2 Pictures of Quantum Dynamics

In quantum mechanics, physical predictions—expectation values, transition probabilities, scattering amplitudes—must be independent of how we choose to describe the system mathematically. This freedom leads to several equivalent "pictures" of quantum dynamics, each distributing time dependence differently between states $|\psi\rangle$ and operators $\hat{A}$. The **Schrödinger picture** places all time dependence in the states, the **Heisenberg picture** transfers it entirely to the operators, and the **interaction (Dirac) picture** provides a hybrid approach essential for perturbation theory. Understanding the equivalence and transformations between these pictures is a prerequisite for perturbation theory and for grasping how scattering processes are computed in quantum field theory.

Throughout this chapter we work with a Hilbert space $\mathcal{H}$ and a (possibly time-dependent) Hermitian Hamiltonian $H(t) = H(t)^\dagger$, which ensures real energy eigenvalues and unitary time evolution. We fix a reference time t_0 (often taken to be zero) and follow the conventions of standard references [3, 4, 5, 6, 7]. We keep $\hbar$ explicit in this chapter to make the picture transformations transparent; in later perturbative QFT formulas we return to natural units, $\hbar = c = 1$.

2.1 The Schrödinger picture

The Schrödinger picture is historically the first formulation of quantum dynamics, introduced by Erwin Schrödinger in 1926. Its defining feature is that the quantum state $|\psi_S(t)\rangle$ evolves in time according to a first-order differential equation—the Schrödinger equation—while operators representing physical observables remain constant (unless they have explicit time dependence). This picture is conceptually natural: we think of the "wavefunction" as describing the state of the system, and we watch it spread, oscillate, or evolve as time progresses.

The Schrödinger equation. In this picture, states carry the time dependence $|\psi_S(t)\rangle$, while operators A_S are time-independent unless they have explicit time dependence $A_S(t)$. Time evolution of states is governed by

$$i\hbar \frac{d}{dt} |\psi_S(t)\rangle = H(t) |\psi_S(t)\rangle . \tag{2.1}$$

Box 2.1 A note on d/dt vs $\partial/\partial t$ in quantum mechanics

In the literature, the Schrödinger equation is written with either the total derivative d/dt or the partial derivative $\partial/\partial t$. The same ambiguity appears in the Heisenberg equation of motion.

The distinction:

- $\dfrac{\partial}{\partial t}$ denotes the *explicit* time derivative: the derivative with respect to t appearing explicitly in the functional form, holding all other variables fixed.
- $\dfrac{d}{dt}$ denotes the *total* time derivative: it includes both explicit time dependence and any implicit time dependence through other variables.

For the state vector $|\psi(t)\rangle$: The state depends on time only through its argument t; there are no other variables. Hence d/dt and $\partial/\partial t$ are equivalent, and both notations are correct.

For operators $A(t)$: An operator may have explicit time dependence $A_S(t)$ (e.g., a time-varying external field) and, in the Heisenberg picture, also implicit time dependence through the transformation $A_H(t) = U^\dagger(t,t_0)A_S U(t,t_0)$. The Heisenberg equation

$$i\hbar\frac{d}{dt}A_H(t) = [A_H(t), H_H(t)] + i\hbar\left(\frac{\partial A_S}{\partial t}\right)_H \tag{2.2}$$

distinguishes between the total derivative (left-hand side) and the explicit time derivative (last term on the right).

Convention in these notes: We use d/dt for the Schrödinger equation and the Heisenberg equation (total time derivative), and reserve $\partial/\partial t$ for explicit time dependence when the distinction matters.

The evolution operator. We define the evolution operator $U(t,t_0)$ by

$$|\psi_S(t)\rangle = U(t,t_0)\,|\psi_S(t_0)\rangle, \qquad U(t_0,t_0) = \mathbb{1}\,. \tag{2.3}$$

Differentiating with respect to t and comparing with the Schrödinger equation shows that $U(t,t_0)$ satisfies

$$i\hbar\frac{\partial}{\partial t}U(t,t_0) = H(t)\,U(t,t_0), \qquad U(t_0,t_0) = \mathbb{1}\,. \tag{2.4}$$

The Hermiticity of the Hamiltonian ensures that $U(t,t_0)$ is unitary. Indeed, computing the time derivative of $U^\dagger U$ using $\partial_t U = -\frac{i}{\hbar}HU$ and $\partial_t U^\dagger = \frac{i}{\hbar}U^\dagger H$, one finds

$$\frac{\partial}{\partial t}(U^\dagger U) = \frac{i}{\hbar}U^\dagger HU - \frac{i}{\hbar}U^\dagger HU = 0\,, \tag{2.5}$$

so $U^\dagger U$ is constant in time. Since $U(t_0,t_0) = \mathbb{1}$, we have $U(t,t_0)^\dagger = U(t,t_0)^{-1}$ for all t.

The evolution operator satisfies a **composition property**:

$$U(t_2,t_0) = U(t_2,t_1)\,U(t_1,t_0)\,. \tag{2.6}$$

Proof. By definition, $U(t,t_0)$ maps $|\psi_S(t_0)\rangle$ into $|\psi_S(t)\rangle$. Therefore

$$|\psi_S(t_1)\rangle = U(t_1,t_0)\,|\psi_S(t_0)\rangle\,.$$

Using t_1 as the initial time, the same definition gives

$$|\psi_S(t_2)\rangle = U(t_2,t_1)\,|\psi_S(t_1)\rangle = U(t_2,t_1)\,U(t_1,t_0)\,|\psi_S(t_0)\rangle\,.$$

On the other hand, evolving directly from t_0 to t_2,

$$|\psi_S(t_2)\rangle = U(t_2,t_0)\,|\psi_S(t_0)\rangle\,.$$

Since the initial state $|\psi_S(t_0)\rangle$ is arbitrary, the operator identity (2.6) follows. □

As an immediate corollary, setting $t_2 = t_0$ in (2.6) yields $\mathbb{1} = U(t_0,t_1)\,U(t_1,t_0)$, whence

$$U(t_0,t) = U(t,t_0)^{-1} = U(t,t_0)^\dagger\,. \tag{2.7}$$

Expectation values. For an operator A_S with no explicit time dependence, the expectation value at time t is

$$\langle A\rangle(t) = \langle\psi_S(t)|A_S|\psi_S(t)\rangle = \langle\psi_S(t_0)|\,U(t,t_0)^\dagger A_S\,U(t,t_0)\,|\psi_S(t_0)\rangle\,. \tag{2.8}$$

2.2 The Heisenberg picture

The Heisenberg picture, developed by Werner Heisenberg in 1925 (actually slightly before Schrödinger's formulation), shifts all time dependence from states to operators. This is reminiscent of classical Hamiltonian mechanics, where we track how observables change in time along a phase-space trajectory. The Heisenberg picture is particularly powerful for studying algebraic structures (the Heisenberg equation mirrors the classical Poisson-bracket formulation), for quantum field theory (where field operators $\phi(x,t)$ carry spacetime dependence), and for proving conservation laws.

Definition. In the Heisenberg picture, states are time-independent:

$$|\psi_H\rangle \equiv |\psi_S(t_0)\rangle\,, \tag{2.9}$$

and the time dependence is transferred to operators by defining

$$A_H(t) \equiv U(t,t_0)^\dagger\, A_S\, U(t,t_0)\,. \tag{2.10}$$

Expectation values then agree with the Schrödinger picture: $\langle A\rangle(t) = \langle\psi_H|A_H(t)|\psi_H\rangle$.

Heisenberg equation of motion. Differentiating the definition of $A_H(t)$ and using the Schrödinger equation for the time-evolution operator, $\partial_t U = -\frac{i}{\hbar}H_S U$ and $\partial_t U^\dagger = \frac{i}{\hbar}U^\dagger H_S$ (where H_S is the Hamiltonian in the Schrödinger picture), one obtains

$$\frac{d}{dt}A_H(t) = \frac{i}{\hbar}\,U^\dagger[H_S, A_S]\,U + U^\dagger\Big(\frac{\partial A_S}{\partial t}\Big)U\,. \tag{2.11}$$

Defining the Heisenberg-picture Hamiltonian $H_H(t) \equiv U^\dagger H_S(t)\,U$ and using the identity $U^\dagger[X,Y]U = [U^\dagger XU,\, U^\dagger YU]$, this becomes

$$i\hbar\,\frac{d}{dt}A_H(t) = [A_H(t), H_H(t)] + i\hbar\left(\frac{\partial A}{\partial t}\right)_H\,. \tag{2.12}$$

If A_S has no explicit time dependence, the last term vanishes and one has

$$i\hbar\,\frac{d}{dt}A_H(t) = [A_H(t), H_H(t)]\,. \tag{2.13}$$

2.3 The interaction (Dirac) picture

The *interaction picture* (also called the Dirac picture) is a hybrid between the Schrödinger and Heisenberg pictures: part of the time evolution is carried by states, part by operators. This split is tailored to perturbation theory and is the workhorse of quantum field theory.

In quantum field theory, the full Hamiltonian $H_S(t) = H_0(t) + V(t)$ consists of a "free" part H_0 (describing non-interacting particles, for which we can solve exactly) and an interaction part V (describing scattering, pair creation, etc.). The idea of the interaction picture is to absorb the *known* free evolution into the operators, leaving only the *unknown* interaction to drive the evolution of states. The interaction-picture evolution operator $U_I(t,t_0)$ can then be expanded perturbatively in powers of V—this is the Dyson series.

Setup. Fix a reference time t_0 and decompose the full Schrödinger-picture Hamiltonian as

$$H_S(t) = H_0(t) + V(t)\,, \tag{2.14}$$

with $H_S(t)$, $H_0(t)$, and $V(t)$ self-adjoint for each time t.

Box 2.2 Notation for H_0 and V

Strictly speaking, H_0 and V are operators that exist in every picture, and we should write $(H_0)_S$, V_S for the Schrödinger picture and $(H_0)_I$, V_I for the interaction picture.
However, in practice:

- **Convention:** We write H_0 and V (without subscript) to denote the *Schrödinger-picture* operators, which are the "original" operators appearing in the Hamiltonian.
- The subscript I in V_I explicitly indicates the interaction-picture version: $V_I(t) = U_0^\dagger(t,t_0)\,V(t)\,U_0(t,t_0)$.
- For the free Hamiltonian, we have $(H_0)_I(t) = U_0^\dagger(t,t_0)H_0(t)U_0(t,t_0)$. If H_0 is time-independent, then $[H_0, U_0] = 0$ and therefore $(H_0)_I = H_0$. If H_0 is time-dependent, one must keep the transformed operator $(H_0)_I(t)$ explicitly.
- In contrast, $V_I(t) \neq V(t)$ in general, because the free evolution generated by H_0 acts nontrivially on the interaction.

Summary:

	Schrödinger	**Interaction**
Free Ham.	$H_0(t) \equiv (H_0)_S(t)$	$(H_0)_I(t) = U_0^\dagger H_0(t) U_0$
Interaction	$V(t) \equiv V_S(t)$	$V_I(t) = U_0^\dagger V(t) U_0$

Throughout these notes, H_0 and V without subscript denote Schrödinger-picture operators.

Define the full evolution operator $U(t,t_0)$ by

$$i\hbar\frac{\partial}{\partial t}U(t,t_0) = H_S(t)\,U(t,t_0), \qquad U(t_0,t_0) = \mathbb{1}, \tag{2.15}$$

and the "free" evolution operator $U_0(t,t_0)$ by

$$i\hbar\frac{\partial}{\partial t}U_0(t,t_0) = H_0(t)\,U_0(t,t_0), \qquad U_0(t_0,t_0) = \mathbb{1}. \tag{2.16}$$

Both are unitary by virtue of Hermiticity.

Interaction-picture states. The interaction-picture state is obtained by "removing" the free evolution from the Schrödinger-picture state:

$$|\psi_I(t)\rangle \equiv U_0^\dagger(t,t_0)\,|\psi_S(t)\rangle\,. \tag{2.17}$$

Equivalently, $|\psi_S(t)\rangle = U_0(t,t_0)|\psi_I(t)\rangle$. Note that $|\psi_I(t_0)\rangle = |\psi_S(t_0)\rangle$: all pictures coincide at the reference time.

Interaction-picture operators. For any Schrödinger-picture operator $A_S(t)$, define

$$A_I(t) \equiv U_0^\dagger(t,t_0)\, A_S(t)\, U_0(t,t_0)\,. \tag{2.18}$$

In particular, the interaction Hamiltonian in the interaction picture is

$$V_I(t) \equiv U_0^\dagger(t,t_0)\, V(t)\, U_0(t,t_0)\,. \tag{2.19}$$

Picture independence of expectation values. Expectation values are the same in all pictures:

$$\begin{aligned}\langle\psi_I(t)|A_I(t)|\psi_I(t)\rangle &= \langle\psi_S(t)|U_0\,U_0^\dagger A_S U_0\,U_0^\dagger|\psi_S(t)\rangle \\ &= \langle\psi_S(t)|A_S(t)|\psi_S(t)\rangle\,.\end{aligned} \tag{2.20}$$

Box 2.3 Equation of motion for interaction-picture states

Differentiate the definition $|\psi_I(t)\rangle = U_0^\dagger(t,t_0)|\psi_S(t)\rangle$:

$$i\hbar\frac{d}{dt}|\psi_I(t)\rangle = i\hbar\frac{dU_0^\dagger}{dt}|\psi_S\rangle + U_0^\dagger\, i\hbar\frac{d}{dt}|\psi_S\rangle\,. \tag{2.21}$$

From $i\hbar\,\partial_t U_0 = H_0(t)U_0$, we get $i\hbar\,\partial_t U_0^\dagger = -U_0^\dagger H_0(t)$. Using the Schrödinger equation $i\hbar\,\partial_t|\psi_S\rangle = H_S|\psi_S\rangle$:

$$\begin{aligned}i\hbar\frac{d}{dt}|\psi_I(t)\rangle &= -U_0^\dagger H_0(t)|\psi_S\rangle + U_0^\dagger H_S(t)|\psi_S\rangle \\ &= U_0^\dagger(H_S(t)-H_0(t))|\psi_S\rangle \\ &= U_0^\dagger V(t)|\psi_S\rangle\,.\end{aligned} \tag{2.22}$$

Writing $|\psi_S\rangle = U_0|\psi_I\rangle$:

$$i\hbar\frac{d}{dt}|\psi_I(t)\rangle = U_0^\dagger V(t) U_0\,|\psi_I\rangle = V_I(t)\,|\psi_I(t)\rangle\,. \tag{2.23}$$

Equation of motion for interaction-picture states. The result of the derivation is:

$$\boxed{i\hbar\frac{d}{dt}|\psi_I(t)\rangle = V_I(t)\,|\psi_I(t)\rangle} \tag{2.24}$$

States evolve with the *interaction alone*—the free Hamiltonian has been absorbed into the operators.

Box 2.4 Derivation of the equation of motion for interaction-picture operators

Differentiate the definition $A_I(t) = U_0^\dagger(t,t_0)\, A_S(t)\, U_0(t,t_0)$:

$$\frac{dA_I}{dt} = \frac{dU_0^\dagger}{dt} A_S(t) U_0 + U_0^\dagger \frac{\partial A_S}{\partial t} U_0 + U_0^\dagger A_S(t) \frac{dU_0}{dt}. \tag{2.25}$$

Using $i\hbar\,\partial_t U_0 = H_0(t) U_0$ and $i\hbar\,\partial_t U_0^\dagger = -U_0^\dagger H_0(t)$:

$$\frac{dA_I}{dt} = \frac{i}{\hbar} U_0^\dagger H_0(t) A_S(t) U_0 + \left(\frac{\partial A_S}{\partial t}\right)_I - \frac{i}{\hbar} U_0^\dagger A_S(t) H_0(t) U_0 . \tag{2.26}$$

Inserting $\mathbb{1} = U_0 U_0^\dagger$ between $H_0(t)$ and $A_S(t)$, and rearranging to match the Heisenberg-style $[A,H]$ ordering used elsewhere in this chapter:

$$i\hbar \frac{dA_I}{dt} = [A_I, H_{0,I}] + i\hbar \left(\frac{\partial A_S}{\partial t}\right)_I , \tag{2.27}$$

where $H_{0,I}(t) = U_0^\dagger H_0(t) U_0$.

Equation of motion for interaction-picture operators.

$$\boxed{i\hbar \frac{d}{dt} A_I(t) = [A_I(t), H_{0,I}(t)] + i\hbar \left(\frac{\partial A_S}{\partial t}\right)_I} \tag{2.28}$$

Operators evolve with the transformed free Hamiltonian $H_{0,I}$, plus any explicit time dependence already present in $A_S(t)$. This mirrors the Heisenberg equation (2.12), except that the relevant Hamiltonian is the *free* one. Once the free problem has been solved, the dynamical evolution of A_I is exact. Moreover, as discussed in the box "Notation for H_0 and V", when H_0 is time-independent (the standard situation in QFT) one has $[H_0, U_0] = 0$ and therefore $H_{0,I} = H_0$: the free Hamiltonian is the same in the Schrödinger and interaction pictures. In this case the boxed equation simplifies to

$$i\hbar \frac{d}{dt} A_I(t) = [A_I(t), H_0] + i\hbar \left(\frac{\partial A_S}{\partial t}\right)_I , \tag{2.29}$$

and we shall use this lighter notation from now on.

The interaction-picture evolution operator and factorization. Define $U_I(t,t_0)$ by

$$|\psi_I(t)\rangle = U_I(t,t_0)\, |\psi_I(t_0)\rangle, \qquad U_I(t_0,t_0) = \mathbb{1}. \tag{2.30}$$

Then U_I satisfies

$$i\hbar \frac{\partial}{\partial t} U_I(t,t_0) = V_I(t)\, U_I(t,t_0), \qquad U_I(t_0,t_0) = \mathbb{1}. \tag{2.31}$$

The full evolution operator factorizes as

$$\boxed{U(t,t_0) = U_0(t,t_0)\, U_I(t,t_0)} \tag{2.32}$$

Equivalently, $U_I(t,t_0) = U_0^\dagger(t,t_0)\, U(t,t_0)$.

Physical interpretation: The full evolution U is split into the exactly-known free evolution U_0 (which generates propagators) and the interaction evolution U_I (which generates vertices in Feynman diagrams).

Connecting the Heisenberg and interaction pictures. Using the factorization (2.32) and the definition of Heisenberg-picture operators $A_H = U^\dagger A_S U$:

$$A_H(t) = U_I^\dagger(t,t_0)\, A_I(t)\, U_I(t,t_0)\,. \tag{2.33}$$

This provides an efficient bridge between the interaction picture (where A_I evolves with the simple free Hamiltonian) and Heisenberg observables.

Summary: the interaction picture at a glance.

States	evolve with V_I only
Operators	evolve with $H_{0,I}$, plus explicit time dependence
Transformation	$\lvert\psi_I\rangle = U_0^\dagger \lvert\psi_S\rangle, \quad A_I = U_0^\dagger A_S U_0$
Evolution operator	$U = U_0\, U_I$

Remark. The interaction picture is a special case of a general framework in which *any* picture is specified by a choice of unitary transformation $\Omega(t)$. The Schrödinger picture corresponds to $\Omega = \mathbb{1}$, the Heisenberg picture to $\Omega = U^\dagger$, and the interaction picture to $\Omega = U_0^\dagger$. This unified viewpoint is developed in Section 2.4.

2.4 A unified framework: the general picture

The Schrödinger and Heisenberg pictures are two extreme choices: in the former, states carry all the time dependence; in the latter, operators do. In fact, infinitely many intermediate pictures exist, each distributing time dependence differently between states and operators. The key insight is that *any* such picture is connected to the Schrödinger picture by a unitary transformation, and the equations of motion in each picture follow from a single master formula.

General setup. Let $\Omega(t)$ be an arbitrary time-dependent unitary operator, $\Omega(t)^\dagger \Omega(t) = \mathbb{1}$. Define the "$\Omega$-picture" by

$$\lvert\psi_\Omega(t)\rangle \equiv \Omega(t)\, \lvert\psi_S(t)\rangle\,, \qquad A_\Omega(t) \equiv \Omega(t)\, A_S(t)\, \Omega^\dagger(t)\,. \tag{2.34}$$

Expectation values are automatically picture-independent:

$$\langle\psi_\Omega|A_\Omega|\psi_\Omega\rangle = \langle\psi_S|\,\Omega^\dagger\,\Omega\,A_S(t)\,\Omega^\dagger\,\Omega\,|\psi_S\rangle = \langle\psi_S|A_S(t)|\psi_S\rangle\,.$$

Note also that the transformed Hamiltonian operator

$$H_\Omega(t) = \Omega(t)H_S(t)\Omega^\dagger(t)$$

has the *same functional form* in terms of Ω-picture operators as H_S has in terms of Schrödinger-picture operators: one can verify this by expanding H_S as a power series in p_S, q_S and repeatedly inserting $\Omega^\dagger\Omega = \mathbb{1}$ to convert each p_S, q_S into p_Ω, q_Ω. For a time-dependent change of picture, however, the actual generator of state evolution is the combination $H_\Omega + i\hbar\,\dot\Omega\,\Omega^\dagger$ appearing in (2.36), not H_Ω alone.

Equation of motion for states. Differentiating $|\psi_\Omega\rangle = \Omega|\psi_S\rangle$ and using the Schrödinger equation (2.1):

$$\begin{aligned} i\hbar\frac{d}{dt}|\psi_\Omega(t)\rangle &= i\hbar\,\dot\Omega\,|\psi_S\rangle + \Omega\, i\hbar\frac{d}{dt}|\psi_S\rangle \\ &= i\hbar\,\dot\Omega\,\Omega^\dagger\,|\psi_\Omega\rangle + \Omega\,H_S(t)\,\Omega^\dagger\,|\psi_\Omega\rangle\,. \end{aligned} \tag{2.35}$$

Therefore the general evolution equation for states reads

$$\boxed{i\hbar\frac{d}{dt}|\psi_\Omega(t)\rangle = \left(H_\Omega + i\hbar\,\dot\Omega\,\Omega^\dagger\right)|\psi_\Omega(t)\rangle} \tag{2.36}$$

Equation of motion for operators. Differentiating $A_\Omega = \Omega\,A_S(t)\,\Omega^\dagger$ and using $\frac{d}{dt}(\Omega\Omega^\dagger) = 0$, which implies $\dot\Omega\,\Omega^\dagger = -\Omega\,\dot\Omega^\dagger$:

$$\begin{aligned} i\hbar\frac{d}{dt}A_\Omega &= i\hbar\,\dot\Omega\,A_S(t)\,\Omega^\dagger + i\hbar\,\Omega\left(\frac{\partial A_S}{\partial t}\right)\Omega^\dagger + i\hbar\,\Omega\,A_S(t)\,\dot\Omega^\dagger \\ &= i\hbar\left(\frac{\partial A_S}{\partial t}\right)_\Omega + i\hbar\left[\dot\Omega\,\Omega^\dagger,\,A_\Omega\right], \end{aligned} \tag{2.37}$$

where in the last step we inserted $\Omega^\dagger\Omega$ and recognised the commutator. The general evolution equation for operators is

$$\boxed{i\hbar\frac{d}{dt}A_\Omega(t) = \left[A_\Omega,\,-i\hbar\,\dot\Omega\,\Omega^\dagger\right] + i\hbar\left(\frac{\partial A_S}{\partial t}\right)_\Omega} \tag{2.38}$$

Recovering the Schrödinger picture. Choose $\Omega(t) = \mathbb{1}$. Then $\dot\Omega = 0$ and equation (2.36) becomes $i\hbar\frac{d}{dt}|\psi_S\rangle = H_S|\psi_S\rangle$, the Schrödinger equation. The operator equation (2.38) gives $\frac{d}{dt}A_S = \partial_t A_S$: operators have no dynamical time dependence.

Recovering the Heisenberg picture. Choose $\Omega(t) = U(t,t_0)^\dagger$, the adjoint of the full evolution operator. Since $i\hbar\,\partial_t U = H_S U$ implies $i\hbar\,\partial_t U^\dagger = -U^\dagger H_S$, we have $i\hbar\,\dot\Omega = -U^\dagger H_S$ and therefore

$$i\hbar\,\dot\Omega\,\Omega^\dagger = -U^\dagger H_S\,U = -H_H\,.$$

The combination $H_\Omega + i\hbar\,\dot\Omega\,\Omega^\dagger = H_H - H_H = 0$: the state equation (2.36) yields $\frac{d}{dt}|\psi_H\rangle = 0$, confirming that Heisenberg-picture states are frozen. The operator equation (2.38) gives $i\hbar\frac{d}{dt}A_H = [A_H, H_H] + i\hbar(\partial_t A_S)_H$, which is the Heisenberg equation (2.12).

Recovering the interaction picture. Setting $\Omega(t) = U_0(t,t_0)^\dagger$ gives $i\hbar\,\dot\Omega\,\Omega^\dagger = -H_{0,I}(t)$, so:

- *States:* $H_\Omega + i\hbar\,\dot\Omega\,\Omega^\dagger = H_I - H_{0,I} = V_I$: states evolve with the interaction alone, as derived in Section 2.3.
- *Operators:* $-i\hbar\,\dot\Omega\,\Omega^\dagger = H_{0,I}$: operators evolve with the free Hamiltonian, plus any explicit time dependence of the original Schrödinger-picture operator.

This unified viewpoint makes transparent that the choice of picture amounts to choosing $\Omega(t)$, and that all pictures give the same physics—the same matrix elements, the same transition probabilities.

Remark: from quantum mechanics to quantum field theory. The formulas above were derived for ordinary quantum-mechanical operators, but the same structure carries over to QFT essentially unchanged. In canonical field theory the replacement rule is

$$q_i \longrightarrow \phi_k(\vec{x},t)\,, \qquad p_i \longrightarrow \pi_k(\vec{x},t) = \frac{\partial \mathcal{L}}{\partial \dot\phi_k}\,, \tag{2.39}$$

i.e. the discrete index i becomes the pair $(k,\vec{x})$, with k a discrete field label and $\vec{x}$ a continuous spatial argument. Equal-time canonical relations become

$$[\phi_k(\vec{x},t),\,\pi_l(\vec{y},t)] = i\hbar\,\delta_{kl}\,\delta^{(3)}(\vec{x}-\vec{y})\,, \tag{2.40}$$

and the Hamiltonian $H(p,q)$ becomes a spatial integral of a Hamiltonian density,

$$H[\phi,\pi] = \int d^3x\,\mathcal{H}(\phi,\pi)\,, \qquad V(t) = \int d^3x\,\mathcal{H}_{\rm int}(\phi,\pi)\,. \tag{2.41}$$

In the interaction picture the local interaction Hamiltonian density acquires its free time evolution. With $U_0 = U_0(t,t_0)$ (and writing $V(t)$ in case the Schrödinger-picture interaction also has explicit time dependence),

$$V_I(t) = U_0^\dagger(t,t_0)\,V(t)\,U_0(t,t_0) = \int d^3x\,\mathscr{H}_I(\vec{x},t)\,, \qquad \mathscr{H}_I(\vec{x},t) = U_0^\dagger(t,t_0)\,\mathcal{H}_{\rm int}\big(\phi(\vec{x}),\pi(\vec{x});t\big)\,U_0($$

(2.42)

Substituting into the Dyson series (3.10) and grouping the temporal and spatial integrations into a single $\int d^4x$ converts the perturbative expansion into a sum over T-products of local Hamiltonian densities,

$$U_I = T\exp\left[-\frac{i}{\hbar}\int d^4x\,\mathscr{H}_I(x)\right], \tag{2.43}$$

which is the form used throughout QFT and the starting point of the covariant Feynman rules. The structural lessons of this chapter—picture independence, the factorisation $U = U_0U_I$, the Dyson series, the existence of an S-matrix—all carry over without modification; only the specific dynamical content of $\mathscr{H}_I$ changes from problem to problem.

2.5 Exponential representations of the evolution operators

The evolution operators introduced above can be written as exponentials, ordinary or time-ordered depending on whether the Hamiltonian commutes with itself at different times. These exponential forms follow directly from solving the Schrödinger equation for the evolution operator.

Box 2.5 Origin of the exponential representation

The evolution operator $U(t,t_0)$ satisfies the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}U(t,t_0) = H(t)\,U(t,t_0)\,, \qquad U(t_0,t_0) = \mathbb{1}\,.$$

For a **time-independent** Hamiltonian H, this first-order ODE can be integrated directly:

$$U(t,t_0) = e^{-\frac{i}{\hbar}H(t-t_0)}\,.$$

For a **time-dependent** Hamiltonian that commutes with itself at different times, $[H(t_1),H(t_2)] = 0$, the solution generalizes to

$$U(t,t_0) = \exp\left[-\frac{i}{\hbar}\int_{t_0}^{t} dt'\,H(t')\right].$$

However, when $[H(t_1),H(t_2)] \neq 0$, the naive exponential of the integral is *not* correct—operator ordering matters. The proper solution involves the **time-ordered exponential**, which is defined through the Dyson series. The detailed derivation is presented in Chapter 3 (The Dyson Series).

General case (time-dependent, noncommuting Hamiltonian). If $[H(t_1), H(t_2)] \neq 0$ in general, the full evolution operator is given by the time-ordered exponential

$$U(t,t_0) = T\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t} dt'\, H(t')\right], \tag{2.44}$$

where T denotes **time ordering**: operators at later times appear to the left. This formula is written for forward evolution, $t > t_0$; backward evolution follows from unitarity and the composition law, or equivalently from anti-time ordering. This compact notation is shorthand for the infinite Dyson series derived in the next chapter. Similarly,

$$U_0(t,t_0) = T\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t} dt'\, H_0(t')\right], \tag{2.45}$$

$$U_I(t,t_0) = T\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t} dt'\, V_I(t')\right]. \tag{2.46}$$

Commuting case. If $[H(t_1), H(t_2)] = 0$ for all t_1, t_2, time ordering is unnecessary:

$$U(t,t_0) = \exp\left[-\frac{i}{\hbar}\int_{t_0}^{t} dt'\, H(t')\right]. \tag{2.47}$$

In particular, if H is time-independent,

$$U(t,t_0) = \exp\left[-\frac{i}{\hbar}H(t-t_0)\right]. \tag{2.48}$$

Time-independent case. When $H = H_0 + V$ with all terms time-independent, the evolution operators simplify to

$$U(t,t_0) = e^{-\frac{i}{\hbar}(H_0+V)(t-t_0)}, \qquad U_0(t,t_0) = e^{-\frac{i}{\hbar}H_0(t-t_0)}, \tag{2.49}$$

and the interaction-picture quantities become

$$V_I(t) = e^{\frac{i}{\hbar}H_0(t-t_0)}\, V\, e^{-\frac{i}{\hbar}H_0(t-t_0)}, \qquad U_I(t,t_0) = e^{\frac{i}{\hbar}H_0(t-t_0)}\, U(t,t_0)\,. \tag{2.50}$$

Note that $V_I(t)$ is still time-dependent unless $[H_0, V] = 0$. In the special commuting case $[H_0, V] = 0$, one has $V_I(t) = V$ and $U_I(t,t_0) = e^{-\frac{i}{\hbar}V(t-t_0)}$.

2.6 Summary: comparison of pictures

The following table summarizes the three pictures of quantum dynamics. In all cases, the full Hamiltonian is $H_S(t) = H_0(t) + V(t)$ and physical observables (expectation values, transition amplitudes) are picture-independent.

	Schrödinger	**Heisenberg**	
Transformation Ω	$\mathbb{1}$	$U^\dagger$	
State	$\lvert\psi_S(t)\rangle$	$\lvert\psi_H\rangle = \lvert\psi_S(t_0)\rangle$	
Operator	$A_S(t)$	$A_H = U^\dagger A_S(t) U$	
State equation	$i\hbar \dfrac{d}{dt}\lvert\psi_S\rangle = H_S\lvert\psi_S\rangle$	$\dfrac{d}{dt}\lvert\psi_H\rangle = 0$	
Operator equation	$\dfrac{dA_S}{dt} = \dfrac{\partial A_S}{\partial t}$	$i\hbar \dfrac{dA_H}{dt} = [A_H, H_H] + i\hbar \left(\dfrac{\partial A_S}{\partial t}\right)_H$	
Evolution operator	$U(t,t_0) = e^{-iH_S(t-t_0)/\hbar}$ (only if H_S is time-indep.)	—	
States evolve with	full H_S	nothing (frozen)	
Operators evolve with	explicit time dependence only	full H_H	f

Key relations:

- Factorization: $U(t,t_0) = U_0(t,t_0)\, U_I(t,t_0)$
- Heisenberg from interaction: $A_H(t) = U_I^\dagger(t,t_0)\, A_I(t)\, U_I(t,t_0)$
- For time-independent H_0: $U_0(t,t_0) = e^{-iH_0(t-t_0)/\hbar}$
- Interaction-picture operators, valid only when H_0 is time-independent and A_S has no residual explicit time dependence:

$$A_I(t) = e^{+iH_0(t-t_0)/\hbar}\, A_S\, e^{-iH_0(t-t_0)/\hbar}.$$

  In the general case keep the operator form $A_I(t) = U_0^\dagger(t,t_0)\, A_S(t)\, U_0(t,t_0)$ as in (2.18).

3 The Dyson Series

In 1949, Freeman Dyson published two landmark papers [8] that unified the perturbative approaches of Feynman, Schwinger, and Tomonaga and established the modern framework for computing S-matrix elements in quantum electrodynamics. The key mathematical tool is the Dyson series—a systematic expansion of the interaction-picture evolution operator $U_I(t,t_0)$ in powers of the interaction-picture operator $V_I(t)$.

The physical content of the Dyson series is profound. The n-th term in the series involves n powers of V_I, integrated over all possible orderings of the n interaction times, corresponding to processes with n vertices in Feynman diagrams. The time-ordering operator T ensures that operators at later times appear to the left; together with locality and microcausality, this is compatible with Lorentz-covariant scattering amplitudes. When the interaction density involves products of field operators, Wick's theorem converts the time-ordered products into sums of normal-ordered products plus contractions, with each contraction corresponding to a propagator line in a Feynman diagram. Thus the Dyson series is the bridge connecting the abstract operator formalism of QFT to the concrete computational rules that make predictions possible.

Natural units. From this chapter onward, we adopt **natural units** $\hbar = c = 1$, which is standard in particle physics and simplifies all formulas considerably.

3.1 Derivation of the series

Setup. Fix t_0 and split the Schrödinger-picture Hamiltonian as $H_S(t) = H_0(t) + V(t)$. The interaction-picture evolution operator $U_I(t,t_0)$ satisfies

$$i\frac{\partial}{\partial t}U_I(t,t_0) = V_I(t)\,U_I(t,t_0), \qquad U_I(t_0,t_0) = \mathbb{1}\,, \tag{3.1}$$

where $V_I(t) = U_0^\dagger(t,t_0)\,V(t)\,U_0(t,t_0)$.

Integral equation. Integrating the differential equation from t_0 to t with the initial condition gives the Volterra integral equation

$$U_I(t,t_0) = \mathbb{1} - i\int_{t_0}^{t} dt_1\, V_I(t_1)\,U_I(t_1,t_0)\,. \tag{3.2}$$

Picard iteration. Define iterates starting from $U_I^{(0)} = \mathbb{1}$ and recursively

$$U_I^{(n+1)}(t,t_0) = \mathbb{1} - i\int_{t_0}^{t} dt_1\, V_I(t_1)\, U_I^{(n)}(t_1,t_0)\,. \tag{3.3}$$

The first few iterates are:

$$U_I^{(1)} = \mathbb{1} - i\int_{t_0}^{t} dt_1\, V_I(t_1), \tag{3.4}$$

$$U_I^{(2)} = \mathbb{1} - i\int_{t_0}^{t} dt_1\, V_I(t_1) + (-i)^2\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, V_I(t_1)V_I(t_2)\,. \tag{3.5}$$

The general n-th term. Continuing the iteration, the formal solution is

$$U_I(t,t_0) = \mathbb{1} + \sum_{n=1}^{\infty}(-i)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n\, V_I(t_1)V_I(t_2)\cdots V_I(t_n)\,. \tag{3.6}$$

The nested integration limits enforce $t_0 \le t_n \le \cdots \le t_2 \le t_1 \le t$, so the operator product automatically appears with later times on the left.

3.2 Time ordering and the $1/n!$ factor

The time-ordering operator. Define the time-ordering operator T by

$$T\{V_I(t_1)V_I(t_2)\} = \begin{cases} V_I(t_1)V_I(t_2), & t_1 > t_2, \\ V_I(t_2)V_I(t_1), & t_2 > t_1\,, \end{cases} \tag{3.7}$$

and analogously for n factors: T permutes the operators so that the largest time argument stands leftmost. Equal-time cases can be assigned any fixed convention in the present integrals, since they form a set of measure zero. For elementary fermionic fields, time ordering is graded and includes the usual minus sign for odd permutations; the interaction Hamiltonian density itself is bosonic, so no additional sign appears in the products of V_I written here.

Geometric identity. The hypercube $C = [t_0,t]^n$ can be partitioned, up to measure-zero boundaries, into $n!$ regions corresponding to the $n!$ strict orderings of the times. The nested domain $t_0 \le t_n \le \cdots \le t_1 \le t$ is precisely one such region. On each ordering region, the time-ordered integrand rearranges the factors into decreasing time order; after relabeling dummy integration variables, every region

gives the same contribution as the nested one. Therefore integrating over the full hypercube with time ordering gives $n!$ times the nested integral:

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n \, V_I(t_1) \cdots V_I(t_n) = \frac{1}{n!} \int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t} dt_n \, T\{V_I(t_1) \cdots V_I(t_n)\} . \quad (3.8)$$

Explicit example for $n = 2$. Consider the square $C = [t_0, t]^2$. It decomposes into two triangular regions: $\mathcal{R}_{12} = \{t_1 \geq t_2\}$ and $\mathcal{R}_{21} = \{t_2 \geq t_1\}$. The nested integral covers $\mathcal{R}_{12}$. Since $T\{V_I(t_1)V_I(t_2)\}$ equals $V_I(t_1)V_I(t_2)$ on $\mathcal{R}_{12}$ and $V_I(t_2)V_I(t_1)$ on $\mathcal{R}_{21}$, and since these two contributions are equal after relabeling, one has

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \, V_I(t_1)V_I(t_2) = \frac{1}{2!} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \, T\{V_I(t_1)V_I(t_2)\} . \quad (3.9)$$

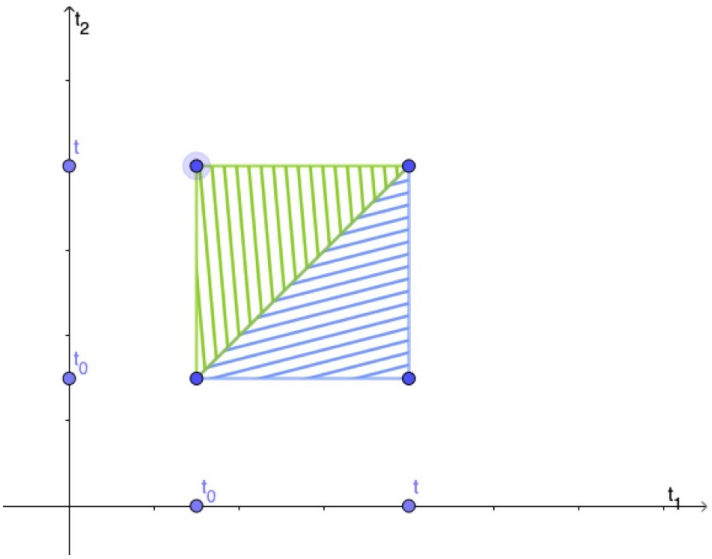

Figure 3.1. Decomposition of the integration square $C = [t_0, t]^2$ into two triangular regions $\mathcal{R}_{12}$ (blue, $t_1 \geq t_2$) and $\mathcal{R}_{21}$ (green, $t_2 \geq t_1$), separated by the diagonal $t_1 = t_2$.

3.3 The time-ordered exponential

For forward evolution $t > t_0$, substituting the identity relating nested integrals to time-ordered integrals gives

$$U_I(t, t_0) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t} dt_n \, T\{V_I(t_1) \cdots V_I(t_n)\} . \quad (3.10)$$

This is precisely the series expansion of the time-ordered exponential:

$$\boxed{U_I(t,t_0) = T\exp\left[-i\int_{t_0}^{t} dt'\, V_I(t')\right]} \tag{3.11}$$

For $t < t_0$, the inverse evolution is obtained from unitarity and composition, equivalently by using anti-time ordering.

Verification. Differentiating the series term by term with respect to t (using the fundamental theorem of calculus for the outermost integral) confirms that the Dyson series satisfies

$$\frac{d}{dt}U_I(t,t_0) = -i\,V_I(t)\,U_I(t,t_0), \qquad U_I(t_0,t_0) = \mathbb{1}\,. \tag{3.12}$$

Unitarity. If $V_I(t)$ is self-adjoint, the solution is unitary. Using the evolution equation and its adjoint, one finds $\partial_t(U_I^\dagger U_I) = 0$, so $U_I^\dagger U_I = \mathbb{1}$ for all t.

3.4 Definition of the S-matrix

The perturbation formalism developed above is particularly suitable for describing **scattering processes**. In a collision, particles approach each other, interact, and then separate. We now define the S-matrix, which encodes all the information about such transitions.

Physical setup. Consider a scattering event. Long before the collision ($t \to -\infty$), the particles are far apart and propagate as free particles—they constitute the **initial state** $|i\rangle$. After the collision ($t \to +\infty$), the resulting particles are again far apart and propagate freely—they constitute the **final state**. The equation of motion (3.1) determines how the initial state $|\Phi(-\infty)\rangle = |i\rangle$ evolves into the state $|\Phi(\infty)\rangle$ at $t = +\infty$.

Definition. The S-matrix (scattering matrix) is defined as the operator that relates the final state to the initial state:

$$\boxed{|\Phi(\infty)\rangle = S\,|\Phi(-\infty)\rangle = S\,|i\rangle} \tag{3.13}$$

Formally, and with the large-time limit understood through an adiabatic or equivalent LSZ prescription, $|\Phi(t)\rangle = U_I(t,t_0)|\Phi(t_0)\rangle$ implies

$$S \;=\; U_I(+\infty,-\infty) \;\equiv\; \lim_{\substack{t\to+\infty(1-i\epsilon)\\ t_0\to-\infty(1-i\epsilon)}} U_I(t,t_0)\,, \tag{3.14}$$

where the slight tilt of the asymptotic times into the lower complex plane, $\pm\infty(1-i\epsilon)$ with $\epsilon \to 0^+$, is the standard *adiabatic prescription*: it damps oscillations exponentially as $|t| \to \infty$ and selects the interacting ground state and isolated

one-particle states. The same $i\epsilon$ prescription underlies the Feynman propagator and is operationally equivalent to the adiabatic switching $V \to V\,g(t)$ discussed below; in the LSZ framework (Chapter 8) it returns as the choice of contour for the Fourier transform of the propagator.

The Dyson expansion of the S-matrix. From the time-ordered exponential representation (3.11), the S-matrix has the expansion

$$\begin{aligned} S &= T\exp\left[-i\int_{-\infty}^{+\infty} dt'\, V_I(t')\right] \\ &= \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{+\infty} dt_1\cdots\int_{-\infty}^{+\infty} dt_n\, T\{V_I(t_1)\cdots V_I(t_n)\}. \end{aligned} \tag{3.15}$$

This is the **Dyson expansion of the S-matrix**. It forms the starting point for perturbative calculations in quantum field theory.

Covariant form. Writing the interaction in terms of the interaction Hamiltonian density $\mathscr{H}_I(x)$ via

$$V_I(t) = \int d^3x\, \mathscr{H}_I(t,\mathbf{x})\,,$$

the Dyson expansion takes the manifestly Lorentz-covariant form, provided $\mathscr{H}_I(x)$ is a local Lorentz-scalar interaction density:

$$\boxed{S = \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int d^4x_1\cdots\int d^4x_n\, T\{\mathscr{H}_I(x_1)\cdots\mathscr{H}_I(x_n)\}} \tag{3.16}$$

where the integrations are over all spacetime.

Box 3.1 Why time ordering is compatible with Lorentz covariance

In the covariant form (3.16), the integration measure d^4x and the domain (all of spacetime) are manifestly Lorentz-invariant. The only potential obstacle is the time-ordered product $T\{\mathscr{H}_I(x_1)\cdots\mathscr{H}_I(x_n)\}$: since "which event comes first" depends on the observer, T could introduce frame dependence.
This does not happen for local bosonic interaction densities satisfying **microcausality**. Consider two spacetime points x_1 and x_2:

- *Timelike separation,* $(x_1-x_2)^2 > 0$: all Lorentz observers agree on the temporal ordering, so $T\{\cdots\}$ is unambiguous.
- *Spacelike separation,* $(x_1-x_2)^2 < 0$: different frames may disagree on the ordering. However, the microcausality condition requires that local bosonic densities commute at spacelike separation,

$$[\mathscr{H}_I(x_1), \mathscr{H}_I(x_2)] = 0 \quad \text{for } (x_1-x_2)^2 < 0\,,$$

so the order is irrelevant and $T\{\cdots\}$ gives the same result in every frame.

In contrast, the original nested-integral form of the Dyson series (equation (3.6)) singles out a particular time coordinate and is *not* manifestly covariant. The passage to unrestricted integrals with time ordering is what makes the Lorentz invariance of the S-matrix transparent.

Transition amplitudes. The state $|\Phi(\infty)\rangle$ contains all possible outcomes of the collision. Expanding symbolically in a complete set of final states $|f\rangle$:

$$|\Phi(\infty)\rangle = \sum_f |f\rangle\langle f|\Phi(\infty)\rangle = \sum_f |f\rangle\, S_{fi}\,, \tag{3.17}$$

where the **S-matrix element**

$$S_{fi} = \langle f|S|i\rangle \tag{3.18}$$

is the probability amplitude for the transition $|i\rangle \to |f\rangle$. For continuum scattering states, the sum in (3.17) stands for the appropriate mixture of sums and phase-space integrals.

In practice one separates the trivial "no-scattering" identity contribution by writing

$$S \;=\; \mathbb{1} + iT\,, \tag{3.19}$$

which defines the **transition operator** T (often called the T-matrix). For momentum eigenstates, conservation of total four-momentum is then factored out explicitly:

$$\langle f|T|i\rangle \;=\; (2\pi)^4\,\delta^{(4)}(P_f - P_i)\,\mathcal{M}_{fi}\,, \tag{3.20}$$

with $\mathcal{M}_{fi}$ the *invariant amplitude* (or "reduced" matrix element). The connected, fully amputated, on-shell version of $\mathcal{M}_{fi}$ is what the LSZ reduction formula computes from n-point Green functions (Chapter 8); physical probabilities, rates, and cross sections are then extracted from $|\mathcal{M}_{fi}|^2$ via the standard invariant phase-space formulae (developed in the chapter on cross sections), rather than naively from $|S_{fi}|^2$.

Properties of the S-matrix. From the unitarity of U_I, the S-matrix satisfies:

$$S^\dagger S = S S^\dagger = \mathbb{1}\,. \tag{3.21}$$

For a discrete orthonormal basis this implies probability conservation in the form

$$\sum_f |S_{fi}|^2 = \sum_f \langle i|S^\dagger|f\rangle\langle f|S|i\rangle = \langle i|S^\dagger S|i\rangle = 1\,. \tag{3.22}$$

The adiabatic hypothesis. A subtlety arises in interpreting the initial and final states. In the perturbation formalism, $|i\rangle$ and $|f\rangle$ are eigenstates of the *free* Hamiltonian H_0—they describe "bare" particles without interactions. But physical particles, even when far apart, are surrounded by their self-interactions (an electron by its photon cloud, for example).

This issue is addressed by the **adiabatic hypothesis**: we replace the interaction $V(t)$ by $V(t)g(t)$, where $g(t)$ is a switching function satisfying $g(t) = 1$ for $|t| \leq T$ and $g(t) \to 0$ smoothly as $t \to \pm\infty$. This is a formal device used to define asymptotic in/out states and control the large-time limit. In this picture:

- At $t \ll -T$, the interaction is "switched off," and the system is described by bare-particle states.
- During $-T \leq t \leq T$, the full interaction is active and produces the scattering process.
- At $t \gg T$, the interaction switches off again.

The physical assumption is that observable scattering results are independent of the detailed switching procedure in the remote past and future. One takes the limit $T \to \infty$ at the end. In realistic relativistic QFTs, especially in gauge theories with infrared subtleties, this statement must be understood with care and is often replaced or refined by the LSZ framework.

In many tree-level calculations one often suppresses the explicit switching function and works directly with $V(t)$, with the understanding that adiabatic switching (or an equivalent LSZ prescription) is needed to define asymptotic in/out states and make the S-matrix well defined.