5 QED Scattering Processes

In this chapter, we apply the Feynman rules of quantum electrodynamics to compute cross sections for fundamental scattering processes. These calculations illustrate the power and elegance of the diagrammatic approach developed by Feynman, and provide quantitative predictions that have been tested to extraordinary precision.

5.1 Overview of QED processes

The QED interaction vertex

The fundamental interaction in QED is the coupling of a photon to a charged fermion.

Charge conventions. We adopt the following conventions throughout these notes:

- The symbol e denotes the **positron charge**, a positive quantity: $e > 0$. Numerically, $e \simeq 0.303$ in natural units, or equivalently $e = \sqrt{4\pi\alpha}$ with $\alpha \simeq 1/137$.
- The **electron** carries charge $-e$ and the **positron** carries charge $+e$.
- Similarly, μ^- has charge $-e$ and μ^+ has charge $+e$.

The QED Lagrangian. For a Dirac fermion field ψ with electric charge q, the QED Lagrangian is:

$$\mathscr{L}_{\text{QED}} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{5.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength tensor and D_μ is the gauge-covariant derivative.

Box 5.1 Sign conventions in QED: a systematic treatment

There are **two independent convention choices** that are often confused:

1. Whether the symbol e denotes the **positive magnitude** ($e > 0$) or the **signed** electron charge ($e < 0$).
2. Whether the covariant derivative is written with $+iq$ or $-iq$ in front of the gauge field.

The fundamental starting point: interaction energy. For a static point charge q in an external electrostatic potential $\phi(\vec{x})$, the potential energy is:

$$V = q\,\phi(\vec{x})\,. \tag{5.2}$$

Since $L = T - V$, the interaction Lagrangian is $L_{\text{int}} = -V = -q\phi$. In field theory with the Dirac current $j^\mu = \bar{\psi}\gamma^\mu\psi$ (the probability/number current, *without* the charge factor), the covariant generalization is:

$$\boxed{\mathscr{L}_{\text{int}} = -q\,j^\mu A_\mu = -q\,\bar{\psi}\gamma^\mu\psi\,A_\mu}\,. \tag{5.3}$$

Mnemonic: *Potential energy is $+q\phi$, so Lagrangian interaction is $-q\phi$.* This pins down $\mathscr{L}_{\text{int}} = -q\,j^\mu A_\mu$.
Covariant derivative conventions. Two common conventions exist for implementing gauge invariance:

Convention A (used in these notes, consistent with energy argument):

$$\boxed{D_\mu = \partial_\mu + iq\,A_\mu}\,, \tag{5.4}$$

with gauge transformations $\psi \to e^{-iq\alpha(x)}\psi$ and $A_\mu \to A_\mu + \partial_\mu\alpha$.
Expanding the Dirac Lagrangian:

$$\begin{aligned}\bar{\psi}(i\gamma^\mu D_\mu - m)\psi &= \bar{\psi}(i\partial\!\!\!/ - m)\psi + \bar{\psi}(i\gamma^\mu \cdot iqA_\mu)\psi \\ &= \bar{\psi}(i\partial\!\!\!/ - m)\psi - q\,\bar{\psi}\gamma^\mu\psi\,A_\mu\,.\end{aligned} \tag{5.5}$$

Hence:

$$\boxed{\mathscr{L}_{\text{int}} = -q\,\bar{\psi}\gamma^\mu\psi\,A_\mu \qquad \text{(Convention A)}}\,. \tag{5.6}$$

This matches exactly the energy-based derivation eq. (5.3).

Convention B (used in some texts):

$$D_\mu = \partial_\mu - iq\,A_\mu\,. \tag{5.7}$$

This gives $\mathscr{L}_{\text{int}} = +q\,\bar{\psi}\gamma^\mu\psi\,A_\mu$, with opposite sign.
Key point: The sign is a convention; physics is unchanged. If you switch from A to B, you can compensate by $q \to -q$ or $A_\mu \to -A_\mu$.
Our convention: electron charge $q_e = -e$ with $e > 0$. We define:

- $e > 0$ is the **magnitude** of the electron charge (the positron charge)
- The **electron** has charge $q_e = -e < 0$

- The **positron** has charge $q_{e^+} = +e > 0$

With Convention A and $q = -e$ for the electron:

$$D_\mu = \partial_\mu + i(-e)A_\mu = \partial_\mu - ie\,A_\mu\,, \tag{5.8}$$

$$\boxed{\mathscr{L}_{\text{int}} = +e\,\bar{\psi}\gamma^\mu\psi\,A_\mu}, \qquad e > 0, \quad q_e = -e\,. \tag{5.9}$$

From Lagrangian to Feynman rule. The S-matrix expansion is:

$$S = T\exp\left(i\int d^4x\,\mathscr{L}_{\text{int}}(x)\right). \tag{5.10}$$

The vertex factor is obtained by: *take i times the coefficient of $A_\mu\bar{\psi}\psi$ in $\mathscr{L}_{int}$, attach γ^μ.*
From eq. (5.6) with charge q:

$$\boxed{\text{vertex} = i\times(-q)\times\gamma^\mu = -iq\,\gamma^\mu}. \tag{5.11}$$

For the **electron** ($q = -e$):

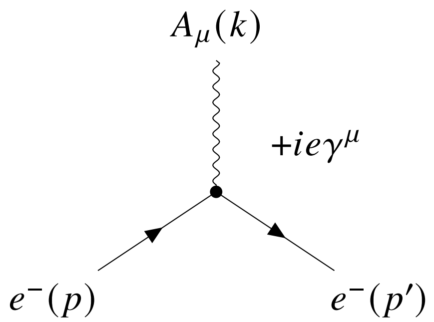
$$\boxed{\text{vertex} = -i(-e)\gamma^\mu = +ie\,\gamma^\mu \qquad \text{(our convention)}}. \tag{5.12}$$

Memory rule: Vertex $= i\times$ (coefficient of $A_\mu\bar{\psi}\psi$ in $\mathscr{L}_{\text{int}}$) $\times\gamma^\mu$.
Comparison with other textbooks. Different books package the same physics differently. The key is: *what is $\mathscr{L}_{int}$ and hence the vertex?*

Text	D_μ	$\mathscr{L}_{\text{int}}$	**Vertex**
These notes (with $q_e = -e$)	$\partial_\mu + iqA_\mu$ $= \partial_\mu - ieA_\mu$	$-q\bar{\psi}\gamma^\mu\psi A_\mu$ $= +e\bar{\psi}\gamma^\mu\psi A_\mu$	$-iq\gamma^\mu$ $= +ie\gamma^\mu$
Peskin–Schroeder [6]	$\partial_\mu + ieA_\mu$	$-e\bar{\psi}\gamma^\mu\psi A_\mu$	$-ie\gamma^\mu$
Greiner et al. [13]	$\partial_\mu + ieA_\mu$	$-e\bar{\psi}\gamma^\mu\psi A_\mu$	$-ie\gamma^\mu$
Bjorken–Drell [10]	$\partial_\mu + ieA_\mu$	$-e\bar{\psi}\gamma^\mu\psi A_\mu$	$-ie\gamma^\mu$
Mandl–Shaw [1]	$\partial_\mu - ieA_\mu$	$+e\bar{\psi}\gamma^\mu\psi A_\mu$	$+ie\gamma^\mu$

Note: Peskin–Schroeder [6] write $D_\mu = \partial_\mu + ieA_\mu$ and $\mathscr{L}_{\text{int}} = -e\bar{\psi}\gamma^\mu\psi A_\mu$, with $e > 0$ denoting the magnitude of the electron charge. This is a different sign convention from the one adopted in these notes, but it describes the same physics once used consistently. Our convention (and Mandl–Shaw's [1]) is directly consistent with the energy argument.

Feynman rule for the vertex. From eq. (5.9), the Feynman rule for the electron-photon vertex is:



The Feynman rule is $\boxed{+ie\gamma^\mu}$ for electrons and muons (charge $q = -e$), with momentum conservation $p + k = p'$ at the vertex. For a general fermion of charge q, the vertex is $-iq\gamma^\mu$.

Classification of tree-level processes

QED processes can be classified according to the number and type of external particles. At tree level (lowest order in $\alpha = e^2/4\pi$), the fundamental $2 \to 2$ scattering processes are:

Process	**Name**	**Channel**
$e^-e^- \to e^-e^-$	Møller scattering	t, u
$e^+e^- \to e^+e^-$	Bhabha scattering	s, t
$e^+e^- \to \mu^+\mu^-$	Muon pair production	s
$e^+e^- \to \gamma\gamma$	Pair annihilation	t, u
$e^-\gamma \to e^-\gamma$	Compton scattering	s, u
$e^-p \to e^-p$	Electron-proton scattering	t

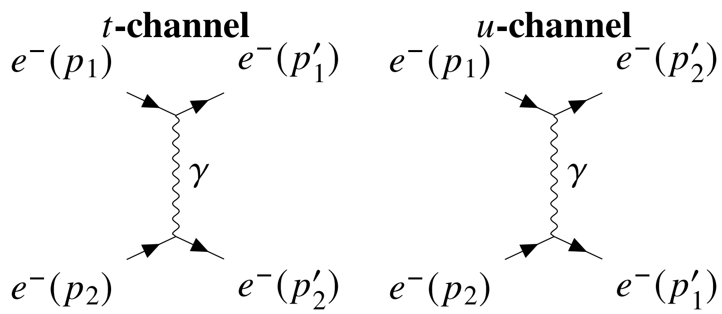
The "channel" indicates which Mandelstam variable (s, t, or u) appears in the propagator denominator, corresponding to different physical mechanisms:

- s**-channel**: particle-antiparticle annihilation into a virtual particle
- t**-channel**: exchange of a virtual particle between the two scattering particles
- u**-channel**: like t-channel but with crossed final-state particles

Feynman diagrams for classic QED processes

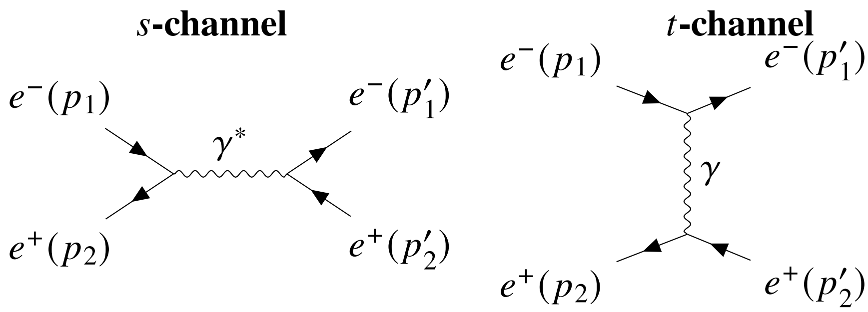
Møller scattering ($e^-e^- \to e^-e^-$)

Two electrons scatter via photon exchange. There are two diagrams at tree level (the u-channel diagram accounts for the indistinguishability of the final-state electrons):



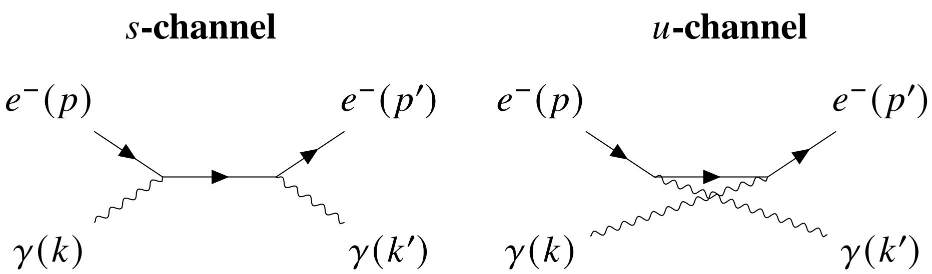
Bhabha scattering ($e^+e^- \to e^+e^-$)

Electron-positron elastic scattering proceeds via both annihilation (s-channel) and exchange (t-channel):



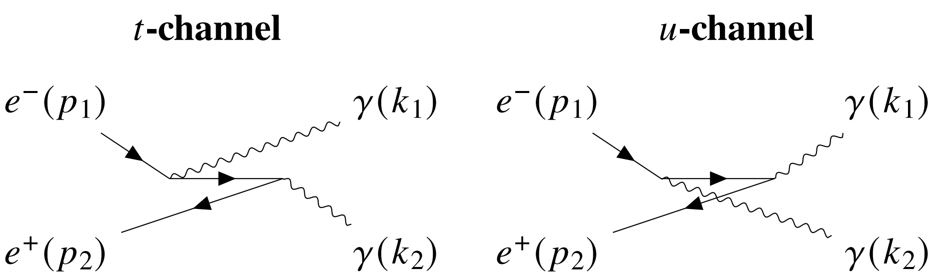
Compton scattering ($e^-\gamma \to e^-\gamma$)

The scattering of a photon by an electron involves two diagrams:



Pair annihilation ($e^+e^- \to \gamma\gamma$)

An electron-positron pair annihilates into two photons:



Remark. [Crossing symmetry] These processes are related by *crossing symmetry*: replacing an incoming particle with an outgoing antiparticle (or vice versa) transforms one process into another. For example, $e^+e^- \to \gamma\gamma$ is the crossed version of Compton scattering $e^-\gamma \to e^-\gamma$.

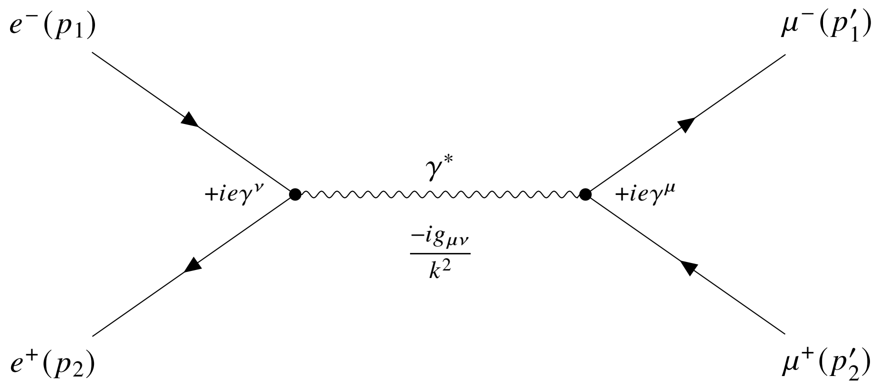
5.2 Muon pair production: $e^+e^- \to \mu^+\mu^-$

We now present a detailed calculation of the cross section for the process $e^+e^- \to \mu^+\mu^-$. This is one of the cleanest QED processes to analyze because: (i) only the s-channel contributes at tree level, since QED contains no vertex that converts an electron into a muon; (ii) it provides a "standard candle" for measuring luminosity at e^+e^- colliders; (iii) the theoretical prediction agrees with experiment to remarkable precision.

Kinematics and Feynman diagram. Consider the process:

$$e^-(p_1) + e^+(p_2) \to \mu^-(p_1') + \mu^+(p_2') \,. \tag{5.13}$$

At tree level, the only contributing diagram is the s-channel annihilation through a virtual photon:



Scattering amplitude. Applying the Feynman rules, the scattering amplitude is:

$$\begin{aligned} i\mathcal{M} = {} & \bar{u}(p_1', s_1')(+ie\gamma^\mu)v(p_2', s_2') \cdot \frac{-ig_{\mu\nu}}{k^2} \\ & \cdot \bar{v}(p_2, s_2)(+ie\gamma^\nu)u(p_1, s_1) \,. \end{aligned} \tag{5.14}$$

Simplifying (note that $(+ie)^2 = -e^2$):

$$\boxed{\mathcal{M} = \frac{e^2}{k^2} \left[\bar{u}(p_1')\gamma_\nu v(p_2')\right] \left[\bar{v}(p_2)\gamma^\nu u(p_1)\right]} \tag{5.15}$$

where $k = p_1 + p_2 = p_1' + p_2'$ is the four-momentum of the virtual photon, and $k^2 = s$ is the center-of-mass energy squared.

BD normalization factors. In the BD convention each external fermion field contributes a factor $\sqrt{m/E}$ from the mode expansion (3.25). For this process there are four external fermions, so the BD amplitude carries an overall factor

$$\mathcal{N}_{\mathrm{BD}} = \sqrt{\frac{m_e^2 m_\mu^2}{E_1 E_2 E_1' E_2'}}\,. \tag{5.16}$$

As explained in Section 1.1, this factor combines with the $\prod(2m_i) = (2m_e)^2(2m_\mu)^2$ in the BD cross section formula (3.55) and the $1/(2m)^2$ from the BD spin sums (3.58), and all mass factors cancel in the final cross section. In the spin sums below we work with the BD completeness relations; the $1/(2m)^2$ prefactors are carried explicitly through the leptonic tensors until the cancellation is shown at the level of the cross section.

Spin-averaged amplitude squared. For unpolarized beams, we average over initial spins (factor $\frac{1}{4}$) and sum over final spins:

$$\overline{|\mathcal{M}|^2} \equiv \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2\,. \tag{5.17}$$

From eq. (5.15), we have:

$$\mathcal{M} = \frac{e^2}{s}\left[\bar{u}_\mu(p_1')\gamma_\nu v_\mu(p_2')\right]\left[\bar{v}_e(p_2)\gamma^\nu u_e(p_1)\right], \tag{5.18}$$

where we suppress spin labels for clarity and used $k^2 = s$. The complex conjugate is:

$$\mathcal{M}^* = \frac{e^2}{s}\left[\bar{v}_\mu(p_2')\gamma_\rho u_\mu(p_1')\right]\left[\bar{u}_e(p_1)\gamma^\rho v_e(p_2)\right], \tag{5.19}$$

where we used $(\bar{\psi}_1\gamma_\nu\psi_2)^* = \bar{\psi}_2\gamma_\nu\psi_1$.

The amplitude squared is therefore:

$$\begin{aligned}|\mathcal{M}|^2 &= \mathcal{M}\mathcal{M}^* \\ &= \frac{e^4}{s^2}\left[\bar{u}_\mu(p_1')\gamma_\nu v_\mu(p_2')\right]\left[\bar{v}_e(p_2)\gamma^\nu u_e(p_1)\right] \\ &\quad\times\left[\bar{v}_\mu(p_2')\gamma_\rho u_\mu(p_1')\right]\left[\bar{u}_e(p_1)\gamma^\rho v_e(p_2)\right].\end{aligned} \tag{5.20}$$

From spinor products to traces. To perform the spin sums, we use the **complete-**

ness relations appropriate to the BD normalization:

$$\boxed{\sum_s u_\sigma(p,s)\bar{u}_\tau(p,s) = \frac{(\not{p}+m)_{\sigma\tau}}{2m}} \tag{5.21}$$

$$\boxed{\sum_s v_\sigma(p,s)\bar{v}_\tau(p,s) = \frac{(\not{p}-m)_{\sigma\tau}}{2m}}, \tag{5.22}$$

where σ, τ are spinor indices. These $1/(2m)$ factors can be carried explicitly or absorbed into external-state normalizations; below we keep them explicit.

We now show in detail how the spin sum converts the product of spinor bilinears into traces. Consider first the muon sector. We can write the spinor bilinears with explicit spinor indices:

$$\left[\bar{u}_\mu(p_1')\gamma_\nu v_\mu(p_2')\right] = \bar{u}_{\mu,\sigma}(p_1')(\gamma_\nu)_\sigma{}^\tau\, v_{\mu,\tau}(p_2') \tag{5.23}$$

$$\left[\bar{v}_\mu(p_2')\gamma_\rho u_\mu(p_1')\right] = \bar{v}_{\mu,\xi}(p_2')(\gamma_\rho)_\xi{}^\zeta\, u_{\mu,\zeta}(p_1')\,. \tag{5.24}$$

Their product, summed over muon spins s_1' and s_2', is:

$$\begin{aligned}
&\sum_{s_1',s_2'}\left[\bar{u}_\mu(p_1')\gamma_\nu v_\mu(p_2')\right]\left[\bar{v}_\mu(p_2')\gamma_\rho u_\mu(p_1')\right] \\
&= \sum_{s_1',s_2'} \bar{u}_{\mu,\sigma}(p_1')(\gamma_\nu)_\sigma{}^\tau\, v_{\mu,\tau}(p_2')\,\bar{v}_{\mu,\xi}(p_2')(\gamma_\rho)_\xi{}^\zeta\, u_{\mu,\zeta}(p_1') \\
&= (\gamma_\nu)_\sigma{}^\tau\,(\gamma_\rho)_\xi{}^\zeta \sum_{s_2'} v_{\mu,\tau}(p_2')\bar{v}_{\mu,\xi}(p_2') \sum_{s_1'} u_{\mu,\zeta}(p_1')\bar{u}_{\mu,\sigma}(p_1')\,.
\end{aligned} \tag{5.25}$$

Using the completeness relations:

$$\sum_{s_2'} v_{\mu,\tau}(p_2')\bar{v}_{\mu,\xi}(p_2') = \frac{(\not{p}_2' - m_\mu)_{\tau\xi}}{2m_\mu} \tag{5.26}$$

$$\sum_{s_1'} u_{\mu,\zeta}(p_1')\bar{u}_{\mu,\sigma}(p_1') = \frac{(\not{p}_1' + m_\mu)_{\zeta\sigma}}{2m_\mu}\,. \tag{5.27}$$

Substituting:

$$\begin{aligned}
&\sum_{s_1',s_2'}\left[\bar{u}_\mu(p_1')\gamma_\nu v_\mu(p_2')\right]\left[\bar{v}_\mu(p_2')\gamma_\rho u_\mu(p_1')\right] \\
&= \frac{1}{(2m_\mu)^2}(\gamma_\nu)_\sigma{}^\tau\,(\not{p}_2' - m_\mu)_{\tau\xi}\,(\gamma_\rho)_\xi{}^\zeta\,(\not{p}_1' + m_\mu)_{\zeta\sigma} \\
&= \frac{1}{(2m_\mu)^2}\left[(\not{p}_1' + m_\mu)\gamma_\nu(\not{p}_2' - m_\mu)\gamma_\rho\right]_{\sigma\sigma} \\
&= \frac{1}{(2m_\mu)^2}\mathrm{Tr}\left[(\not{p}_1' + m_\mu)\gamma_\nu(\not{p}_2' - m_\mu)\gamma_\rho\right],
\end{aligned} \tag{5.28}$$

where in the last step we recognized the sum over the repeated index σ as a trace.

By exactly the same procedure for the electron sector:

$$\sum_{s_1,s_2}\left[\bar{v}_e(p_2)\gamma^\nu u_e(p_1)\right]\left[\bar{u}_e(p_1)\gamma^\rho v_e(p_2)\right]$$
$$= \frac{1}{(2m_e)^2}\mathrm{Tr}\left[(\not{p}_2 - m_e)\gamma^\nu(\not{p}_1 + m_e)\gamma^\rho\right]. \tag{5.29}$$

Therefore, the spin-averaged amplitude squared is:

$$\boxed{\overline{|\mathcal{M}|^2} = \frac{e^4}{4s^2}L^{(\mu)}_{\nu\rho}\cdot L^{(e)\nu\rho}}, \tag{5.30}$$

where we define the **leptonic tensors** (retaining the BD normalization factors):

$$L^{(\mu)}_{\nu\rho} \equiv \frac{1}{(2m_\mu)^2}\mathrm{Tr}\left[(\not{p}'_1 + m_\mu)\gamma_\nu(\not{p}'_2 - m_\mu)\gamma_\rho\right], \tag{5.31}$$

$$L^{(e)\nu\rho} \equiv \frac{1}{(2m_e)^2}\mathrm{Tr}\left[(\not{p}_2 - m_e)\gamma^\nu(\not{p}_1 + m_e)\gamma^\rho\right]. \tag{5.32}$$

The $1/(2m)^2$ prefactors originate from the BD completeness relations (5.21)–(5.22). They will cancel exactly against the external-state normalization factors when computing the cross section (see below, eq. (5.71)).

Evaluation of the muon trace. Expanding the product $(\not{p}'_1 + m_\mu)\gamma_\nu(\not{p}'_2 - m_\mu)\gamma_\rho$ inside the trace in eq. (5.31):

$$L^{(\mu)}_{\nu\rho} = \frac{1}{(2m_\mu)^2}\Big[\mathrm{Tr}(\not{p}'_1\gamma_\nu\not{p}'_2\gamma_\rho) - m_\mu\mathrm{Tr}(\not{p}'_1\gamma_\nu\gamma_\rho)$$
$$+ m_\mu\mathrm{Tr}(\gamma_\nu\not{p}'_2\gamma_\rho) - m_\mu^2\mathrm{Tr}(\gamma_\nu\gamma_\rho)\Big]. \tag{5.33}$$

The second and third terms vanish because they contain an odd number of gamma matrices (three each). Thus:

$$L^{(\mu)}_{\nu\rho} = \frac{1}{(2m_\mu)^2}\Big[\mathrm{Tr}(\not{p}'_1\gamma_\nu\not{p}'_2\gamma_\rho) - m_\mu^2\mathrm{Tr}(\gamma_\nu\gamma_\rho)\Big]. \tag{5.34}$$

For the first term, we write $\not{p}'_1 = p'_{1\alpha}\gamma^\alpha$ and $\not{p}'_2 = p'_{2\beta}\gamma^\beta$:

$$\mathrm{Tr}(\not{p}'_1\gamma_\nu\not{p}'_2\gamma_\rho) = p'_{1\alpha}p'_{2\beta}\mathrm{Tr}(\gamma^\alpha\gamma_\nu\gamma^\beta\gamma_\rho). \tag{5.35}$$

Using the trace formula:

$$\mathrm{Tr}(\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta) = 4(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\delta}g^{\beta\gamma}), \tag{5.36}$$

we have:

$$\begin{aligned}\mathrm{Tr}(\gamma^\alpha\gamma_\nu\gamma^\beta\gamma_\rho) &= 4(g^\alpha{}_\nu g^\beta{}_\rho - g^{\alpha\beta}g_{\nu\rho} + g^\alpha{}_\rho g^\beta{}_\nu)\\ &= 4(\delta^\alpha_\nu\delta^\beta_\rho - g^{\alpha\beta}g_{\nu\rho} + \delta^\alpha_\rho\delta^\beta_\nu)\,. \end{aligned} \tag{5.37}$$

Contracting with $p'_{1\alpha}p'_{2\beta}$:

$$p'_{1\alpha}p'_{2\beta}\mathrm{Tr}(\gamma^\alpha\gamma_\nu\gamma^\beta\gamma_\rho) = 4\big(p'_{1\nu}p'_{2\rho} - (p'_1\cdot p'_2)g_{\nu\rho} + p'_{1\rho}p'_{2\nu}\big)\,. \tag{5.38}$$

For the second term: $\mathrm{Tr}(\gamma_\nu\gamma_\rho) = 4g_{\nu\rho}$.
Combining:

$$\boxed{L^{(\mu)}_{\nu\rho} = \frac{1}{m_\mu^2}\big[p'_{1\nu}p'_{2\rho} + p'_{1\rho}p'_{2\nu} - (p'_1\cdot p'_2 + m_\mu^2)g_{\nu\rho}\big]}\,. \tag{5.39}$$

Evaluation of the electron trace. By the same calculation for the electron tensor $L^{(e)\nu\rho}$:

$$\boxed{L^{(e)\nu\rho} = \frac{1}{m_e^2}\big[p_2^\nu p_1^\rho + p_2^\rho p_1^\nu - (p_1\cdot p_2 + m_e^2)g^{\nu\rho}\big]}\,. \tag{5.40}$$

Contraction of the leptonic tensors. The spin-averaged amplitude squared is:

$$\overline{|\mathcal{M}|^2} = \frac{e^4}{4s^2}L^{(\mu)}_{\nu\rho}L^{(e)\nu\rho} = \frac{e^4}{4s^2}\cdot\frac{1}{(2m_\mu)^2(2m_e)^2}\mathrm{Tr}[\mu]\,\mathrm{Tr}[e]\,. \tag{5.41}$$

The contraction of the two traces (without the $1/(2m)^2$ prefactors) proceeds as follows. Writing:

$$\mathrm{Tr}[\mu]_{\nu\rho} = 4\big[p'_{1\nu}p'_{2\rho} + p'_{1\rho}p'_{2\nu} - A\,g_{\nu\rho}\big] \quad \text{where } A = p'_1\cdot p'_2 + m_\mu^2 \tag{5.42}$$

$$\mathrm{Tr}[e]^{\nu\rho} = 4\big[p_2^\nu p_1^\rho + p_2^\rho p_1^\nu - B\,g^{\nu\rho}\big] \quad \text{where } B = p_1\cdot p_2 + m_e^2\,. \tag{5.43}$$

The contraction gives:

$$\begin{aligned}L^{(\mu)}_{\nu\rho}L^{(e)\nu\rho} = 16\Big[&(p'_{1\nu}p'_{2\rho})(p_2^\nu p_1^\rho) + (p'_{1\nu}p'_{2\rho})(p_2^\rho p_1^\nu)\\ &+ (p'_{1\rho}p'_{2\nu})(p_2^\nu p_1^\rho) + (p'_{1\rho}p'_{2\nu})(p_2^\rho p_1^\nu)\\ &- A(p_2^\nu p_1^\rho g_{\nu\rho}) - A(p_2^\rho p_1^\nu g_{\nu\rho})\\ &- B(p'_{1\nu}p'_{2\rho}g^{\nu\rho}) - B(p'_{1\rho}p'_{2\nu}g^{\nu\rho}) + 4AB\Big]\,. \end{aligned} \tag{5.44}$$

Evaluating each term using $p'_{1\nu}p_2^\nu = p'_1 \cdot p_2$, etc.:

$$(p'_{1\nu}p'_{2\rho})(p_2^\nu p_1^\rho) = (p'_1 \cdot p_2)(p'_2 \cdot p_1) \tag{5.45}$$
$$(p'_{1\nu}p'_{2\rho})(p_2^\rho p_1^\nu) = (p'_1 \cdot p_1)(p'_2 \cdot p_2) \tag{5.46}$$
$$(p'_{1\rho}p'_{2\nu})(p_2^\nu p_1^\rho) = (p'_2 \cdot p_2)(p'_1 \cdot p_1) \tag{5.47}$$
$$(p'_{1\rho}p'_{2\nu})(p_2^\rho p_1^\nu) = (p'_1 \cdot p_2)(p'_2 \cdot p_1) \tag{5.48}$$
$$p_2^\nu p_1^\rho g_{\nu\rho} = p_1 \cdot p_2 \tag{5.49}$$
$$p'_{1\nu}p'_{2\rho}g^{\nu\rho} = p'_1 \cdot p'_2 . \tag{5.50}$$

Collecting terms:

$$\begin{aligned} L^{(\mu)}_{\nu\rho}L^{(e)\nu\rho} = 16\Big[&2(p'_1 \cdot p_2)(p'_2 \cdot p_1) + 2(p'_1 \cdot p_1)(p'_2 \cdot p_2) \\ &- 2A(p_1 \cdot p_2) - 2B(p'_1 \cdot p'_2) + 4AB\Big] . \end{aligned} \tag{5.51}$$

Substituting $A = p'_1 \cdot p'_2 + m_\mu^2$ and $B = p_1 \cdot p_2 + m_e^2$, and simplifying:

$$\begin{aligned} \mathrm{Tr}[\mu]_{\nu\rho}\,\mathrm{Tr}[e]^{\nu\rho} = 32\Big[&(p_1 \cdot p'_1)(p_2 \cdot p'_2) + (p_1 \cdot p'_2)(p_2 \cdot p'_1) \\ &+ m_\mu^2(p_1 \cdot p_2) + m_e^2(p'_1 \cdot p'_2) + 2m_e^2 m_\mu^2\Big] . \end{aligned} \tag{5.52}$$

Including the BD normalization prefactors:

$$\boxed{\overline{|\mathcal{M}|^2} = \frac{e^4}{4s^2} \cdot \frac{1}{(2m_\mu)^2(2m_e)^2}\,\mathrm{Tr}[\mu]_{\nu\rho}\,\mathrm{Tr}[e]^{\nu\rho}} . \tag{5.53}$$

Center-of-mass kinematics. In the center-of-mass (CM) frame, the momenta are:

$$p_1 = (E, \vec{p}), \qquad p'_1 = (E, \vec{p}\,') \tag{5.54}$$
$$p_2 = (E, -\vec{p}), \qquad p'_2 = (E, -\vec{p}\,') , \tag{5.55}$$

where energy conservation requires all particles to have the same energy E. The relevant scalar products are:

$$p_1 \cdot p_2 = E^2 + |\vec{p}|^2 \tag{5.56}$$
$$p'_1 \cdot p'_2 = E^2 + |\vec{p}\,'|^2 \tag{5.57}$$
$$p_1 \cdot p'_1 = E^2 - |\vec{p}||\vec{p}\,'|\cos\theta \tag{5.58}$$
$$p_1 \cdot p'_2 = E^2 + |\vec{p}||\vec{p}\,'|\cos\theta , \tag{5.59}$$

where θ is the scattering angle between $\vec{p}$ and $\vec{p}\,'$, $|\vec{p}| = \sqrt{E^2 - m_e^2}$, and $|\vec{p}\,'| = \sqrt{E^2 - m_\mu^2}$.

Mandelstam variables. For a $2 \to 2$ scattering process, the Mandelstam variables are:

$$s \equiv (p_1 + p_2)^2, \qquad t \equiv (p_1 - p_1')^2, \qquad u \equiv (p_1 - p_2')^2 . \tag{5.60}$$

These satisfy $s + t + u = 2m_e^2 + 2m_\mu^2$. In the CM frame: $s = 4E^2$.

Ultrarelativistic limit. In the limit $E \gg m_e, m_\mu$, we have $|\vec{p}| \approx |\vec{p}\,'| \approx E$, and the Mandelstam variables simplify to:

$$s = 4E^2, \qquad t = -\frac{s}{2}(1 - \cos\theta), \qquad u = -\frac{s}{2}(1 + \cos\theta) . \tag{5.61}$$

The scalar products become:

$$p_1 \cdot p_2 = p_1' \cdot p_2' = \frac{s}{2}, \qquad p_1 \cdot p_1' = -\frac{t}{2}, \qquad p_1 \cdot p_2' = -\frac{u}{2} . \tag{5.62}$$

Substituting into eq. (5.52) with $m_e = m_\mu = 0$:

$$\mathrm{Tr}[\mu]_{\nu\rho}\,\mathrm{Tr}[e]^{\nu\rho} = 32\left[\frac{t^2}{4} + \frac{u^2}{4}\right] = 8(t^2 + u^2) . \tag{5.63}$$

Expressing in terms of the scattering angle:

$$t^2 + u^2 = \frac{s^2}{4}\left[(1 - \cos\theta)^2 + (1 + \cos\theta)^2\right] = \frac{s^2}{2}(1 + \cos^2\theta) . \tag{5.64}$$

Therefore:

$$\overline{|\mathcal{M}|^2} = \frac{e^4}{4s^2} \cdot \frac{8(t^2 + u^2)}{(2m_\mu)^2(2m_e)^2} = \frac{2e^4(t^2 + u^2)}{(2m_\mu)^2(2m_e)^2\, s^2} . \tag{5.65}$$

Note that $\overline{|\mathcal{M}|^2}$ still carries the BD normalization factors $1/(2m)^2$; these will cancel in the cross section.

Differential cross section and cancellation of BD normalization factors. We now derive $d\sigma/d\Omega$ in the CM frame from the BD master formula (3.55). The two-body Lorentz-invariant phase space (3.45) reads

$$d\Phi_2 = (2\pi)^4 \delta^{(4)}(p_A + p_B - p_1' - p_2') \frac{d^3 p_1'}{(2\pi)^3 2E_1'} \frac{d^3 p_2'}{(2\pi)^3 2E_2'} . \tag{5.66}$$

In the CM frame $\vec{p}_A + \vec{p}_B = 0$, so the three-momentum delta function enforces $\vec{p}_2\,' = -\vec{p}_1\,'$. Writing $|\vec{p}_1\,'| = |\vec{p}_2\,'| \equiv |\vec{p}\,'|$ and $d^3 p_1' = |\vec{p}\,'|^2\, d|\vec{p}\,'|\, d\Omega$, integration over $\vec{p}_2\,'$ yields

$$d\Phi_2 = \frac{1}{(2\pi)^2} \frac{|\vec{p}\,'|^2\, d|\vec{p}\,'|\, d\Omega}{4E_1' E_2'}\, \delta(\sqrt{s} - E_1' - E_2') . \tag{5.67}$$

To perform the $|\vec{p}\,'|$ integration we use $\delta(f(x)) = \delta(x-x_0)/|f'(x_0)|$, which requires the Jacobian $d(E_1'+E_2')/d|\vec{p}\,'|$. Since both final-state energies depend on the *same* variable $|\vec{p}\,'|$ through $E_i' = \sqrt{|\vec{p}\,'|^2 + m_i'^2}$, differentiation gives

$$\frac{dE_i'}{d|\vec{p}\,'|} = \frac{|\vec{p}\,'|}{E_i'}, \tag{5.68}$$

so that

$$\frac{d(E_1'+E_2')}{d|\vec{p}\,'|} = \frac{|\vec{p}\,'|}{E_1'} + \frac{|\vec{p}\,'|}{E_2'} = |\vec{p}\,'|\frac{E_2'+E_1'}{E_1'E_2'} = \frac{|\vec{p}\,'|\sqrt{s}}{E_1'E_2'}, \tag{5.69}$$

where in the last step we used $E_1'+E_2' = \sqrt{s}$ on the support of the delta function. The delta function identity then gives

$$d\Phi_2\Big|_{\text{CM}} = \frac{1}{(2\pi)^2}\frac{|\vec{p}\,'|^2}{4E_1'E_2'}\frac{E_1'E_2'}{|\vec{p}\,'|\sqrt{s}}\,d\Omega = \frac{|\vec{p}\,'|}{16\pi^2\sqrt{s}}\,d\Omega. \tag{5.70}$$

Substituting into the master formula (3.55) with the CM-frame flux factor $\mathcal{F} = 4|\vec{p}\,|\sqrt{s}$ (cf. eq. (3.53)):

$$\begin{aligned}\left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} &= \frac{1}{4|\vec{p}\,|\sqrt{s}}\left(\prod_{i\in\text{fermions}} 2m_i\right)\overline{|\mathcal{M}_{\text{BD}}|^2}\,\frac{|\vec{p}\,'|}{16\pi^2\sqrt{s}} \\ &= \frac{1}{64\pi^2 s}\frac{|\vec{p}\,'|}{|\vec{p}\,|}\left(\prod_{i\in\text{fermions}} 2m_i\right)\overline{|\mathcal{M}_{\text{BD}}|^2}.\end{aligned} \tag{5.71}$$

For $e^+e^- \to \mu^+\mu^-$ with four external fermions: $\prod(2m_i) = (2m_e)^2(2m_\mu)^2$.

Using eq. (5.53), the factor $(2m_e)^2(2m_\mu)^2$ from the cross-section formula *exactly cancels* the $1/(2m_e)^2(2m_\mu)^2$ from the BD completeness relations in the leptonic tensors:

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \frac{1}{64\pi^2 s}\frac{|\vec{p}\,'|}{|\vec{p}\,|}\cdot\cancel{(2m_e)^2(2m_\mu)^2}\cdot\frac{e^4}{4s^2}\cdot\frac{\text{Tr}[\mu]\,\text{Tr}[e]}{\cancel{(2m_\mu)^2(2m_e)^2}} \\ &= \frac{1}{64\pi^2 s}\frac{|\vec{p}\,'|}{|\vec{p}\,|}\cdot\frac{e^4}{4s^2}\cdot\text{Tr}[\mu]\,\text{Tr}[e].\end{aligned} \tag{5.72}$$

All mass factors from the BD normalisation drop out, as they must: the cross section is a physical observable independent of the spinor normalisation convention.

In the ultrarelativistic limit ($|\vec{p}\,'| = |\vec{p}\,|$, $m_e = m_\mu = 0$), using eq. (5.63) and $\alpha = e^2/4\pi$:

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}(1+\cos^2\theta)}. \tag{5.73}$$

The factor $(1+\cos^2\theta)$ reflects the spin-1 nature of the intermediate photon. At $\theta = 0$ (forward) and $\theta = \pi$ (backward), the cross section is maximized, while at $\theta = \pi/2$ it has a minimum.

Total cross section. Integrating over the solid angle:

$$\begin{aligned}\sigma_{\text{tot}} &= \int \frac{d\sigma}{d\Omega}\, d\Omega = 2\pi \cdot \frac{\alpha^2}{4s} \int_{-1}^{+1} (1+x^2)\, dx \\ &= \frac{\pi\alpha^2}{2s} \cdot \frac{8}{3}.\end{aligned} \tag{5.74}$$

Therefore:

$$\boxed{\sigma_{\text{tot}}(e^+e^- \to \mu^+\mu^-) = \frac{4\pi\alpha^2}{3s}}. \tag{5.75}$$

Numerical estimate. Using $\hbar c \simeq 0.197$ GeV·fm and $\alpha \simeq 1/137$, at $\sqrt{s} = 1$ GeV:

$$\sigma \simeq 87\ \text{nb}. \tag{5.76}$$

A useful practical formula is:

$$\boxed{\sigma(e^+e^- \to \mu^+\mu^-) \simeq \frac{87\ \text{nb}}{s/\text{GeV}^2}}. \tag{5.77}$$

Remark. [The "point cross section"] The result $\sigma \propto 1/s$ is characteristic of point-like electromagnetic scattering. Any deviation from this behavior signals the presence of structure (form factors) or new physics. This cross section serves as the normalization standard for the R-ratio:

$$R \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}, \tag{5.78}$$

which provides crucial information about quark charges and the number of colors.

Comparison with experiment. The QED prediction $\sigma = 4\pi\alpha^2/3s$ has been tested at numerous e^+e^- colliders. Figure 5.1 shows a comparison between the theoretical cross section and experimental measurements from various facilities.

The agreement between theory and experiment is remarkable, typically at the percent level or better. This precision test of QED provides strong evidence that the electron and muon are indeed point-like particles with no internal structure down to scales of $\sim 10^{-18}$ m.

Summary of results. We collect the main results of this calculation in the following table:

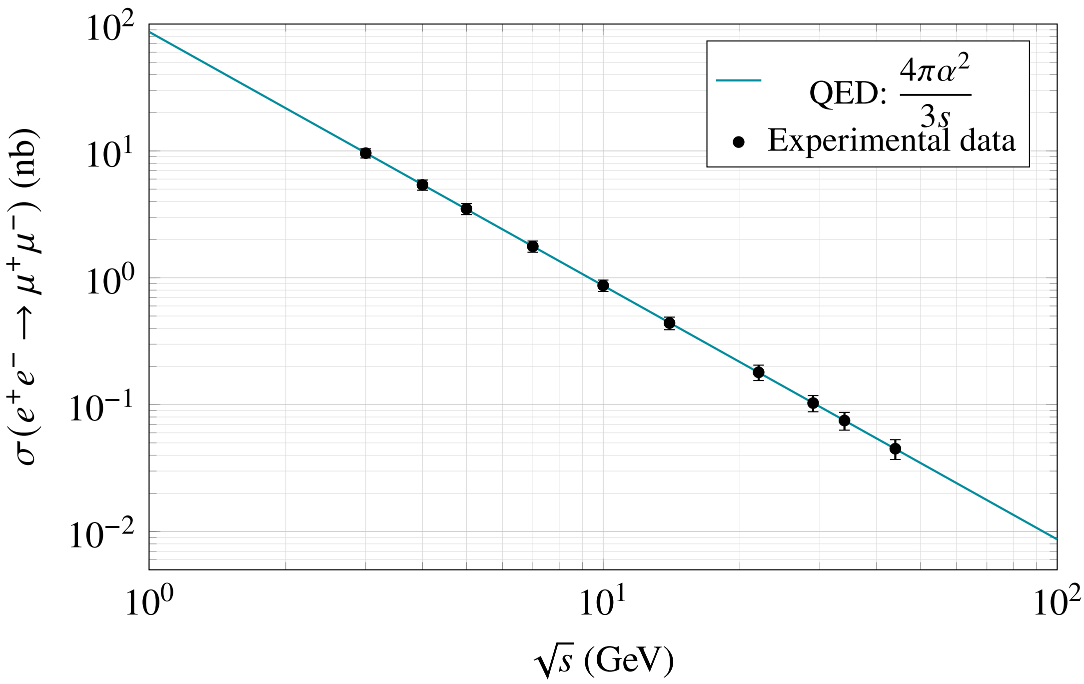


Figure 5.1. Total cross section for $e^+e^- \to \mu^+\mu^-$ as a function of center-of-mass energy $\sqrt{s}$. The solid line shows the QED prediction $\sigma = 4\pi\alpha^2/3s \simeq 86.8$ nb$/(\sqrt{s}/\text{GeV})^2$. The data points represent measurements from various e^+e^- colliders (SPEAR, DORIS, PETRA, PEP, TRISTAN). The excellent agreement over more than two orders of magnitude in energy demonstrates the validity of QED as a description of electromagnetic interactions.