3.5 Cross sections in the Bjorken–Drell conventions

This section develops the master formula for the differential cross section of a generic scattering process $A+B \longrightarrow 1+2+\cdots+n$ within the Bjorken–Drell (BD) conventions [9, 10]. The derivation proceeds systematically from the normalization of states through the definition of the invariant amplitude to the final expression suitable for practical calculations. For a modern treatment, see also Peskin and Schroeder [6]. A concise explanation of how the field expansion, (anti)commutators, and state normalization are tied together is given in Section 1.1.

BD normalization of states. The BD conventions employ non-covariant normalization for one-particle states:

$$\langle \mathbf{p}', s' | \mathbf{p}, s \rangle = \delta^{(3)}(\mathbf{p}-\mathbf{p}')\,\delta_{ss'}, \qquad E_{\mathbf{p}} = \sqrt{\mathbf{p}^2+m^2}\,. \tag{3.23}$$

This normalization is not Lorentz invariant, but the final cross-section formula can be expressed in terms of the invariant phase-space measure $d^3p/[(2\pi)^3 2E_{\mathbf{p}}]$.

Field expansions and commutation relations. The BD normalization propagates into the mode expansions of quantum fields. For a real scalar field,

$$\phi(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2E_{\mathbf{p}}}} \Big(a(\mathbf{p}) e^{-ip\cdot x} + a^\dagger(\mathbf{p}) e^{+ip\cdot x} \Big), \qquad p^0 = E_{\mathbf{p}}, \tag{3.24}$$

with the commutation relation $[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = \delta^{(3)}(\mathbf{p}-\mathbf{p}')$ ensuring that the state $|\mathbf{p}\rangle = a^\dagger(\mathbf{p})|0\rangle$ satisfies the BD normalization. For a Dirac field, the expansion reads

$$\psi(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{m}{E_{\mathbf{p}}}} \sum_s \Big(b_s(\mathbf{p}) u_s(\mathbf{p}) e^{-ip\cdot x} + d_s^\dagger(\mathbf{p}) v_s(\mathbf{p}) e^{+ip\cdot x} \Big), \tag{3.25}$$

with anticommutation relations

$$\{b_s(\mathbf{p}), b_{s'}^\dagger(\mathbf{p}')\} = \{d_s(\mathbf{p}), d_{s'}^\dagger(\mathbf{p}')\} = \delta^{(3)}(\mathbf{p}-\mathbf{p}')\,\delta_{ss'}\,. \tag{3.26}$$

The BD spinor normalizations are $\bar{u}_s(\mathbf{p})u_{s'}(\mathbf{p}) = \delta_{ss'}$ and $u_s^\dagger(\mathbf{p})u_{s'}(\mathbf{p}) = (E_{\mathbf{p}}/m)\,\delta_{ss'}$, with analogous relations for the v-spinors. For a massive vector field,

$$A_\mu(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2E_{\mathbf{p}}}} \sum_\lambda \Big(a_\lambda(\mathbf{p}) \epsilon_\mu(\mathbf{p},\lambda) e^{-ip\cdot x} + a_\lambda^\dagger(\mathbf{p}) \epsilon_\mu^*(\mathbf{p},\lambda) e^{+ip\cdot x} \Big), \tag{3.27}$$

where the polarization sum satisfies $\sum_\lambda \epsilon_\mu(\mathbf{p},\lambda)\epsilon_\nu^*(\mathbf{p},\lambda) = -g_{\mu\nu} + p_\mu p_\nu/m^2$.

The S-matrix and the invariant amplitude. For a scattering process $A(p_A)+B(p_B) \to 1(p_1)+\cdots+n(p_n)$, we define the invariant amplitude $\mathcal{M}$ through the decomposition

$$S_{fi} \equiv \langle f|S|i\rangle = \langle f|i\rangle + i(2\pi)^4\delta^{(4)}\left(p_A+p_B-\sum_{k=1}^{n} p_k\right)\mathcal{M}_{fi}\,. \tag{3.28}$$

The first term represents the possibility that no interaction occurs, while the second contains the dynamics encoded in $\mathcal{M}$. The momentum-conserving delta function is factored out explicitly, leaving $\mathcal{M}$ as a function only of the kinematic invariants.

Transition probability and the $\delta^{(4)}(0)$ regularization. Squaring the transition amplitude yields

$$|S_{fi}-\delta_{fi}|^2 = (2\pi)^8\,\delta^{(4)}(P_i-P_f)^2\,|\mathcal{M}|^2\,. \tag{3.29}$$

The square of the delta function is mathematically ill-defined and must be regularized. Introducing a spacetime box of volume V and duration T, one interprets

$$\delta^{(4)}(P)^2 = \delta^{(4)}(P)\,\delta^{(4)}(0), \qquad \delta^{(4)}(0) = \frac{VT}{(2\pi)^4}, \tag{3.30}$$

so that the transition probability becomes

$$|S_{fi}-\delta_{fi}|^2 = (2\pi)^4\,\delta^{(4)}(P_i-P_f)\,(VT)\,|\mathcal{M}|^2\,. \tag{3.31}$$

The factors of V and T will cancel against normalization factors, yielding a finite cross section.

Box 3.2 Derivation of $\delta^{(4)}(0) = VT/(2\pi)^4$

The four-dimensional delta function has the integral representation

$$\delta^{(4)}(P) = \int \frac{d^4x}{(2\pi)^4}\,e^{iP\cdot x}\,. \tag{3.32}$$

Setting $P=0$:

$$\delta^{(4)}(0) = \int \frac{d^4x}{(2\pi)^4} = \frac{1}{(2\pi)^4}\int d^4x\,. \tag{3.33}$$

In infinite spacetime this integral diverges, reflecting the fact that $\delta(0)$ is undefined as a distribution.

Box regularization. We regularize by confining the system to a finite spacetime region: a spatial box of volume $V=L^3$ and a time interval T. Then

$$\int d^4x = \int_0^T dt\int_V d^3x = VT\,, \tag{3.34}$$

and therefore

$$\boxed{\delta^{(4)}(0) = \frac{VT}{(2\pi)^4}} \tag{3.35}$$

Physical interpretation. This regularization has a clear physical meaning:

- The delta function $\delta^{(4)}(P_i - P_f)$ enforces energy-momentum conservation.
- The factor $\delta^{(4)}(0) = VT/(2\pi)^4$ represents the "phase space volume" available for the conserved total momentum—it counts the number of ways the system can satisfy $P_i = P_f$ within the finite spacetime region.
- As $V, T \to \infty$, individual transition probabilities diverge, but observable quantities like cross sections (probability per unit time per unit flux) remain finite after proper normalization.

The one-dimensional analogue. For a single momentum component, $\delta(p) = \int dx\, e^{ipx}/(2\pi)$, so $\delta(0) = L/(2\pi)$ in a box of length L. The four-dimensional case is simply the product of four such factors.

Phase space and the transition rate. To compute physical transition probabilities from the BD-normalized amplitudes, we must carefully account for the normalization of states.

The Mandl–Shaw convention: $V = (2\pi)^3$. In the BD convention, the continuous-momentum states satisfy $\langle \mathbf{p}'|\mathbf{p}\rangle = \delta^{(3)}(\mathbf{p} - \mathbf{p}')$. To define probabilities we need a finite normalization volume V. Following Mandl and Shaw [1], we set $V = (2\pi)^3$. This is not a physical restriction—V cancels in all observables—but it greatly simplifies the algebra, since the density of states becomes

$$\frac{V\, d^3p}{(2\pi)^3} = d^3p\,, \tag{3.36}$$

and the replacement $\delta^{(3)}(\mathbf{0}) \to V/(2\pi)^3 = 1$ removes all explicit factors of V from intermediate expressions.

Normalization factors per external leg. With $V = (2\pi)^3$, the BD field expansions (3.24)–(3.27) give a specific normalization factor for each external particle in a Feynman amplitude. These factors depend on the *spin* of the field:

Particle type	**Factor per leg**	**Source**
Scalar, photon	$\dfrac{1}{\sqrt{2E}}$	$\dfrac{1}{\sqrt{2EV}} = \dfrac{1}{\sqrt{2E\,(2\pi)^3}}$
Dirac fermion (BD)	$\sqrt{\dfrac{m}{E}}$	$\dfrac{1}{\sqrt{V}}\sqrt{\dfrac{m}{E}} = \dfrac{1}{(2\pi)^{3/2}}\sqrt{\dfrac{m}{E}}$

The crucial difference: a BD fermion carries $\sqrt{m/E}$, not $1/\sqrt{2E}$. The ratio is

$$\frac{\sqrt{m/E}}{1/\sqrt{2E}} = \sqrt{2m}\,. \tag{3.37}$$

This factor $\sqrt{2m}$ per external fermion leg is the price of using BD spinors ($\bar{u}u = 1$) instead of covariantly normalized ones ($\bar{u}u = 2m$).

Squared amplitude in the box. For a process with n_b external bosons and n_f external fermions, the squared amplitude between box-normalized states is:

$$|\mathcal{M}|^2_{\text{box}} = |\mathcal{M}_{\text{BD}}|^2 \times \underbrace{\prod_{i\in\text{bosons}} \frac{1}{2E_i}}_{n_b \text{ factors}} \times \underbrace{\prod_{j\in\text{fermions}} \frac{m_j}{E_j}}_{n_f \text{ factors}}\,. \tag{3.38}$$

Each fermion factor can be rewritten as $m_j/E_j = 2m_j/(2E_j)$. Factoring out $1/(2E)$ from every leg:

$$|\mathcal{M}|^2_{\text{box}} = \left(\prod_{i\in\text{fermions}} 2m_i\right) \times |\mathcal{M}_{\text{BD}}|^2 \times \prod_{\text{all legs}} \frac{1}{2E}\,. \tag{3.39}$$

The extra factor $\prod(2m_i)$ is present for fermions in BD normalization and absent for scalars/photons.

Phase space and the transition rate. Each final-state particle contributes a density of states d^3p_k (with $V = (2\pi)^3$, eq. (3.36)) times the normalization factor $1/(2E_k)$:

$$d^3p_k \times \frac{1}{2E_k} = \frac{d^3p_k}{2E_k}\,, \tag{3.40}$$

which is manifestly Lorentz invariant (the restriction of d^4p to the mass shell).

The total number of transitions into the phase-space element is therefore

$$dN = (2\pi)^4\delta^{(4)}(P_i - P_f)\,T\,\frac{|\mathcal{M}_{\text{BD}}|^2}{4E_AE_B}\left(\prod_{i\in\text{fermions}} 2m_i\right)\prod_{k=1}^{n}\frac{d^3p_k}{2E_k}\,, \tag{3.41}$$

where the factor T comes from $\delta^{(4)}(0) = T/(2\pi)^4 \times (2\pi)^3 = T$ with $V = (2\pi)^3$, and the $1/(4E_A E_B)$ from the two initial-state normalization factors.

Dividing (3.41) by T gives the transition rate dN/T. The cross section is defined as this rate divided by the incident flux. With $V = (2\pi)^3$ the factor T cancels and no explicit V remains.

The incident flux. With one particle of each type in the volume $V = (2\pi)^3$, the incident flux is

$$\text{flux} = \frac{|\mathbf{v}_A - \mathbf{v}_B|}{V} = \frac{|\mathbf{v}_A - \mathbf{v}_B|}{(2\pi)^3}, \qquad \mathbf{v}_i = \frac{\mathbf{p}_i}{E_i}. \tag{3.42}$$

The differential cross section is then:

$$d\sigma = \frac{dN/T}{\text{flux}} = \frac{(2\pi)^3}{|\mathbf{v}_A - \mathbf{v}_B|} \cdot \frac{1}{4E_A E_B} \left(\prod_{i \in \text{ferm.}} 2m_i \right) |\mathcal{M}_{\text{BD}}|^2 \cdot (2\pi)^4 \delta^{(4)} \prod_{k=1}^{n} \frac{d^3 p_k}{2E_k}. \tag{3.43}$$

Where do the energies E_A, E_B come from? They arise from the normalisation of the **initial-state particles**: each particle in the box contributes a factor $1/(2E)$ (for bosons) or $m/E = 2m/(2E)$ (for BD fermions) to the squared amplitude (eq. (3.38)). The product of these two initial-state factors gives $1/(2E_A \cdot 2E_B) = 1/(4E_A E_B)$ in (3.41). So the combination that appears in the cross section is:

$$\frac{1}{4E_A E_B} \cdot \frac{1}{\text{flux}} = \frac{1}{4E_A E_B} \cdot \frac{V}{|\mathbf{v}_A - \mathbf{v}_B|} = \frac{1}{4E_A E_B\, |\mathbf{v}_A - \mathbf{v}_B|/(2\pi)^3}. \tag{3.44}$$

We see that $E_A E_B$ is *not* part of the flux: it comes from the initial-state normalisation. The physical flux is just $|\mathbf{v}_A - \mathbf{v}_B|/V$.

The Lorentz-invariant phase-space element. It is convenient to define

$$d\Phi_n(P_i; p_1, \ldots, p_n) \equiv (2\pi)^4 \delta^{(4)} \left(P_i - \sum_{k=1}^{n} p_k \right) \prod_{k=1}^{n} \frac{d^3 p_k}{(2\pi)^3\, 2E_k}, \tag{3.45}$$

which transforms as a scalar under Lorentz transformations (each factor $d^3p/(2E)$ is the restriction of d^4p to the forward mass shell, and the delta function is Lorentz-invariant since $|\det \Lambda| = 1$).

Comparing (3.43) with this definition, the factor $(2\pi)^4 \delta^{(4)} \prod d^3 p_k/(2E_k)$ in (3.43) differs from $d\Phi_n$ by $(2\pi)^{3n}$. Adding the $(2\pi)^3$ from 1/flux, we collect a total of $(2\pi)^{3n+3}$ which must be absorbed somewhere. Explicitly, for $n = 2$:

$$\frac{(2\pi)^3}{|\mathbf{v}|} \cdot (2\pi)^4 \delta^{(4)} \prod_{k=1}^{2} \frac{d^3 p_k}{2E_k} = \frac{(2\pi)^3}{|\mathbf{v}|} \cdot (2\pi)^6 \cdot d\Phi_2 = \frac{(2\pi)^9}{|\mathbf{v}|} d\Phi_2. \tag{3.46}$$

Where do the $(2\pi)^9$ go? They are absorbed into the *box-to-stripped amplitude conversion* that was already performed in passing from $|\mathcal{M}|^2_{\text{box}}$ to $|\mathcal{M}_{\text{BD}}|^2$. Specifically:

- Each external particle's box-normalised wavefunction carries a factor $1/\sqrt{(2\pi)^3}$ (from $V = (2\pi)^3$, see eq. (3.36)). For a $2 \to 2$ process with 4 external legs, $|\mathcal{M}|^2_{\text{box}}$ therefore contains $(2\pi)^{-12}$ relative to the *stripped* amplitude $|\mathcal{M}_{\text{BD}}|^2$ that appears in the master formula (the latter is computed with spinors $\bar{u}u = 1$ and *no* explicit wavefunction prefactors).
- The replacement $\delta^{(3)}(\mathbf{0}) \to V/(2\pi)^3 = 1$ in the transition rate dN/T removes a further factor $(2\pi)^3$, and the box volume $V = (2\pi)^3$ in the flux denominator removes another $(2\pi)^{-3}$.

The net effect: the $(2\pi)^9$ that appeared above is matched, term by term, by the combined effect of $V = (2\pi)^3$ in the density of states, in the flux, and in the $\delta^{(3)}(0)$ replacement – i.e. by the conversion of the box-normalised matrix element $\mathcal{M}_{\text{box}}$ into the stripped BD amplitude $\mathcal{M}_{\text{BD}}$. *$\mathcal{M}_{BD}$ itself contains no (2π) wavefunction factors – they have already been peeled off when one writes eq. (3.38). Crucially, the (2π) factors do **not** enter the kinematic flux factor $\mathcal{F}$.*

Convention choice for the rest of these notes. From this point on we keep the BD convention consistently. The master formula will be written as

$$d\sigma = \frac{1}{\mathcal{F}} \Big(\prod_{i \in \text{ferm.}} 2m_i \Big) |\mathcal{M}_{\text{BD}}|^2 \, d\Phi_n ,$$

where $|\mathcal{M}_{\text{BD}}|^2$ uses BD-normalised spinors ($\bar{u}u = 1$), and the kinematic factor $\mathcal{F} = 4E_A E_B |\mathbf{v}|$ carries no (2π) powers. (Readers who prefer the covariant convention ($\bar{u}u = 2m$) can convert at any time via $\mathcal{M}_{\text{cov}} = \prod_{\text{ferm.}} \sqrt{2m_i}\, \mathcal{M}_{\text{BD}}$, which absorbs the explicit $\prod 2m_i$ factor and yields the equivalent form $d\sigma = |\mathcal{M}_{\text{cov}}|^2 d\Phi_n/\mathcal{F}$.)

Box 3.3 Dimensional analysis in natural units ($\hbar = c = 1$)

Every factor has a well-defined mass dimension:

Quantity	Dimension	Why
$\lvert\mathbf{v}_A - \mathbf{v}_B\rvert$	0	velocity is dimensionless
$E_A E_B$	+2	product of two energies
$\mathcal{F} = 4E_A E_B \lvert\mathbf{v}\rvert$	+2	energy^2
$\lvert\mathcal{M}_{\text{BD}}\rvert^2 \prod 2m_i$	0	dimensionless for $2 \to 2$
$d\Phi_2$	0	$(E^{-4})(E^{+2})^2 = E^0$
$d\sigma$	−2	area = length^2 = energy^{-2} ✓

$\mathcal{F}$ has dimensions of energy^2 and carries *no* (2π) factors. The two powers of energy come from the initial-state normalisation $(1/(2E_A) \cdot 1/(2E_B))$ in the box amplitude), not from the flux.

The invariant flux factor. The kinematic factor introduced above is

$$\mathcal{F} = 4E_A E_B\,|\mathbf{v}_A - \mathbf{v}_B|\,, \tag{3.47}$$

with $4E_A E_B$ coming from the initial-state normalisation and $|\mathbf{v}_A - \mathbf{v}_B|$ from the physical flux.

We now show that this can be written as a Lorentz invariant. Since $\mathbf{v}_i = \mathbf{p}_i/E_i$:

$$E_A E_B\,|\mathbf{v}_A - \mathbf{v}_B| = E_A E_B\left|\frac{\mathbf{p}_A}{E_A} - \frac{\mathbf{p}_B}{E_B}\right| = |E_B\,\mathbf{p}_A - E_A\,\mathbf{p}_B|\,. \tag{3.48}$$

Squaring:

$$(E_A E_B)^2\,|\mathbf{v}_A - \mathbf{v}_B|^2 = E_B^2\,|\mathbf{p}_A|^2 + E_A^2\,|\mathbf{p}_B|^2 - 2E_A E_B\,(\mathbf{p}_A \cdot \mathbf{p}_B)\,. \tag{3.49}$$

Now, $p_A \cdot p_B = E_A E_B - \mathbf{p}_A \cdot \mathbf{p}_B$, so $\mathbf{p}_A \cdot \mathbf{p}_B = E_A E_B - p_A \cdot p_B$. Also, $|\mathbf{p}_i|^2 = E_i^2 - m_i^2$. Substituting:

$$\begin{aligned}
&= E_B^2(E_A^2 - m_A^2) + E_A^2(E_B^2 - m_B^2) - 2E_A E_B(E_A E_B - p_A \cdot p_B)\\
&= 2E_A^2 E_B^2 - E_B^2 m_A^2 - E_A^2 m_B^2 - 2E_A^2 E_B^2 + 2E_A E_B(p_A \cdot p_B)\\
&= 2E_A E_B(p_A \cdot p_B) - m_A^2 E_B^2 - m_B^2 E_A^2\,.
\end{aligned} \tag{3.50}$$

We want to compare this with $(p_A \cdot p_B)^2 - m_A^2 m_B^2$. Expanding:

$$\begin{aligned}
(p_A \cdot p_B)^2 &= (E_A E_B - \mathbf{p}_A \cdot \mathbf{p}_B)^2\\
&= E_A^2 E_B^2 - 2E_A E_B(\mathbf{p}_A \cdot \mathbf{p}_B) + (\mathbf{p}_A \cdot \mathbf{p}_B)^2\,.
\end{aligned} \tag{3.51}$$

Therefore:

$$(p_A \cdot p_B)^2 - m_A^2 m_B^2 = E_A^2 E_B^2 - 2E_A E_B(\mathbf{p}_A \cdot \mathbf{p}_B) + (\mathbf{p}_A \cdot \mathbf{p}_B)^2 - m_A^2 m_B^2\,. \tag{3.52}$$

This expression is *not* equal to (3.50) in general, because of the $(\mathbf{p}_A \cdot \mathbf{p}_B)^2$ term. The identity $E_A E_B|\mathbf{v}_A - \mathbf{v}_B| = \sqrt{(p_A \cdot p_B)^2 - m_A^2 m_B^2}$ holds exactly only when $\mathbf{p}_A$ and $\mathbf{p}_B$ are *collinear* (parallel or antiparallel), which is always the case in the CM frame ($\mathbf{p}_B = -\mathbf{p}_A$) and in any collinear frame. In that case $(\mathbf{p}_A \cdot \mathbf{p}_B)^2 = |\mathbf{p}_A|^2|\mathbf{p}_B|^2$, and one can verify that (3.52) reduces to (3.50).

The **invariant flux factor** is defined as the Lorentz-invariant quantity that equals $4E_A E_B|\mathbf{v}_A - \mathbf{v}_B|$ in any collinear frame:

$$\boxed{\mathcal{F} \equiv 4\sqrt{(p_A \cdot p_B)^2 - m_A^2 m_B^2}}\,. \tag{3.53}$$

In the **center-of-mass frame**, $\mathbf{p}_A = -\mathbf{p}_B \equiv \vec{p}$, so $|\mathbf{v}_A - \mathbf{v}_B| = |\vec{p}\,|(1/E_A + 1/E_B) = |\vec{p}\,|(E_A + E_B)/(E_A E_B)$, and therefore:

$$\mathcal{F} = 4E_A E_B \cdot \frac{|\vec{p}\,|(E_A + E_B)}{E_A E_B} = 4|\vec{p}\,|(E_A + E_B) = 4|\vec{p}\,|\sqrt{s}\,. \tag{3.54}$$

The BD master formula for the cross section. Combining the transition rate (3.41) with the flux, the cross section is:

$$\boxed{d\sigma(AB \to 1\cdots n) = \frac{1}{\mathcal{F}} \left(\prod_{i \in \text{fermions}} 2m_i \right) |\mathcal{M}_{\text{BD}}|^2 \, d\Phi_n} \, . \tag{3.55}$$

Here $\mathcal{M}_{\text{BD}}$ is the amplitude computed with BD spinors ($\bar{u}u = 1$). The product $\prod(2m_i)$ runs over *all* external fermion legs (initial and final). For processes involving only scalars or photons it equals 1.

Box 3.4 Origin and cancellation of the $\prod(2m_i)$ factor

The factor $\prod(2m_i)$ arises because the BD Dirac field expansion (3.25) has $\sqrt{m/E}$ per external fermion, whereas the covariant normalization ($\bar{u}u = 2m$) would give $1/\sqrt{2E}$. Their ratio is $\sqrt{2m}$, so at the amplitude-squared level each fermion contributes a factor $2m$.
When computing $\overline{|\mathcal{M}_{\text{BD}}|^2}$ using the BD spin sums $\sum_s u_s \bar{u}_s = (\not{p} + m)/(2m)$, each fermion–antifermion pair produces a factor $1/(2m)^2$ in the leptonic tensors. For a process with n_f external fermions (always paired), the product of these factors is $1/\prod(2m_i)$.
The $(2m)$ factors in the cross section formula and the $1/(2m)$ factors in the spin sums **cancel exactly**:

$$\prod_{i \in \text{fermions}} 2m_i \times \overline{|\mathcal{M}_{\text{BD}}|^2}\Bigg|_{\text{contains } \prod 1/(2m_i)} = (\text{pure traces, no mass prefactors})\, . \tag{3.56}$$

This cancellation is guaranteed: the cross section is a physical observable, independent of the spinor normalization convention. Equivalently, defining the covariantly normalized amplitude $\mathcal{M}_{\text{cov}} = \prod \sqrt{2m_i}\, \mathcal{M}_{\text{BD}}$, the master formula reduces to the standard form $d\sigma = |\mathcal{M}_{\text{cov}}|^2\, d\Phi_n/\mathcal{F}$.

Spin and color averages. The kinematic prefactors derived above are universal, determined solely by the BD normalization. The dependence on the specific field content enters through the invariant amplitude $\mathcal{M}$ and through the treatment of discrete quantum numbers. For unpolarized initial states, one averages over the initial spins, colors, and polarizations:

$$|\mathcal{M}|^2 \longrightarrow \overline{|\mathcal{M}|^2} = \frac{1}{g_A g_B} \sum_{\text{spins, colors, pols}} |\mathcal{M}|^2, \qquad g_i = (2s_i + 1) \times (\text{color mult.}) \times \cdots \, . \tag{3.57}$$

The BD spin sums for external fermions are

$$\sum_s u_s(p)\bar{u}_s(p) = \frac{\not{p}+m}{2m}, \qquad \sum_s v_s(p)\bar{v}_s(p) = \frac{\not{p}-m}{2m}, \tag{3.58}$$

where each $1/(2m)$ originates from the BD normalization $\bar{u}u = 1$. These factors, together with the $\prod(2m_i)$ in eq. (3.55), cancel in the final cross section (see the box above). For massive vectors the polarization sum was given above. For massless gauge bosons, appropriate physical polarization prescriptions or gauge-fixing conditions must be employed.

Identical particles and symmetry factors. When the final state contains identical particles, the phase-space integration overcounts physically indistinguishable configurations. If among the n final particles there are groups of identical particles with multiplicities $n_1, n_2, \ldots$, the cross section must include the symmetry factor

$$d\sigma \;\to\; \frac{1}{n_1!\,n_2!\cdots}\,d\sigma\,. \tag{3.59}$$

The complete formula. Collecting all factors, the differential cross section for an unpolarized scattering process with possible identical particles in the final state is

$$\boxed{d\sigma = \frac{1}{\mathcal{F}}\cdot\frac{1}{n_1!\,n_2!\cdots}\cdot\left(\prod_{i\in\text{fermions}} 2m_i\right)\cdot\overline{|\mathcal{M}_{\text{BD}}|^2}\cdot d\Phi_n} \tag{3.60}$$

This expression serves as the starting point for all cross-section calculations in this course.

4 Feynman Rules

The Feynman rules provide a systematic algorithm for computing scattering amplitudes in quantum field theory. Rather than evaluating complicated operator products and time-ordered integrals directly, we translate each term in the perturbative expansion into a pictorial *Feynman diagram*, and then associate mathematical factors with each element of the diagram. This chapter derives the Feynman rules from first principles, applying the Dyson series [8] (developed in Chapter 3) together with Wick's theorem. For complementary and particularly clear presentations, see also Peskin and Schroeder [6], Srednicki [11], and Schwartz [12].

4.1 The perturbative expansion of the S-matrix

As derived in the previous chapter, the S-matrix in the interaction picture is given by the Dyson series:

$$S = T\exp\left(-i\int d^4x\,\mathcal{H}_I(x)\right) = \sum_{n=0}^{\infty} S^{(n)}, \tag{4.1}$$

where the n-th order contribution is:

$$\boxed{S^{(n)} = \frac{(-i)^n}{n!}\int d^4x_1\cdots d^4x_n\,T\{\mathcal{H}_I(x_1)\cdots\mathcal{H}_I(x_n)\}}\,. \tag{4.2}$$

For QED, the interaction Hamiltonian density is:

$$\mathcal{H}_I(x) = -\mathcal{L}_I(x) = -e\,:\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x):, \tag{4.3}$$

where the normal ordering $:\cdots:$ is implicit in the definition of $\mathcal{H}_I$. The coupling constant $e > 0$ is the positron charge (in natural units, $e = \sqrt{4\pi\alpha} \approx 0.303$).

To compute $\langle f|S|i\rangle$ for a scattering process, we must evaluate vacuum expectation values of time-ordered products of field operators. This can be seen as follows. The initial and final states $|i\rangle$, $|f\rangle$ are multi-particle states created by acting with creation operators on the vacuum: for instance, for a real scalar field, $|i\rangle = a^\dagger(\vec{p}_1)a^\dagger(\vec{p}_2)|0\rangle$, where $a^\dagger(\vec{p})$ is the creation operator appearing in the mode expansion (1.13) (and analogously for spinor and vector fields). The $S^{(n)}$ term in the Dyson series contains a time-ordered product of n interaction Hamiltonians, each of which is built from field operators. Since the fields themselves are sums of creation and annihilation operators, the matrix element $\langle f|S^{(n)}|i\rangle$ reduces to a vacuum expectation value: the creation operators in $|i\rangle$ and $\langle f|$ can be traded for additional field operators using the relation $a^\dagger(\vec{p})|0\rangle = (2\pi)^{3/2}\sqrt{2E_p}\,\phi^{(-)}(\vec{p})|0\rangle$, so that ultimately we need to evaluate expressions of the form $\langle 0|T\{\text{fields}\}|0\rangle$. This is accomplished by **Wick's theorem**.

4.2 Wick's theorem

As argued above, the computation of scattering amplitudes $\langle f|S|i\rangle$ ultimately requires the evaluation of vacuum expectation values of time-ordered products of field operators. Wick's theorem is the fundamental tool that makes this computation feasible: it reduces complex operator products to sums of c-numbers (propagators) and normal-ordered terms, whose vacuum expectation value vanishes. The mode decompositions below are written in the BD convention; see Section 1.1 for how the expansion factors, (anti)commutators, and state normalizations fit together.

Normal ordering

Definition. For any product of field operators, the *normal-ordered* form $:\mathcal{O}:$ is obtained by moving all creation operators to the left of all annihilation operators, *ignoring* any (anti)commutators that would arise. For fermionic fields, each transposition contributes a factor (-1).

Decomposition of fields. Any free field can be written as $\phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x)$, where:

- $\phi^{(+)}(x)$ contains annihilation operators (positive frequency: $e^{-ip\cdot x}$)
- $\phi^{(-)}(x)$ contains creation operators (negative frequency: $e^{+ip\cdot x}$)

For a real scalar field:

$$\phi^{(+)}(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2E_p}}\, a(\vec{p})\, e^{-ip\cdot x} \tag{4.4}$$

$$\phi^{(-)}(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2E_p}}\, a^\dagger(\vec{p})\, e^{+ip\cdot x}\,. \tag{4.5}$$

For a Dirac field:

$$\psi^{(+)}(x) = \int \frac{d^3p}{(2\pi)^{3/2}}\sqrt{\frac{m}{E_p}}\sum_s b_s(\vec{p})\, u_s(p)\, e^{-ip\cdot x}\,, \tag{4.6}$$

$$\psi^{(-)}(x) = \int \frac{d^3p}{(2\pi)^{3/2}}\sqrt{\frac{m}{E_p}}\sum_s d_s^\dagger(\vec{p})\, v_s(p)\, e^{+ip\cdot x}\,. \tag{4.7}$$

Examples of normal ordering.

$$\begin{aligned}
:\phi(x)\phi(y): &=: (\phi^{(+)}(x) + \phi^{(-)}(x))(\phi^{(+)}(y) + \phi^{(-)}(y)): \\
&=: \phi^{(-)}(x)\phi^{(-)}(y) + \phi^{(-)}(x)\phi^{(+)}(y) \\
&\quad + \phi^{(+)}(x)\phi^{(-)}(y) + \phi^{(+)}(x)\phi^{(+)}(y): \\
&= \phi^{(-)}(x)\phi^{(-)}(y) + \phi^{(-)}(x)\phi^{(+)}(y) \\
&\quad + \phi^{(-)}(y)\phi^{(+)}(x) + \phi^{(+)}(x)\phi^{(+)}(y)\,.
\end{aligned} \tag{4.8}$$

In the second step we have simply expanded the product, but the normal-ordering symbol $:\cdots:$ still applies to the whole expression. In the third step, we *execute* the normal ordering: the first, second, and fourth terms are already normal-ordered (creation operators $\phi^{(-)}$ appear to the left of annihilation operators $\phi^{(+)}$), while the third term $\phi^{(+)}(x)\phi^{(-)}(y)$ is not. Normal ordering means we rewrite it as $\phi^{(-)}(y)\phi^{(+)}(x)$ by moving the creation operator $\phi^{(-)}(y)$ to the left, *without* introducing any commutator (this is what "ignoring commutators" means in the definition of normal ordering).

For fermions, with $\psi = \psi^{(+)} + \psi^{(-)}$ and $\bar{\psi} = \bar{\psi}^{(+)} + \bar{\psi}^{(-)}$:

$$:\bar{\psi}\psi: = \bar{\psi}^{(-)}\psi^{(-)} + \bar{\psi}^{(-)}\psi^{(+)} - \psi^{(-)}\bar{\psi}^{(+)} + \bar{\psi}^{(+)}\psi^{(+)}\,. \tag{4.9}$$

The minus sign arises from anticommuting $\bar{\psi}^{(+)}$ past $\psi^{(-)}$.

Key property.

$$\boxed{\langle 0|:\mathcal{O}:|0\rangle = 0}\,. \tag{4.10}$$

This holds because all annihilation operators act on $|0\rangle$ giving zero, and all creation operators act on $\langle 0|$ giving zero.

Contraction

Definition. The *contraction* of two fields is defined as:

$$\phi(x)\phi(y) \equiv T\{\phi(x)\phi(y)\} - :\phi(x)\phi(y):\ . \tag{4.11}$$

Explicit calculation for scalar fields. The key identity relating the T-product and normal ordering emerges from a straightforward calculation. We present it for both time orderings to show that the result is universal.

Case 1: $x^0 > y^0$. In this case $T\{\phi(x)\phi(y)\} = \phi(x)\phi(y)$. Expanding:

$$\begin{aligned}
\phi(x)\phi(y) &= \left(\phi^{(+)}(x) + \phi^{(-)}(x)\right)\left(\phi^{(+)}(y) + \phi^{(-)}(y)\right) \\
&= \phi^{(+)}(x)\phi^{(+)}(y) + \phi^{(-)}(x)\phi^{(+)}(y) \\
&\quad + \phi^{(+)}(x)\phi^{(-)}(y) + \phi^{(-)}(x)\phi^{(-)}(y)\,.
\end{aligned} \tag{4.12}$$

The first, second, and fourth terms are already normal ordered. The third term is not: it has an annihilation operator ($\phi^{(+)}$) to the left of a creation operator ($\phi^{(-)}$). We reorder it using the commutator:

$$\phi^{(+)}(x)\phi^{(-)}(y) = \phi^{(-)}(y)\phi^{(+)}(x) + [\phi^{(+)}(x), \phi^{(-)}(y)]\,. \tag{4.13}$$

Since $[\phi^{(+)}(x), \phi^{(-)}(y)]$ is a c-number (as shown in the box below), we find:

Box 4.1 The commutator $[\phi^{(+)}(x), \phi^{(-)}(y)]$ is a c-number

Using the mode expansions (4.4) and (4.5):

$$\begin{aligned}[\phi^{(+)}(x), \phi^{(-)}(y)] &= \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2E_p}} \int \frac{d^3p'}{(2\pi)^{3/2}\sqrt{2E_{p'}}} \\ &\quad \times \underbrace{[a(\vec{p}), a^\dagger(\vec{p}\,')]}_{=\delta^{(3)}(\vec{p}-\vec{p}\,')} \, e^{-ip\cdot x}\, e^{+ip'\cdot y} \\ &= \int \frac{d^3p}{(2\pi)^3\, 2E_p} e^{-ip\cdot(x-y)}\,. \end{aligned} \tag{4.14}$$

This is a pure number (a c-number): it commutes with all operators, depends only on the spacetime separation $x-y$, and contains no creation or annihilation operators. It is, in fact, the positive-frequency Wightman function $D^{(+)}(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle$ (only the $\phi^{(+)}\phi^{(-)}$ piece survives in the VEV). The key point is that the commutator $[a(\vec{p}), a^\dagger(\vec{p}\,')]$ produces a delta function, which eliminates one of the two momentum integrals and leaves a purely numerical function of the coordinates.

$$T\{\phi(x)\phi(y)\} =: \phi(x)\phi(y) : + [\phi^{(+)}(x), \phi^{(-)}(y)] \qquad (x^0 > y^0)\,. \tag{4.15}$$

Case 2: $y^0 > x^0$. Now $T\{\phi(x)\phi(y)\} = \phi(y)\phi(x)$ (for bosons). Repeating the same steps with $x \leftrightarrow y$:

$$\begin{aligned} T\{\phi(x)\phi(y)\} &=: \phi(y)\phi(x) : + [\phi^{(+)}(y), \phi^{(-)}(x)] \\ &=: \phi(x)\phi(y) : + [\phi^{(+)}(y), \phi^{(-)}(x)]\,, \end{aligned} \tag{4.16}$$

where in the last step we used the fact that bosonic fields can be freely reordered inside the normal ordering.

Combining both cases. Equations (4.15) and (4.16) can be written as a single formula:

$$\begin{aligned} T\{\phi(x)\phi(y)\} &=: \phi(x)\phi(y) : \\ &\quad + [\phi^{(+)}(x), \phi^{(-)}(y)]\,\theta(x^0-y^0) + [\phi^{(+)}(y), \phi^{(-)}(x)]\,\theta(y^0-x^0)\,. \end{aligned} \tag{4.17}$$

We have established that

$$T\{\phi(x)\phi(y)\} =: \phi(x)\phi(y): + \underbrace{c\text{-number function of } (x-y)}_{\text{contraction } \phi(x)\phi(y)} . \tag{4.18}$$

We now show that the c-number remainder equals the vacuum expectation value of the time-ordered product. The argument is simple: take the vacuum expectation value of both sides of (4.18). On the left-hand side we get $\langle 0|T\{\phi(x)\phi(y)\}|0\rangle$. On the right-hand side, the normal-ordered term vanishes because in a normal-ordered product all annihilation operators stand to the right and destroy $|0\rangle$:

$$\langle 0| : \phi(x)\phi(y) : |0\rangle = 0 , \tag{4.19}$$

while the c-number passes through the VEV unchanged: $\langle 0|c|0\rangle = c \cdot \langle 0|0\rangle = c$. Therefore:

$$\boxed{\phi(x)\phi(y) = T\{\phi(x)\phi(y)\} - :\phi(x)\phi(y): = \langle 0|T\{\phi(x)\phi(y)\}|0\rangle} \tag{4.20}$$

This is a key result: the contraction of two fields—defined algebraically as what remains after subtracting the normal-ordered product from the time-ordered product—is equal to the **Feynman propagator**, which has a direct physical interpretation as the amplitude for a particle to propagate from y to x (or from x to y).

The contraction equals the Feynman propagator.

$$\boxed{\phi(x)\phi(y) = \langle 0|T\{\phi(x)\phi(y)\}|0\rangle = i\Delta_F(x-y)} . \tag{4.21}$$

Explicit calculation for Dirac fields. For fermions, the calculation parallels the scalar case but with anticommutators replacing commutators. Consider $x^0 > y^0$, so $T\{\psi_\alpha(x)\bar{\psi}_\beta(y)\} = \psi_\alpha(x)\bar{\psi}_\beta(y)$. Expanding:

$$\begin{aligned}\psi_\alpha(x)\bar{\psi}_\beta(y) &= \left(\psi_\alpha^{(+)}(x) + \psi_\alpha^{(-)}(x)\right)\left(\bar{\psi}_\beta^{(+)}(y) + \bar{\psi}_\beta^{(-)}(y)\right) \\ &= \psi_\alpha^{(+)}\bar{\psi}_\beta^{(+)} + \psi_\alpha^{(-)}\bar{\psi}_\beta^{(+)} + \psi_\alpha^{(+)}\bar{\psi}_\beta^{(-)} + \psi_\alpha^{(-)}\bar{\psi}_\beta^{(-)} .\end{aligned} \tag{4.22}$$

The problematic term is $\psi_\alpha^{(+)}\bar{\psi}_\beta^{(-)}$. To normal-order it, we use the anticommutator (note the sign):

$$\begin{aligned}\psi_\alpha^{(+)}(x)\bar{\psi}_\beta^{(-)}(y) = -\bar{\psi}_\beta^{(-)}(y)\psi_\alpha^{(+)}(x) \\ + \{\psi_\alpha^{(+)}(x), \bar{\psi}_\beta^{(-)}(y)\} .\end{aligned} \tag{4.23}$$

The anticommutator is again a c-number. Collecting all terms:

$$
\begin{aligned}
T\{\psi_\alpha(x)\bar{\psi}_\beta(y)\} = &:\psi_\alpha(x)\bar{\psi}_\beta(y): \\
&+ \{\psi_\alpha^{(+)}(x), \bar{\psi}_\beta^{(-)}(y)\} \qquad (x^0 > y^0)\,.
\end{aligned} \tag{4.24}
$$

As in the scalar case, combining both time orderings and taking the vacuum expectation value gives:

$$
\overbracket{\psi_\alpha(x)\bar{\psi}_\beta}(y) = \langle 0|T\{\psi_\alpha(x)\bar{\psi}_\beta(y)\}|0\rangle = iS_{F,\alpha\beta}(x-y)\,. \tag{4.25}
$$

An important sign arises for the reversed contraction: $\overbracket{\bar{\psi}_\beta(y)\psi_\alpha}(x) = -\overbracket{\psi_\alpha(x)\bar{\psi}_\beta}(y)$, because fermionic fields must be anticommuted to bring them into the standard order.

Contractions for QED fields. The non-vanishing contractions are:

Contraction	**Propagator**
$\overbracket{\phi(x)\phi}(y)$	$i\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)}$
$\overbracket{\psi_\alpha(x)\bar{\psi}_\beta}(y)$	$iS_{F,\alpha\beta}(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p}+m)_{\alpha\beta}}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)}$
$\overbracket{A_\mu(x)A_\nu}(y)$	$iD_F^{\mu\nu}(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{-ig^{\mu\nu}}{p^2 + i\epsilon} e^{-ip\cdot(x-y)}$

Important: $\overbracket{\bar{\psi}(x)\psi}(y) = -\overbracket{\psi(y)\bar{\psi}}(x) = -iS_F(y-x)$ due to fermionic anticommutation.

Vanishing contractions. The following contractions are zero:

$$
\overbracket{\psi\psi} = 0, \quad \overbracket{\bar{\psi}\bar{\psi}} = 0, \quad \overbracket{\psi A} = 0, \quad \overbracket{\bar{\psi}A} = 0\,. \tag{4.26}
$$

These vanish because the corresponding vacuum two-point functions are zero: there is no non-vanishing propagator connecting those field species and charge assignments.

Statement of Wick's theorem

For n field operators $\phi_1, \phi_2, \ldots, \phi_n$ (each may be ϕ, ψ, $\bar{\psi}$, A_μ, etc.):

$$
\begin{aligned}
T\{\phi_1 \cdots \phi_n\} = &: \phi_1 \cdots \phi_n : \\
&+ \sum_{i<j} (-1)^{P_{ij}} \, \overbrace{\phi_i \phi_j} \; : \prod_{\substack{k=1 \\ k \neq i,j}}^{n} \phi_k : \\
&+ \sum_{\substack{(i_1<j_1),(i_2<j_2) \\ \text{all distinct}}} (-1)^P \, \overbrace{\phi_{i_1} \phi_{j_1}} \, \overbrace{\phi_{i_2} \phi_{j_2}} \; : \prod_{k \notin \{i_1,j_1,i_2,j_2\}} \phi_k : \\
&+ \cdots
\end{aligned}
\tag{4.27}
$$

where $(-1)^P$ is the sign of the permutation of *fermionic* fields needed to bring each contracted pair together from the original ordering. For purely bosonic fields, all signs are $+1$.

Interpretation:

- The first line is the fully normal-ordered product (no contractions).
- The second line sums over all $\binom{n}{2}$ ways of choosing one pair: the pair is replaced by its contraction (c-number = propagator), while the remaining $n-2$ fields are normal-ordered. The sign $(-1)^{P_{ij}}$ counts the number of fermionic transpositions required to move ϕ_i and ϕ_j next to each other.
- The third line sums over all ways of choosing two disjoint pairs, and so on.
- Continue until all fields are contracted (n even) or one field remains (n odd).

Box 4.2 Proof of Wick's theorem

We prove by induction on the number of fields n.

Base case ($n = 2$). By definition of contraction:

$$
T\{\phi_1 \phi_2\} = : \phi_1 \phi_2 : + \overbrace{\phi_1 \phi_2} \, . \tag{4.28}
$$

This is exactly Wick's theorem for two fields. ✓

Inductive hypothesis. Assume Wick's theorem holds for $n-1$ fields.

Inductive step. We must prove it for n fields. Without loss of generality, assume t_n is the *earliest* time (if not, relabel). With the convention that time ordering puts the later field to the left, the earliest field is rightmost, so we can pull ϕ_n out on the right:

$$
T\{\phi_1 \cdots \phi_n\} = T\{\phi_1 \cdots \phi_{n-1}\} \cdot \phi_n \, , \tag{4.29}
$$

where ϕ_n now sits to the right of the time-ordered product of the remaining $n-1$ fields.

Decompose the field into creation and annihilation parts: $\phi_n = \phi_n^{(+)} + \phi_n^{(-)}$ where $\phi_n^{(+)}$ contains annihilation operators and $\phi_n^{(-)}$ contains creation operators. By the inductive hypothesis:

$$T\{\phi_1 \cdots \phi_{n-1}\} =: \phi_1 \cdots \phi_{n-1} : + (\text{terms with contractions})\,. \tag{4.30}$$

Consider the normal-ordered term $:\phi_1 \cdots \phi_{n-1}:$ multiplied by ϕ_n:

$$:\phi_1 \cdots \phi_{n-1} : \cdot \phi_n =: \phi_1 \cdots \phi_{n-1} : \cdot (\phi_n^{(+)} + \phi_n^{(-)}) \tag{4.31}$$

The $\phi_n^{(+)}$ part: Annihilation operators can be moved inside the normal ordering:

$$:\phi_1 \cdots \phi_{n-1} : \cdot \phi_n^{(+)} =: \phi_1 \cdots \phi_{n-1} \phi_n^{(+)} : \,. \tag{4.32}$$

The $\phi_n^{(-)}$ part: Creation operators must be moved to the left. For each field ϕ_j in the normal-ordered product:

$$\phi_j \phi_n^{(-)} = \phi_n^{(-)} \phi_j \pm [\phi_j, \phi_n^{(-)}]_{\pm}\,, \tag{4.33}$$

where $[\cdot,\cdot]_+$ is the anticommutator for fermions and $[\cdot,\cdot]_-$ is the commutator for bosons.
The (anti)commutator $[\phi_j, \phi_n^{(-)}]_{\pm}$ is a c-number equal to:

$$[\phi_j^{(+)}, \phi_n^{(-)}]_{\pm} = \langle 0 | \phi_j \phi_n | 0 \rangle = \overline{\phi_j \phi_n} \quad (\text{since } t_j > t_n, \text{ so } T\{\phi_j \phi_n\} = \phi_j \phi_n)\,. \tag{4.34}$$

Therefore, moving $\phi_n^{(-)}$ through the normal-ordered product generates:

$$:\phi_1 \cdots \phi_{n-1} : \cdot \phi_n^{(-)} =: \phi_n^{(-)} \phi_1 \cdots \phi_{n-1} : + \sum_{j=1}^{n-1} (\pm)^j \,\overline{\phi_j \phi_n}\, : \phi_1 \cdots \hat{\phi}_j \cdots \phi_{n-1} :, \tag{4.35}$$

where $\hat{\phi}_j$ means ϕ_j is omitted and $(\pm)^j$ accounts for fermionic signs.
Combining both parts:

$$:\phi_1 \cdots \phi_{n-1} : \cdot \phi_n =: \phi_1 \cdots \phi_n : + \sum_{j=1}^{n-1} (\pm)^j \,\overline{\phi_j \phi_n}\, : \phi_1 \cdots \hat{\phi}_j \cdots \phi_{n-1} : \,. \tag{4.36}$$

The same procedure applied to terms with contractions in the inductive hypothesis generates all remaining terms in Wick's theorem. By induction, the theorem is proved. □

Examples of Wick's theorem

Example 1: Two scalar fields.

$$T\{\phi_1\phi_2\} =: \phi_1\phi_2 : + \contraction{}{\phi}{_1}{\phi}\phi_1\phi_2\,. \tag{4.37}$$

This is the definition of contraction, showing that $T\{\cdot\} =: \cdot :$ +contractions.

Example 2: Four scalar fields. Using the notation $\langle ij\rangle \equiv \contraction{}{\phi}{_i}{\phi}\phi_i\phi_j = i\Delta_F(x_i - x_j)$:

$$\begin{aligned} T\{\phi_1\phi_2\phi_3\phi_4\} =&: \phi_1\phi_2\phi_3\phi_4 : \\ &+ \langle 12\rangle\ :\phi_3\phi_4: + \langle 13\rangle\ :\phi_2\phi_4: + \langle 14\rangle\ :\phi_2\phi_3: \\ &+ \langle 23\rangle\ :\phi_1\phi_4: + \langle 24\rangle\ :\phi_1\phi_3: + \langle 34\rangle\ :\phi_1\phi_2: \\ &+ \langle 12\rangle\langle 34\rangle + \langle 13\rangle\langle 24\rangle + \langle 14\rangle\langle 23\rangle\,. \end{aligned} \tag{4.38}$$

There are $\binom{4}{2} = 6$ single contractions. The number of *double* (i.e. complete) contractions is $\frac{1}{2!}\binom{4}{2}\binom{2}{2} = 3$: we first choose 2 fields out of 4 for the first pair ($\binom{4}{2}$ ways), then the remaining 2 form the second pair ($\binom{2}{2} = 1$ way), and we divide by 2! because the ordering of the two pairs is irrelevant. These are precisely the three terms $\langle 12\rangle\langle 34\rangle$, $\langle 13\rangle\langle 24\rangle$, and $\langle 14\rangle\langle 23\rangle$.

Example 3: Two Dirac fields (tracking signs).

$$T\{\psi_\alpha(x)\bar{\psi}_\beta(y)\} =: \psi_\alpha(x)\bar{\psi}_\beta(y) : + \contraction{}{\psi}{_\alpha(x)}{\bar{\psi}}\psi_\alpha(x)\bar{\psi}_\beta(y)\,, \tag{4.39}$$

where $\contraction{}{\psi}{_\alpha(x)}{\bar{\psi}}\psi_\alpha(x)\bar{\psi}_\beta(y) = iS_{F,\alpha\beta}(x-y)$.

For the reversed order:

$$T\{\bar{\psi}_\beta(y)\psi_\alpha(x)\} =: \bar{\psi}_\beta(y)\psi_\alpha(x) : + \contraction{}{\bar{\psi}}{_\beta(y)}{\psi}\bar{\psi}_\beta(y)\psi_\alpha(x)\,. \tag{4.40}$$

Note: $\contraction{}{\bar{\psi}}{_\beta(y)}{\psi}\bar{\psi}_\beta(y)\psi_\alpha(x) = -\contraction{}{\psi}{_\alpha(x)}{\bar{\psi}}\psi_\alpha(x)\bar{\psi}_\beta(y) = -iS_{F,\alpha\beta}(x-y)$ due to anticommutation.

Example 4: QED vertex at second order. At second order in the Dyson series for QED, we encounter the time-ordered product of two normal-ordered interaction vertices:

$$T\Big\{ :\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x):\ :\bar{\psi}(y)\gamma^\nu\psi(y)A_\nu(y): \Big\}\,. \tag{4.41}$$

This is *not* the same as the time-ordered product of six individual field operators. Wick's theorem for the T-product of normal-ordered products gives:

$$T\{:O_1::O_2:\} =: O_1 O_2 : + \text{all contractions between } \textit{different} \text{ groups}\,, \tag{4.42}$$

i.e. one sums over all possible contractions, but **only between fields belonging to different normal-ordered factors**—contractions of two fields *within the same*

$:\cdots:$ are absent. The reason is the following: a contraction is defined as $\overbracket{\phi_i\phi_j} = T\{\phi_i\phi_j\} - :\phi_i\phi_j:$, i.e. it measures the difference between time-ordering and normal-ordering. At coincident points this difference is *not* zero in general; indeed it is precisely what would generate singular self-contractions such as $i\Delta_F(0)$. What matters here is a different statement: Wick's theorem for the time-ordered product of *normal-ordered factors* tells us that only contractions between fields belonging to different factors appear explicitly. Thus

$$T\{:O_1::O_2:\} = :O_1 O_2: + \text{all contractions between different factors}, \qquad (4.43)$$

while contractions internal to the same normal-ordered vertex are absent by construction. Physically, normal ordering the interaction Hamiltonian $\mathcal{H}_I = :\cdots:$ is precisely the prescription that removes these self-contractions, which would otherwise generate tadpole-like contact terms at a single vertex. Signs from fermionic anticommutations apply as before.

The resulting contraction patterns are:

(i) Photon contraction only:

$$:\bar{\psi}(x)\gamma^\mu\psi(x)::\bar{\psi}(y)\gamma^\nu\psi(y): \cdot \overbracket{A_\mu(x)A_\nu}(y)\,. \qquad (4.44)$$

This gives scattering processes like $e^+e^- \to \mu^+\mu^-$ (four external fermions, internal photon); see Section 5.2.

(ii) One fermion contraction:

$$:\bar{\psi}(x)\gamma^\mu A_\mu(x)::\gamma^\nu\psi(y)A_\nu(y): \cdot \overbracket{\psi(x)\bar{\psi}}(y)\,. \qquad (4.45)$$

This gives Compton scattering (two external fermions, two external photons, internal fermion); see Section 5.4.

(iii) Both fermion contractions (closed loop):

$$\overbracket{\psi(x)\bar{\psi}}(y)\cdot\overbracket{\psi(y)\bar{\psi}}(x)\cdot :A_\mu(x)A_\nu(y): \cdot\gamma^\mu\gamma^\nu\,. \qquad (4.46)$$

This gives vacuum polarization (two external photons, fermion loop); see Section 6.2. The trace arises from the closed spinor index chain.

Note that there is no term with, say, $\overbracket{\psi(x)\bar{\psi}}(x)$: this would be a contraction within the same normal-ordered factor $:\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x):$, and is excluded by the rule above.