4.3 Scalar field theory: $\lambda\phi^4$

Before tackling the complexities of QED, we develop the Feynman rules for the simplest interacting quantum field theory: a real scalar field with quartic self-interaction. This theory, while lacking direct phenomenological application, serves as the ideal pedagogical laboratory for understanding perturbation theory, renormalization, and the renormalization group.

The Lagrangian

The Lagrangian density for $\lambda\phi^4$ theory is:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\,, \tag{4.47}$$

where $\phi(x)$ is a real scalar field, m is the mass, and $\lambda > 0$ is a dimensionless coupling constant. The factor $1/4!$ is conventional and simplifies combinatorics.

The interaction Hamiltonian density is:

$$\mathcal{H}_I = -\mathcal{L}_I = \frac{\lambda}{4!}\phi^4\,. \tag{4.48}$$

Dimensional analysis. In $d = 4$ spacetime dimensions:

- $[\phi] = 1$ (mass dimension)
- $[\partial_\mu] = 1$
- $[m] = 1$
- $[\lambda] = 0$ (dimensionless)

The dimensionlessness of λ makes ϕ^4 theory *renormalizable* in $d = 4$; the precise meaning of this statement and the general classification of interactions based on power counting are discussed in Section 6.7.

Why ϕ^4 and not ϕ^3? One might wonder why the simplest scalar interaction is taken to be quartic rather than cubic. A ϕ^3 coupling would actually be *super-renormalizable* in $d = 4$ (the coupling has mass dimension $[\lambda_3] = 1$), so renormalizability is not the issue. The real problem is **stability**: the potential $V(\phi) = \frac{\lambda_3}{3!}\phi^3$ is unbounded from below—for $\phi \to -\infty$ (if $\lambda_3 > 0$) the energy decreases without limit, so the vacuum is unstable and the theory has no ground state. By contrast, $V(\phi) = \frac{\lambda}{4!}\phi^4$ with $\lambda > 0$ is bounded from below and possesses a stable vacuum. Moreover, ϕ^4 theory enjoys a discrete $\mathbb{Z}_2$ symmetry $\phi \to -\phi$, which forbids odd-power interactions such as ϕ^3 and ϕ and ensures that the vacuum at $\phi = 0$ is a true minimum.

Derivation of the Feynman rules for $\lambda\phi^4$

We now derive the Feynman rules systematically from the Dyson series and Wick's theorem.

The S-matrix expansion. The S-matrix is given by the Dyson series (4.1):

$$S = T\exp\left(-i\int d^4x\,\mathcal{H}_I(x)\right) = \mathbb{1} + S^{(1)} + S^{(2)} + \cdots\,, \tag{4.49}$$

where for ϕ^4 theory:

$$S^{(n)} = \frac{(-i)^n}{n!}\left(\frac{\lambda}{4!}\right)^n \int d^4x_1\cdots d^4x_n\,T\{\phi^4(x_1)\cdots\phi^4(x_n)\}\,. \tag{4.50}$$

First order: complete classification of $S^{(1)}$. At first order,

$$S^{(1)} = -i\frac{\lambda}{4!}\int d^4x\,T\{\phi^4(x)\}\,. \tag{4.51}$$

Because all four fields sit at the *same* spacetime point, a first-order Wick contraction can occur in two different ways:

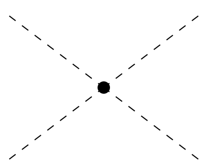
- a field contracts with an external particle state;
- two fields at the same vertex contract with each other, producing a factor $i\Delta_F(0)$.

If $r = 0, 1, 2$ self-contractions occur at the vertex, then $2r$ fields are used internally and the number of external legs is

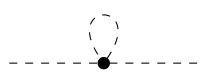
$$E = 4 - 2r\,. \tag{4.52}$$

Therefore the *full* first-order operator $S^{(1)}$ contains exactly three topologies:

- **$r = 0$ ($E = 4$):** the elementary quartic contact vertex,

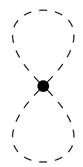


  which underlies the matrix elements $0 \to 4$, $1 \to 3$, $2 \to 2$, $3 \to 1$, $4 \to 0$.
- **$r = 1$ ($E = 2$):** the tadpole topology,



which contributes to two-leg amplitudes (for example the $1 \to 1$ matrix element, or equivalently the two-point function).

- $r = 2$ **(**$E = 0$**):** the vacuum bubble ("figure-eight"),



which contributes to the vacuum-to-vacuum amplitude $\langle 0|S^{(1)}|0\rangle$.

Among these three first-order topologies, the elementary interaction vertex is the four-leg contribution with no self-contractions. We now isolate this term and derive its factor explicitly. If one normal-orders the interaction Hamiltonian, the tadpole and vacuum-bubble topologies are absent from the outset.

Derivation of the vertex factor. Consider the amplitude with four incoming particles:

$$\langle 0|S^{(1)}|p_1, p_2, p_3, p_4\rangle = \frac{-i\lambda}{4!}\int d^4x \times \langle 0|T\{\phi^4(x)\}|p_1, p_2, p_3, p_4\rangle\,. \tag{4.53}$$

The initial state is $|p_1, p_2, p_3, p_4\rangle = a^\dagger(\vec{p}_1)a^\dagger(\vec{p}_2)a^\dagger(\vec{p}_3)a^\dagger(\vec{p}_4)|0\rangle$. We must contract all four field operators $\phi(x)$ with the four creation operators.

Using the mode expansion $\phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x)$ where $\phi^{(+)}$ contains the annihilation operators:

$$\phi^{(+)}(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2E_k}}\, a(\vec{k})\, e^{-ik\cdot x}\,, \tag{4.54}$$

the **contraction** of $\phi^{(+)}(x)$ with a single one-particle state $|p_i\rangle = a^\dagger(\vec{p}_i)|0\rangle$ is:

$$\langle 0|\phi^{(+)}(x)|p_i\rangle = \frac{e^{-ip_i\cdot x}}{(2\pi)^{3/2}\sqrt{2E_{p_i}}}\,. \tag{4.55}$$

This is the basic building block: each field $\phi^{(+)}(x)$, when contracted with a specific particle of momentum p_i, contributes a plane wave $e^{-ip_i\cdot x}$ (times the normalization factor).

For $\phi^4(x)$ to annihilate all four particles, we need all four $\phi^{(+)}$ components, each contracted with a different particle. There are $4!$ ways to assign the four fields to the four particles; each assignment gives the same result:

$$\langle 0|\phi^{(+)}(x)\phi^{(+)}(x)\phi^{(+)}(x)\phi^{(+)}(x)|p_1, p_2, p_3, p_4\rangle = 4!\prod_{i=1}^{4}\frac{1}{(2\pi)^{3/2}\sqrt{2E_{p_i}}}\, e^{-i(p_1+p_2+p_3+p_4)\cdot x}\,. \tag{4.56}$$

Therefore, denoting the external-leg normalization by $\mathcal{N} \equiv \prod_{i=1}^{4}[(2\pi)^{3/2}\sqrt{2E_{p_i}}]^{-1}$:

$$\langle 0|S^{(1)}|p_1,p_2,p_3,p_4\rangle = \frac{-i\lambda}{4!}\cdot 4!\,\mathcal{N}\int d^4x\, e^{-i(p_1+p_2+p_3+p_4)\cdot x} \\ = -i\lambda\,\mathcal{N}\,(2\pi)^4\delta^{(4)}(p_1+p_2+p_3+p_4)\,. \tag{4.57}$$

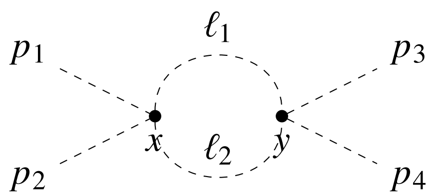
The product $\prod_i[(2\pi)^{3/2}\sqrt{2E_{p_i}}]^{-1}$ is the standard external-leg normalization factor that accompanies every S-matrix element in the Bjorken–Drell convention; it will reappear systematically in the cross-section formula (Chapter 5) and cancel against the flux and phase-space factors.

Result: The vertex factor is $\boxed{-i\lambda}$ and momentum is conserved at the vertex. The 4! from the Lagrangian is exactly cancelled by the 4! ways of contracting four external lines with four fields.

Derivation of the propagator. Consider the second-order contribution to $2 \to 2$ scattering with two vertices at x and y. To generate the connected bubble topology, we must perform (for example) two cross-vertex contractions, i.e. two fields at x contracted with two fields at y:

$$\langle 0|T\{\phi_a(x)\phi_b(x)\phi_c(y)\phi_d(y)\}|0\rangle \\ \supset \underbrace{\langle 0|T\{\phi_a(x)\phi_c(y)\}|0\rangle}_{i\Delta_F(x-y)}\,\underbrace{\langle 0|T\{\phi_b(x)\phi_d(y)\}|0\rangle}_{i\Delta_F(x-y)} = \big[i\Delta_F(x-y)\big]^2\,. \tag{4.58}$$

The alternative pairing $c \leftrightarrow d$ gives the same integrand; this exchange is precisely the origin of the $S = 2$ symmetry factor for the bubble.



Each contraction contributes one propagator. In momentum space, the pair of contractions gives:

$$\big[i\Delta_F(x-y)\big]^2 = \prod_{j=1}^{2}\int \frac{d^4\ell_j}{(2\pi)^4}\,\frac{i}{\ell_j^2-m^2+i\epsilon}\,e^{-i\ell_j\cdot(x-y)}\,. \tag{4.59}$$

When we integrate over the vertex positions x and y, the exponentials combine with those from external lines to produce momentum-conserving delta functions.

Let us see this explicitly for the s-channel diagram, where the two incoming particles are annihilated at x and the two outgoing ones are created at y, with two internal propagators connecting the vertices.

Suppressing normalization and combinatorial prefactors, the spacetime integrals are:

$$\int d^4x\, d^4y\ \underbrace{e^{-ip_1\cdot x}\, e^{-ip_2\cdot x}}_{\text{incoming at } x}\ \underbrace{e^{+ip_3\cdot y}\, e^{+ip_4\cdot y}}_{\text{outgoing at } y} \prod_{j=1}^{2} \int \frac{d^4\ell_j}{(2\pi)^4} \frac{i}{\ell_j^2 - m^2 + i\epsilon}\, e^{-i\ell_j\cdot(x-y)} . \tag{4.60}$$

Each propagator $e^{-i\ell_j\cdot(x-y)}$ contributes a phase $e^{-i\ell_j\cdot x}$ at vertex x and $e^{+i\ell_j\cdot y}$ at vertex y. Collecting all x-dependent and y-dependent phases:

$$\text{integrate over } x: \quad \int d^4x\, e^{-i(p_1+p_2+\ell_1+\ell_2)\cdot x} = (2\pi)^4\, \delta^{(4)}(p_1+p_2+\ell_1+\ell_2)\,, \tag{4.61}$$

$$\text{integrate over } y: \quad \int d^4y\, e^{+i(p_3+p_4+\ell_1+\ell_2)\cdot y} = (2\pi)^4\, \delta^{(4)}(p_3+p_4+\ell_1+\ell_2)\,. \tag{4.62}$$

Each delta function enforces **momentum conservation at its vertex**: the total four-momentum flowing into the vertex equals the total momentum flowing out.

Using (4.61) to eliminate $\ell_2 = -(p_1+p_2) - \ell_1$ and substituting into (4.62):

$$(2\pi)^4\, \delta^{(4)}\big(p_3+p_4-(p_1+p_2)\big)\,, \tag{4.63}$$

which is **overall** four-momentum conservation. One loop momentum ($\ell \equiv \ell_1$) remains a free integration variable—this is the loop integral $\int d^4\ell/(2\pi)^4$ in the Feynman rules.

In general, V vertex integrations produce V delta functions, of which one enforces overall momentum conservation and the remaining $V-1$ fix internal momenta, leaving $L = I - V + 1$ independent loop momenta to be integrated over (see eq. (4.64) below).

After all vertex integrations are performed, what remains for each internal line is simply the momentum-space propagator:

Result: The propagator factor is $\boxed{\dfrac{i}{p^2 - m^2 + i\epsilon}}$ where p is determined by momentum conservation.

Loop integrals and momentum conservation. For a diagram with V vertices and I internal lines:

- Each vertex contributes an integral $\int d^4x_i$ and a factor $(-i\lambda/4!)$
- Each internal line contributes a propagator with momentum p_j
- The integrals over vertex positions produce V momentum-conserving delta functions

The number of independent loop momenta is:

$$L = I - V + 1\,. \tag{4.64}$$

(This is Euler's formula for connected graphs.) Each loop momentum requires an integration $\int d^4\ell/(2\pi)^4$.

Symmetry factors. Let us understand where overcounting arises. The Dyson series at order n contains a factor $1/n!$, and each ϕ^4 vertex carries $1/4!$. When we enumerate all Wick contractions we generate every possible Feynman diagram—but some contractions that *look* different at the algebraic level produce *the same* diagram. This happens whenever the diagram possesses an internal symmetry: a permutation of internal lines or vertices that maps the diagram onto itself, leaving all external legs fixed. Each such permutation corresponds to a distinct Wick contraction that gives an identical integral, so the sum over contractions overcounts by a factor equal to the number of these permutations.

More concretely, consider the nth-order term in the Dyson series:

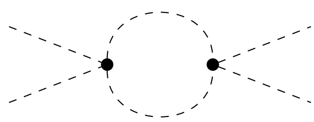
$$S^{(n)} = \frac{1}{n!}\left(\frac{-i\lambda}{4!}\right)^n \int d^4x_1 \cdots d^4x_n\, T\{\phi^4(x_1)\cdots\phi^4(x_n)\}\,. \tag{4.65}$$

The factor $1/n!$ compensates for the $n!$ permutations of the vertex labels $x_1, \ldots, x_n$ (the time-ordered product is symmetric). Similarly, the $(4!)^n$ in the denominator compensates for the 4! permutations of the four fields at each vertex. However, if a diagram has an additional symmetry—for instance, two internal lines connecting the same pair of vertices can be swapped, or a loop can be traversed in either direction—then some of those permutations produce identical integrands. The Wick sum counts them as separate contributions, so the result is too large by a factor S.

The **symmetry factor** S is the order of the group of internal automorphisms of the diagram (permutations of internal lines and vertices that leave the diagram and all external legs unchanged). The correct amplitude is:

$$\text{Amplitude} = \frac{1}{S} \times (\text{naive Feynman rules})\,. \tag{4.66}$$

Example 1: For the s-channel bubble diagram in $2 \to 2$ scattering:



The two internal lines can be exchanged $\Rightarrow S = 2$.

Example 2: The “figure-eight” vacuum diagram.

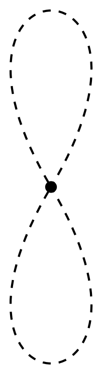
This diagram arises from the vacuum-to-vacuum amplitude at first order:

$$\langle 0|S^{(1)}|0\rangle = \frac{-i\lambda}{4!}\int d^4x\,\langle 0|T\{\phi(x)\phi(x)\phi(x)\phi(x)\}|0\rangle\,. \tag{4.67}$$

Since there are no external particles, all four fields at the vertex must be contracted among themselves. By Wick’s theorem (Example 2 in (4.38) with $x_1 = x_2 = x_3 = x_4 = x$), there are 3 complete contractions, each consisting of two propagators that start and end at the same point:

$$\langle 0|T\{\phi^4(x)\}|0\rangle = 3\left[i\Delta_F(0)\right]^2. \tag{4.68}$$

Each factor $i\Delta_F(0) = \overbracket{\phi(x)\phi(x)}$ represents a closed loop—a propagator from x back to x—giving the characteristic figure-eight shape with two loops meeting at a single vertex:



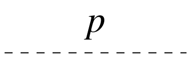
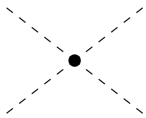
The symmetry factor is $S = 8$: each of the two loops can be traversed in two directions ($2\times 2 = 4$), and the two loops can be exchanged ($\times 2$). Equivalently, $S = 4!/3 = 8$, since the 4! from the Lagrangian overcounts by a factor of 3 (the number of Wick contractions) relative to the single diagram.

The full first-order vacuum amplitude is therefore

$$\langle 0|S^{(1)}|0\rangle = \frac{-i\lambda}{4!}\cdot 3\cdot\left[i\Delta_F(0)\right]^2\int d^4x = \frac{-i\lambda}{8}\left[i\Delta_F(0)\right]^2\cdot VT\,, \tag{4.69}$$

where $VT = \int d^4x$ is the spacetime volume. Note that $\Delta_F(0)$ is UV-divergent—this is a first example of a divergence that must be handled by renormalization.

Summary: Feynman rules for $\lambda\phi^4$

Element	**Diagram**	**Factor**
Propagator		$\dfrac{i}{p^2 - m^2 + i\epsilon}$
Vertex		$-i\lambda$
External line	in/out	1

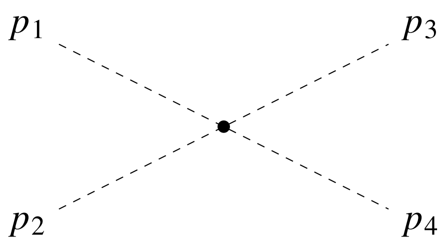
Procedural rules:

1. Draw all topologically distinct connected diagrams
2. Label external momenta p_i (incoming positive) and internal momenta q_j
3. Write $-i\lambda$ for each vertex
4. Write $\dfrac{i}{q_j^2 - m^2 + i\epsilon}$ for each internal line
5. Impose momentum conservation at each vertex: include $(2\pi)^4\delta^{(4)}(\sum p)$
6. Integrate $\displaystyle\int \frac{d^4\ell}{(2\pi)^4}$ over each independent loop momentum
7. Divide by the symmetry factor S
8. External lines contribute factor 1 (for scalars)

Remark on the (2π) *factors.* Each internal line brings $\int d^4q_j/(2\pi)^4$, while each vertex brings $(2\pi)^4\delta^{(4)}(\sum p)$. Of the V delta functions, $V-1$ are used to fix $V-1$ internal momenta, cancelling the corresponding integrals and $(2\pi)^4$ factors. The remaining delta function enforces overall momentum conservation, $(2\pi)^4\delta^{(4)}(p_{\text{in}} - p_{\text{out}})$, and is factored out in the definition of $i\mathcal{M}$. After this reduction, only $L = I - V + 1$ independent loop integrals $\int d^4\ell/(2\pi)^4$ survive, with all (2π) factors exactly accounted for.

Tree-level scattering: $\phi\phi \to \phi\phi$

The simplest scattering process is $2 \to 2$ scattering. At tree level (order λ), there is a single diagram:



The amplitude is simply:[1]

$$\boxed{i\mathcal{M}^{(1)} = -i\lambda}. \tag{4.70}$$

This is remarkably simple: the tree-level amplitude is just the coupling constant (times $-i$).

The Mandelstam variables for this process are:

$$s = (p_1 + p_2)^2, \qquad t = (p_1 - p_3)^2, \qquad u = (p_1 - p_4)^2, \tag{4.71}$$

with the constraint $s + t + u = 4m^2$. At tree level, $\mathcal{M}$ is independent of these kinematic variables.

One-loop corrections: systematic calculation

At order λ^2, we have contributions from the second-order term in the Dyson series:

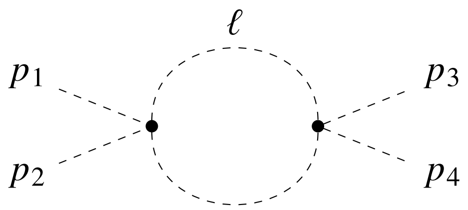
$$S^{(2)} = \frac{(-i)^2}{2!}\left(\frac{\lambda}{4!}\right)^2 \int d^4x\, d^4y\, T\{\phi^4(x)\phi^4(y)\}. \tag{4.72}$$

Applying Wick's theorem to $T\{\phi^4(x)\phi^4(y)\}$ generates all possible contractions. For $2 \to 2$ scattering ($p_1, p_2 \to p_3, p_4$), the connected diagrams at one loop are three **bubble diagrams**, distinguished by *which pair of external particles meets at each vertex*. The three possibilities correspond to the three Mandelstam channels:

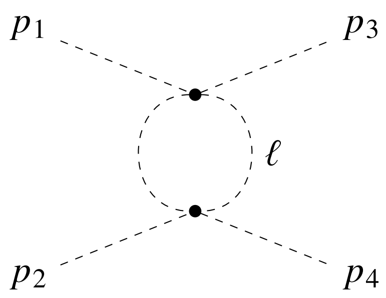
- **s-channel**: Both incoming particles (p_1, p_2) enter the *same* vertex; both outgoing (p_3, p_4) leave from the other. The total momentum flowing through the internal lines is $p_1 + p_2$, so the bubble depends on $s = (p_1 + p_2)^2$.
- **t-channel**: One incoming and one outgoing particle meet at each vertex—specifically, p_1 and p_3 at one vertex, p_2 and p_4 at the other. The momentum transfer through the bubble is $p_1 - p_3$, so it depends on $t = (p_1 - p_3)^2$.
- **u-channel**: Same topology as the t-channel, but with the outgoing particles swapped: p_1 and p_4 meet at one vertex, p_2 and p_3 at the other. The momentum transfer is $p_1 - p_4$, and the bubble depends on $u = (p_1 - p_4)^2$.

[1] By convention, the connected part of the S-matrix element is written as $\langle f|S|i\rangle = i(2\pi)^4\delta^{(4)}(p_f - p_i)\,\mathcal{M}$, so the Feynman rules directly compute $i\mathcal{M}$. Therefore here $\mathcal{M}^{(1)} = -\lambda$, while the diagram gives $i\mathcal{M}^{(1)} = -i\lambda$.

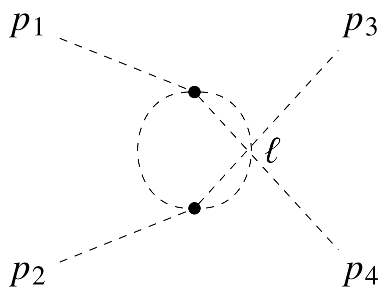
The diagrams are:



s-channel
$p_1{+}p_2$ through bubble



t-channel
$p_1{-}p_3$ through bubble



u-channel
$p_1{-}p_4$ through bubble

Note the key visual difference: in the s-channel, the bubble is *horizontal* (connecting the incoming side to the outgoing side), while in the t- and u-channels it is *vertical* (connecting top to bottom). The t- and u-channel diagrams have the same topology—they differ only in which outgoing particle (p_3 or p_4) is paired with p_1. Physically, the s-channel probes the *total energy* of the collision, while the t- and u-channels probe the *momentum transfer* between specific pairs of particles.

The amplitudes are:

$$i\mathcal{M}_s = \frac{(-i\lambda)^2}{2}\int \frac{d^4\ell}{(2\pi)^4}\frac{i}{\ell^2-m^2+i\epsilon}\cdot\frac{i}{(\ell-p_1-p_2)^2-m^2+i\epsilon}\,, \tag{4.73}$$

$$i\mathcal{M}_t = \frac{(-i\lambda)^2}{2}\int \frac{d^4\ell}{(2\pi)^4}\frac{i}{\ell^2-m^2+i\epsilon}\cdot\frac{i}{(\ell-p_1+p_3)^2-m^2+i\epsilon}\,, \tag{4.74}$$

$$i\mathcal{M}_u = \frac{(-i\lambda)^2}{2}\int \frac{d^4\ell}{(2\pi)^4}\frac{i}{\ell^2-m^2+i\epsilon}\cdot\frac{i}{(\ell-p_1+p_4)^2-m^2+i\epsilon}\,. \tag{4.75}$$

In each case, the factor $1/2$ is the **symmetry factor**: the two internal propagators can be exchanged without changing the diagram. Each amplitude is a function of one Mandelstam variable: $\mathcal{M}_s = \mathcal{M}_s(s)$, $\mathcal{M}_t = \mathcal{M}_t(t)$, $\mathcal{M}_u = \mathcal{M}_u(u)$, reflecting which kinematic invariant flows through the bubble.

The total one-loop correction to $2 \to 2$ scattering is:

$$i\mathcal{M}^{(2)}_{2\to 2} = i\mathcal{M}_s + i\mathcal{M}_t + i\mathcal{M}_u\,. \tag{4.76}$$

Evaluation of the loop integral

Consider the generic bubble integral:

$$I(p^2) = \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{(\ell^2 - m^2 + i\epsilon)[(\ell - p)^2 - m^2 + i\epsilon]} . \tag{4.77}$$

This integral is **logarithmically divergent**. To see this, note that for large $|\ell|$:

$$I(p^2) \sim \int^{\Lambda} \frac{d^4\ell}{\ell^4} \sim \int^{\Lambda} \frac{\ell^3 d\ell}{\ell^4} = \int^{\Lambda} \frac{d\ell}{\ell} \sim \ln\Lambda . \tag{4.78}$$

Dimensional regularization. The standard modern approach is to analytically continue to $d = 4 - \varepsilon$ dimensions:

$$\int \frac{d^4\ell}{(2\pi)^4} \to \mu^{\varepsilon} \int \frac{d^d\ell}{(2\pi)^d} , \tag{4.79}$$

where μ is an arbitrary mass scale introduced to keep λ dimensionless.

Using Feynman parameters and standard techniques (see Appendix), one finds:

$$I(p^2) = \frac{i}{16\pi^2} \left[\frac{2}{\varepsilon} - \gamma_E + \ln(4\pi) - \int_0^1 dx \ln \frac{m^2 - x(1-x)p^2}{\mu^2} \right] , \tag{4.80}$$

where $\gamma_E \approx 0.5772$ is the Euler-Mascheroni constant.

The total one-loop amplitude. Each channel contributes a copy of this integral with p^2 replaced by the corresponding Mandelstam variable: $\mathcal{M}_s$ depends on $I(s)$, $\mathcal{M}_t$ on $I(t)$, and $\mathcal{M}_u$ on $I(u)$. Summing all three channels, the total one-loop amplitude (4.76) becomes

$$i\mathcal{M}^{(\text{1-loop})} = -\frac{i\lambda^2}{32\pi^2} \left[\frac{6}{\varepsilon} + \text{finite}(s,t,u) \right] , \tag{4.81}$$

where the factor of 6 arises because each of the three channels contributes $2/\varepsilon$, and the finite part is a calculable function of s, t, u (and m^2/μ^2) from the logarithmic integrals in (4.80).

The $6/\varepsilon$ pole means the one-loop amplitude is UV divergent: it blows up as $\varepsilon \to 0$ ($d \to 4$). This divergence must be cancelled by a **coupling counterterm** δ_λ of order λ^2—this is the subject of Chapter 6 (Renormalization). The finite remainder is physical and contains, among other things, the logarithmic dependence on the energy scale that drives the *running* of the coupling constant.

Box 4.3 **Summary: $\lambda\phi^4$ Feynman rules**

Propagator: $\dfrac{i}{p^2 - m^2 + i\epsilon}$

Vertex: $-i\lambda$

Rules:

- Momentum conservation at each vertex
- $\displaystyle\int \frac{d^4\ell}{(2\pi)^4}$ for each loop
- Divide by symmetry factor S
- No fermion signs (bosonic theory)

Key results:

- Tree-level $2 \to 2$: $\mathcal{M} = -\lambda$
- One-loop integrals contain UV divergences (see Chapter on Renormalization)

4.4 Feynman rules for QED

We now turn to the physically relevant case of quantum electrodynamics. With the sign conventions adopted in these notes (Convention A, see the detailed discussion in Chapter 5), the QED interaction Lagrangian density for the electron field is

$$\mathscr{L}_{\text{int}} = +e\,\bar{\psi}\gamma^{\mu}\psi\,A_{\mu}\,, \tag{4.82}$$

where $e > 0$ is the positron charge (the electron has charge $q_e = -e$) and the coupling constant $\alpha = e^2/(4\pi) \simeq 1/137$ is dimensionless, making QED a renormalizable theory (see Section 6.7). The S-matrix is then

$$S = T\exp\left(i\int d^4x\,\mathscr{L}_{\text{int}}(x)\right) = T\exp\left(ie\int d^4x\,:\bar{\psi}\gamma^{\mu}\psi\,A_{\mu}:\right)\,. \tag{4.83}$$

We derive the Feynman rules by applying Wick's theorem order by order, starting from the first-order term—which, as we shall see, gives no physical contribution—and then analysing the second-order term, from which all leading QED processes originate.

First-order contributions: why they vanish

The first-order term in the expansion (4.83) is:

$$S^{(1)} = ie\int d^4x\,:\bar{\psi}(x)\gamma^{\mu}\psi(x)\,A_{\mu}(x):\,. \tag{4.84}$$

Let us systematically examine all possible matrix elements $\langle f|S^{(1)}|i\rangle$.

The QED vertex structure. The interaction $:\bar{\psi}\gamma^{\mu}\psi A_{\mu}:$ contains:

- One $\bar{\psi}$ field (can create e^- or annihilate e^+)
- One ψ field (can annihilate e^- or create e^+)
- One A_{μ} field (can create or annihilate γ)

At a single vertex, the interaction $:\bar{\psi}\gamma^{\mu}\psi A_{\mu}:$ involves exactly three field operators, each of which can create or annihilate a particle. This gives $2^3 = 8$ possible first-order processes.

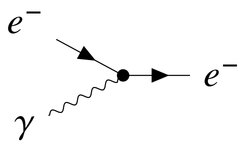
Diagram conventions.

Time direction. In these notes, **time flows from left to right**: initial-state particles enter from the left, and final-state particles exit to the right. This is the most common convention in particle physics textbooks (e.g. Peskin–Schroeder [6], Schwartz [12]). Other conventions exist in the literature: Srednicki [11] draws time flowing upward, while some condensed-matter and older texts use time flowing downward or to the left. The physics is of course independent of the choice.

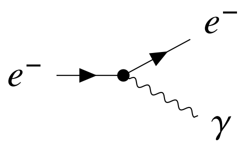
Vertex dots. Each interaction vertex is marked with a **dot** (•) to distinguish it from a mere crossing of lines. The dot represents the spacetime point x over which we integrate in the Dyson series.

Fermion arrows. The arrow on a fermion line follows the flow of *fermion number* (equivalently, the flow of negative charge), **not** the direction of the particle's momentum. For electrons, the arrow points along the direction of propagation (left to right for initial-state electrons, right for final-state); for positrons, the arrow points *opposite* to the direction of propagation. This convention arises naturally from the structure of the interaction: $\bar{\psi}$ creates an e^- or annihilates an e^+ at the arrow's tail, while ψ annihilates an e^- or creates an e^+ at the arrow's head. Thus, an incoming positron is represented by an outgoing arrow, and an outgoing positron by an incoming arrow.

Electron processes:

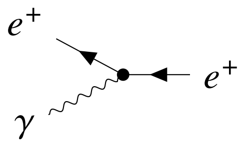


(a) $e^-\gamma \to e^-$

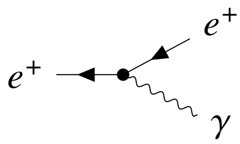


(b) $e^- \to e^-\gamma$

Positron processes:

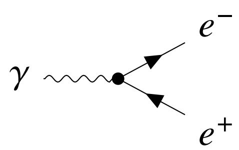


(c) $e^+\gamma \to e^+$

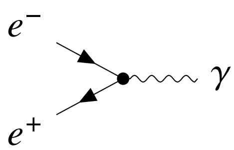


(d) $e^+ \to e^+\gamma$

Pair creation/annihilation:

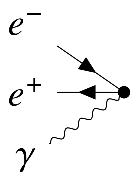


(e) $\gamma \to e^+e^-$

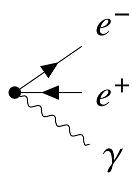


(f) $e^+e^- \to \gamma$

All particles in/out (vacuum transitions):



(g) $e^+e^-\gamma \to 0$



(h) $0 \to e^+e^-\gamma$

Why all first-order processes are forbidden. Each vertex conserves four-momentum. Let us show this explicitly for diagram (a), $e^-(p) + \gamma(k) \to e^-(p')$.

The amplitude is obtained by sandwiching $S^{(1)}$ between the initial and final states. The mode expansions (4.4)–(4.5) of the fields ψ, $\bar{\psi}$, and A_μ assign a plane-wave factor $e^{\pm iq\cdot x}$ to each external particle at the vertex located at x:

- the incoming electron (ψ annihilates it): factor $e^{-ip\cdot x}$,
- the incoming photon (A_μ annihilates it): factor $e^{-ik\cdot x}$,
- the outgoing electron ($\bar{\psi}$ creates it): factor $e^{+ip'\cdot x}$.

The amplitude therefore contains the spacetime integral:

$$\int d^4x\, e^{-ip\cdot x}\, e^{-ik\cdot x}\, e^{+ip'\cdot x} = \int d^4x\, e^{i(p'-p-k)\cdot x} . \tag{4.85}$$

We now use the integral representation of the Dirac delta:

$$\int d^4x\, e^{iq\cdot x} = (2\pi)^4 \delta^{(4)}(q) , \tag{4.86}$$

which holds because each of the four components gives $\int dx^\mu\, e^{iq_\mu x^\mu} = 2\pi\,\delta(q_\mu)$. Applying this with $q = p' - p - k$:

$$\int d^4x\, e^{i(p'-p-k)\cdot x} = (2\pi)^4 \delta^{(4)}(p' - p - k) , \tag{4.87}$$

which enforces $p + k = p'$, i.e. four-momentum conservation at the vertex. The same mechanism operates at every vertex and for every process: the integral over each vertex position produces a momentum-conserving delta function.

Consider diagram (a): $e^-(p) + \gamma(k) \to e^-(p')$. Momentum conservation requires:

$$p + k = p' . \tag{4.88}$$

But for on-shell particles: $p^2 = p'^2 = m_e^2$ and $k^2 = 0$. Then:

$$p'^2 = (p+k)^2 = p^2 + 2p\cdot k + k^2 = m_e^2 + 2p\cdot k . \tag{4.89}$$

This equals m_e^2 only if $p\cdot k = 0$. But for physical photons, $p\cdot k = E_e E_\gamma - |\vec{p}||\vec{k}|\cos\theta = E_e E_\gamma (1 - \beta_e \cos\theta) > 0$ (where $\beta_e = |\vec{p}|/E_e$) unless $E_\gamma = 0$.

Conclusion: All first-order QED processes violate energy-momentum conservation for on-shell external particles. Therefore:

$$\langle f|S^{(1)}|i\rangle = 0 \quad \text{for all physical processes} . \tag{4.90}$$

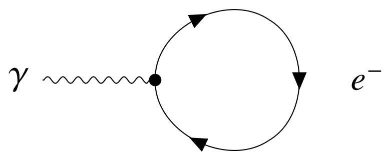
The leading non-trivial contributions come from $S^{(2)}$.

Vacuum diagrams at first order. One might consider the vacuum-to-vacuum amplitude $\langle 0|S^{(1)}|0\rangle$. However:

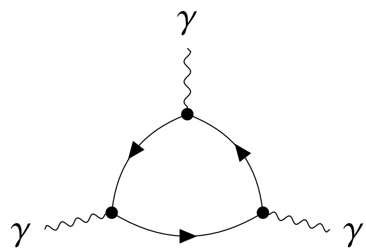
$$\langle 0| :\bar{\psi}\gamma^\mu\psi A_\mu: |0\rangle = 0\,, \tag{4.91}$$

because normal-ordered products have zero vacuum expectation value. There are no first-order vacuum diagrams in QED.

Tadpole diagrams and Furry's theorem. At higher orders, one might consider *tadpole diagrams*: a closed fermion loop attached to a single external photon line.



Tadpole



3-photon vertex (also zero)

This diagram corresponds to a one-point function $\langle 0|j^\mu(x)|0\rangle$ where $j^\mu = \bar{\psi}\gamma^\mu\psi$ is the electromagnetic current. Such diagrams **vanish identically in QED** due to *Furry's theorem*:

Furry's theorem: Any Feynman diagram with a closed fermion loop and an *odd* number of photon vertices vanishes.

Proof. QED is invariant under **charge conjugation** C, the discrete symmetry that exchanges particles with antiparticles. On the fields, it acts as:

$$\psi \xrightarrow{C} C\bar{\psi}^\top\,, \qquad \bar{\psi} \xrightarrow{C} -\psi^\top C^{-1}\,, \qquad A_\mu \xrightarrow{C} -A_\mu\,, \tag{4.92}$$

where $C = i\gamma^0\gamma^2$ is the **charge conjugation matrix**, satisfying the defining property $C^{-1}\gamma^\mu C = -(\gamma^\mu)^\top$ (see eq. (9.153)). The photon field changes sign because the electromagnetic current j^μ changes sign when all charges are reversed.

One can verify that the QED interaction $\mathscr{L}_{\text{int}} = e\bar{\psi}\gamma^\mu\psi A_\mu$ is invariant: the current picks up a minus sign (shown below) and so does A_μ, giving an overall $(-1)^2 = +1$.

Under C, the electromagnetic current transforms as:

$$j^\mu = \bar{\psi}\gamma^\mu\psi \xrightarrow{C} -\bar{\psi}\gamma^\mu\psi = -j^\mu\,. \tag{4.93}$$

Since the vacuum is C-invariant: $C|0\rangle = |0\rangle$, we have:

$$\langle 0|j^\mu|0\rangle = \langle 0|C^\dagger j^\mu C|0\rangle = -\langle 0|j^\mu|0\rangle = 0\,. \tag{4.94}$$

More generally, a fermion loop with n photon insertions contributes a factor $(-1)^n$ under charge conjugation. For odd n, this gives a sign flip, forcing the amplitude to vanish.

Consequence: Tadpole diagrams, photon three-point vertices, and all diagrams with an odd number of photons attached to a single fermion loop are zero in QED. This significantly simplifies higher-order calculations.

4.5 Second-order contributions: complete derivation

The second-order term is:

$$S^{(2)} = \frac{(ie)^2}{2!} \int d^4x\, d^4y\, T\{:\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x)::\bar{\psi}(y)\gamma^\nu\psi(y)A_\nu(y):\}\,. \quad (4.95)$$

We must evaluate $\langle f|S^{(2)}|i\rangle$ for various initial and final states using Wick's theorem.

Field content at second order. The operator $:\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x):\,:\bar{\psi}(y)\gamma^\nu\psi(y)A_\nu(y):$ contains:

- Two $\bar{\psi}$ fields: $\bar{\psi}(x)$ and $\bar{\psi}(y)$
- Two ψ fields: $\psi(x)$ and $\psi(y)$
- Two A_μ fields: $A_\mu(x)$ and $A_\nu(y)$

Complete Wick expansion. Applying Wick's theorem to the time-ordered product, we obtain all possible contractions. Using the shorthand notation:

$$\bar{\psi}_x \equiv \bar{\psi}(x), \quad \psi_x \equiv \psi(x), \quad A_x \equiv A_\mu(x), \quad \text{etc.}, \quad (4.96)$$

The complete expansion is (omitting γ^μ, γ^ν for clarity):

$$T\{:\bar{\psi}_x\psi_x A_x::\bar{\psi}_y\psi_y A_y:\} =$$

(0) No contractions:

$$:\bar{\psi}_x\psi_x A_x\bar{\psi}_y\psi_y A_y:$$

(1) Single contractions:

$$\overbracket{A_x A_y}\cdot :\bar{\psi}_x\psi_x\bar{\psi}_y\psi_y:$$
$$\overbracket{\psi_x\bar{\psi}_y}\cdot :\bar{\psi}_x A_x\psi_y A_y:$$
$$\overbracket{\psi_y\bar{\psi}_x}\cdot :\psi_x A_x\bar{\psi}_y A_y:$$

(2) Double contractions:

$$\overbracket{A_x A_y}\cdot\overbracket{\psi_x\bar{\psi}_y}\cdot :\bar{\psi}_x\psi_y:$$
$$\overbracket{A_x A_y}\cdot\overbracket{\psi_y\bar{\psi}_x}\cdot :\psi_x\bar{\psi}_y:$$
$$\overbracket{\psi_x\bar{\psi}_y}\cdot\overbracket{\psi_y\bar{\psi}_x}\cdot :A_x A_y:$$

(3) Triple contractions:

$$\overbracket{A_x A_y}\cdot\overbracket{\psi_x\bar{\psi}_y}\cdot\overbracket{\psi_y\bar{\psi}_x} \quad \text{(fully contracted)}\,. \tag{4.97}$$

Physical interpretation of each term:

- **(0) No contractions**: No fields are contracted between the two vertices x and y. Each vertex then acts independently, producing what amounts to two *first-order* processes happening separately. The result is a **disconnected diagram**—the product of two independent first-order diagrams. Since each first-order process is forbidden by energy-momentum conservation (Section 4.4), this contribution vanishes. More generally, disconnected diagrams factorise and are absorbed into the vacuum-to-vacuum amplitude; they do not contribute to scattering amplitudes (this is formalised by the linked-cluster theorem)
- **(1a) Photon contraction only**: $\overbracket{A_x A_y}$ → internal photon, 4 external fermions (e.g., $e^+e^- \to \mu^+\mu^-$, Bhabha, Møller)
- **(1b,c) Single fermion contraction**: $\overbracket{\psi\bar{\psi}}$ → internal fermion, 2 external photons + 2 external fermions (Compton)
- **(2a,b) Photon + fermion**: → internal photon + fermion, 2 external fermions (self-energy correction)
- **(2c) Two fermion contractions**: $\overbracket{\psi_x\bar{\psi}_y}\cdot\overbracket{\psi_y\bar{\psi}_x}$ → fermion loop, 2 external photons (vacuum polarization)
- **(3) Fully contracted**: → vacuum bubble (no external lines, contributes to $\langle 0|S|0\rangle$)

Wick's theorem tells us to sum over all contractions. Different contraction patterns give different physical processes.

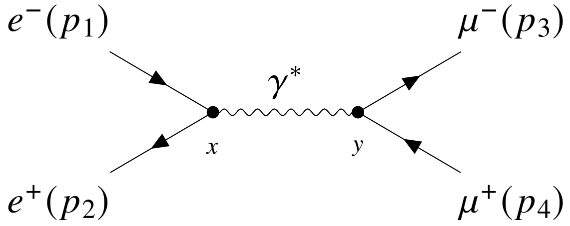
Classification of second-order diagrams

Case 1: Contract both photon fields (internal photon line).

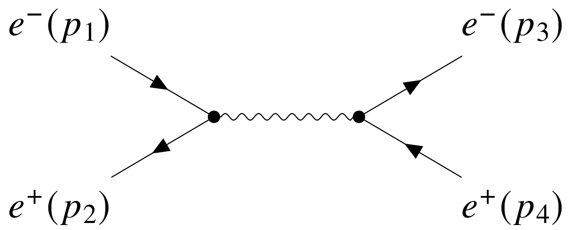
$$\overline{A_\mu(x)A_\nu(y)} = iD_F^{\mu\nu}(x-y)\,. \tag{4.98}$$

The remaining fermion fields $\bar\psi(x), \psi(x), \bar\psi(y), \psi(y)$ connect to external states.

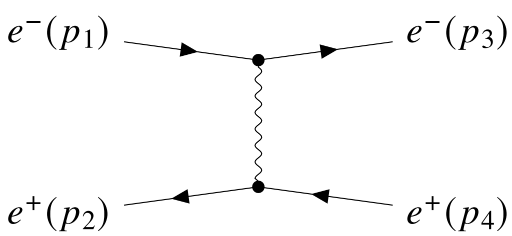
This gives processes with **two fermion lines connected by a photon**:



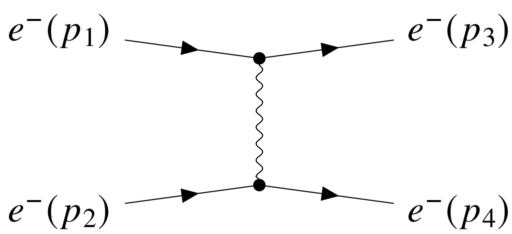
$e^+e^- \to \mu^+\mu^-$ (s-channel)



Bhabha s-channel



Bhabha t-channel

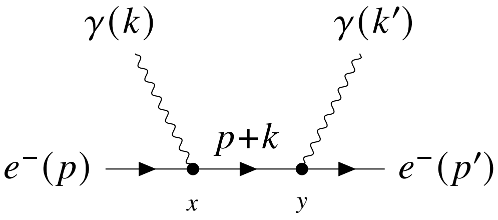


Møller scattering

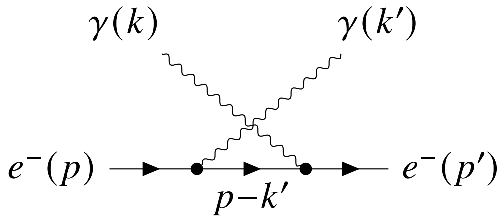
Case 2: Contract one fermion pair (internal fermion line).

$$\overline{\psi(x)\bar\psi(y)} = iS_F(x-y) \quad \text{or} \quad \overline{\psi(y)\bar\psi(x)} = iS_F(y-x)\,. \tag{4.99}$$

The remaining fields connect to external states, including two external photons.



Compton (s-channel)

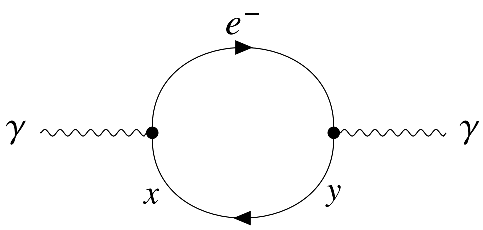


Compton (u-channel)

Case 3: Contract both fermion pairs (vacuum polarization).

$$\psi(x)\bar{\psi}(y)\cdot\psi(y)\bar{\psi}(x) = iS_F(x-y)\cdot iS_F(y-x)\,. \tag{4.100}$$

This creates a closed fermion loop with two external photons:



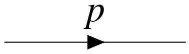
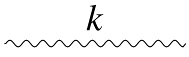
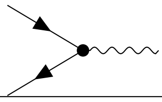
Vacuum polarization (photon self-energy)

Case 4: No contractions between different vertices. If we don't contract fields between x and y, we get disconnected diagrams or vacuum fluctuations. These contribute to the overall phase but not to scattering amplitudes.

4.6 Feynman rules in momentum space

The calculation above illustrates the general pattern. We now summarize the Feynman rules.

Graphical elements

Element	**Diagram**	**Factor**
Fermion propagator	 p	$\dfrac{i(\not{p}+m)}{p^2-m^2+i\epsilon}$
Photon propagator	 k	$\dfrac{-ig_{\mu\nu}}{k^2+i\epsilon}$
QED vertex		$+ie\gamma^\mu$

External line factors

e^- in $= u(p,s)$	e^- out $= \bar{u}(p,s)$
e^+ in $= \bar{v}(p,s)$	e^+ out $= v(p,s)$
γ in $= \epsilon_\mu(k,\lambda)$	γ out $= \epsilon^*_\mu(k,\lambda)$

Derivation of external line factors

These factors arise from contracting field operators with external states.

Incoming electron. The state $|e^-(p,s)\rangle = b_s^\dagger(\vec{p})|0\rangle$. The field $\psi(x)$ acting on this state, using the BD field expansion (eq. (**??**)) and the anticommutator $\{b_{s'}(\vec{k}), b_s^\dagger(\vec{p})\} = \delta^{(3)}(\vec{k}-\vec{p})\delta_{s's}$:

$$\begin{aligned}\psi(x)|e^-(p,s)\rangle &= \int \frac{d^3k}{(2\pi)^{3/2}}\sqrt{\frac{m}{E_k}}\sum_{s'} b_{s'}(\vec{k})u_{s'}(k)e^{-ik\cdot x}\cdot b_s^\dagger(\vec{p})|0\rangle \\ &= \underbrace{\frac{1}{(2\pi)^{3/2}}\sqrt{\frac{m}{E_p}}}_{\text{BD wavefunction norm.}} \quad u_s(p)\,e^{-ip\cdot x}\,|0\rangle\,. \end{aligned} \tag{4.101}$$

Stripping the BD normalisation. The prefactor $(2\pi)^{-3/2}\sqrt{m/E_p}$ is the Bjorken–Drell wavefunction normalisation. By convention, when writing the Feynman rule for the external line this factor is *stripped off*: it is absorbed into the conversion from the box-normalised matrix element $\mathcal{M}_{\text{box}}$ to the stripped BD amplitude $\mathcal{M}_{\text{BD}}$ that appears in the cross-section master formula (cf. eq. (3.38) and the discussion in Chapter 3). The Feynman rule for an incoming electron is then simply $u(p,s)$. The position integral $\int d^4x\, e^{-ip\cdot x}$ at the vertex produces the momentum-conserving delta function. What remains as the external-line factor is therefore $u(p,s)$, with the understanding that the BD wavefunction normalisation has been moved into the box-to-stripped conversion. The same convention is applied to all four external-fermion factors below.

Outgoing electron. The state $\langle e^-(p',s')| = \langle 0|b_{s'}(\vec{p}')$. Acting with $\bar{\psi}(x)$:

$$\langle e^-(p',s')|\bar{\psi}(x) = \langle 0|\bar{u}_{s'}(p')e^{+ip'\cdot x} \,. \tag{4.102}$$

Factor: $\bar{u}(p',s')$.

Incoming positron. $|e^+(p,s)\rangle = d_s^\dagger(\vec{p})|0\rangle$. The field $\bar{\psi}(x)$ annihilates positrons when acting on kets:

$$\bar{\psi}(x)|e^+(p,s)\rangle = \bar{v}_s(p)e^{+ip\cdot x}|0\rangle \,. \tag{4.103}$$

Factor: $\bar{v}(p,s)$.

Outgoing positron. $\langle e^+(p',s')| = \langle 0|d_{s'}(\vec{p}')$:

$$\langle e^+(p',s')|\psi(x) = \langle 0|v_{s'}(p')e^{-ip'\cdot x} \,. \tag{4.104}$$

Factor: $v(p',s')$.

Additional rules

1. **Momentum conservation at each vertex.**
   Origin: Each vertex carries a spacetime integral $\int d^4x$. When fields are replaced by their Fourier representations, the position integral produces:

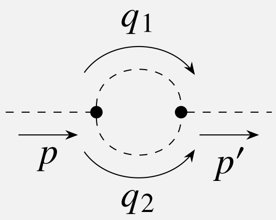
   $$\int d^4x\, e^{i(p_1+p_2+p_3)\cdot x} = (2\pi)^4\delta^{(4)}(p_1+p_2+p_3)\,, \tag{4.105}$$

   where all momenta are defined as *incoming*. This delta function enforces momentum conservation at every vertex.
   Rule: Assign a factor $(2\pi)^4\delta^{(4)}(\sum_i p_i^{\text{in}})$ to each vertex, with the sign convention that momenta flowing into the vertex are positive.
2. **Loop integration.**
   Origin: After imposing momentum conservation at all vertices, there remain undetermined momenta circulating in closed loops. These arise because the diagram has more internal lines than independent momentum constraints.
   For a diagram with I internal lines and V vertices, the number of independent loop momenta is:

   $$L = I - V + 1\,. \tag{4.106}$$

Box 4.4 **Why $L = I - V + 1$**

Idea. Each internal line carries a momentum we must determine. Each vertex fixes one relation among these momenta (conservation). So: L = (unknowns) − (independent equations).
Concrete example. Consider the s-channel bubble in $2 \to 2$ scattering ($V = 2$, $I = 2$):



There are $I = 2$ unknown internal momenta: q_1 and q_2. Vertex 1 gives $p = q_1 + q_2$, vertex 2 gives $q_1 + q_2 = p'$. But these two equations are not independent: the second is just "$p = p'$" once you substitute the first. So there is only $V - 1 = 1$ independent equation, leaving $L = 2 - 1 = 1$ free momentum. We can choose $q_1 = \ell$ (the loop momentum), and then $q_2 = p - \ell$ is fixed.
General argument.

(a) *Unknowns*: I internal momenta.
(b) *Equations*: V vertices, each imposing 4-momentum conservation.
(c) *Redundancy*: one equation is always redundant, because if we *add* the conservation laws of all V vertices, every internal momentum cancels (it enters one vertex and exits another with opposite sign). What remains is just $\sum p_{\text{ext,in}} = \sum p_{\text{ext,out}}$—overall conservation of external momenta, which tells us nothing about the internal ones.
(d) *Result*: $L = I - (V - 1) = I - V + 1$.

Examples.

- *Tree diagrams*: $I = V - 1 \Rightarrow L = 0$. All internal momenta are fixed—no integrals.
- *Figure-eight*: $V = 1, I = 2 \Rightarrow L = 2$. Two independent loop momenta.

Rule: For each independent loop momentum ℓ, include an integration:

$$\int \frac{d^4\ell}{(2\pi)^4}. \tag{4.107}$$

The factor $(2\pi)^{-4}$ matches the $(2\pi)^4$ from the momentum-conserving delta functions.

3. **Fermion line ordering.**

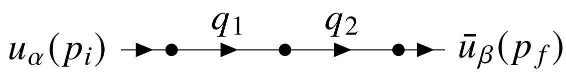
Origin: Matrix multiplication of spinor indices. Consider a generic S-matrix element for a fermion scattering through three vertices. From the Dyson series we have schematically:

$$\langle p_f|S|p_i\rangle \sim \langle 0|\, b_{p_f}\, \bar{\psi}(x_3)\,\gamma\; \underbrace{\psi(x_3)\,\bar{\psi}(x_2)}_{\text{contract}\to S_F(q_2)}\; \gamma\; \underbrace{\psi(x_2)\,\bar{\psi}(x_1)}_{\text{contract}\to S_F(q_1)}\; \gamma\,\psi(x_1)\; b^\dagger_{p_i}|0\rangle\,. \tag{4.108}$$

Each object carries spinor indices ($\alpha,\beta,\cdots = 1,\ldots,4$). Making them explicit:

- $b^\dagger_{p_i}|0\rangle$ produces $u_\alpha(p_i)$ from $\psi_\alpha(x_1)$;
- the vertex at x_1 contributes $(\gamma^{\mu_1})_{\zeta\alpha}$, with α contracted with u_α;
- the Wick contraction $\psi_\epsilon(x_2)\bar{\psi}_\zeta(x_1) = S_{F,\epsilon\zeta}(q_1)$, with ζ contracted with the vertex $\gamma_{\zeta\alpha}$;
- the vertex at x_2 contributes $(\gamma^{\mu_2})_{\delta\epsilon}$, with ϵ contracted with $S_{F,\epsilon\zeta}$;
- the contraction $\psi_\gamma(x_3)\bar{\psi}_\delta(x_2) = S_{F,\gamma\delta}(q_2)$;
- the vertex at x_3 contributes $(\gamma^{\mu_3})_{\beta\gamma}$;
- $\langle 0|b_{p_f}$ produces $\bar{u}_\beta(p_f)$ from $\bar{\psi}_\beta(x_3)$.

Assembling all factors and summing over repeated indices, we obtain for the fermion line:



$$\bar{u}_\beta(p_f)\cdot(\gamma)_{\beta\gamma}\cdot S_{\gamma\delta}(q_2)\cdot(\gamma)_{\delta\epsilon}\cdot S_{\epsilon\zeta}(q_1)\cdot(\gamma)_{\zeta\alpha}\cdot u_\alpha(p_i)\,. \tag{4.109}$$

This is simply matrix multiplication: each repeated index is summed, and the result is a scalar in spinor space.
Rule: Read spinor factors along the fermion line *opposite* to the arrow direction:

$$\bar{u}(p_f)\,\gamma^{\mu_3}\,S_F(q_2)\,\gamma^{\mu_2}\,S_F(q_1)\,\gamma^{\mu_1}\,u(p_i)\,. \tag{4.110}$$

4. **Closed fermion loop.**
Origin: A closed loop involves a contraction pattern like:

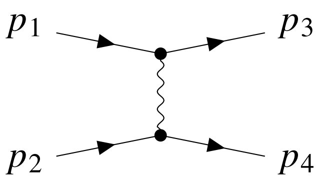
$$\psi_\alpha(x_1)\bar{\psi}_\beta(x_2)\,\psi_\beta(x_2)\bar{\psi}_\gamma(x_3)\cdots\psi_\omega(x_n)\bar{\psi}_\alpha(x_1)\,. \tag{4.111}$$

The spinor indices form a closed chain: $\alpha\to\beta\to\gamma\to\cdots\to\omega\to\alpha$.
This closed chain of matrix multiplications yields a *trace* over spinor indices:

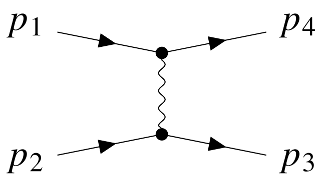
$$\mathrm{Tr}\big[\gamma^{\mu_1}S_F(x_1-x_2)\gamma^{\mu_2}S_F(x_2-x_3)\cdots\gamma^{\mu_n}S_F(x_n-x_1)\big]\,. \tag{4.112}$$

The factor of (-1) arises from anticommuting fermionic fields to form the closed contraction pattern (see Section 4.7).
Rule: For each closed fermion loop, include a factor (-1) and take the trace over Dirac indices.

5. **Relative signs between diagrams.**
Origin: When computing $\langle f|S|i\rangle$, the external fermion operators must be contracted with the field operators. Different diagrams may require different orderings of these contractions.
Consider $e^-e^- \to e^-e^-$ (Møller scattering). The initial state $|i\rangle = b^\dagger(p_1)\, b^\dagger(p_2)|0\rangle$ involves two identical fermions, and we contract with $b(p_3)\, b(p_4)$ in the final state. Two diagrams exist:



Direct: $1 \to 3$, $2 \to 4$



Exchange: $1 \to 4$, $2 \to 3$

The exchange diagram requires permuting the final-state labels: $(p_3, p_4) \to (p_4, p_3)$. Since $b(p_3)b(p_4) = -b(p_4)b(p_3)$, the two diagrams have opposite signs.
Rule: Diagrams related by exchange of two identical external fermions differ by a factor (-1).

6. **Symmetry factor.**
Origin: The factor $1/n!$ in $S^{(n)}$ is cancelled by the $n!$ ways of assigning the interaction vertices to the spacetime integration points $x_1, \ldots, x_n$. To see how this works explicitly, consider Compton scattering at second order ($n = 2$). The Dyson series gives:

$$S^{(2)} = \frac{(-i)^2}{2!} \int d^4x_1\, d^4x_2\, T\{H_I(x_1)\, H_I(x_2)\}\,. \tag{4.113}$$

Applying Wick's theorem to the time-ordered product, each Feynman diagram (say, the s-channel) arises from *two* distinct contractions:

- *Contraction A:* the incoming photon is absorbed at x_1, the outgoing photon is emitted at x_2;
- *Contraction B:* the incoming photon is absorbed at x_2, the outgoing photon is emitted at x_1.

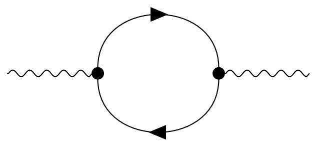
Since x_1 and x_2 are dummy integration variables, both contractions yield the *same* integral. The factor of 2 from these two equivalent contractions cancels the $1/2!$ from the Dyson series:

$$\frac{1}{2!} \times 2 = 1\,. \tag{4.114}$$

The same cancellation occurs for the u-channel diagram. This is a general feature: at order n, the $n!$ permutations of the vertices $x_1, \dots, x_n$ produce $n!$ identical Wick contractions for each diagram topology, exactly cancelling the $1/n!$ prefactor. This is *why the Feynman rules do not include the* $1/n!$ *factor* from the Dyson series.

However, if the diagram has internal symmetries (automorphisms), some of these $n!$ assignments produce identical contributions even *before* relabelling, so the cancellation is only partial.

Example: Vacuum polarization has a symmetry under exchanging the two fermion propagators:



The two internal fermion lines can be exchanged without changing the diagram $\Rightarrow S = 1$ (the exchange is equivalent to $x \leftrightarrow y$, already accounted for).

General rule: S equals the number of permutations of internal lines and vertices that leave the diagram unchanged (its automorphism group order). For most tree-level QED diagrams, $S = 1$.

4.7 The fermion minus sign: detailed proof

Origin. The minus signs arise from the anticommutation of fermionic field operators in Wick's theorem.

Closed loop. Consider a fermion loop connecting n vertices at $x_1, x_2, \dots, x_n$. The corresponding Feynman-diagram expression is:

$$(-1)\,\mathrm{Tr}\big[iS_F(x_1-x_2)\gamma^{\mu_2}\cdot iS_F(x_2-x_3)\gamma^{\mu_3}\cdots iS_F(x_n-x_1)\gamma^{\mu_1}\big], \tag{4.115}$$

where the overall factor of (-1) is the fermion loop sign we wish to derive. The trace arises because the spinor indices form a closed chain.

Box 4.5 Proof of the (-1) sign for a closed fermion loop

Warm-up: $n = 2$ (vacuum polarization). The relevant piece of the Wick expansion is (suppressing γ matrices):

$$\bar\psi_1\psi_1\bar\psi_2\psi_2,$$

where the subscripts label vertices. To form the loop, we need the contractions $\psi_1\bar\psi_2$ and $\psi_2\bar\psi_1$. Starting from the natural (normal-ordering) sequence

$\bar{\psi}_1\psi_1\bar{\psi}_2\psi_2$, we must bring ψ_1 next to $\bar{\psi}_2$, and ψ_2 next to $\bar{\psi}_1$. In detail:

$$\begin{aligned}&\bar{\psi}_1\psi_1\bar{\psi}_2\psi_2\\ &=\bar{\psi}_1\underbrace{\psi_1\bar{\psi}_2}_{\text{swap}}\psi_2=-\bar{\psi}_1\bar{\psi}_2\psi_1\psi_2 \qquad (1\text{ swap}\to -1)\end{aligned}$$

Now contract: $\overbracket{\psi_1\bar{\psi}_2}$ is already adjacent (reading $\bar{\psi}_2\psi_1$ from right to left gives $\psi_1\bar{\psi}_2$), and $\overbracket{\psi_2\bar{\psi}_1}$ closes the loop. The single swap gives one factor of (-1).

Compare with an open fermion line, where we would contract $\overbracket{\psi_1\bar{\psi}_2}$ only, leaving $\bar{\psi}_1$ and ψ_2 as external legs—no rearrangement needed, no sign.

General case: n vertices. For an open fermion line through n vertices, the contractions are:

$$\overbracket{\psi_1\bar{\psi}_2}\,\overbracket{\psi_2\bar{\psi}_3}\cdots\overbracket{\psi_{n-1}\bar{\psi}_n}\,,$$

with $\bar{\psi}_1$ and ψ_n connecting to external states. Call the sign required to bring the fields into this contracted order σ_{open}.

For a *closed* loop, we need the additional contraction $\overbracket{\psi_n\bar{\psi}_1}$: the field $\bar{\psi}_1$ must now also be contracted with ψ_n. Starting from the open-line arrangement, $\bar{\psi}_1$ sits at the far left. To bring it next to ψ_n, we must move it past the $n-1$ other fermionic fields ($\psi_1,\bar{\psi}_2,\psi_2,\ldots$ are already paired, but $\bar{\psi}_1$ must anticommute past each contracted pair). A careful counting shows that closing the loop always requires an *odd* number of additional transpositions relative to the open line, giving one extra factor of (-1):

$$\sigma_{\text{loop}}=(-1)\times\sigma_{\text{open}}\,.$$

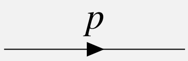
Since σ_{open} is already absorbed into the standard Feynman rules for open fermion lines, the only new ingredient for a closed loop is the additional factor of $\boxed{(-1)}$.

Exchange diagrams. For processes involving identical external fermions, diagrams that differ by exchanging two such external fermion lines carry a relative minus sign. This is the situation in Møller scattering discussed above. In Bhabha scattering, the relative sign between the s- and t-channel contributions is obtained by keeping a consistent ordering of the external fermionic operators in the Wick/LSZ reduction.

4.8 Summary: QED Feynman rules

Box 4.6 Complete QED Feynman rules (momentum space)

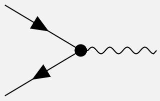
Propagators:



$$= \frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}$$

$$= \frac{-ig_{\mu\nu}}{k^2+i\epsilon}$$

Vertex:



$$= +ie\gamma^\mu$$

External lines:

 $u(p,s)$  $\bar{u}(p,s)$

 $\bar{v}(p,s)$  $v(p,s)$

 ϵ_μ ϵ^*_μ

Rules:

- Conserve momentum at each vertex
- Integrate $\int \frac{d^4\ell}{(2\pi)^4}$ over each loop
- Factor (-1) for each closed fermion loop
- Relative (-1) for fermion exchange
- Divide by symmetry factor S

Part II

Quantum Electrodynamics