j$ gives $M^{ij}\phi_j + M^{ji}\phi_j = 2M^{ij}\phi_j$ upon differentiating with respect to ϕ_i, and similarly $\frac{1}{6}y^{ijk}\phi_i\phi_j\phi_k$ gives $\frac{3}{6}y^{ijk}\phi_j\phi_k = \frac{1}{2}y^{ijk}\phi_j\phi_k$ by total symmetry.)

Step 3: The derivative terms. Next we consider those parts of $\delta_\epsilon\mathcal{L}_{\rm int}$ containing one spacetime derivative ∂_μ. These come from:

- $\delta_\epsilon F_i = -i\bar\epsilon\bar\sigma^\mu\partial_\mu\chi_i$ acting on W^iF_i
- $\delta_\epsilon\chi_i = -i\sigma^\mu\bar\epsilon\,\partial_\mu\phi_i$ acting on $-\frac{1}{2}W^{ij}\chi_i\chi_j$

Explicitly:

$$\delta_\epsilon\mathcal{L}\big|_\partial = W^i(-i\bar\epsilon\bar\sigma^\mu\partial_\mu\chi_i) - \frac{1}{2}W^{ij}\Big[(-i\sigma^\mu\bar\epsilon\,\partial_\mu\phi_i)\chi_j + \chi_i(-i\sigma^\mu\bar\epsilon\,\partial_\mu\phi_j)\Big] + \text{h.c.}\,. \tag{10.86}$$

Let us manipulate the last two terms in (10.86) (those containing $\delta_\epsilon\chi$). In the second term, $(-i\sigma^\mu\bar\epsilon\,\partial_\mu\phi_i)$ is a left-handed spinor multiplied into χ_j; in the third term, χ_i is multiplied into $(-i\sigma^\mu\bar\epsilon\,\partial_\mu\phi_j)$. Since the spinor product is symmetric ($\alpha\beta = \beta\alpha$ for left-handed spinors) and $W^{ij} = W^{ji}$, the two terms are equal:

$$-\frac{1}{2}W^{ij}\Big[(-i\sigma^\mu\bar\epsilon\,\partial_\mu\phi_i)\chi_j + \chi_i(-i\sigma^\mu\bar\epsilon\,\partial_\mu\phi_j)\Big] = -W^{ij}(-i\sigma^\mu\bar\epsilon\,\partial_\mu\phi_j)\chi_i\,. \tag{10.87}$$

Keeping the NW-SE contractions and the Grassmann signs in the product $(-i\sigma^\mu\bar\epsilon)\chi_i$ gives the Aitchison identity

$$-W^{ij}(-i\sigma^\mu\bar\epsilon\,\partial_\mu\phi_j)\chi_i = -iW^{ij}\,\bar\epsilon\bar\sigma^\mu\chi_i\,\partial_\mu\phi_j\,. \tag{10.88}$$

Now we use the chain rule. Since $W^{ij} = \partial^2 W/\partial\phi_i\partial\phi_j$:

$$W^{ij}\,\partial_\mu\phi_j = \frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\,\partial_\mu\phi_j = \partial_\mu\left(\frac{\partial W}{\partial\phi_i}\right), \tag{10.89}$$

where the last step follows because W depends on x^μ only through the fields $\phi_j(x)$, and $\partial W^{ij}/\partial(\partial_\mu\phi_j) = 0$.

Collecting the first term in (10.86) (from $\delta_\epsilon F_i$) and the result (10.88):

$$\delta_\epsilon\mathcal{L}\big|_\partial = -iW^i\bar\epsilon\bar\sigma^\mu\partial_\mu\chi_i - i\,\bar\epsilon\bar\sigma^\mu\chi_i\,\partial_\mu\left(\frac{\partial W}{\partial\phi_i}\right) + \text{h.c.}\,. \tag{10.90}$$

We want this to be a total derivative. Using the product rule,

$$\partial_\mu(f\,\bar\epsilon\bar\sigma^\mu\chi_i) = (\partial_\mu f)\,\bar\epsilon\bar\sigma^\mu\chi_i + f\,\bar\epsilon\bar\sigma^\mu\partial_\mu\chi_i\,, \tag{10.91}$$

we see that (10.90) equals $\partial_\mu(-iW^i\,\bar\epsilon\bar\sigma^\mu\chi_i)+\text{h.c.}$—which is a total derivative—*if and only if* the coefficient W^i of the first term equals $\partial W/\partial\phi_i$ from the second term:

$$\boxed{W^i = \frac{\partial W}{\partial\phi_i}}\,. \tag{10.92}$$

With this identification, (10.90) becomes

$$\delta_\epsilon\mathcal{L}\big|_\partial = \partial_\mu\left(-i\frac{\partial W}{\partial\phi_i}\,\bar\epsilon\bar\sigma^\mu\chi_i\right)+\text{h.c.}\,, \tag{10.93}$$

which is indeed a total derivative and does not affect the action.

Let us verify the consistency of (10.92) with the superpotential (10.84). Computing:

$$W^i = \frac{\partial W}{\partial\phi_i} = M^{ij}\phi_j + \frac{1}{2}y^{ijk}\phi_j\phi_k\,, \tag{10.94}$$

which is indeed at most quadratic in ϕ, as required by $[W^i]\le 2$. ✓

Holomorphicity of W^i is automatic. Although we proved the holomorphicity constraint (10.81) only for W^{ij} (from the four-spinor terms in Step 1), the relation $W^i=\partial W/\partial\phi_i$ established in Step 3 *forces* W^i to be holomorphic as well: it is the derivative of a holomorphic function $W(\phi)$, and therefore depends only on ϕ and not on $\phi^\dagger$. From the explicit form (10.94), this is manifest — no $\phi^\dagger$ appears.

Equivalently, had we kept an antiholomorphic dependence in the original ansatz $W^i(\phi,\phi^\dagger)$, the SUSY variation of W^iF_i via $\delta_\epsilon\phi^{\dagger k}=\bar\epsilon\bar\chi^k$ would have produced uncancelled terms with structure $\bar\epsilon\bar\chi^kF_i$ (Step 4 below shows that all such F-terms cancel only when the holomorphic structure is used). So the SUSY-invariance argument enforces holomorphicity of both W^{ij} and W^i in a single, mutually consistent way — ultimately because both descend from one holomorphic function W.

Step 4: The F-terms. The remaining terms in $\delta_\epsilon\mathcal{L}_{\text{int}}$ are those containing F_i or $F^{\dagger i}$ (but no spacetime derivatives). They arise from three sources:

1. **From $\delta_\epsilon\phi_k=\epsilon\chi_k$ acting on W^iF_i:** since W^i depends on ϕ,

$$\delta_\epsilon(W^iF_i)\big|_F = \frac{\partial W^i}{\partial\phi_k}(\epsilon\chi_k)F_i = W^{ik}(\epsilon\chi_k)F_i\,, \tag{10.95}$$

where we used $\partial W^i/\partial\phi_k=\partial^2W/\partial\phi_i\partial\phi_k=W^{ik}$.

2. **From $\delta_\epsilon \chi_i = \epsilon F_i$ acting on $-\frac{1}{2}W^{ij}\chi_i\chi_j$:** varying the first χ_i gives ϵF_i, and similarly for χ_j; using $W^{ij} = W^{ji}$ and $\chi_i\chi_j = \chi_j\chi_i$, both contributions are equal:

$$-\frac{1}{2}W^{ij}\Big[(\epsilon F_i)\chi_j + \chi_i(\epsilon F_j)\Big] = -W^{ij}(\epsilon\chi_j)F_i\,. \tag{10.96}$$

3. **Hermitian conjugate terms:** the h.c. of $\mathcal{L}_{\text{int}}$ produces analogous terms with $F^{\dagger i}$, $\bar{\epsilon}$, and $\bar{\chi}^k$.

Adding (10.95) and (10.96):

$$\delta_\epsilon \mathcal{L}_{\text{int}}\big|_F = W^{ik}(\epsilon\chi_k)F_i - W^{ij}(\epsilon\chi_j)F_i + \text{h.c.} \tag{10.97}$$

Since W^{ik} and W^{ij} are the same object (both equal $\partial^2 W/\partial\phi_i\partial\phi_{k \text{ or } j}$) and the dummy summation indices k and j run over the same range, the two terms cancel exactly:

$$\delta_\epsilon \mathcal{L}_{\text{int}}\big|_F = 0\,. \quad \checkmark \tag{10.98}$$

This completes the proof that $\mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}}$ is invariant under SUSY, provided $W^i = \partial W/\partial\phi_i$ and $W^{ij} = \partial^2 W/\partial\phi_i\partial\phi_j$ with W holomorphic.

Summary: the superpotential

Collecting the results: SUSY invariance forces W to be holomorphic and fixes $W^i = \partial W/\partial\phi_i$ and $W^{ij} = \partial^2 W/\partial\phi_i\partial\phi_j$. For a renormalizable theory, W is at most cubic in the fields (see (10.84)), with M^{ij} symmetric and y^{ijk} totally symmetric.

The superpotential is *not* an ordinary potential. The name "superpotential" is somewhat misleading. In ordinary classical or quantum mechanics, the *potential* $V(\phi)$ is a real function of the fields that enters the Lagrangian as $\mathcal{L} \supset -V(\phi)$ and whose minima define stable vacua. The superpotential W is different in two essential ways:

1. **W is complex (holomorphic).** A real potential $V(\phi, \phi^\dagger)$ depends on both ϕ and $\phi^\dagger$. The superpotential $W(\phi)$, by contrast, is a function of the *holomorphic* fields ϕ_i alone — it does not depend on $\phi^{\dagger i}$. This is a powerful constraint forced by SUSY invariance (proved in the holomorphicity argument above) and is what underlies the famous *non-renormalization theorems*: W receives no perturbative quantum corrections, term by term in ϕ. In particular, if $W = 0$ at tree level, it stays zero to all orders in perturbation theory.

2. W **does *not* appear directly in the Lagrangian as $-W$.** It enters only through its derivatives W^i and W^{ij}:

$$\mathcal{L}_{\text{int}} = W^i F_i - \tfrac{1}{2} W^{ij} \chi_i \chi_j + \text{h.c.}$$

Eliminating the auxiliary field F_i via its equation of motion ($F_i = -W^{i*}$, see Section 10.5 below) produces a *real* scalar potential

$$V(\phi, \phi^\dagger) = \sum_i \left| \frac{\partial W}{\partial \phi_i} \right|^2 = |W^i|^2 ,$$

which is manifestly real and non-negative ($V \geq 0$, with $V = 0$ only at SUSY-preserving vacua). Thus W is the "square root" of the physical scalar potential: $V = |W^i|^2$, not $V = W$.

In short: W is a complex, holomorphic "generating function" from which the real physical potential, the fermion mass matrix, and the Yukawa couplings are all derived in a SUSY-consistent way. The word "super-" is the only hint that it lives in a different category from ordinary potentials.

Remark. [Linear terms] A linear term $L^i \phi_i$ can be added to W without spoiling SUSY. Such terms are only gauge-invariant when ϕ_i is a gauge singlet, and play a role in certain models of spontaneous SUSY breaking (e.g., the O'Raifeartaigh model).

Eliminating the auxiliary fields

The part of the Lagrangian containing the auxiliary fields is

$$\mathcal{L}_F = F^{\dagger i} F_i + W^i F_i + W_i^\dagger F^{\dagger i} . \tag{10.99}$$

Since this contains no derivatives, the Euler-Lagrange equations for F_i and $F^{\dagger i}$ are simply

$$\frac{\partial \mathcal{L}}{\partial F_i} = F^{\dagger i} + W^i = 0 \quad \Rightarrow \quad F^{\dagger i} = -W^i = -\frac{\partial W}{\partial \phi_i} , \tag{10.100}$$

and similarly $F_i = -W_i^\dagger$. These relations involve no derivatives, so the canonical commutation relations are unaffected and we may substitute them directly into the Lagrangian.

The complete **Wess-Zumino Lagrangian** after eliminating auxiliary fields is

$$\boxed{\mathcal{L}_{\text{WZ}} = \partial^\mu \phi^{\dagger i} \partial_\mu \phi_i + \bar{\chi}^i i \bar{\sigma}^\mu \partial_\mu \chi_i - \frac{1}{2} \left(W^{ij} \chi_i \chi_j + W_{ij}^\dagger \bar{\chi}^i \bar{\chi}^j \right) - W^i W_i^\dagger .} \tag{10.101}$$

The **scalar potential** is

$$\boxed{V(\phi,\phi^\dagger) = W^i W_i^\dagger = F^{\dagger i} F_i = \sum_i \left|\frac{\partial W}{\partial \phi_i}\right|^2 \geq 0}\,. \tag{10.102}$$

This is the famous **F-term contribution**. Since V is a sum of absolute squares, it is automatically non-negative for all field configurations—a general feature of supersymmetric theories.

Explicit form of the Lagrangian

Substituting $W = \frac{1}{2}M^{ij}\phi_i\phi_j + \frac{1}{6}y^{ijk}\phi_i\phi_j\phi_k$, we have

$$W^i = M^{ij}\phi_j + \frac{1}{2}y^{ijk}\phi_j\phi_k\,, \tag{10.103}$$

$$W^{ij} = M^{ij} + y^{ijk}\phi_k\,. \tag{10.104}$$

The full Lagrangian becomes

$$\begin{aligned}\mathcal{L} = {} & \partial^\mu\phi^{\dagger i}\partial_\mu\phi_i + \bar\chi^i i\bar\sigma^\mu\partial_\mu\chi_i - V(\phi,\phi^\dagger) \\ & - \tfrac{1}{2}M^{ij}\chi_i\chi_j - \tfrac{1}{2}M^*_{ij}\bar\chi^i\bar\chi^j \\ & - \tfrac{1}{2}y^{ijk}\phi_k\chi_i\chi_j - \tfrac{1}{2}y^*_{ijk}\phi^{\dagger k}\bar\chi^i\bar\chi^j\,,\end{aligned} \tag{10.105}$$

with scalar potential

$$\begin{aligned}V = {} & M^*_{ik}M^{kj}\phi^{\dagger i}\phi_j + \tfrac{1}{2}M^{in}y^*_{jkn}\phi_i\phi^{\dagger j}\phi^{\dagger k} \\ & + \tfrac{1}{2}M^*_{in}y^{jkn}\phi^{\dagger i}\phi_j\phi_k + \tfrac{1}{4}y^{ijn}y^*_{kln}\phi_i\phi_j\phi^{\dagger k}\phi^{\dagger l}\,.\end{aligned} \tag{10.106}$$

Mass degeneracy within supermultiplets

The linearized equations of motion (keeping only mass terms) are:

$$\partial^\mu\partial_\mu\phi_i = -M^*_{ik}M^{kj}\phi_j + \cdots\,, \tag{10.107}$$

$$i\bar\sigma^\mu\partial_\mu\chi_i = M^*_{ij}\bar\chi^j + \cdots\,, \qquad i\sigma^\mu\partial_\mu\bar\chi^i = M^{ij}\chi_j + \cdots\,. \tag{10.108}$$

Eliminating $\bar\chi$ from the fermion equations (using $\bar\sigma^\mu\sigma^\nu + \bar\sigma^\nu\sigma^\mu = 2g^{\mu\nu}$):

$$\partial^\mu\partial_\mu\chi_i = -M^*_{ik}M^{kj}\chi_j + \cdots\,. \tag{10.109}$$

Both ϕ_i and χ_i satisfy the *same* wave equation with squared-mass matrix $(M^2)_i{}^j = M^*_{ik}M^{kj}$. Diagonalizing this matrix gives supermultiplets with *identical* boson and fermion masses, as required by the superalgebra.

Box 10.3 Interactions from the Superpotential

The superpotential W encodes *all* non-gauge interactions:

- **Fermion masses:** $-\frac{1}{2}M^{ij}\chi_i\chi_j + \text{h.c.}$
- **Scalar masses:** $M^*_{ik}M^{kj}\phi^{\dagger i}\phi_j$
- **Yukawa couplings:** $-\frac{1}{2}y^{ijk}\phi_k\chi_i\chi_j + \text{h.c.}$
- **Scalar cubics:** from $M^{in}y^*_{jkn}\phi_i\phi^{\dagger j}\phi^{\dagger k} + \text{h.c.}$
- **Scalar quartics:** $\frac{1}{4}y^{ijn}y^*_{kln}\phi_i\phi_j\phi^{\dagger k}\phi^{\dagger l}$

Worked example: a single chiral supermultiplet

Following Aitchison [38], §5.1, we specialise to a *single* chiral supermultiplet (ϕ, χ, F) to make the interaction structure concrete. With a single field the superpotential (10.84) reduces to

$$W = \tfrac{1}{2}M\phi^2 + \tfrac{1}{6}y\phi^3\,, \tag{10.110}$$

where M is a mass parameter and y a dimensionless Yukawa coupling (both taken real for simplicity). The derivatives are

$$W' \equiv \frac{\partial W}{\partial \phi} = M\phi + \tfrac{1}{2}y\phi^2\,, \qquad W'' \equiv \frac{\partial^2 W}{\partial \phi^2} = M + y\phi\,. \tag{10.111}$$

The on-shell Wess–Zumino Lagrangian (10.101) becomes

$$\mathcal{L} = \partial^\mu\phi^\dagger\partial_\mu\phi + \bar{\chi}\,i\bar{\sigma}^\mu\partial_\mu\chi - \underbrace{\left|M\phi + \tfrac{1}{2}y\phi^2\right|^2}_{V(\phi,\phi^\dagger)} - \tfrac{1}{2}\left[(M + y\phi)\,\chi\chi + \text{h.c.}\right]. \tag{10.112}$$

Mass terms. The quadratic terms in (10.112) are

$$\mathcal{L}_{\text{mass}} = -M^2\phi^\dagger\phi \;-\; \tfrac{1}{2}M\left(\chi\chi + \bar{\chi}\bar{\chi}\right). \tag{10.113}$$

Both the complex scalar and the Weyl fermion have the *same* mass $|M|$, confirming the general mass-degeneracy result within a single supermultiplet.

Interaction terms. Expanding the scalar potential $V = |W'|^2$ beyond the mass terms and collecting the Yukawa coupling, we identify three distinct interactions:

(i) **Scalar cubic** (from the cross term in $|W'|^2$):

$$\mathcal{L}_{\text{cubic}} = -\tfrac{1}{2}My\left(\phi^2\phi^\dagger + \phi^{\dagger 2}\phi\right). \tag{10.114}$$

(ii) **Scalar quartic** (from $|\frac{1}{2}y\phi^2|^2$):

$$\mathcal{L}_{\text{quartic}} = -\tfrac{1}{4}y^2\,|\phi|^4\,. \tag{10.115}$$

(iii) **Yukawa** (from $-\frac{1}{2}W''\chi\chi + \text{h.c.}$, keeping only the interaction part):

$$\mathcal{L}_{\text{Yukawa}} = -\tfrac{1}{2}y\big(\phi\,\chi\chi + \phi^\dagger\bar{\chi}\bar{\chi}\big)\,. \tag{10.116}$$

Coupling relations. The three interaction couplings are *not* independent—they are all determined by the single parameter y appearing in the superpotential:

Interaction	**Coupling**	**Relation**
Yukawa $\phi\chi\chi$	y	–
Scalar cubic $\phi^2\phi^\dagger$	$\frac{1}{2}My$	$= \frac{1}{2}M \times y_{\text{Yuk}}$
Scalar quartic $\lvert\phi\rvert^4$	$\frac{1}{4}y^2$	$= \frac{1}{4}y^2_{\text{Yuk}}$

In particular, the quartic scalar coupling is fixed by the *square* of the Yukawa coupling:

$$\boxed{\lambda_{\text{quartic}} = \frac{1}{4}\,y^2_{\text{Yukawa}}\,.} \tag{10.117}$$

This relationship is *not* imposed by hand—it is *forced* by the holomorphicity of W and the SUSY algebra. It is precisely the condition needed for the cancellation of quadratic divergences in the scalar self-energy at one loop: the bosonic loop (quartic vertex, coupling $\sim\lambda$) and the fermionic loop (two Yukawa vertices, coupling $\sim y^2$) contribute with opposite signs and cancel when $\lambda = y^2/4$. This will be demonstrated explicitly in Section 5.2 of Aitchison.

Remark. [Decomposition into real fields] Writing $\phi = (A + iB)/\sqrt{2}$ with A, B real (cf. Aitchison, eq. (4.123)) and defining $g \equiv y/(2\sqrt{2})$, the interactions take the form familiar from the Majorana formulation:

$$\mathcal{L}_{\text{cubic}} = -Mg\,A(A^2 + B^2)\,, \tag{10.118}$$

$$\mathcal{L}_{\text{quartic}} = -\tfrac{1}{2}g^2(A^2 + B^2)^2\,, \tag{10.119}$$

$$\mathcal{L}_{\text{Yukawa}} = -g\left[A\,\bar{\Psi}_{\text{M}}\Psi_{\text{M}} + iB\,\bar{\Psi}_{\text{M}}\gamma_5\Psi_{\text{M}}\right], \tag{10.120}$$

where Ψ_{M} is the four-component Majorana spinor built from χ (see Section 2.5 of Aitchison). Here A is a scalar and B a pseudoscalar under parity, and the same coupling g governs all three interactions. The coefficient $g^2/2$ in (10.119) is Aitchison's real-field normalization of the same relation: after the vertex combinatorics are included, it is equivalent to the complex-field statement

$$\lambda_{\text{quartic}} = y^2/4.$$

Physical significance. The fact that a single holomorphic function W determines *all* interactions—fixing the relative strengths of Yukawa, cubic, and quartic couplings—is one of the most powerful consequences of supersymmetry. It has two far-reaching implications:

1. **Cancellation of quadratic divergences.** The relation $\lambda = y^2/4$ ensures that the Λ^2-divergent contributions to the scalar mass from bosonic and fermionic loops cancel order by order in perturbation theory. This is the technical solution to the hierarchy problem that originally motivated SUSY (cf. Section 1.1 of Aitchison).
2. **Non-renormalization.** The holomorphicity of W implies that the superpotential itself receives no perturbative renormalization. Physical masses and couplings can still run through wavefunction/Kähler-potential renormalization, but W is radiatively stable.

Part V

Appendices

A Dirac gamma matrices

This section collects the essential identities involving Dirac gamma matrices that are needed for QED cross-section calculations. We work in $d = 4 - 2\epsilon$ dimensions for compatibility with dimensional regularization.

Box A.1 Convention notice: $d = 4 - 2\epsilon$ vs $d = 4 - \varepsilon$

Important. Two slightly different conventions for the dimensional regulator coexist in these notes. To avoid factor-of-two confusion in UV poles, here is the explicit dictionary.

Definitions:

- **This appendix** (Aitchison/Peskin–Schroeder style): $d = 4 - 2\epsilon_{\text{app}}$, so $\epsilon_{\text{app}} = (4-d)/2$.
- **Chapters 6–7 (renormalization) and Sec. 4 of Ch. 4 (Feynman rules)**: $d = 4 - \varepsilon_{\text{ch}}$, so $\varepsilon_{\text{ch}} = 4 - d$.

Explicit translation bridge:

$$\boxed{\varepsilon_{\text{ch}} = 2\,\epsilon_{\text{app}}\,, \qquad \frac{1}{\epsilon_{\text{app}}} = \frac{2}{\varepsilon_{\text{ch}}}} \tag{A.1}$$

Equivalently, a residue/log expansion in the appendix written as $\Gamma(\epsilon_{\text{app}}) = 1/\epsilon_{\text{app}} - \gamma_E + O(\epsilon_{\text{app}})$ becomes $\Gamma(\varepsilon_{\text{ch}}/2) = 2/\varepsilon_{\text{ch}} - \gamma_E + O(\varepsilon_{\text{ch}})$ in the renormalization chapters.

Practical rule. When importing a formula from the appendix into a renormalization calculation (or vice versa), substitute $\epsilon_{\text{app}} \to \varepsilon_{\text{ch}}/2$ (and accordingly $1/\epsilon_{\text{app}} \to 2/\varepsilon_{\text{ch}}$). The $\ln(\mu^2/m^2)$ accompanying the pole is *unaffected* by the rescaling (the factor of 2 in the pole residue is compensated by the half-shift in d when expanding μ^{d-4}).

Clifford algebra and basic properties

The gamma matrices γ^μ satisfy the **Clifford algebra**:

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\mathbb{1}}\,. \tag{A.2}$$

In d dimensions, $g^{\mu\nu} = \mathrm{diag}(+1, -1, -1, \ldots, -1)$ with $g^\mu{}_\mu = d$.

Hermiticity properties (in the Dirac representation):

$$(\gamma^0)^\dagger = \gamma^0, \qquad (\gamma^i)^\dagger = -\gamma^i \quad (i = 1,2,3)\,, \tag{A.3}$$

or equivalently:

$$\boxed{(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0}\,. \tag{A.4}$$

Slash notation: For any four-vector a^μ, we define

$$\not{a} \equiv a_\mu\gamma^\mu = a^0\gamma^0 - a^1\gamma^1 - a^2\gamma^2 - a^3\gamma^3\,. \tag{A.5}$$

From the Clifford algebra, it follows that:

$$\boxed{\not{a}\not{b} + \not{b}\not{a} = 2(a\cdot b)\mathbb{1}}\,, \tag{A.6}$$

and in particular:

$$\not{a}\not{a} = a^2\mathbb{1}\,. \tag{A.7}$$

Contraction identities

The following identities are essential for simplifying products of gamma matrices. We derive them systematically from the Clifford algebra (A.2).

Identity 1: $\gamma^\mu\gamma_\mu = d$
Proof. Contract the Clifford algebra with $g_{\mu\nu}$:

$$\begin{aligned}\gamma^\mu\gamma_\mu &= g_{\mu\nu}\gamma^\mu\gamma^\nu = \tfrac{1}{2}g_{\mu\nu}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)\\ &= \tfrac{1}{2}g_{\mu\nu}\cdot 2g^{\mu\nu} = g^\mu{}_\mu = d\,.\end{aligned}$$

□

Identity 2: $\gamma^\mu\gamma^\nu\gamma_\mu = (2-d)\gamma^\nu$
Proof. Use the Clifford algebra to move γ_μ past γ^ν:

$$\begin{aligned}\gamma^\mu\gamma^\nu\gamma_\mu &= \gamma^\mu(2g^\nu{}_\mu - \gamma_\mu\gamma^\nu)\\ &= 2\gamma^\nu - \gamma^\mu\gamma_\mu\gamma^\nu = 2\gamma^\nu - d\gamma^\nu = (2-d)\gamma^\nu\,.\end{aligned}$$

□

Identity 3: $\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4g^{\nu\rho} - (4-d)\gamma^\nu\gamma^\rho$
Proof. Apply Identity 2 twice:

$$\begin{aligned}\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu &= \gamma^\mu\gamma^\nu(2g^\rho{}_\mu - \gamma_\mu\gamma^\rho) = 2\gamma^\rho\gamma^\nu - \gamma^\mu\gamma^\nu\gamma_\mu\gamma^\rho \\ &= 2\gamma^\rho\gamma^\nu - (2-d)\gamma^\nu\gamma^\rho \\ &= 2(2g^{\nu\rho} - \gamma^\nu\gamma^\rho) - (2-d)\gamma^\nu\gamma^\rho \\ &= 4g^{\nu\rho} - 2\gamma^\nu\gamma^\rho - (2-d)\gamma^\nu\gamma^\rho = 4g^{\nu\rho} - (4-d)\gamma^\nu\gamma^\rho. \qquad \square\end{aligned}$$

Identity 4: $\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu + (4-d)\gamma^\nu\gamma^\rho\gamma^\sigma$
Proof. Apply the previous identities:

$$\begin{aligned}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu &= \gamma^\mu\gamma^\nu\gamma^\rho(2g^\sigma{}_\mu - \gamma_\mu\gamma^\sigma) = 2\gamma^\sigma\gamma^\nu\gamma^\rho - \gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu\gamma^\sigma \\ &= 2\gamma^\sigma\gamma^\nu\gamma^\rho - [4g^{\nu\rho} - (4-d)\gamma^\nu\gamma^\rho]\gamma^\sigma \\ &= 2\gamma^\sigma\gamma^\nu\gamma^\rho - 4g^{\nu\rho}\gamma^\sigma + (4-d)\gamma^\nu\gamma^\rho\gamma^\sigma. \qquad \text{(A.8)}\end{aligned}$$

Now use $\gamma^\sigma\gamma^\nu\gamma^\rho = 2g^{\sigma\nu}\gamma^\rho - \gamma^\nu\gamma^\sigma\gamma^\rho = 2g^{\sigma\nu}\gamma^\rho - 2g^{\sigma\rho}\gamma^\nu + \gamma^\nu\gamma^\rho\gamma^\sigma$, but a simpler approach is to note that $\gamma^\sigma\gamma^\nu\gamma^\rho = -\gamma^\sigma\gamma^\rho\gamma^\nu + 2g^{\nu\rho}\gamma^\sigma$. Thus:

$$\begin{aligned}&= 2(-\gamma^\sigma\gamma^\rho\gamma^\nu + 2g^{\nu\rho}\gamma^\sigma) - 4g^{\nu\rho}\gamma^\sigma + (4-d)\gamma^\nu\gamma^\rho\gamma^\sigma \\ &= -2\gamma^\sigma\gamma^\rho\gamma^\nu + (4-d)\gamma^\nu\gamma^\rho\gamma^\sigma. \qquad \square\end{aligned}$$

In $d = 4$ dimensions, these simplify to:

$$\boxed{\begin{aligned}\gamma^\mu\gamma_\mu &= 4 \\ \gamma^\mu\gamma^\nu\gamma_\mu &= -2\gamma^\nu \\ \gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu &= 4g^{\nu\rho} \\ \gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu &= -2\gamma^\sigma\gamma^\rho\gamma^\nu.\end{aligned}} \qquad \text{(A.9)}$$

Contractions with slashed vectors follow immediately by replacing $\gamma^\nu \to \not{a} = a_\nu\gamma^\nu$:

$$\gamma^\mu\not{a}\gamma_\mu = (2-d)\not{a} = -2\not{a} \quad (\text{in } d=4) \qquad \text{(A.10)}$$

$$\gamma^\mu\not{a}\not{b}\gamma_\mu = 4(a\cdot b) - (4-d)\not{a}\not{b} = 4(a\cdot b) \quad (\text{in } d=4) \qquad \text{(A.11)}$$

$$\begin{aligned}\gamma^\mu\not{a}\not{b}\not{c}\gamma_\mu &= -2\not{c}\not{b}\not{a} + (4-d)\not{a}\not{b}\not{c} \\ &= -2\not{c}\not{b}\not{a} \quad (\text{in } d=4). \qquad \text{(A.12)}\end{aligned}$$

Trace identities

The trace of gamma matrices is fundamental for computing spin-summed amplitudes squared. The gamma matrices are 4×4 in four dimensions (or $2^{d/2}\times2^{d/2}$ in d dimensions, analytically continued).

Basic trace rules:

$$\boxed{\begin{aligned} \mathrm{Tr}(\mathbb{1}) &= 4 \quad (\text{or } 2^{d/2} \text{ in } d \text{ dim.}) \\ \mathrm{Tr}(\text{odd \# of } \gamma\text{'s}) &= 0 \end{aligned}} \tag{A.13}$$

Proof that the trace of an odd number of gamma matrices vanishes. In $d = 4$, we use the fact that γ^5 anticommutes with all γ^μ and $(\gamma^5)^2 = \mathbb{1}$. For any string Γ of n gamma matrices:

$$\mathrm{Tr}(\Gamma) = \mathrm{Tr}(\gamma^5\gamma^5\Gamma) = \mathrm{Tr}(\gamma^5\Gamma\gamma^5) = (-1)^n\mathrm{Tr}(\gamma^5\gamma^5\Gamma) = (-1)^n\mathrm{Tr}(\Gamma), \tag{A.14}$$

where we used the cyclic property of the trace and moved γ^5 past n gamma matrices, picking up $(-1)^n$. For odd n, this gives $\mathrm{Tr}(\Gamma) = -\mathrm{Tr}(\Gamma)$, hence $\mathrm{Tr}(\Gamma) = 0$. □

Trace of two gamma matrices:

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}\,. \tag{A.15}$$

Proof. From the Clifford algebra, $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\mathbb{1}$. Taking the trace and using $\mathrm{Tr}(\gamma^\mu\gamma^\nu) = \mathrm{Tr}(\gamma^\nu\gamma^\mu)$ (cyclic property):

$$2\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 2g^{\mu\nu}\mathrm{Tr}(\mathbb{1}) = 8g^{\mu\nu} \implies \mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}. \qquad \square$$

Trace of four gamma matrices:

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})\,. \tag{A.16}$$

Proof. Use the Clifford algebra to anticommute γ^μ through:

$$\begin{aligned} \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= \mathrm{Tr}[(2g^{\mu\nu} - \gamma^\nu\gamma^\mu)\gamma^\rho\gamma^\sigma] \\ &= 2g^{\mu\nu}\mathrm{Tr}(\gamma^\rho\gamma^\sigma) - \mathrm{Tr}(\gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma). \end{aligned} \tag{A.17}$$

Continue anticommuting γ^μ in the second term:

$$\begin{aligned} \mathrm{Tr}(\gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma) &= 2g^{\mu\rho}\,\mathrm{Tr}(\gamma^\nu\gamma^\sigma) \\ &\quad - \mathrm{Tr}(\gamma^\nu\gamma^\rho\gamma^\mu\gamma^\sigma) \\ &= 2g^{\mu\rho}\,\mathrm{Tr}(\gamma^\nu\gamma^\sigma) \\ &\quad - 2g^{\mu\sigma}\,\mathrm{Tr}(\gamma^\nu\gamma^\rho) \\ &\quad + \mathrm{Tr}(\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu). \end{aligned} \tag{A.18}$$

By the cyclic property of the trace,

$$\mathrm{Tr}(\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu) = \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma)\,.$$

Denoting $T \equiv \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma)$:

$$T = 2g^{\mu\nu}\cdot 4g^{\rho\sigma} - 2g^{\mu\rho}\cdot 4g^{\nu\sigma} + 2g^{\mu\sigma}\cdot 4g^{\nu\rho} - T. \tag{A.19}$$

Solving: $2T = 8(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$, hence the result. □

The result (A.16) can be written more symmetrically as:

$$\boxed{\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right)}. \tag{A.20}$$

Traces with slashed vectors follow by contraction with the momenta:

$$\mathrm{Tr}(\not{a}\not{b}) = a_\mu b_\nu \mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4a_\mu b_\nu g^{\mu\nu} = 4(a\cdot b) \tag{A.21}$$

$$\mathrm{Tr}(\not{a}\not{b}\not{c}\not{d}) = 4\big[(a\cdot b)(c\cdot d) - (a\cdot c)(b\cdot d) + (a\cdot d)(b\cdot c)\big]. \tag{A.22}$$

Remark. [Recursive trace formula] The trace of $2n$ gamma matrices can be computed recursively:

$$\mathrm{Tr}(\gamma^{\mu_1}\cdots\gamma^{\mu_{2n}}) = \sum_{k=2}^{2n}(-1)^k g^{\mu_1\mu_k}\mathrm{Tr}(\gamma^{\mu_2}\cdots\hat{\gamma}^{\mu_k}\cdots\gamma^{\mu_{2n}}) \tag{A.23}$$

where $\hat{\gamma}^{\mu_k}$ means that γ^{μ_k} is omitted. This follows by repeatedly anticommuting γ^{μ_1} through the string using the Clifford algebra, then using the cyclic property of the trace.

Trace of six gamma matrices can be computed using the recursive formula. The result is:

$$\begin{aligned}\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\alpha\gamma^\beta) = 4\big[&g^{\mu\nu}(g^{\rho\sigma}g^{\alpha\beta} - g^{\rho\alpha}g^{\sigma\beta} + g^{\rho\beta}g^{\sigma\alpha})\\ &- g^{\mu\rho}(g^{\nu\sigma}g^{\alpha\beta} - g^{\nu\alpha}g^{\sigma\beta} + g^{\nu\beta}g^{\sigma\alpha})\\ &+ g^{\mu\sigma}(g^{\nu\rho}g^{\alpha\beta} - g^{\nu\alpha}g^{\rho\beta} + g^{\nu\beta}g^{\rho\alpha})\\ &- g^{\mu\alpha}(g^{\nu\rho}g^{\sigma\beta} - g^{\nu\sigma}g^{\rho\beta} + g^{\nu\beta}g^{\rho\sigma})\\ &+ g^{\mu\beta}(g^{\nu\rho}g^{\sigma\alpha} - g^{\nu\sigma}g^{\rho\alpha} + g^{\nu\alpha}g^{\rho\sigma})\big]\end{aligned} \tag{A.24}$$

The γ^5 matrix

In four dimensions, the chiral matrix γ^5 is defined as:

$$\boxed{\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma}, \tag{A.25}$$

where $\epsilon^{0123} = +1$, consistently with Chapter 9.

Properties of γ^5:

$$\boxed{\begin{aligned} (\gamma^5)^2 &= \mathbb{1} \\ (\gamma^5)^\dagger &= \gamma^5 \\ \{\gamma^5, \gamma^\mu\} &= 0 \quad \text{(anticommutes with all } \gamma^\mu\text{)} \end{aligned}} \tag{A.26}$$

Proof of $(\gamma^5)^2 = \mathbb{1}$. Direct computation:

$$(\gamma^5)^2 = (i\gamma^0\gamma^1\gamma^2\gamma^3)(i\gamma^0\gamma^1\gamma^2\gamma^3) = -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3. \tag{A.27}$$

Move γ^0 to the left using $\gamma^3\gamma^0 = -\gamma^0\gamma^3$, etc. Each anticommutation gives a minus sign, and we need to move γ^0 past $\gamma^3, \gamma^2, \gamma^1$: three minus signs. Then $\gamma^0\gamma^0 = \mathbb{1}$:

$$= -(-1)^3\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 = \gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3. \tag{A.28}$$

Similarly, move γ^1 past γ^3, γ^2 (two minus signs) and use $\gamma^1\gamma^1 = -\mathbb{1}$:

$$= (-1)^2(-1)\gamma^2\gamma^3\gamma^2\gamma^3 = -\gamma^2\gamma^3\gamma^2\gamma^3. \tag{A.29}$$

Move γ^2 past γ^3 (one minus sign) and use $\gamma^2\gamma^2 = -\mathbb{1}$:

$$= -(-1)(-1)\gamma^3\gamma^3 = -\gamma^3\gamma^3 = -(-\mathbb{1}) = \mathbb{1}. \qquad \square$$

Proof of $(\gamma^5)^\dagger = \gamma^5$. Using $(\gamma^0)^\dagger = \gamma^0$ and $(\gamma^i)^\dagger = -\gamma^i$:

$$\begin{aligned} (\gamma^5)^\dagger &= (i\gamma^0\gamma^1\gamma^2\gamma^3)^\dagger = -i(\gamma^3)^\dagger(\gamma^2)^\dagger(\gamma^1)^\dagger(\gamma^0)^\dagger \\ &= -i(-\gamma^3)(-\gamma^2)(-\gamma^1)(\gamma^0) = i\gamma^3\gamma^2\gamma^1\gamma^0. \end{aligned} \tag{A.30}$$

Now anticommute to restore the original order. Moving γ^0 to the front requires passing it through $\gamma^1, \gamma^2, \gamma^3$ (three anticommutations, giving $(-1)^3$):

$$= i(-1)^3\gamma^0\gamma^3\gamma^2\gamma^1 = -i\gamma^0\gamma^3\gamma^2\gamma^1. \tag{A.31}$$

Moving γ^1 to the second position (past γ^2, γ^3, two anticommutations):

$$= -i(-1)^2\gamma^0\gamma^1\gamma^3\gamma^2 = -i\gamma^0\gamma^1\gamma^3\gamma^2. \tag{A.32}$$

Finally, swap γ^2 and γ^3:

$$= -i(-1)\gamma^0\gamma^1\gamma^2\gamma^3 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5. \qquad \square$$

Proof of $\{\gamma^5, \gamma^\mu\} = 0$. For $\mu = 0$:

$$\gamma^5\gamma^0 = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0 = i\gamma^0\gamma^1\gamma^2(-\gamma^0\gamma^3) = \cdots = -i\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^0\gamma^5. \tag{A.33}$$

Each spatial gamma anticommutes with γ^0, giving three minus signs total for the three anticommutations needed to bring γ^0 to the front. For $\mu = i$ (spatial), the factor γ^i already appears in γ^5. Moving γ^i through γ^5 produces an odd number of anticommutations (with the other spatial gammas and γ^0), giving an overall minus sign. Thus $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$ for all μ. □

Chirality projectors:

$$P_L = \frac{1-\gamma^5}{2}, \qquad P_R = \frac{1+\gamma^5}{2}, \tag{A.34}$$

satisfying $P_L + P_R = \mathbb{1}$, $P_L^2 = P_L$, $P_R^2 = P_R$, and $P_L P_R = 0$.
Proof of projector properties. The sum is obvious. For P_L^2:

$$P_L^2 = \frac{(1-\gamma^5)^2}{4} = \frac{1-2\gamma^5+(\gamma^5)^2}{4} = \frac{2-2\gamma^5}{4} = \frac{1-\gamma^5}{2} = P_L. \qquad \square$$

Trace identities involving γ^5:

$$\mathrm{Tr}(\gamma^5) = 0 \tag{A.35}$$

$$\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu) = 0 \tag{A.36}$$

$$\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = -4i\epsilon^{\mu\nu\rho\sigma}. \tag{A.37}$$

Proof of $Tr(\gamma^5) = 0$. Write $\gamma^5 = \gamma^5\gamma^\mu\gamma_\mu/4$ (using $\gamma^\mu\gamma_\mu = 4$). Then:

$$\mathrm{Tr}(\gamma^5) = \frac{1}{4}\mathrm{Tr}(\gamma^5\gamma^\mu\gamma_\mu) = \frac{1}{4}\mathrm{Tr}(\gamma_\mu\gamma^5\gamma^\mu) = -\frac{1}{4}\mathrm{Tr}(\gamma^5\gamma_\mu\gamma^\mu) = -\mathrm{Tr}(\gamma^5), \tag{A.38}$$

where we used the cyclic property and $\gamma_\mu\gamma^5 = -\gamma^5\gamma_\mu$. Hence $\mathrm{Tr}(\gamma^5) = 0$. □
Proof of $Tr(\gamma^5\gamma^\mu\gamma^\nu) = 0$. The trace has three gamma matrices plus $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, giving a total of 7 gamma matrices. Since the trace of an odd number of gamma matrices vanishes (proven above), this trace is zero. □
Proof of $Tr(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = -4i\epsilon^{\mu\nu\rho\sigma}$. The trace vanishes unless all four indices are distinct (otherwise, use the Clifford algebra to reduce the number of gamma matrices, giving a trace with γ^5 and fewer than 4 gammas). For distinct indices, the trace is completely antisymmetric (anticommuting two gammas past each other gives a minus sign). The only completely antisymmetric tensor is $\epsilon^{\mu\nu\rho\sigma}$, so:

$$\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = c\cdot\epsilon^{\mu\nu\rho\sigma}. \tag{A.39}$$

To find c, evaluate $\mathrm{Tr}(\gamma^5\gamma^0\gamma^1\gamma^2\gamma^3)$:

$$\mathrm{Tr}(\gamma^5\gamma^0\gamma^1\gamma^2\gamma^3) = \mathrm{Tr}\Big(i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3\Big) = -i\,\mathrm{Tr}(\mathbb{1}) = -4i. \tag{A.40}$$

Since $\epsilon^{0123} = +1$, we have $c = -4i$. □

More generally, traces with γ^5 and fewer than four other gamma matrices vanish:

$$\mathrm{Tr}(\gamma^5 \gamma^{\mu_1} \cdots \gamma^{\mu_n}) = 0 \quad \text{for } n < 4\,. \tag{A.41}$$

With slashed vectors:

$$\boxed{\mathrm{Tr}(\gamma^5 \not{a} \not{b} \not{c} \not{d}) = -4i\epsilon^{\mu\nu\rho\sigma} a_\mu b_\nu c_\rho d_\sigma = -4i\epsilon(a,b,c,d)} \tag{A.42}$$

where $\epsilon(a,b,c,d) \equiv \epsilon^{\mu\nu\rho\sigma} a_\mu b_\nu c_\rho d_\sigma$ is the fully antisymmetric contraction.

Remark. [γ^5 in dimensional regularization] The treatment of γ^5 in $d \neq 4$ dimensions is subtle because γ^5 is intrinsically four-dimensional. Common prescriptions include:

- **Naive dimensional regularization (NDR):** Assume $\{\gamma^5, \gamma^\mu\} = 0$ holds in d dimensions. This is algebraically inconsistent but works for many calculations.
- **'t Hooft–Veltman scheme:** Split $\gamma^\mu = \hat{\gamma}^\mu + \tilde{\gamma}^\mu$ where $\hat{\gamma}^\mu$ is 4-dimensional and $\tilde{\gamma}^\mu$ lives in the extra $(d-4)$ dimensions. Then $\{\gamma^5, \hat{\gamma}^\mu\} = 0$ but $[\gamma^5, \tilde{\gamma}^\mu] = 0$.

For QED calculations without axial currents, these subtleties can often be avoided.

Gordon identity and related decompositions

The **Gordon identity** expresses a vector current in terms of a convection current and a spin current:

$$\boxed{\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{(p'+p)^\mu}{2m} + \frac{i\sigma^{\mu\nu}(p'-p)_\nu}{2m}\right] u(p)}\,, \tag{A.43}$$

where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$ and the spinors satisfy the Dirac equation.

Proof: Start from $\bar{u}(p')(\not{p}' - m) = 0$ and $(\not{p} - m)u(p) = 0$, which imply $\bar{u}(p')\not{p}' = m\bar{u}(p')$ and $\not{p}\, u(p) = m\, u(p)$. Then:

$$\begin{aligned}
\bar{u}(p')\gamma^\mu u(p) &= \frac{1}{2m}\bar{u}(p')(\not{p}'\gamma^\mu + \gamma^\mu \not{p})u(p) \\
&= \frac{1}{2m}\bar{u}(p')\left[p'^\mu + p^\mu - \frac{1}{2}[\gamma^\mu, \gamma^\nu](p'-p)_\nu\right] u(p) \\
&= \bar{u}(p')\left[\frac{(p'+p)^\mu}{2m} + \frac{i\sigma^{\mu\nu}(p'-p)_\nu}{2m}\right] u(p).
\end{aligned} \tag{A.44}$$

In the last step we used $[\gamma^\mu, \gamma^\nu] = -2i\sigma^{\mu\nu}$, which follows from $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$.

Generalized Gordon identities:

$$\bar{u}(p')\gamma^\mu\gamma^5 u(p) = \bar{u}(p')\left[\frac{(p'-p)^\mu}{2m}\gamma^5 + \frac{i\sigma^{\mu\nu}(p'+p)_\nu}{2m}\gamma^5\right]u(p)\,. \tag{A.45}$$

Fierz identities

The **Fierz identities** allow reordering of spinor bilinears. The key observation is that the 16 matrices

$$\Gamma_A \in \{\mathbb{1}, \gamma^\mu, \sigma^{\mu\nu}, \gamma^\mu\gamma^5, \gamma^5\} \quad (A = 1, \ldots, 16)\,, \tag{A.46}$$

form a complete basis for 4×4 matrices, satisfying the orthogonality relation $\mathrm{Tr}(\Gamma_A\Gamma_B) = 4\delta_{AB}$ (with appropriate index handling).

Derivation. Any 4×4 matrix M can be expanded as $M = \sum_A c_A\Gamma_A$ with $c_A = \frac{1}{4}\mathrm{Tr}(M\Gamma_A)$. In particular, applying this to the outer product $(\psi_4)_\alpha(\bar{\psi}_3)_\beta$ (viewed as a matrix in spinor space):

$$(\psi_4)_\alpha(\bar{\psi}_3)_\beta = \frac{1}{4}\sum_A (\Gamma_A)_{\alpha\beta}(\bar{\psi}_3\Gamma_A\psi_4)\,. \tag{A.47}$$

Contracting with $(\bar{\psi}_1)^\alpha(\Gamma_B)^\beta{}_\gamma(\psi_2)^\gamma$ yields the general Fierz rearrangement:

$$(\bar{\psi}_1\Gamma_B\psi_2)(\bar{\psi}_3\Gamma^A\psi_4) = \sum_C C_{BC}(\bar{\psi}_1\Gamma_C\psi_4)(\bar{\psi}_3\Gamma^C\psi_2)\,, \tag{A.48}$$

where the Fierz coefficients C_{BC} are determined by the algebra of gamma matrices.

The Fierz matrix for vector currents is particularly useful:

$$(\bar{\psi}_1\gamma^\mu\psi_2)(\bar{\psi}_3\gamma_\mu\psi_4) = -(\bar{\psi}_1\gamma^\mu\psi_4)(\bar{\psi}_3\gamma_\mu\psi_2) + \text{scalar and tensor terms} \tag{A.49}$$

Summary: most frequently used identities

For quick reference, here are the identities used most often in QED calculations.

Identity	Result
Contraction identities (in $d = 4$)	
$\gamma^\mu \gamma_\mu$	4
$\gamma^\mu \gamma^\nu \gamma_\mu$	$-2\gamma^\nu$
$\gamma^\mu \not{a} \gamma_\mu$	$-2\not{a}$
$\gamma^\mu \not{a} \not{b} \gamma_\mu$	$4(a \cdot b)$
Trace identities	
$\mathrm{Tr}(\mathbb{1})$	4
$\mathrm{Tr}(\gamma^\mu \gamma^\nu)$	$4g^{\mu\nu}$
$\mathrm{Tr}(\not{a} \not{b})$	$4(a \cdot b)$
$\mathrm{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma)$	$4(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho})$
$\mathrm{Tr}(\not{a} \not{b} \not{c} \not{d})$	$4[(a \cdot b)(c \cdot d) - (a \cdot c)(b \cdot d) + (a \cdot d)(b \cdot c)]$
Traces with γ^5	
$\mathrm{Tr}(\text{odd \# of } \gamma)$	0
$\mathrm{Tr}(\gamma^5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma)$	$-4i\epsilon^{\mu\nu\rho\sigma}$

B Choice of representation for Dirac matrices

The Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}$ defines the gamma matrices only up to a similarity transformation: if γ^μ satisfies the algebra, so does $S\gamma^\mu S^{-1}$ for any invertible matrix S. This freedom leads to several standard *representations* (or *bases*), each with distinct advantages. In these notes we adopt the **Weyl (chiral) representation**, which is optimal for relativistic and high-energy physics. This appendix explains the main representations and justifies our choice.

The Dirac (standard) representation

The Dirac representation was historically the first to be introduced and remains common in textbooks emphasizing non-relativistic limits. The matrices are:

$$\gamma^0_{\mathrm{D}} = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \qquad \gamma^i_{\mathrm{D}} = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \qquad \gamma^5_{\mathrm{D}} = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}, \tag{B.1}$$

where σ^i are the Pauli matrices. The key property is that γ^0 is diagonal.

Advantages:

- In the non-relativistic limit, the upper two components of a positive-energy Dirac spinor dominate ("large components"), while the lower two are suppressed by $O(v/c)$ ("small components"). This makes the connection to Pauli spinors transparent.
- The Dirac equation in this basis directly exhibits the non-relativistic reduction to the Pauli equation.

Disadvantages:

- The chiral projectors $P_{L,R} = \frac{1}{2}(1 \mp \gamma^5)$ mix upper and lower components, obscuring the separation of left- and right-handed degrees of freedom.
- Less natural for ultra-relativistic particles or massless fermions.

The Weyl (chiral) representation

The Weyl representation is defined by making γ^5 diagonal:

$$\boxed{\gamma^0_{\rm W} = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}, \qquad \gamma^i_{\rm W} = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \qquad \gamma^5_{\rm W} = \begin{pmatrix} -\mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix}.} \tag{B.2}$$

This can be written compactly using the sigma matrices $\sigma^\mu = (\mathbb{1}_2, \vec{\sigma})$ and $\bar{\sigma}^\mu = (\mathbb{1}_2, -\vec{\sigma})$:

$$\gamma^\mu_{\rm W} = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}. \tag{B.3}$$

The Dirac spinor decomposes into two-component Weyl spinors as:

$$\Psi = \begin{pmatrix} \chi_\alpha \\ \eta^{\dagger\dot{\alpha}} \end{pmatrix} \equiv \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \tag{B.4}$$

where $\psi_L = \chi_\alpha$ (upper, undotted indices) is left-handed and transforms under $(\frac{1}{2}, 0)$, while $\psi_R = \eta^{\dagger\dot{\alpha}}$ (lower, dotted indices) is right-handed and transforms under $(0, \frac{1}{2})$. The chiral projectors act simply:

$$P_L \Psi = \frac{1}{2}(1 - \gamma^5)\Psi = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix}, \qquad P_R \Psi = \frac{1}{2}(1 + \gamma^5)\Psi = \begin{pmatrix} 0 \\ \psi_R \end{pmatrix}. \tag{B.5}$$

Box B.1 Convention choices in the Weyl representation

There are two common conventions in the literature for the Weyl representation, differing in which chirality occupies the upper block:

Convention A (used in these notes, matching Chapters 9–10):

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} -\mathbb{1}_2 & 0 \\ 0 & +\mathbb{1}_2 \end{pmatrix}, \qquad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

Convention B (used in some QFT textbooks, e.g., Peskin & Schroeder):

$$\Psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} +\mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \qquad \gamma^\mu = \begin{pmatrix} 0 & \bar{\sigma}^\mu \\ \sigma^\mu & 0 \end{pmatrix}$$

The two conventions are related by the unitary transformation $\Psi_B = \gamma^0 \Psi_A$ (which swaps upper and lower blocks). Physical observables are identical in both conventions, but intermediate expressions differ. When consulting external references, always verify which convention is being used.
Convention A is natural when building the Dirac spinor from Weyl spinors with standard index notation: the left-handed spinor χ_α carries undotted (lower) indices and sits in the upper block, while the right-handed spinor $\eta^{\dagger\dot{\alpha}}$ carries dotted (upper) indices and sits in the lower block.

Advantages of the Weyl representation:

1. **Chirality is manifest.** With γ^5 diagonal, the left- and right-handed components of a Dirac spinor are cleanly separated. This is essential for:
   - The electroweak sector of the Standard Model, where $\mathrm{SU}(2)_L$ couples only to left-handed fermions
   - Theories with chiral gauge symmetries
   - Supersymmetry, which is naturally formulated in terms of Weyl spinors
2. **Natural for massless and ultra-relativistic fermions.** In the massless limit, the Dirac equation decouples into two independent Weyl equations:

$$i\bar{\sigma}^\mu \partial_\mu \psi_L = 0, \qquad i\sigma^\mu \partial_\mu \psi_R = 0\,. \tag{B.6}$$

   The mass term $m\bar{\Psi}\Psi = m(\psi_R^\dagger \psi_L + \psi_L^\dagger \psi_R)$ is the *only* term that couples left and right components. For high-energy processes where $E \gg m$, the chiral components propagate nearly independently.
3. **Direct connection to the Lorentz group.** The two-component Weyl spinors transform under definite representations of $\mathrm{SL}(2,\mathbb{C})$:

$$\psi_L = \chi_\alpha \sim \left(\tfrac{1}{2}, 0\right) \text{ under } M, \qquad \psi_R = \eta^{\dagger\dot{\alpha}} \sim \left(0, \tfrac{1}{2}\right) \text{ under } (M^{-1})^\dagger\,. \tag{B.7}$$

   This makes the spinor transformation properties under boosts and rotations transparent (see Chapter 9).
4. **Simplified structure for QED and Standard Model calculations.** In the Weyl basis, the vertex $\bar{\Psi}\gamma^\mu\Psi$ decomposes as:

$$\bar{\Psi}\gamma^\mu\Psi = \psi_R^\dagger \sigma^\mu \psi_R + \psi_L^\dagger \bar{\sigma}^\mu \psi_L\,. \tag{B.8}$$

   The off-diagonal form of γ^μ is compensated by the Dirac adjoint $\bar{\Psi} = (\psi_R^\dagger, \psi_L^\dagger)$, so the vector current is diagonal in chirality. For chiral currents (involving P_L or P_R), each term involves only one chirality, simplifying the bookkeeping in Feynman diagram calculations.

5. **Essential for supersymmetry.** The SUSY algebra naturally pairs a Weyl spinor with a complex scalar into a chiral supermultiplet. The superspace formalism and superpotential are most elegantly expressed using two-component notation.

Disadvantages:

- In the non-relativistic limit, neither the upper nor lower component dominates; both contribute comparably. The reduction to Pauli spinors is less direct.
- Some older textbooks and atomic physics literature use the Dirac representation, requiring translation.

The Majorana representation

The Majorana representation is defined by requiring all gamma matrices to be purely imaginary:

$$\gamma^0_{\mathrm{M}} = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \quad \gamma^1_{\mathrm{M}} = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix}, \quad \gamma^2_{\mathrm{M}} = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \quad \gamma^3_{\mathrm{M}} = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix}. \tag{B.9}$$

Advantages:

- The Dirac operator $i\partial\!\!\!/ - m$ is purely real, so the Dirac equation admits real solutions. These correspond to **Majorana fermions**, particles that are their own antiparticles.
- Essential for studying Majorana mass terms and see-saw mechanisms in neutrino physics.

Disadvantages:

- Chirality is not manifest (γ^5 is not diagonal).
- Less convenient for standard QED and Standard Model calculations.

Transformation between representations

The representations are related by unitary similarity transformations. For Convention A (left-handed upper, as used in these notes):

$$U_{\mathrm{D}\to\mathrm{W}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1}_2 & -\mathbb{1}_2 \\ \mathbb{1}_2 & \mathbb{1}_2 \end{pmatrix}, \tag{B.10}$$

we have $\gamma^\mu_{\mathrm{W}} = U_{\mathrm{D}\to\mathrm{W}}\,\gamma^\mu_{\mathrm{D}}\,U^\dagger_{\mathrm{D}\to\mathrm{W}}$. The spinors transform correspondingly: $\Psi_{\mathrm{W}} = U_{\mathrm{D}\to\mathrm{W}}\Psi_{\mathrm{D}}$.

Remark. [Transformation to Convention B] For Convention B (right-handed upper, as in Peskin & Schroeder), the transformation matrix is instead:

$$U_{\mathrm{D}\to\mathrm{W}_B} = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1}_2 & \mathbb{1}_2 \\ -\mathbb{1}_2 & \mathbb{1}_2 \end{pmatrix}. \tag{B.11}$$

The two conventions are related by $U_{\mathrm{D}\to\mathrm{W}_A} = \gamma^0_{\mathrm{D}} \cdot U_{\mathrm{D}\to\mathrm{W}_B}$, which swaps the upper and lower blocks.

Physical observables are representation-independent. Traces, contractions, and matrix elements computed in any representation yield identical results. The choice of representation is purely a matter of convenience for the problem at hand.

Summary: why we use the Weyl representation

Property	Dirac	Weyl	Majorana
γ^5 diagonal	No	**Yes**	No
γ^0 diagonal	Yes	No	No
All γ^μ imaginary	No	No	Yes
Chiral components separated	No	**Yes**	No
Non-relativistic limit clear	Yes	No	No
Massless limit natural	No	**Yes**	No
SUSY/Weyl spinor connection	Indirect	**Direct**	Indirect

For relativistic quantum field theory, the electroweak Standard Model, and supersymmetry, the Weyl representation is the natural choice. The diagonal γ^5 makes chirality explicit, the decomposition into Weyl spinors is immediate, and the massless limit is transparent. We adopt this convention throughout these notes.

Explicit identities in the Weyl basis

For reference, we collect useful identities specific to the Weyl representation with Convention A (left-handed upper).

Charge conjugation matrix:

$$C = i\gamma^2\gamma^0 = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{pmatrix} = \begin{pmatrix} -\epsilon & 0 \\ 0 & \epsilon \end{pmatrix}, \tag{B.12}$$

where $\epsilon = -i\sigma^2$ is the antisymmetric 2×2 matrix with $\epsilon^{12} = +1$.

Dirac adjoint:

$$\bar{\Psi} = \Psi^\dagger\gamma^0 = (\psi_L^\dagger, \psi_R^\dagger)\begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix} = (\psi_R^\dagger, \psi_L^\dagger)\,. \tag{B.13}$$

Mass term:

$$m\bar{\Psi}\Psi = m(\psi_R^\dagger\psi_L + \psi_L^\dagger\psi_R)\,. \tag{B.14}$$

The mass couples left to right; massless fermions have independent chiral components.

Vector current:

$$\bar{\Psi}\gamma^\mu\Psi = \psi_R^\dagger\sigma^\mu\psi_R + \psi_L^\dagger\bar{\sigma}^\mu\psi_L\,. \tag{B.15}$$

Axial current:

$$\bar{\Psi}\gamma^\mu\gamma^5\Psi = \psi_R^\dagger\sigma^\mu\psi_R - \psi_L^\dagger\bar{\sigma}^\mu\psi_L\,. \tag{B.16}$$

Left and right currents:

$$\bar{\Psi}\gamma^\mu P_L\Psi = \psi_L^\dagger\bar{\sigma}^\mu\psi_L, \qquad \bar{\Psi}\gamma^\mu P_R\Psi = \psi_R^\dagger\sigma^\mu\psi_R\,. \tag{B.17}$$

These are the currents that couple to the W and Z bosons in the electroweak theory: $\mathrm{SU}(2)_L$ gauge interactions involve only $\bar{\Psi}\gamma^\mu P_L\Psi$.

Remark. [Verification of the vector current] Using the explicit forms $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$ and $\bar{\Psi} = (\psi_R^\dagger, \psi_L^\dagger)$:

$$\bar{\Psi}\gamma^\mu\Psi = (\psi_R^\dagger, \psi_L^\dagger)\begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \psi_R^\dagger\sigma^\mu\psi_R + \psi_L^\dagger\bar{\sigma}^\mu\psi_L$$

This confirms consistency with the chiral currents above: $\bar{\Psi}\gamma^\mu\Psi = \bar{\Psi}\gamma^\mu P_L\Psi +$

$$\bar{\Psi}\gamma^{\mu}P_{R}\Psi.$$

C Weyl spinor identities

We collect here fundamental identities involving the matrices σ^μ and $\bar{\sigma}^\mu$:

$$\sigma^\mu_{\alpha\dot{\alpha}} = \varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\sigma}^{\mu\dot{\beta}\beta} \tag{C.1}$$

$$\sigma^\mu_{\alpha\dot{\alpha}}\bar{\sigma}^{\dot{\beta}\beta}_\mu = 2\delta^\beta_\alpha\delta^{\dot{\beta}}_{\dot{\alpha}} \tag{C.2}$$

$$[\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu]_\alpha{}^\beta = 2g^{\mu\nu}\delta^\beta_\alpha \tag{C.3}$$

$$[\bar{\sigma}^\mu\sigma^\nu + \bar{\sigma}^\nu\sigma^\mu]^{\dot{\alpha}}{}_{\dot{\beta}} = 2g^{\mu\nu}\delta^{\dot{\alpha}}_{\dot{\beta}}\,. \tag{C.4}$$

The matrix $p^\mu\sigma_\mu$ is Hermitian and its determinant is $|p^\mu\sigma_\mu| = p^2$. For $M \in SL(2,\mathbb{C})$, the transformation $Mp^\mu\sigma_\mu M^\dagger$ produces another four-vector p'^μ connected to p^μ by a Lorentz transformation:

$$M\sigma_\mu M^\dagger = \sigma_\nu\Lambda^\nu{}_\mu\,. \tag{C.5}$$

Object	Transformation	Index
χ_α (left-handed, $(\frac{1}{2},0)$)	M	undotted lower
χ^α	$(M^{-1})^\top$	undotted upper
$\bar{\chi}^{\dot{\alpha}}$ (right-handed, $(0,\frac{1}{2})$)	$(M^{-1})^\dagger$	dotted upper
$\bar{\chi}_{\dot{\alpha}}$	M^*	dotted lower

where $M = \exp\left(-\frac{i}{2}\vec{\theta}\cdot\vec{\sigma} - \frac{1}{2}\vec{\zeta}\cdot\vec{\sigma}\right)$ for rotation $\vec{\theta}$ and boost rapidity $\vec{\zeta}$, and $(M^{-1})^\dagger = \exp\left(-\frac{i}{2}\vec{\theta}\cdot\vec{\sigma} + \frac{1}{2}\vec{\zeta}\cdot\vec{\sigma}\right)$. Note that rotations act identically on both chiralities, while boosts act with opposite signs—this is the defining difference between $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ representations.

Contractions:

- $\chi\psi = \chi^\alpha\psi_\alpha$ (undotted: upper-lower)
- $\bar{\chi}\bar{\psi} = \bar{\chi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}$ (dotted: lower-upper)
- $\chi\sigma^\mu\bar{\psi} = \chi^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}$ (vector)
- $\bar{\chi}\bar{\sigma}^\mu\psi = \bar{\chi}_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\psi_\alpha$ (vector)

Hermitian conjugation: $(\chi\psi)^\dagger = \bar{\psi}\bar{\chi}$

D Mandelstam variables

Consider a generic $2 \to 2$ scattering process

$$a(p_1) + b(p_2) \longrightarrow c(p_3) + d(p_4)\,, \tag{D.1}$$

where all momenta are on-shell: $p_i^2 = m_i^2$. Four-momentum conservation reads $p_1 + p_2 = p_3 + p_4$.

Definition

The three **Mandelstam variables** are defined as

$$\boxed{s \equiv (p_1 + p_2)^2\,, \qquad t \equiv (p_1 - p_3)^2\,, \qquad u \equiv (p_1 - p_4)^2\,.} \tag{D.2}$$

They are Lorentz scalars constructed from the external momenta alone, and therefore frame-independent.

Physical interpretation

Each variable is the squared invariant mass of the four-momentum flowing through one of the three possible channels:

- $s = (p_1 + p_2)^2 = (p_3 + p_4)^2$ is the squared centre-of-mass energy. A propagator carrying momentum $p_1 + p_2$ has $q^2 = s$ (s**-channel**: annihilation/creation).
- $t = (p_1 - p_3)^2 = (p_2 - p_4)^2$ is the squared four-momentum transfer. A propagator carrying momentum $p_1 - p_3$ has $q^2 = t$ (t**-channel**: exchange).
- $u = (p_1 - p_4)^2 = (p_2 - p_3)^2$ is the crossed momentum transfer. A propagator carrying momentum $p_1 - p_4$ has $q^2 = u$ (u**-channel**: crossed exchange).

Constraint

The three variables are not independent. Adding the definitions and using $p_i^2 = m_i^2$:

$$s + t + u = (p_1 + p_2)^2 + (p_1 - p_3)^2 + (p_1 - p_4)^2\,. \tag{D.3}$$

Expanding each square and using momentum conservation $p_2 = p_3 + p_4 - p_1$ to eliminate cross terms:

$$\boxed{s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2 .} \tag{D.4}$$

For equal-mass particles ($m_i = m$) this reduces to $s + t + u = 4m^2$.

Proof. Expand each term:

$$s = p_1^2 + 2\,p_1 \cdot p_2 + p_2^2 = m_1^2 + m_2^2 + 2\,p_1 \cdot p_2 , \tag{D.5}$$

$$t = p_1^2 - 2\,p_1 \cdot p_3 + p_3^2 = m_1^2 + m_3^2 - 2\,p_1 \cdot p_3 , \tag{D.6}$$

$$u = p_1^2 - 2\,p_1 \cdot p_4 + p_4^2 = m_1^2 + m_4^2 - 2\,p_1 \cdot p_4 . \tag{D.7}$$

Summing: $s + t + u = 3m_1^2 + m_2^2 + m_3^2 + m_4^2 + 2\,p_1 \cdot (p_2 - p_3 - p_4)$. Momentum conservation gives $p_2 - p_3 - p_4 = -p_1$, so the last term becomes $2\,p_1 \cdot (-p_1) = -2m_1^2$, yielding the result. □

Counting degrees of freedom

A $2 \to 2$ process with fixed external masses has $4 \times 4 = 16$ momentum components, constrained by:

- 4 on-shell conditions ($p_i^2 = m_i^2$): removes 4,
- 4 momentum-conservation equations: removes 4,
- 6 Lorentz transformations (3 boosts + 3 rotations): removes 6.

This leaves $16 - 4 - 4 - 6 = 2$ independent Lorentz-invariant kinematic variables. Since $s + t + u$ is fixed, any two of $\{s, t, u\}$ form a complete set. A common choice is (s, t), with u determined by the constraint.

Centre-of-mass frame expressions

In the CM frame, $\vec{p}_1 + \vec{p}_2 = 0$, so $p_1 = (E_1, \vec{p})$ and $p_2 = (E_2, -\vec{p})$. Then:

$$s = (E_1 + E_2)^2 = E_{\mathrm{cm}}^2 . \tag{D.8}$$

For the final state: $p_3 = (E_3, \vec{p}\,')$ and $p_4 = (E_4, -\vec{p}\,')$. Defining θ as the scattering angle between $\vec{p}$ and $\vec{p}\,'$:

$$t = m_1^2 + m_3^2 - 2(E_1 E_3 - |\vec{p}\,||\vec{p}\,'|\cos\theta) , \tag{D.9}$$

$$u = m_1^2 + m_4^2 - 2(E_1 E_4 + |\vec{p}\,||\vec{p}\,'|\cos\theta) . \tag{D.10}$$

Ultrarelativistic limit

When all masses are negligible ($E \gg m_i$), the kinematics simplifies dramatically. In the CM frame with $|\vec{p}\,| \approx |\vec{p}\,'| \approx E_{\rm cm}/2$:

$$\boxed{s = 4E^2\,, \qquad t = -\frac{s}{2}(1-\cos\theta)\,, \qquad u = -\frac{s}{2}(1+\cos\theta)\,,} \tag{D.11}$$

with the constraint $s+t+u=0$. The physical scattering region is

$$s \geq 0\,, \qquad t \leq 0\,, \qquad u \leq 0\,. \tag{D.12}$$

Scalar products in terms of s, t, u

All independent scalar products of external momenta can be expressed as:

$$\boxed{p_1\cdot p_2 = \tfrac{1}{2}(s - m_1^2 - m_2^2)\,, \qquad p_1\cdot p_3 = \tfrac{1}{2}(m_1^2 + m_3^2 - t)\,, \qquad p_1\cdot p_4 = \tfrac{1}{2}(m_1^2 + m_4^2 - u)\,.} \tag{D.13}$$

The remaining products follow from momentum conservation: $p_2\cdot p_3 = \frac{1}{2}(m_2^2 + m_3^2 - u)$, $p_2\cdot p_4 = \frac{1}{2}(m_2^2 + m_4^2 - t)$, $p_3\cdot p_4 = \frac{1}{2}(s - m_3^2 - m_4^2)$. In the massless limit these reduce to $p_1\cdot p_2 = s/2$, $p_1\cdot p_3 = -t/2$, $p_1\cdot p_4 = -u/2$.

Physical regions

For the *physical* s-channel process $a\,b \to c\,d$, the ranges are:

$$s \geq (m_3+m_4)^2\,, \qquad t_- \leq t \leq t_+\,, \qquad u \text{ fixed by the constraint}\,, \tag{D.14}$$

where $t_\pm$ correspond to $\cos\theta = \pm 1$ (forward/backward scattering).

The same three variables also describe the *crossed* processes:

- s-channel: $a\,b \to c\,d$ (physical region: $s \geq (m_3+m_4)^2$, $t \leq 0$, $u \leq 0$)
- t-channel: $a\,\bar{c} \to \bar{b}\,d$ (physical region: $t \geq (m_1+m_3)^2$, $s \leq 0$, $u \leq 0$)
- u-channel: $a\,\bar{d} \to \bar{b}\,c$ (physical region: $u \geq (m_1+m_4)^2$, $s \leq 0$, $t \leq 0$)

This is the basis of **crossing symmetry**: the amplitude $\mathcal{M}(s,t,u)$ is a single analytic function of which the three physical processes occupy different kinematic regions.

Propagator denominators

In Feynman diagrams, the Mandelstam variables appear naturally in propagator denominators. For a particle of mass M exchanged in the three channels:

$$s\text{-channel: } \frac{i}{s-M^2}\,, \qquad t\text{-channel: } \frac{i}{t-M^2}\,, \qquad u\text{-channel: } \frac{i}{u-M^2}\,. \tag{D.15}$$

This is why amplitudes are naturally expressed as functions of s, t, u.

E Single-spin projectors and helicity

When computing polarized cross sections, one needs the outer product $u(\vec{p},\lambda)\bar{u}(\vec{p},\lambda)$ for a *single* helicity state, rather than the unpolarized spin sum $\sum_s u\bar{u} = (\not{p}+m)/(2m)$. The result is:

$$\boxed{u(\vec{p},\lambda)\,\bar{u}(\vec{p},\lambda) = \Lambda^{+}(\vec{p})\,\Pi^{\lambda}(\vec{p})} \tag{E.1}$$

where $\Lambda^{+}(p) = (\not{p}+m)/(2m)$ is the positive-energy projector and $\Pi^{\lambda}(\vec{p}) = \frac{1}{2}(1+\lambda\gamma^5\not{n}_p)$ is the helicity projector. We now prove this identity step by step.

The covariant spin four-vector

To write helicity in a Lorentz-covariant form, we introduce the **spin four-vector** n_p^μ, defined by three conditions:

$$n_p\cdot p = 0\,, \qquad n_p^2 = -1\,, \qquad n_p^\mu \xrightarrow{\text{rest frame}} (0,\hat{p})\,. \tag{E.2}$$

The first condition ensures that n_p^μ is a pure spatial direction in the rest frame; the second normalizes it. The unique Lorentz boost from the rest frame gives:

$$n_p^\mu = \left(\frac{|\vec{p}|}{m},\ \frac{E_p}{m}\hat{p}\right). \tag{E.3}$$

Verification:

$$n_p\cdot p = \frac{|\vec{p}|}{m}E_p - \frac{E_p}{m}|\vec{p}| = 0\,, \quad \checkmark \tag{E.4}$$

$$n_p^2 = \frac{|\vec{p}|^2}{m^2} - \frac{E_p^2}{m^2} = \frac{|\vec{p}|^2 - E_p^2}{m^2} = \frac{-m^2}{m^2} = -1\,. \quad \checkmark \tag{E.5}$$

The eigenvalue equation $\gamma^5\not{n}_p\,u = \lambda\,u$

The operator $\gamma^5\not{n}_p$ is the covariant generalization of the helicity operator. To prove the eigenvalue equation, work in the rest frame where $p^\mu = (m,\vec{0})$ and $n_p^\mu = (0,\hat{p})$. Then:

$$\not{n}_p = n_{p,\mu}\gamma^\mu = -\vec{\gamma}\cdot\hat{p} = \gamma^0(-\gamma^0\vec{\gamma}\cdot\hat{p})\,. \tag{E.6}$$

The spin operator is $\vec{\Sigma} = \gamma^0\vec{\gamma}\gamma^5$, so $\gamma^0\vec{\gamma} = \vec{\Sigma}\gamma^5$ and:

$$\gamma^5 \not{n}_p = \gamma^5\gamma^0(-\gamma^0\vec{\gamma}\cdot\hat{p}) = -\gamma^5\gamma^0\gamma^0\vec{\gamma}\cdot\hat{p} = -\gamma^5\vec{\gamma}\cdot\hat{p}\,. \tag{E.7}$$

Using $\gamma^5\gamma^i = -\gamma^i\gamma^5$ and $\Sigma^i = \gamma^0\gamma^i\gamma^5$, one can show that in the rest frame:

$$\gamma^5 \not{n}_p = \vec{\Sigma}\cdot\hat{p} = 2\hat{h}\,, \tag{E.8}$$

where $\hat{h} = \vec{\Sigma}\cdot\hat{p}/2$ is the helicity operator. Since $u(\vec{p},\lambda)$ has helicity $\lambda/2$ (i.e. $\hat{h}\,u = (\lambda/2)\,u$):

$$\boxed{\gamma^5 \not{n}_p\, u(\vec{p},\lambda) = \lambda\, u(\vec{p},\lambda)} \tag{E.9}$$

Because this is a Lorentz-covariant equation, it holds in *any* frame, not just the rest frame. ✓

The helicity projector

Define $\Pi^\lambda(\vec{p}) = \frac{1}{2}(1 + \lambda\gamma^5\not{n}_p)$. Using eq. (E.9):

$$\Pi^\lambda u(\vec{p},\lambda') = \frac{1}{2}(1+\lambda\lambda')\,u(\vec{p},\lambda') = \delta_{\lambda\lambda'}\,u(\vec{p},\lambda')\,. \tag{E.10}$$

So Π^λ selects the helicity-λ spinor and annihilates the other. It satisfies the projector properties $(\Pi^\lambda)^2 = \Pi^\lambda$ and $\Pi^+ + \Pi^- = \mathbb{1}$.

Proof of eq. (E.1)

Starting from the completeness relation:

$$\sum_{s=1}^{2} u(\vec{p},s)\,\bar{u}(\vec{p},s) = \frac{\not{p}+m}{2m} = \Lambda^+(\vec{p})\,, \tag{E.11}$$

we apply Π^λ from the right. On the left-hand side, Π^λ acts on $\bar{u}$ from the right, but since $\bar{u}(\vec{p},s)\Pi^\lambda = \overline{\Pi^\lambda u(\vec{p},s)} = \delta_{s\lambda}\bar{u}(\vec{p},\lambda)$, only the $s=\lambda$ term survives:

$$\sum_{s} u(\vec{p},s)\,\bar{u}(\vec{p},s)\,\Pi^\lambda = u(\vec{p},\lambda)\,\bar{u}(\vec{p},\lambda)\,. \tag{E.12}$$

On the right-hand side, we simply get $\Lambda^+\Pi^\lambda$. Therefore:

$$u(\vec{p},\lambda)\,\bar{u}(\vec{p},\lambda) = \Lambda^+(\vec{p})\,\Pi^\lambda(\vec{p})\,. \quad ✓ \tag{E.13}$$

Ultra-relativistic limit

In the ultra-relativistic limit $E \gg m$, the spin four-vector (E.3) becomes $n_p^\mu \approx p^\mu/m$ (since $|\vec{p}| \approx E$). Then $\gamma^5 \not{n}_p \approx \gamma^5 \not{p}/m$. Acting on $u(\vec{p})$, the Dirac equation gives $\not{p}\, u = m\, u$, so $\gamma^5 \not{n}_p\, u \approx \gamma^5 u$. The helicity projector therefore simplifies to:

$$\Pi^\lambda(\vec{p}) \xrightarrow{E \gg m} \frac{1}{2}(1 + \lambda \gamma^5)\,, \tag{E.14}$$

which is the **chirality projector**. This is the mathematical statement of the fact that helicity and chirality coincide for massless (or ultra-relativistic) fermions.

F Special functions for loop integrals

The evaluation of Feynman integrals in dimensional regularization makes repeated use of a small number of special functions whose origins predate quantum field theory by more than two centuries. We collect here their definitions, key properties, and the identities most frequently needed in these notes.

The Gamma function

The Gamma function was introduced by Leonhard Euler in 1729 in a letter to Christian Goldbach, motivated by the problem of extending the factorial $n!$ from the positive integers to arbitrary (real or complex) values of the argument. Euler's original definition used an infinite product; the integral representation below is due to Euler himself (1730s) and was later formalized by Adrien-Marie Legendre, who also introduced the notation Γ.

Beyond its role as a generalized factorial, the Gamma function appears throughout physics and mathematics: in the normalization of probability distributions (the χ^2 distribution involves $\Gamma(n/2)$), in the volume of hyperspheres ($\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$, cf. Appendix G), in the Stirling approximation for statistical mechanics, and—centrally for us—in the master formula for dimensionally regularized loop integrals, where the poles of Γ at non-positive integers encode the UV divergences of quantum field theory.

Definition. For $\mathrm{Re}(s) > 0$:

$$\Gamma(s) \equiv \int_0^\infty dt\, t^{s-1} e^{-t}\,. \tag{F.1}$$

This integral converges at $t \to \infty$ for all s, and at $t \to 0$ when $\mathrm{Re}(s) > 0$.

Fundamental recursion. Integration by parts gives

$$\Gamma(s+1) = s\,\Gamma(s)\,. \tag{F.2}$$

Proof. $\Gamma(s+1) = \int_0^\infty t^s e^{-t}\,dt = \left[-t^s e^{-t}\right]_0^\infty + s\int_0^\infty t^{s-1}e^{-t}\,dt = s\,\Gamma(s)$.
Since $\Gamma(1) = \int_0^\infty e^{-t}\,dt = 1$, repeated application gives

$$\Gamma(n+1) = n! \qquad \text{for } n = 0, 1, 2, \ldots \tag{F.3}$$

The Gamma function is thus the analytic continuation of the factorial to the complex plane.

Analytic continuation and poles. The recursion (F.2) can be rewritten as $\Gamma(s) = \Gamma(s+1)/s$. Since the right-hand side is defined for $\mathrm{Re}(s) > -1$ (except $s = 0$), this extends Γ to $\mathrm{Re}(s) > -1$. Iterating:

$$\Gamma(s) = \frac{\Gamma(s+n+1)}{s(s+1)\cdots(s+n)}, \tag{F.4}$$

which defines $\Gamma(s)$ for all $s \in \mathbb{C}$ except $s = 0, -1, -2, \ldots$, where it has simple poles. Near $s = -n$ (with $n = 0, 1, 2, \ldots$):

$$\Gamma(-n+\epsilon) = \frac{(-1)^n}{n!}\left[\frac{1}{\epsilon} - \gamma_E + H_n + O(\epsilon)\right], \tag{F.5}$$

where $H_n = \sum_{k=1}^{n} 1/k$ is the n-th harmonic number ($H_0 = 0$). This is the formula that generates the $1/\epsilon$ poles in dimensional regularization.

Expansion around positive integers. In loop calculations, we frequently need Γ near $s = \epsilon$ (i.e. near the pole at $s = 0$) and near $s = 1+\epsilon$, $s = 2+\epsilon$, etc. The key expansions are:

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + O(\epsilon), \tag{F.6}$$

$$\Gamma(1+\epsilon) = 1 - \gamma_E\,\epsilon + O(\epsilon^2), \tag{F.7}$$

$$\Gamma(2+\epsilon) = 1 + (1-\gamma_E)\epsilon + O(\epsilon^2), \tag{F.8}$$

where $\gamma_E \approx 0.5772$ is the **Euler–Mascheroni constant** (introduced by Euler in 1734 and later computed to higher precision by Lorenzo Mascheroni in 1790; it remains unknown whether γ_E is rational or irrational), defined by

$$\gamma_E \equiv \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right) = -\Gamma'(1). \tag{F.9}$$

The combination $1/\epsilon - \gamma_E + \ln(4\pi)$ appears so ubiquitously that it defines the $\overline{\mathrm{MS}}$ subtraction (cf. Chapter G).

Special values.

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}, \qquad \Gamma\left(\tfrac{3}{2}\right) = \tfrac{1}{2}\sqrt{\pi}, \qquad \Gamma\left(\tfrac{5}{2}\right) = \tfrac{3}{4}\sqrt{\pi}. \tag{F.10}$$

More generally, $\Gamma(n+\frac{1}{2}) = \frac{(2n)!}{4^n\,n!}\sqrt{\pi}$.

Reflection formula (Euler).

$$\Gamma(s)\,\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}. \tag{F.11}$$

Setting $s = 1/2$ gives $[\Gamma(1/2)]^2 = \pi$, hence $\Gamma(1/2) = \sqrt{\pi}$. This formula is also useful for relating $\Gamma(1 - d/2)$ (which appears in the tadpole J_1) to more familiar quantities.

Duplication formula (Legendre).

$$\Gamma(s)\,\Gamma\!\left(s + \tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2s-1}}\,\Gamma(2s)\,. \tag{F.12}$$

The Beta function

The Beta function was also studied by Euler (1730s) and Legendre, originally in connection with the evaluation of integrals of powers of trigonometric functions. The key result $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$, which reduces the Beta function to a ratio of Gamma functions, was proved by Euler and later refined by Legendre.

In physics, the Beta function appears whenever an angular or Feynman-parameter integral can be reduced to the standard form $\int_0^1 t^{a-1}(1-t)^{b-1}\,dt$. This happens in the radial part of loop integrals (after switching to hyperspherical coordinates), in phase-space integrations for particle decays and scattering cross sections, and in statistical mechanics (where it normalizes the Beta distribution). Historically, the "Veneziano amplitude" (1968)—one of the founding results of string theory—was written as a ratio of Beta functions, chosen precisely to have the crossing symmetry and Regge pole structure expected of strong-interaction amplitudes.

Definition. For $\mathrm{Re}(a) > 0$ and $\mathrm{Re}(b) > 0$:

$$B(a,b) \;\equiv\; \int_0^1 dt\; t^{a-1}(1-t)^{b-1}\,. \tag{F.13}$$

Relation to the Gamma function. The fundamental identity is

$$\boxed{B(a,b) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}\,.} \tag{F.14}$$

This is the result used in evaluating the radial part of loop integrals (cf. Step 3 in the derivation of J_n, Appendix G).

Proof. Write $\Gamma(a)\,\Gamma(b)$ as a double integral and change variables:

$$\begin{aligned}\Gamma(a)\,\Gamma(b) &= \int_0^\infty du\; u^{a-1}e^{-u} \int_0^\infty dv\; v^{b-1}e^{-v} \\ &= \int_0^\infty \int_0^\infty du\,dv\; u^{a-1}v^{b-1}\,e^{-(u+v)}\,.\end{aligned} \tag{F.15}$$

Substituting $u = st$, $v = s(1-t)$ with $s \in (0,\infty)$, $t \in (0,1)$, the Jacobian is $|\partial(u,v)/\partial(s,t)| = s$, and $u+v = s$:

$$\begin{aligned}\Gamma(a)\,\Gamma(b) &= \int_0^\infty ds \int_0^1 dt\; (st)^{a-1}\big[s(1-t)\big]^{b-1} e^{-s}\cdot s \\ &= \int_0^\infty ds\; s^{a+b-1}\, e^{-s} \;\cdot\; \int_0^1 dt\; t^{a-1}(1-t)^{b-1} \\ &= \Gamma(a+b)\cdot B(a,b)\,.\end{aligned}$$

□

Symmetry.

$$B(a,b) = B(b,a)\,, \tag{F.16}$$

which is obvious from the substitution $t \longrightarrow 1-t$ in the definition.

Useful alternative forms. The substitution $t = u/(1+u)$ transforms (F.13) into

$$B(a,b) = \int_0^\infty du\; \frac{u^{a-1}}{(1+u)^{a+b}}\,, \tag{F.17}$$

which is the form that arises directly from the radial integral after the substitution $r^2 = \Delta u$ in the derivation of J_n.

Special values.

$$B(a,1) = \frac{1}{a}\,, \qquad B\left(\tfrac{1}{2},\tfrac{1}{2}\right) = \pi\,, \qquad B(n,m) = \frac{(n-1)!(m-1)!}{(n+m-1)!} \quad (n,m \text{ positive integers})\,. \tag{F.18}$$

The digamma function

The digamma function $\psi(s) = \Gamma'(s)/\Gamma(s)$ was studied systematically by Gauss and Legendre in the early 19th century, though particular values had appeared earlier in Euler's work on harmonic series. The name "digamma" comes from the notation ψ (or F), used historically as an analogue of the "di-" prefix applied to the Gamma function (its logarithmic derivative).

In quantum field theory, the digamma function is less prominent than Γ and B, but it enters whenever one needs the $O(\epsilon)$ terms in the expansion of Γ functions—that is, whenever finite parts of loop integrals matter (not just the divergent poles). This is the case in any precision calculation: the anomalous magnetic moment, the Lamb shift, running couplings at two loops, and matching calculations between effective field theories all require the digamma function through its connection $\psi(n) = -\gamma_E + H_{n-1}$ to harmonic numbers.

Definition.

$$\psi(s) \equiv \frac{d}{ds}\ln\Gamma(s) = \frac{\Gamma'(s)}{\Gamma(s)}. \tag{F.19}$$

The digamma function appears when expanding Γ functions around their arguments, which happens systematically in extracting finite parts of dimensionally regularized integrals.

Key values and identities.

$$\psi(1) = -\gamma_E, \tag{F.20}$$

$$\psi(n+1) = -\gamma_E + \sum_{k=1}^{n}\frac{1}{k} = -\gamma_E + H_n \qquad (n \geq 1 \text{ integer}), \tag{F.21}$$

$$\psi(s+1) = \psi(s) + \frac{1}{s}, \tag{F.22}$$

$$\psi(1-s) - \psi(s) = \pi\cot(\pi s) \qquad \text{(reflection formula)}. \tag{F.23}$$

The recursion (F.22) follows immediately from $\Gamma(s+1) = s\Gamma(s)$ by taking the logarithmic derivative.

Connection to ϵ-expansions. When we expand $\Gamma(n+a\epsilon)$ around $\epsilon = 0$ for integer n, the $O(\epsilon)$ term is proportional to $\psi(n)$:

$$\Gamma(n+a\epsilon) = \Gamma(n)\left[1 + a\epsilon\,\psi(n) + O(\epsilon^2)\right]. \tag{F.24}$$

For example, $\Gamma(1+\epsilon) = 1 + \epsilon\,\psi(1) + O(\epsilon^2) = 1 - \gamma_E\epsilon + O(\epsilon^2)$, recovering (F.7).

Summary: the functions at a glance

Box F.1 Special functions: quick reference

Function	Definition	Key identity
$\Gamma(s)$	$\displaystyle\int_0^\infty t^{s-1}e^{-t}\,dt$	$\Gamma(s+1) = s\Gamma(s)$; $\Gamma(n+1) = n!$
$B(a,b)$	$\displaystyle\int_0^1 t^{a-1}(1-t)^{b-1}\,dt$	$B(a,b) = \dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$
$\psi(s)$	$\dfrac{d}{ds}\ln\Gamma(s)$	$\psi(1) = -\gamma_E$; $\psi(s+1) = \psi(s) + 1/s$

Euler–Mascheroni constant: $\gamma_E = -\psi(1) = -\Gamma'(1) \approx 0.5772$

Poles of Γ: $\Gamma(-n+\epsilon) = \dfrac{(-1)^n}{n!}\left[\dfrac{1}{\epsilon} - \gamma_E + H_n + O(\epsilon)\right], \quad H_n = \sum_{k=1}^{n} 1/k$

G Feynman-integral technology

This appendix develops the standard techniques for evaluating one-loop Feynman integrals in dimensional regularization. These methods are used throughout these notes: for the tadpole and bubble in $\lambda\phi^4$ theory (Chapter 4), for the vacuum polarization, electron self-energy, and vertex correction in QED (Chapter 6), and for any other one-loop calculation the reader may encounter.

Feynman parametrization

The central idea is to combine products of propagator denominators into a single denominator raised to a power, at the cost of introducing auxiliary integrations (the *Feynman parameters*).

Two propagators. The simplest case is

$$\frac{1}{AB} = \int_0^1 dx \, \frac{1}{\big[xA + (1-x)B\big]^2} . \tag{G.1}$$

Proof. Writing $xA + (1-x)B = B + x(A-B)$, the integral over x is elementary:

$$\int_0^1 dx \, \frac{1}{[B + x(A-B)]^2} = \left[-\frac{1}{(A-B)\big[B + x(A-B)\big]} \right]_0^1 = -\frac{1}{A-B}\left(\frac{1}{A} - \frac{1}{B}\right) = \frac{1}{AB} .$$

□

Three propagators.

$$\frac{1}{ABC} = 2\int_0^1 dx \int_0^{1-x} dy \, \frac{1}{\big[xA + yB + zC\big]^3} , \qquad z \equiv 1 - x - y . \tag{G.2}$$

This follows by applying (G.1) twice: first combine A and B, then combine the result with C, and change variables so that the integration region becomes the triangular region $x + y + z = 1$ with $x, y, z \geq 0$ (i.e. $0 \leq x \leq 1$, $0 \leq y \leq 1 - x$, $z = 1 - x - y$).

General formula: n propagators.

$$\frac{1}{A_1 A_2 \cdots A_n} = (n-1)! \int_0^1 dx_1 \cdots dx_n \, \delta\Big(1 - \sum_{i=1}^n x_i\Big) \frac{1}{\big[\sum_{i=1}^n x_i A_i\big]^n} . \tag{G.3}$$

The proof is by induction. The δ-function constrains the parameters to the region $x_i \geq 0$, $\sum x_i = 1$ (a segment for $n = 2$, a triangle for $n = 3$, a tetrahedron for $n = 4$, etc.).

Example: the bubble integral (two propagators)

Consider the scalar two-point integral that arises, for instance, in the $\lambda\phi^4$ self-energy or in the QED vacuum polarization:

$$\mathcal{B}(p^2; m_1^2, m_2^2) \equiv \int \frac{d^d k}{(2\pi)^d} \frac{1}{\left[k^2 - m_1^2 + i\epsilon\right]\left[(k-p)^2 - m_2^2 + i\epsilon\right]}. \tag{G.4}$$

Step 1: Feynman parameter. Apply (G.1) with $A = k^2 - m_1^2 + i\epsilon$ and $B = (k-p)^2 - m_2^2 + i\epsilon$:

$$\mathcal{B}(p^2; m_1^2, m_2^2) = \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{1}{\left[x(k^2 - m_1^2) + (1-x)((k-p)^2 - m_2^2) + i\epsilon\right]^2}.$$

We now expand the combined denominator step by step. Writing $D \equiv x(k^2 - m_1^2) + (1-x)((k-p)^2 - m_2^2)$:

$$\begin{aligned} D &= x\,k^2 - x\,m_1^2 + (1-x)\left[k^2 - 2k\cdot p + p^2 - m_2^2\right] \\ &= \underbrace{\left[x + (1-x)\right]}_{=1} k^2 - 2(1-x)\,k\cdot p + (1-x)\,p^2 - x\,m_1^2 - (1-x)\,m_2^2 \\ &= k^2 - 2(1-x)\,k\cdot p + (1-x)\,p^2 - x\,m_1^2 - (1-x)\,m_2^2. \end{aligned} \tag{G.5}$$

Step 2: Complete the square. The first two terms of (G.5) can be rewritten by shifting the loop momentum. Define $\ell \equiv k - (1-x)p$. Then $k = \ell + (1-x)p$ and:

$$\begin{aligned} k^2 - 2(1-x)\,k\cdot p &= \left[\ell + (1-x)p\right]^2 - 2(1-x)\left[\ell + (1-x)p\right]\cdot p \\ &= \ell^2 + 2(1-x)\,\ell\cdot p + (1-x)^2 p^2 - 2(1-x)\,\ell\cdot p - 2(1-x)^2 p^2 \\ &= \ell^2 - (1-x)^2 p^2. \end{aligned} \tag{G.6}$$

Substituting back into (G.5):

$$\begin{aligned} D &= \ell^2 - (1-x)^2 p^2 + (1-x)\,p^2 - x\,m_1^2 - (1-x)\,m_2^2 \\ &= \ell^2 - \left[(1-x)^2 - (1-x)\right] p^2 - x\,m_1^2 - (1-x)\,m_2^2 \\ &= \ell^2 - \left[-(1-x)\,x\right] p^2 - x\,m_1^2 - (1-x)\,m_2^2 \\ &= \ell^2 - \left[x\,m_1^2 + (1-x)\,m_2^2 - x(1-x)\,p^2\right], \end{aligned} \tag{G.7}$$

where in the third line we used $(1-x)^2-(1-x)=(1-x)(1-x-1)=-x(1-x)$. Therefore

$$D \;=\; \ell^2-\Delta_{\rm bub}(x)+i\epsilon\,,$$

with

$$\boxed{\Delta_{\rm bub}(x) \;\equiv\; x\,m_1^2+(1-x)\,m_2^2-x(1-x)\,p^2\,.} \tag{G.8}$$

Note that $\Delta_{\rm bub}$ is a simple quadratic in x that encodes all the dependence on masses and external momentum. Since the integration measure $d^dk=d^d\ell$ (translation invariance),

$$\boxed{\mathcal{B}(p^2;m_1^2,m_2^2) \;=\; \int_0^1 dx\; J_2\big(\Delta_{\rm bub}(x)\big)\,,} \tag{G.9}$$

where $J_n(\Delta)$ is the master scalar integral defined in (G.22) below. For the common case of equal masses ($m_1=m_2=m$), the formula simplifies to $\Delta_{\rm bub}(x)=m^2-x(1-x)\,p^2$.

Example: the triangle integral (three propagators)

A three-propagator integral arises in vertex corrections. The general structure is

$$I^{\{\cdots\}}(p',p) \;\equiv\; \int \frac{d^dk}{(2\pi)^d}\,\frac{\{\,1,\;k^\mu,\;k^\mu k^\nu,\;\ldots\,\}}{\big[k^2-\lambda^2+i\epsilon\big]\,\big[(p'-k)^2-m^2+i\epsilon\big]\,\big[(p-k)^2-m^2+i\epsilon\big]}\,. \tag{G.10}$$

Here $d=4-2\epsilon$ regulates UV divergences and λ is an IR regulator (e.g. a small photon mass). We keep the external momenta general at first.

Feynman parameters. Apply (G.2) with

$$\begin{aligned} A &\equiv (p'-k)^2-m^2+i\epsilon,\\ B &\equiv (p-k)^2-m^2+i\epsilon,\\ C &\equiv k^2-\lambda^2+i\epsilon\,. \end{aligned} \tag{G.11}$$

The combined denominator is

$$xA+yB+zC \;=\; k^2-2k\cdot(xp'+yp)+x(p'^2-m^2)+y(p^2-m^2)-z\lambda^2+i\epsilon\,. \tag{G.12}$$

On-shell specialization. If the external legs are on-shell, $p^2 = p'^2 = m^2$, the terms $x(p'^2 - m^2)$ and $y(p^2 - m^2)$ vanish, simplifying the denominator to

$$xA + yB + zC \;=\; k^2 - 2k\cdot(xp' + yp) - z\lambda^2 + i\epsilon\,. \tag{G.13}$$

Completing the square. Define the shifted momentum

$$\ell \;\equiv\; k - (xp' + yp)\,. \tag{G.14}$$

Then

$$k^2 - 2k\cdot(xp' + yp) \;=\; \ell^2 - (xp' + yp)^2\,, \tag{G.15}$$

so that

$$xA + yB + zC \;=\; \ell^2 - \Delta(x, y; q^2) + i\epsilon\,, \tag{G.16}$$

where (using on-shell conditions and $q \equiv p' - p$)

$$\begin{aligned}(xp' + yp)^2 &= x^2 p'^2 + y^2 p^2 + 2xy\, p'\cdot p \\ &= m^2(x^2 + y^2 + 2xy) - xy\, q^2 \\ &= m^2(x + y)^2 - xy\, q^2\,,\end{aligned} \tag{G.17}$$

and therefore

$$\Delta(x, y; q^2) \;=\; m^2(x + y)^2 \;-\; xy\, q^2 \;+\; z\,\lambda^2\,, \qquad z = 1 - x - y\,. \tag{G.18}$$

At this stage every loop integral becomes

$$I^{\{\cdots\}}(p', p) \;=\; 2\int_0^1 dx \int_0^{1-x} dy \int \frac{d^d\ell}{(2\pi)^d}\, \frac{\{\cdots\}}{\left[\ell^2 - \Delta(x, y; q^2) + i\epsilon\right]^3}\,, \tag{G.19}$$

where the numerator $\{\cdots\}$ is now a polynomial in ℓ plus terms built from p, p' (equivalently q and $p + p'$).

Tensor reduction

Dimensional regularization gives two crucial simplifications:

1. **Odd powers integrate to zero** (by symmetry $\ell \to -\ell$ of the integrand):

$$\int \frac{d^d\ell}{(2\pi)^d}\, \frac{\ell^\mu}{(\ell^2 - \Delta + i\epsilon)^n} \;=\; 0, \qquad \int \frac{d^d\ell}{(2\pi)^d}\, \frac{\ell^\mu \ell^\nu \ell^\rho}{(\ell^2 - \Delta + i\epsilon)^n} \;=\; 0\,. \tag{G.20}$$

2. **Even tensors reduce to the metric** (by Lorentz covariance, $g^{\mu\nu}$ is the only available symmetric tensor):

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^\mu \ell^\nu}{(\ell^2 - \Delta + i\epsilon)^n} = \frac{g^{\mu\nu}}{d} \int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^2}{(\ell^2 - \Delta + i\epsilon)^n}. \tag{G.21}$$

(Contract both sides with $g_{\mu\nu}$ to verify: $g_{\mu\nu}g^{\mu\nu} = d$.) Similarly, a rank-4 tensor reduces to symmetrized products of $g^{\mu\nu}$.

So the whole problem reduces to a small set of *scalar* integrals of the form

$$J_n(\Delta) \equiv \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{(\ell^2 - \Delta + i\epsilon)^n}, \qquad n = 1, 2, 3, \ldots \tag{G.22}$$

and integrals with powers of ℓ^2 in the numerator.

The master integral $J_n(\Delta)$: complete derivation

We now derive the closed-form expression for $J_n(\Delta)$ in $d = 4 - 2\epsilon$ dimensions. The derivation proceeds in four steps.

Step 1: Wick rotation. Write $\ell^\mu = (\ell^0, \vec{\ell})$ in Minkowski space with signature $(+,-,-,-)$. The denominator is

$$\ell^2 - \Delta + i\epsilon = (\ell^0)^2 - |\vec{\ell}|^2 - \Delta + i\epsilon.$$

As a function of ℓ^0, the integrand has poles where $(\ell^0)^2 = |\vec{\ell}|^2 + \Delta - i\epsilon$, i.e. at $\ell^0 = \pm\sqrt{|\vec{\ell}|^2 + \Delta - i\epsilon}$. Writing $\omega^2 \equiv |\vec{\ell}|^2 + \Delta > 0$ and treating ϵ as a small positive number, we have $\ell^0 = \pm\sqrt{\omega^2 - i\epsilon} \approx \pm\omega(1 - i\epsilon/2\omega^2)$, so

$$\ell^0_+ \approx +\omega - i\epsilon' \quad \text{(IV quadrant)}, \qquad \ell^0_- \approx -\omega + i\epsilon' \quad \text{(II quadrant)}, \tag{G.23}$$

where $\epsilon' = \epsilon/(2\omega) > 0$. The $+i\epsilon$ prescription shifts the positive pole *below* the real axis and the negative pole *above* it. This means the first and third quadrants of the complex ℓ^0-plane are free of singularities, and we can rotate the integration contour counterclockwise by 90° (from the real axis to the imaginary axis) without crossing any pole. Setting

$$\ell^0 = i\,\ell^0_E, \qquad d\ell^0 = i\,d\ell^0_E, \tag{G.24}$$

we obtain $\ell^2 = (\ell^0)^2 - |\vec{\ell}|^2 = -(\ell^0_E)^2 - |\vec{\ell}|^2 = -\ell^2_E$, where $\ell^2_E \equiv (\ell^0_E)^2 + |\vec{\ell}|^2$ is the *Euclidean* norm. Since $d^d\ell = i\,d^d\ell_E$ (the factor of i from the ℓ^0 rotation, all spatial components unchanged), the integral becomes

$$J_n(\Delta) = i(-1)^n \int \frac{d^d\ell_E}{(2\pi)^d} \frac{1}{(\ell^2_E + \Delta)^n}. \tag{G.25}$$

The crucial sign: the Minkowski denominator $(\ell^2 - \Delta)^n = (-1)^n(\ell_E^2 + \Delta)^n$, and combined with the factor i from $d^d\ell = i\, d^d\ell_E$, we get the overall prefactor $i(-1)^n$.

Step 2: Hyperspherical coordinates. In d-dimensional Euclidean space, introduce the radial variable $r \equiv |\ell_E|$. The integration measure factorizes as

$$d^d\ell_E \;=\; \Omega_d\, r^{d-1}\, dr\,, \tag{G.26}$$

where $\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$ is the total solid angle (area of the unit $(d-1)$-sphere). Quick checks: $\Omega_2 = 2\pi$ (circle), $\Omega_3 = 4\pi$ (sphere), $\Omega_4 = 2\pi^2$.

Derivation of Ω_d. The trick is to evaluate the Gaussian integral $\int_{-\infty}^{+\infty} d^d x\; e^{-\vec{x}^2}$ in two ways. On one hand, since $\vec{x}^2 = x_1^2 + \cdots + x_d^2$, the integral factorizes into d independent one-dimensional Gaussians:

$$\int_{-\infty}^{+\infty} d^d x\; e^{-\vec{x}^2} = \prod_{i=1}^{d} \int_{-\infty}^{+\infty} dx_i\; e^{-x_i^2} = \left(\sqrt{\pi}\right)^d = \pi^{d/2}\,.$$

On the other hand, using hyperspherical coordinates $d^d x = \Omega_d\, r^{d-1}\, dr$ (where the angular integration has already been performed):

$$\int_{-\infty}^{+\infty} d^d x\; e^{-\vec{x}^2} = \Omega_d \int_0^{\infty} dr\; r^{d-1}\, e^{-r^2}\,.$$

The remaining radial integral is evaluated by the substitution $t = r^2$ ($dr = dt/(2\sqrt{t})$):

$$\int_0^{\infty} dr\; r^{d-1}\, e^{-r^2} = \frac{1}{2}\int_0^{\infty} dt\; t^{d/2-1}\, e^{-t} = \frac{1}{2}\,\Gamma\!\left(\frac{d}{2}\right),$$

where we recognized the definition of the Gamma function $\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t}\, dt$. Equating the two results: $\pi^{d/2} = \frac{1}{2}\Omega_d\,\Gamma(d/2)$, hence

$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}\,. \tag{G.27}$$

Step 3: The radial integral. After angular integration, we need

$$\mathcal{R} \;\equiv\; \int_0^{\infty} dr\; \frac{r^{d-1}}{(r^2 + \Delta)^n}\,. \tag{G.28}$$

Substitute $t = r^2$ (so $dr = dt/(2\sqrt{t}\,)$ and $r^{d-1} = t^{(d-1)/2}$):

$$\mathcal{R} \;=\; \frac{1}{2}\int_0^{\infty} dt\; \frac{t^{d/2-1}}{(t + \Delta)^n}\,.$$

Now let $u = t/\Delta$:

$$\mathcal{R} \;=\; \frac{\Delta^{d/2-n}}{2}\int_0^\infty du\,\frac{u^{d/2-1}}{(1+u)^n} \;=\; \frac{\Delta^{d/2-n}}{2}\,B\!\left(\frac{d}{2},\,n-\frac{d}{2}\right),$$

where $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ is the Euler Beta function. This is valid for $\mathrm{Re}(d/2) > 0$ and $\mathrm{Re}(n-d/2) > 0$; the final answer is extended to all d by analytic continuation. Therefore:

$$\mathcal{R} \;=\; \frac{1}{2}\Delta^{d/2-n}\,\frac{\Gamma(d/2)\,\Gamma(n-d/2)}{\Gamma(n)}\,. \tag{G.29}$$

Step 4: Combining everything. Putting together the Wick-rotation prefactor (G.25), the angular integral $\Omega_d/(2\pi)^d$, and the radial result (G.29):

$$J_n(\Delta) \;=\; i(-1)^n\,\frac{\Omega_d}{(2\pi)^d}\,\mathcal{R} \;=\; i(-1)^n\,\frac{2\pi^{d/2}}{\Gamma(d/2)\,(2\pi)^d}\cdot\frac{1}{2}\,\Delta^{d/2-n}\,\frac{\Gamma(d/2)\,\Gamma(n-d/2)}{\Gamma(n)}\,.$$

The factors of $\Gamma(d/2)$ cancel, and $2\pi^{d/2}/[(2\pi)^d\cdot 2] = 1/[2(4\pi)^{d/2}]\cdot 2\cdot 1/2 = 1/(4\pi)^{d/2}$, giving the master formula:

$$\boxed{J_n(\Delta) \;=\; \frac{i(-1)^n}{(4\pi)^{d/2}}\,\frac{\Gamma\!\left(n-\frac{d}{2}\right)}{\Gamma(n)}\,(\Delta)^{\frac{d}{2}-n}, \qquad \mathrm{Re}(\Delta) > 0\,.} \tag{G.30}$$

The derivation assumed $\mathrm{Re}(n-d/2) > 0$ for UV convergence of the radial integral. For the physically relevant case $d = 4-2\epsilon$ and $n = 2$, this requires $\mathrm{Re}(\epsilon) > 0$—satisfied for any $\epsilon > 0$, no matter how small. The crucial step of **dimensional regularization** is now the following: the right-hand side of (G.30) is a closed-form expression built from Gamma functions and powers of Δ. These are well-defined analytic functions of the *continuous* variable d (or equivalently ϵ), even for values of d where the original integral diverges.

This is the idea of **analytic continuation**: we derived the formula in a region of d where the integral converges, but the *result*—as a function of d—has a natural extension to all $d \in \mathbb{C}$. We *define* the dimensionally regularized integral to be this analytic continuation. The UV divergence of the original $d = 4$ integral now manifests itself as a *pole* of the Gamma function $\Gamma(n-d/2)$ at $n-d/2 = 0,-1,-2,\ldots$ (cf. Appendix F). For example, J_2 at $d = 4$ has $n-d/2 = 0$, producing $\Gamma(\epsilon) = 1/\epsilon - \gamma_E + O(\epsilon)$: the $1/\epsilon$ pole is the regularized version of the logarithmic divergence $\ln\Lambda$.

The power of this approach is that intermediate expressions remain finite for $\epsilon > 0$, standard algebraic manipulations (shifts, symmetry arguments, Dirac algebra) remain valid in d dimensions, and divergences appear only at the very end as isolated $1/\epsilon$ poles that can be systematically cancelled by counterterms.

Special cases: J_1, J_2, J_3

Tadpole: $J_1(\Delta)$. Setting $n = 1$ in (G.30) (with $\Gamma(1) = 1$):

$$J_1(\Delta) \;=\; \frac{-i}{(4\pi)^{d/2}}\,\Gamma\!\left(1-\frac{d}{2}\right)\Delta^{d/2-1}. \tag{G.31}$$

This integral appears in the tadpole diagram of $\lambda\phi^4$ theory. In $d = 4-2\epsilon$, the Gamma function has a pole: $\Gamma(1-d/2) = \Gamma(-1+\epsilon) = -1/\epsilon + (\gamma_E - 1) + O(\epsilon)$, reflecting the quadratic UV divergence (in cutoff language) of the tadpole.

Bubble: $J_2(\Delta)$. Setting $n = 2$ (with $\Gamma(2) = 1$):

$$J_2(\Delta) \;=\; \frac{i}{(4\pi)^{d/2}}\,\Gamma\!\left(2-\frac{d}{2}\right)\Delta^{d/2-2}. \tag{G.32}$$

This is the integral that appears after Feynman parametrization of any two-propagator (bubble) diagram (cf. Eq. (G.9)).

Triangle: $J_3(\Delta)$. Setting $n = 3$ (with $\Gamma(3) = 2$):

$$J_3(\Delta) \;=\; -\frac{i}{2(4\pi)^{d/2}}\,\Gamma\!\left(3-\frac{d}{2}\right)\Delta^{d/2-3}. \tag{G.33}$$

For reference we also record the compact form:

$$\begin{aligned} J_2(\Delta) &= \frac{i}{(4\pi)^{d/2}}\,\Gamma\!\left(2-\frac{d}{2}\right)\Delta^{\frac{d}{2}-2}, \\ J_3(\Delta) &= -\frac{i}{2(4\pi)^{d/2}}\,\Gamma\!\left(3-\frac{d}{2}\right)\Delta^{\frac{d}{2}-3}. \end{aligned} \tag{G.34}$$

Numerators with ℓ^2

Use the algebraic identity

$$\ell^2 \;=\; (\ell^2-\Delta)+\Delta, \tag{G.35}$$

to reduce

$$\begin{aligned} \int \frac{d^d\ell}{(2\pi)^d}\,\frac{\ell^2}{(\ell^2-\Delta+i\epsilon)^n} &= \int \frac{d^d\ell}{(2\pi)^d}\,\frac{1}{(\ell^2-\Delta+i\epsilon)^{n-1}} + \Delta\int \frac{d^d\ell}{(2\pi)^d}\,\frac{1}{(\ell^2-\Delta+i\epsilon)^n} \\ &= J_{n-1}(\Delta) + \Delta\, J_n(\Delta)\,. \end{aligned} \tag{G.36}$$

For example, with $n = 3$: $\int \frac{d^d\ell}{(2\pi)^d}\frac{\ell^2}{(\ell^2-\Delta+i\epsilon)^3} = J_2(\Delta) + \Delta J_3(\Delta)$. Combining (G.21) and (G.36), any rank-2 tensor integral is expressible in terms of J_{n-1} and J_n.

UV structure: expansion around $d = 4$

Set $d = 4 - 2\epsilon$ and introduce a renormalization scale μ so that the coupling constant remains dimensionless. Define the standard combination that collects the $\overline{\text{MS}}$ terms:

$$\Delta_{\text{UV}} \equiv \frac{1}{\epsilon} - \gamma_E + \ln(4\pi) \,. \tag{G.37}$$

Expansion of J_1 (tadpole). Using $\Gamma(-1+\epsilon) = -1/\epsilon + (\gamma_E - 1) + O(\epsilon)$ and $\Delta^{1-\epsilon} = \Delta\,[1 - \epsilon \ln\Delta + O(\epsilon^2)]$:

$$J_1(\Delta) \;=\; \frac{i}{(4\pi)^2}\,\Delta\left[\Delta_{\text{UV}} + 1 - \ln\!\left(\frac{\Delta}{\mu^2}\right) + O(\epsilon)\right] . \tag{G.38}$$

The $1/\epsilon$ pole signals a quadratic UV divergence (in cutoff language, $J_1 \sim \Lambda^2$). This is the divergence encountered in the $\lambda\phi^4$ tadpole.

Expansion of J_2 (bubble). Using $\Gamma(\epsilon) = 1/\epsilon - \gamma_E + O(\epsilon)$ and $\Delta^{-\epsilon} = 1 - \epsilon\ln\Delta + O(\epsilon^2)$:

$$J_2(\Delta) \;=\; \frac{i}{(4\pi)^2}\left[\Delta_{\text{UV}} - \ln\!\left(\frac{\Delta}{\mu^2}\right) + O(\epsilon)\right] . \tag{G.39}$$

The $1/\epsilon$ pole corresponds to a logarithmic UV divergence ($\sim \ln\Lambda$ in cutoff language). This is the divergence that appears in one-loop bubble-type integrals: the $\lambda\phi^4$ four-point (vertex) correction, the QED vacuum polarization, and the electron self-energy.

Expansion of J_3 (triangle). Since $\Gamma(1+\epsilon) = 1 + O(\epsilon)$:

$$J_3(\Delta) \;=\; -\frac{i}{2(4\pi)^2}\,\frac{1}{\Delta}\,[1 + O(\epsilon)] \,. \tag{G.40}$$

This is UV-finite: no $1/\epsilon$ pole. Triangle integrals (and higher-point functions) are UV-convergent at one loop in four dimensions.

General pattern. The integral $J_n(\Delta)$ is UV-divergent when $n \le d/2$, i.e. when $\Gamma(n - d/2)$ has a pole. In $d = 4$ dimensions:

- $n = 1$ (tadpole): divergent (quadratic),
- $n = 2$ (bubble): divergent (logarithmic),
- $n \ge 3$ (triangle, box, . . .): UV-finite.

Box G.1 Summary of one-loop master integrals

Integral	Exact in d dimensions	Expansion ($d = 4 - 2\epsilon$)
$J_1(\Delta)$	$\dfrac{-i}{(4\pi)^{d/2}}\,\Gamma\!\left(1-\dfrac{d}{2}\right)\Delta^{d/2-1}$	$\dfrac{i\Delta}{(4\pi)^2}\left[\Delta_{\rm UV}+1-\ln\dfrac{\Delta}{\mu^2}\right]$
$J_2(\Delta)$	$\dfrac{i}{(4\pi)^{d/2}}\,\Gamma\!\left(2-\dfrac{d}{2}\right)\Delta^{d/2-2}$	$\dfrac{i}{(4\pi)^2}\left[\Delta_{\rm UV}-\ln\dfrac{\Delta}{\mu^2}\right]$
$J_3(\Delta)$	$-\dfrac{i}{2(4\pi)^{d/2}}\,\Gamma\!\left(3-\dfrac{d}{2}\right)\Delta^{d/2-3}$	$-\dfrac{i}{2(4\pi)^2}\,\dfrac{1}{\Delta}$

$\Delta_{\rm UV} \equiv 1/\epsilon - \gamma_E + \ln(4\pi)$. All ϵ-expansions are to leading order; $+O(\epsilon)$ terms are suppressed.

Putting it together: the parameter integral

After completing the Dirac algebra (or other numerator algebra) and applying the tensor-reduction identities above, every one-loop diagram reduces to an integral over Feynman parameters of J_n functions. For a two-propagator diagram (bubble), the integration is over a single Feynman parameter $x \in [0,1]$. For a three-propagator diagram (triangle), the integration is over the triangular region

$$0 \le x \le 1, \qquad 0 \le y \le 1-x, \qquad z = 1-x-y\,. \tag{G.41}$$

Schematically, a typical one-loop contribution (with n propagators combined into J_n) takes the form

$$\text{(one-loop diagram)} \;\sim\; \int [dx_i]\,\Big[\mathcal{A}(x_i)\,J_{n-1}(\Delta) + \mathcal{B}(x_i)\,J_n(\Delta)\Big], \tag{G.42}$$

where $\int [dx_i]$ denotes the Feynman-parameter integral over the appropriate region ($0 \le x \le 1$ for a bubble, the triangle $0 \le x \le 1$, $0 \le y \le 1-x$ for a vertex, etc.), and $\mathcal{A}$, $\mathcal{B}$ are rational functions of the Feynman parameters and external kinematic invariants that come from the numerator algebra. The UV divergence is carried entirely by the lower-n master integral (e.g. J_2 for a triangle diagram, J_1 for a bubble), while the higher-n piece is UV-finite.

H Optical theorem and Cutkosky cutting rules

In Chapter 8 we showed, on general grounds (unitarity of S), that the imaginary part of the exact propagator Δ'_F encodes the multi-particle states into which the field can decay. The corresponding statement at the level of forward scattering amplitudes is the *optical theorem* (8.176). This appendix establishes the perturbative, diagrammatic translation of these results: the **Cutkosky cutting rules**, which allow one to compute $\mathrm{Im}\,\Pi(p^2)$ — and hence the spectral density $\rho(p^2)$ — by "cutting" Feynman diagrams along sets of internal propagators that go simultaneously on shell. We motivate, state, and apply the rules to two concrete examples in scalar field theory: the one-loop bubble and the two-loop sunset.

Motivation: where does $\mathrm{Im}\,\Pi$ come from?

The self-energy $\Pi(p^2)$ is, order by order in perturbation theory, a sum of 1PI Feynman diagrams with two external legs (Fig. H.1). Each such diagram is, individually, an integral over loop momenta of products of Feynman propagators

$$\frac{i}{k^2 - m^2 + i\epsilon}, \tag{H.1}$$

with an overall factor of i from the Feynman rules. At first glance this is a *real* integral with a small imaginary regulator ϵ: one might wonder how $\Pi(p^2)$ ever acquires a finite imaginary part as $\epsilon \to 0^+$.

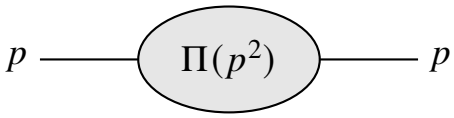


Figure H.1. Diagrammatic representation of the 1PI self-energy $\Pi(p^2)$: sum of all 1PI two-point graphs.

The answer lies in the $+i\epsilon$ prescription itself. By Sokhotski–Plemelj (8.169),

$$\frac{1}{k^2 - m^2 + i\epsilon} = \mathcal{P}\frac{1}{k^2 - m^2} - i\pi\,\delta(k^2 - m^2), \tag{H.2}$$

so each propagator carries a built-in δ-function contribution localized on its mass shell. When the external momentum p is large enough that several internal

propagators can be *simultaneously* on shell with positive energies, the corresponding δ-functions produce an honest, finite imaginary part of the diagram. The set of internal lines that go on shell defines a *cut* of the diagram, and identifies the physical multi-particle intermediate state $|X\rangle$ being produced.

The Cutkosky rules turn this observation into a precise prescription.

Statement of the Cutkosky cutting rules

Let $\mathcal{D}$ be a 1PI two-point Feynman diagram contributing to $\Pi(p^2)$. To compute its imaginary part one proceeds as follows.

1. **Enumerate the cuts.** Identify every way of cutting $\mathcal{D}$ into two connected pieces by slicing through a set of internal propagators $\{k_1, \dots, k_n\}$ such that the cut propagators can be simultaneously put on shell with positive energies summing to $\sqrt{p^2}$. Such a cut is *kinematically allowed* only if

$$p^2 \;\geq\; \big(m_1 + m_2 + \cdots + m_n\big)^2 , \tag{H.3}$$

   where m_i is the mass of the i-th cut line. Equivalently: each cut is in one-to-one correspondence with a physical multi-particle intermediate state $|X\rangle = |k_1, \dots, k_n\rangle$ whose constituents are exactly the particles flowing through the cut lines.
2. **Apply the cut substitution.** For each cut, replace each cut propagator by its on-shell mass-shell projector

$$\frac{i}{k^2 - m^2 + i\epsilon} \;\longrightarrow\; 2\pi\,\delta(k^2 - m^2)\,\theta(k^0) , \tag{H.4}$$

   enforcing that the cut particle is real, on shell, and propagating with positive energy.
3. **Complex-conjugate one side of the cut.** On one side of the cut — conventionally, the right-hand side — replace every uncut propagator and every coupling by its complex conjugate (in practice, flip the signs of all $+i\epsilon$ prescriptions and conjugate all complex couplings).
4. **Sum.** The imaginary part is obtained by summing the contributions of all kinematically allowed cuts of all 1PI diagrams at the given perturbative order:

$$2\,\mathrm{Im}\,\Pi(p^2) \;=\; \sum_{\text{cuts}} \big(\text{cut diagram}\big) . \tag{H.5}$$

This is the diagrammatic incarnation of the optical theorem (8.176): cutting the self-energy along a set of internal lines is the perturbative implementation of

inserting a complete set of states $\sum_X |X\rangle\langle X|$ on the unitarity cut. The two "halves" of the cut diagram are precisely the two factors $\langle 0|\varphi(0)|X\rangle^*$ and $\langle X|\varphi(0)|0\rangle$ that build up the spectral density $\rho(p^2)$ in (8.130).

Why the $i\epsilon$ prescription matters: a one-line derivation

The mathematical heuristic underlying the rules is short. Consider a diagram contributing to $\Pi(p^2)$ with loop momentum k and a single propagator $i/(k^2 - m^2 + i\epsilon)$. Using (H.2), the diagram splits into a real principal-value part and an imaginary δ-function part. For a single propagator there is no kinematical constraint that forces the δ to be saturated, and the imaginary piece vanishes after the loop integration. But if the diagram contains *two or more* propagators that can be put on shell simultaneously — meaning the energy-momentum constraints of the diagram admit a solution where each of those propagators sits at its pole — then their δ-functions reinforce, and a finite imaginary part survives. The condition that the constraints admit such a solution is precisely the kinematical threshold $p^2 \geq (\sum m_i)^2$ in Step 1.

Example 1: the bubble diagram (one-loop self-energy)

Consider the simplest one-loop self-energy in a scalar theory with cubic coupling $g\varphi^2\chi$, where χ is a second scalar of mass μ and φ has mass m. The one-loop φ self-energy contains the *bubble* diagram with two internal χ propagators, Fig. H.2.

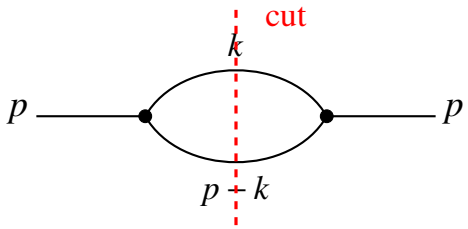


Figure H.2. One-loop bubble contribution to the φ self-energy with two internal χ propagators of mass μ. The dashed red line shows the (unique) Cutkosky cut, which sends both internal lines on shell.

The diagram contributes to $\Pi(p^2)$ as

$$-i\Pi_{\text{bubble}}(p^2) \;=\; (-ig)^2 \int \frac{d^4k}{(2\pi)^4}\, \frac{i}{k^2 - \mu^2 + i\epsilon}\, \frac{i}{(p-k)^2 - \mu^2 + i\epsilon}\,. \tag{H.6}$$

Cuts. The only kinematically allowed cut goes through both internal lines simultaneously. By Step 2,

$$\frac{i}{k^2-\mu^2+i\epsilon} \longrightarrow 2\pi\,\delta(k^2-\mu^2)\,\theta(k^0)\,, \tag{H.7}$$

$$\frac{i}{(p-k)^2-\mu^2+i\epsilon} \longrightarrow 2\pi\,\delta((p-k)^2-\mu^2)\,\theta(p^0-k^0)\,. \tag{H.8}$$

The cut diagram becomes (after also flipping the sign of $i\epsilon$ on the right of the cut, which has no propagators here),

$$2\,\mathrm{Im}\,\Pi_{\text{bubble}}(p^2) \;=\; g^2\int\frac{d^4k}{(2\pi)^2}\,\delta(k^2-\mu^2)\theta(k^0)\,\delta((p-k)^2-\mu^2)\theta(p^0-k^0)\,. \tag{H.9}$$

The two δ-functions are precisely the Lorentz-invariant two-body phase space $d\Pi_2$ for a final state of two on-shell χ's, in agreement with the optical theorem (8.176). *Kinematical threshold.* The cut is allowed only if $\sqrt{p^2}\geq 2\mu$, i.e. $p^2\geq(2\mu)^2=4\mu^2$. Hence $\mathrm{Im}\,\Pi_{\text{bubble}}(p^2)$, and therefore the bubble's contribution to $\rho(p^2)$, switches on at the two-χ production threshold and is identically zero for $p^2<4\mu^2$. This is precisely the multi-particle threshold of the two-χ channel in the spectral density.

Example 2: the sunset diagram (two-loop self-energy in φ^4)

In a $\mathbb{Z}_2$-symmetric φ^4 theory, $\langle 0|\varphi(0)|n\rangle$ vanishes for any state with an even number of particles by parity. The lowest non-trivial multi-particle channel is therefore the three-particle channel, with threshold $3m$. The corresponding 1PI two-point diagram is the two-loop *sunset* (or "sunrise") diagram, Fig. H.3.

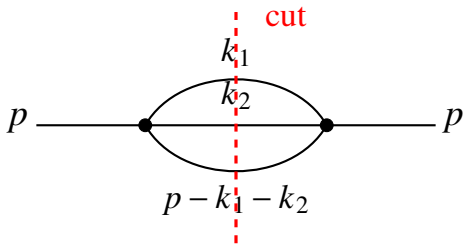


Figure H.3. Two-loop sunset contribution to the φ self-energy in φ^4 theory, with three internal φ propagators. The unique Cutkosky cut goes through all three internal lines, producing a three-particle on-shell intermediate state.

Cuts. The only kinematically allowed cut slices through all three internal φ lines simultaneously. By Step 2, each φ propagator is replaced by its on-shell mass-shell

projector $2\pi\,\delta(k_i^2 - m^2)\theta(k_i^0)$, producing the Lorentz-invariant three-body phase space.

Kinematical threshold. The cut requires $\sqrt{p^2} \geq 3m$, i.e. $p^2 \geq 9m^2$. Hence the sunset diagram contributes to $\mathrm{Im}\,\Pi(p^2)$ and to $\rho_{\mathrm{cont.}}(\sigma^2)$ only for $\sigma^2 \geq 9m^2$, opening the three-particle threshold that is the lowest continuum threshold of the spectral density in this theory — exactly as predicted by the non-perturbative analysis of Chapter 8.

Connection to the spectral density

Putting the pieces together, the diagrammatic procedure can be summarized as:

1. Compute the 1PI self-energy $\Pi(p^2)$ in perturbation theory to the desired order.
2. Take its imaginary part using the Cutkosky cutting rules.
3. Insert into the relation

$$\mathrm{Im}\,\Delta'_F(p^2) \;=\; \frac{\mathrm{Im}\,\Pi(p^2)}{\left|p^2 - m_0^2 - \Pi(p^2)\right|^2}\,, \tag{H.10}$$

   obtained from the Dyson form (8.155), and apply the inversion formula (8.173) to recover the spectral density $\rho(p^2)$.

The result is, by construction, consistent with the non-perturbative spectral structure: each kinematically allowed Cutkosky cut of a self-energy diagram corresponds to a multi-particle on-shell intermediate state, and contributes to $\rho_{\mathrm{cont.}}(\sigma^2)$ only above the threshold $(\sum_i m_i)^2$ of that channel. Higher-order perturbative diagrams open higher-particle thresholds; non-perturbative effects (such as bound states) reveal themselves as additional isolated poles below the continuum, beyond the reach of any finite perturbative resummation.

I Derivation of the Gell-Mann–Low formula

The goal of this appendix is to derive, step by step and pedagogically, the master identity that connects the *Heisenberg-picture* Green's function of the full interacting theory to a vacuum expectation value of *interaction-picture* fields, which is the object that perturbation theory and Wick's theorem actually compute:

$$\boxed{G(x_1,\dots,x_n) \;=\; \langle\Omega|T\big(\varphi(x_1)\cdots\varphi(x_n)\big)|\Omega\rangle \;=\; \frac{\langle 0|T\big\{S\,\varphi_I(x_1)\cdots\varphi_I(x_n)\big\}|0\rangle}{\langle 0|S|0\rangle}} \tag{I.1}$$

This is the *Gell-Mann–Low formula* (sometimes called the Gell-Mann–Low theorem). It is the bridge between the LSZ machinery and the diagrammatic expansion. We follow Peskin & Schroeder, Ch. 4.2–4.3 [6]; the equivalent derivation in Mandl & Shaw, Ch. 6 [1] adopts the same logic with slightly different notation.

Setup: three pictures and the reference time t_0

Recall the three standard pictures of quantum mechanics:

- **Schrödinger picture**: states $|\psi_S(t)\rangle$ carry the time dependence, operators O_S do not. Evolution: $i\partial_t|\psi_S\rangle = H|\psi_S\rangle$.
- **Heisenberg picture**: states $|\psi_H\rangle$ are time-independent, operators $O_H(t) = e^{iH(t-t_0)}O_S e^{-iH(t-t_0)}$ carry the full evolution under the *interacting* Hamiltonian $H = H_0 + H_{\text{int}}$.
- **Interaction picture**: operators $O_I(t) = e^{iH_0(t-t_0)}O_S e^{-iH_0(t-t_0)}$ evolve only with the free Hamiltonian H_0; the residual time evolution is carried by the states, $|\psi_I(t)\rangle = e^{iH_0(t-t_0)}|\psi_S(t)\rangle$.

The three pictures are defined to coincide at one reference time t_0:

$$\varphi_H(t_0,\vec{x}) \;=\; \varphi_I(t_0,\vec{x}) \;=\; \varphi_S(\vec{x})\,. \tag{I.2}$$

This identification at t_0 is the linchpin of the whole construction. We will eventually take t_0 to be arbitrary and the times $\pm T$ at the boundaries to $\pm\infty(1-i\epsilon)$.

The interacting vacuum $|\Omega\rangle$ (ground state of H) is, by definition, time-independent in the Heisenberg picture: $H|\Omega\rangle = E_\Omega|\Omega\rangle$. The free vacuum $|0\rangle$ is the ground state of H_0: $H_0|0\rangle = 0$ (we choose the zero-point energy so that $E_0^{(0)} = 0$).

Step 1: Relating φ_H and φ_I via the operator $U(t,t_0)$

Combining the definitions above,

$$\begin{aligned}\varphi_H(t,\vec{x}) &= e^{iH(t-t_0)}\,\varphi_S(\vec{x})\,e^{-iH(t-t_0)}\\ &= e^{iH(t-t_0)}\,e^{-iH_0(t-t_0)}\,\varphi_I(t,\vec{x})\,e^{iH_0(t-t_0)}\,e^{-iH(t-t_0)}\\ &= U^\dagger(t,t_0)\,\varphi_I(t,\vec{x})\,U(t,t_0)\,,\end{aligned} \tag{I.3}$$

where we have *defined*

$$U(t,t_0) \;\equiv\; e^{iH_0(t-t_0)}\,e^{-iH(t-t_0)}\,. \tag{I.4}$$

$U(t,t_0)$ is the *interaction-picture time-evolution operator*: it advances a state from t_0 to t in the interaction picture.

Key properties of $U(t,t')$.

1. $U(t_0,t_0)=1$ (trivially from (I.4)).
2. $U(t,t_0)$ satisfies the Schrödinger-like equation

$$i\,\partial_t\,U(t,t_0) \;=\; H_I(t)\,U(t,t_0)\,, \tag{I.5}$$

where $H_I(t)\equiv e^{iH_0(t-t_0)}\,H_{\rm int}\,e^{-iH_0(t-t_0)}$ is the interaction Hamiltonian in the interaction picture. Proof: differentiate (I.4) directly. The result is solved by the **Dyson series**:

$$U(t,t_0) \;=\; T\exp\!\left[-i\int_{t_0}^{t} dt'\,H_I(t')\right]. \tag{I.6}$$

3. *Composition law.* For three times t_1,t_2,t_3,

$$U(t_1,t_2)\,U(t_2,t_3) \;=\; U(t_1,t_3)\,, \tag{I.7}$$

where the general two-argument U is

$$U(t,t') \;\equiv\; e^{iH_0(t-t_0)}\,e^{-iH(t-t')}\,e^{-iH_0(t'-t_0)}\,. \tag{I.8}$$

One verifies that this reduces to (I.4) when $t'=t_0$, satisfies the same equation of motion (I.5), and obeys $U^\dagger(t,t')=U(t',t)$.

Iterating (I.7) we obtain the useful identity that will be crucial in the final step:

$$\begin{aligned}U(t_1,t_0)\,U^\dagger(t_2,t_0) &= U(t_1,t_0)\,U(t_0,t_2)\\ &= U(t_1,t_2)\,.\end{aligned} \tag{I.9}$$

Step 2: The interacting vacuum from the free vacuum (adiabatic trick)

We now construct $|\Omega\rangle$ starting from $|0\rangle$. The idea — which goes back to Gell-Mann and Low — is to project onto the ground state of H by "cooling" through imaginary time.

Expand the free vacuum on the basis of eigenstates of the full Hamiltonian, $H|n\rangle = E_n|n\rangle$ (with $|\Omega\rangle$ the unique ground state, $E_\Omega < E_n$ for $n \neq \Omega$):

$$|0\rangle = |\Omega\rangle\langle\Omega|0\rangle + \sum_{n\neq\Omega} |n\rangle\langle n|0\rangle\,. \tag{I.10}$$

Apply e^{-iHT} for some large $T > 0$:

$$e^{-iHT}|0\rangle = e^{-iE_\Omega T}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n\neq\Omega} e^{-iE_n T}|n\rangle\langle n|0\rangle\,. \tag{I.11}$$

On the real axis this is just a sum of oscillating phases. The trick is to rotate T slightly into the lower half plane, $T \to T(1-i\epsilon)$ with $\epsilon \to 0^+$ at the end. Then

$$e^{-iE_n T(1-i\epsilon)} = e^{-iE_n T}\, e^{-E_n T\epsilon}\,, \tag{I.12}$$

and, after factoring out the ground-state contribution, an excited state is suppressed relative to $|\Omega\rangle$ by

$$\exp\!\big[-(E_n - E_\Omega)T\epsilon\big]\,. \tag{I.13}$$

Since $E_n > E_\Omega$ for every excited state, all excited-state components disappear in the limit $T \to \infty$. We get

$$|\Omega\rangle \;=\; \lim_{T\to\infty(1-i\epsilon)} \frac{e^{-iHT}|0\rangle}{e^{-iE_\Omega T}\,\langle\Omega|0\rangle}\,. \tag{I.14}$$

This is the central "$i\epsilon$ rotation" identity. The numerator alone does not converge to $|\Omega\rangle$: the divergent phase factor $e^{-iE_\Omega T}$ and the overlap $\langle\Omega|0\rangle$ are subtracted explicitly so that the ratio is finite and unit-normalised.

Recasting in terms of U. We rewrite the right-hand side as the action of an interaction-picture evolution operator on the free vacuum. Replace $T \to t_0 + T$ in (I.14) (a harmless shift):

$$|\Omega\rangle \;=\; \lim_{T\to\infty(1-i\epsilon)} \frac{e^{-iH(t_0+T)}|0\rangle}{e^{-iE_\Omega(t_0+T)}\,\langle\Omega|0\rangle}\,. \tag{I.15}$$

Since $H_0|0\rangle = 0$, we may freely insert $e^{-iH_0(-T-t_0)}|0\rangle = |0\rangle$ on the right of the numerator:

$$\begin{aligned} e^{-iH(t_0+T)}|0\rangle &= e^{-iH(t_0-(-T))}\, e^{-iH_0(-T-t_0)}|0\rangle \\ &= U(t_0,-T)\,|0\rangle\,, \end{aligned} \tag{I.16}$$

where we used (I.8). Therefore

$$\boxed{|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{U(t_0,-T)\,|0\rangle}{e^{-iE_\Omega(t_0+T)}\,\langle\Omega|0\rangle}} \tag{I.17}$$

By an analogous argument applied to the bra,

$$\boxed{\langle\Omega| = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|\,U(T,t_0)}{e^{-iE_\Omega(T-t_0)}\,\langle 0|\Omega\rangle}} \tag{I.18}$$

Equations (I.17)–(I.18) are the key result of this step: they express the interacting vacuum entirely in terms of objects living in the interaction picture, plus c-number normalisation factors that will cancel in the next step.

Step 3: Putting it together for the n-point function

Consider the Heisenberg correlator

$$G(x_1,\dots,x_n) = \langle\Omega|T\big(\varphi_H(x_1)\cdots\varphi_H(x_n)\big)|\Omega\rangle\,. \tag{I.19}$$

Without loss of generality (the time-ordering symbol takes care of permutations) assume the points are already time-ordered, $x_1^0 > x_2^0 > \cdots > x_n^0$. Drop the T symbol while we manipulate the operators, and use (I.3) on each field:

$$\begin{aligned} &\langle\Omega|\varphi_H(x_1)\varphi_H(x_2)\cdots\varphi_H(x_n)|\Omega\rangle \\ &= \langle\Omega|\,U^\dagger(x_1^0,t_0)\,\varphi_I(x_1)\,U(x_1^0,t_0) \\ &\qquad \times U^\dagger(x_2^0,t_0)\,\varphi_I(x_2)\,U(x_2^0,t_0) \\ &\qquad \times\cdots \\ &\qquad \times U^\dagger(x_n^0,t_0)\,\varphi_I(x_n)\,U(x_n^0,t_0)\,|\Omega\rangle\,. \end{aligned} \tag{I.20}$$

Now collapse the U-pairs using the chain identity (I.9), $U(t_i^0,t_0)\,U^\dagger(t_{i+1}^0,t_0) = U(t_i^0,t_{i+1}^0)$:

$$\begin{aligned} &= \langle\Omega|\,U^\dagger(x_1^0,t_0) \\ &\qquad \times\varphi_I(x_1)\,U(x_1^0,x_2^0)\,\varphi_I(x_2)\,U(x_2^0,x_3^0)\cdots U(x_{n-1}^0,x_n^0)\,\varphi_I(x_n) \\ &\qquad \times U(x_n^0,t_0)\,|\Omega\rangle\,. \end{aligned} \tag{I.21}$$

The structure is now a chain of fields and U-operators sandwiched between $\langle\Omega|U^\dagger(x_1^0,t_0)$ on the left and $U(x_n^0,t_0)|\Omega\rangle$ on the right. Both extremes are absorbed by the vacuum identities of Step 2:

- Right end: from (I.17) and (I.9),

$$\begin{aligned} U(x_n^0,t_0)|\Omega\rangle &= \lim_{T\to\infty(1-i\epsilon)} \frac{U(x_n^0,t_0)\,U(t_0,-T)\,|0\rangle}{e^{-iE_\Omega(t_0+T)}\,\langle\Omega|0\rangle} \\ &= \lim \frac{U(x_n^0,-T)|0\rangle}{e^{-iE_\Omega(t_0+T)}\,\langle\Omega|0\rangle}\,. \end{aligned} \tag{I.22}$$

- Left end: similarly, from (I.18) and (I.9), using $U^\dagger(x_1^0,t_0)=U(t_0,x_1^0)$,

$$\begin{aligned} \langle\Omega|U^\dagger(x_1^0,t_0) &= \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|\,U(T,t_0)\,U(t_0,x_1^0)}{e^{-iE_\Omega(T-t_0)}\,\langle 0|\Omega\rangle} \\ &= \lim \frac{\langle 0|U(T,x_1^0)}{e^{-iE_\Omega(T-t_0)}\,\langle 0|\Omega\rangle}\,. \end{aligned} \tag{I.23}$$

Substituting both into (I.21), the numerator becomes

$$\langle 0|\,U(T,x_1^0)\,\varphi_I(x_1)\,U(x_1^0,x_2^0)\,\varphi_I(x_2)\cdots U(x_{n-1}^0,x_n^0)\,\varphi_I(x_n)\,U(x_n^0,-T)\,|0\rangle\,. \tag{I.24}$$

The denominator carries the c-number $e^{-iE_\Omega\cdot 2T}\,\langle\Omega|0\rangle\langle 0|\Omega\rangle$, which we will deal with below.

Recognising the time-ordered S-operator. With times satisfying $T>x_1^0>x_2^0>\cdots>x_n^0>-T$, every operator in the string displayed in (I.24) is already in chronological order from left (latest) to right (earliest). The time-ordering symbol therefore preserves the string, and we can rewrite it as

$$\begin{aligned} \langle 0|\,T\big\{&\varphi_I(x_1)\cdots\varphi_I(x_n)\,U(T,x_1^0)U(x_1^0,x_2^0) \\ &\cdots U(x_n^0,-T)\big\}|0\rangle\,. \end{aligned} \tag{I.25}$$

Inside the T-symbol, the product of all the U's telescopes (composition law (I.7)) into a single one spanning the full range $[-T,T]$:

$$U(T,x_1^0)\,U(x_1^0,x_2^0)\cdots U(x_n^0,-T) \;=\; U(T,-T)\,. \tag{I.26}$$

Taking the limit $T\to\infty(1-i\epsilon)$ we define the *S-matrix operator in the interaction picture*:

$$S \;\equiv\; \lim_{T\to\infty(1-i\epsilon)} U(T,-T) \;=\; T\exp\!\left[-i\int_{-\infty}^{+\infty} dt\,H_I(t)\right]. \tag{I.27}$$

The numerator of (I.20), multiplied by its diverging prefactor, is therefore

$$\lim_{T\to\infty(1-i\epsilon)} \langle 0|T\{\varphi_I(x_1)\cdots\varphi_I(x_n)\,U(T,-T)\}\,|0\rangle = \langle 0|T\big\{\varphi_I(x_1)\cdots\varphi_I(x_n)\,S\big\}|0\rangle\,. \tag{I.28}$$

Step 4: The denominator and the cancellation of vacuum divergences

Going back to the full ratio: combining the Heisenberg bra and ket with their normalisation factors, we collected the c-number

$$\mathcal{N} \equiv e^{-iE_\Omega \cdot 2T} \langle \Omega|0\rangle \langle 0|\Omega\rangle \tag{I.29}$$

in the denominator. We now show that $\mathcal{N}$ is exactly cancelled by the vacuum-to-vacuum matrix element of S.

Apply the very same construction (I.17)–(I.18) to the trivial correlator $\langle\Omega|\Omega\rangle$ (with no field insertions):

$$1 = \langle\Omega|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|U(T,-T)|0\rangle}{\mathcal{N}} = \frac{\langle 0|S|0\rangle}{\mathcal{N}}. \tag{I.30}$$

Therefore $\mathcal{N} = \langle 0|S|0\rangle$. The diverging normalisation in the denominator of G is nothing but the vacuum-to-vacuum S-matrix element — which itself is a non-trivial divergent quantity (the sum of all disconnected "vacuum bubbles"), but it is exactly the factor needed to cancel them in the numerator.

Putting everything together: the Gell-Mann–Low formula

Combining (I.28) with $\mathcal{N} = \langle 0|S|0\rangle$ from (I.30):

$$\boxed{\begin{aligned} G(x_1,\dots,x_n) &= \langle\Omega|T\big(\varphi_H(x_1)\cdots\varphi_H(x_n)\big)|\Omega\rangle \\ &= \frac{\langle 0|T\big\{\varphi_I(x_1)\cdots\varphi_I(x_n)\,S\big\}|0\rangle}{\langle 0|S|0\rangle}. \end{aligned}} \tag{I.31}$$

This is exactly equation (I.1), the formula used in the main text.

Physical and computational reading

The Gell-Mann–Low formula has both a computational and a conceptual content.

Computational content. The right-hand side is something the perturbative machinery handles directly:

- Expand the interaction-picture S-operator:

$$S = T\exp\left[-i\int H_I\,dt\right] = \sum_{k=0}^{\infty} \frac{(-i)^k}{k!} \int dt_1\cdots dt_k\, T\{H_I(t_1)\cdots H_I(t_k)\}.$$

- Each term is a vacuum expectation of a time-ordered product of *free* (interaction-picture) fields, evaluated on the *free* vacuum $|0\rangle$.
- Wick's theorem reduces it to sums of products of free Feynman propagators Δ_F — i.e. to Feynman diagrams.

Conceptual content: cancellation of vacuum bubbles. The denominator $\langle 0|S|0\rangle$ does crucial bookkeeping. Order by order, the numerator $\langle 0|T\{\varphi_I \cdots \varphi_I S\}|0\rangle$ is a sum of two kinds of diagrams:

- diagrams with no vacuum bubbles, in which every connected component contains at least one external field $\varphi_I(x_i)$;
- disconnected vacuum pieces, in which an interaction sub-graph involves no external $\varphi_I(x_i)$ and just produces a "vacuum bubble".

The vacuum bubbles factorise out and give exactly the expansion of $\langle 0|S|0\rangle$:

$$\langle 0|T\{\varphi_I(x_1)\cdots\varphi_I(x_n)\,S\}|0\rangle \;=\; \Big[\sum(\text{connected w.r.t. at least one } x_i)\Big] \times \langle 0|S|0\rangle\,. \tag{I.32}$$

Hence dividing by $\langle 0|S|0\rangle$ removes the vacuum bubbles, leaving only diagrams in which every component is connected to at least one external point. This is the *linked-cluster theorem*, derived in detail in Peskin Ch. 4.4 and Mandl–Shaw Ch. 6.

The role of the $i\epsilon$ rotation. The slight tilt $T \to \infty(1-i\epsilon)$ is not optional. Without it, $e^{-iHT}|0\rangle$ would oscillate forever and never project onto $|\Omega\rangle$. The tilt is the same one that picks the *Feynman* contour in the propagator, $i/(p^2-m^2+i\epsilon)$, and the same one implicitly used in the LSZ asymptotic conditions to define in/out states. All of these are aspects of one and the same prescription: tilt the contour into the lower half plane, project onto the ground state, and analytically continue back.

Why the choice of t_0 is irrelevant. The intermediate steps depended on a reference time t_0 at which the pictures coincide. But t_0 never appears in the final formula (I.31): it cancels between the bra and ket sides, between U and $U^\dagger$, and between the numerator and the denominator. Physically this expresses the fact that the choice of reference time is gauge in this construction; what matters is the asymptotic structure $\pm T \to \pm\infty(1-i\epsilon)$ that selects the interacting vacuum.

Comparison with the LSZ side. The Gell-Mann–Low formula computes Green's functions $G(x_1,\ldots,x_n)$, which are off-shell objects in coordinate space. The LSZ formula then takes these Green's functions, Fourier transforms them, amputates external propagators, and puts the external momenta on shell, yielding the S-matrix elements $\langle f|S|i\rangle$. Schematically:

$$\langle 0|T\{\varphi_I\cdots\varphi_I S\}|0\rangle/\langle 0|S|0\rangle \xrightarrow{\text{Gell-Mann–Low}} G(x_1,\ldots,x_n) \xrightarrow{\text{LSZ}} S\text{-matrix}.$$

The first arrow takes us from a perturbatively computable expression to the interacting Green's function; the second arrow extracts the physical observable.

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