Remark. [Crossing symmetry] These processes are related by *crossing symmetry*: replacing an incoming particle with an outgoing antiparticle (or vice versa) transforms one process into another. For example, $e^+e^- \to \gamma\gamma$ is the crossed version of Compton scattering $e^-\gamma \to e^-\gamma$.

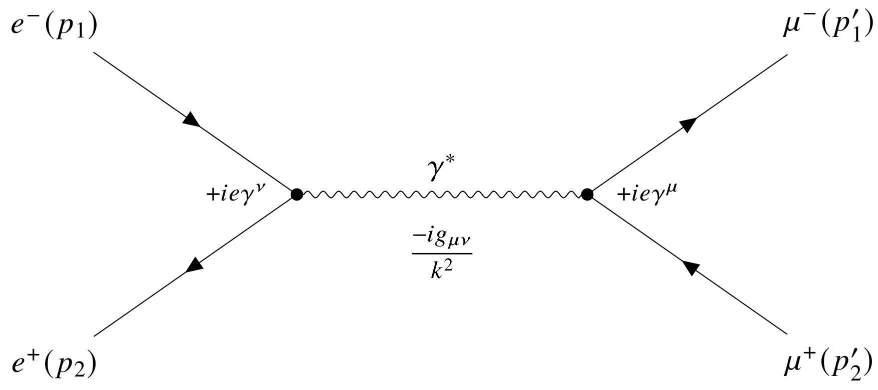
5.2 Muon pair production: $e^+e^- \to \mu^+\mu^-$

We now present a detailed calculation of the cross section for the process $e^+e^- \to \mu^+\mu^-$. This is one of the cleanest QED processes to analyze because: (i) only the s-channel contributes at tree level, since QED contains no vertex that converts an electron into a muon; (ii) it provides a "standard candle" for measuring luminosity at e^+e^- colliders; (iii) the theoretical prediction agrees with experiment to remarkable precision.

Kinematics and Feynman diagram. Consider the process:

$$e^-(p_1) + e^+(p_2) \to \mu^-(p_1') + \mu^+(p_2')\,. \tag{5.13}$$

At tree level, the only contributing diagram is the s-channel annihilation through a virtual photon:



Scattering amplitude. Applying the Feynman rules, the scattering amplitude is:

$$\begin{aligned} i\mathcal{M} = \bar{u}(p_1', s_1')(+ie\gamma^\mu)v(p_2', s_2') \cdot \frac{-ig_{\mu\nu}}{k^2} \\ \cdot \bar{v}(p_2, s_2)(+ie\gamma^\nu)u(p_1, s_1)\,. \end{aligned} \tag{5.14}$$

Simplifying (note that $(+ie)^2 = -e^2$):

$$\boxed{\mathcal{M} = \frac{e^2}{k^2}\left[\bar{u}(p_1')\gamma_\nu v(p_2')\right]\left[\bar{v}(p_2)\gamma^\nu u(p_1)\right]} \tag{5.15}$$

where $k = p_1 + p_2 = p_1' + p_2'$ is the four-momentum of the virtual photon, and $k^2 = s$ is the center-of-mass energy squared.

BD normalization factors. In the BD convention each external fermion field contributes a factor $\sqrt{m/E}$ from the mode expansion (3.25). For this process there are four external fermions, so the BD amplitude carries an overall factor

$$\mathcal{N}_{\mathrm{BD}} = \sqrt{\frac{m_e^2 m_\mu^2}{E_1 E_2 E_1' E_2'}}\,. \tag{5.16}$$

As explained in Section 1.1, this factor combines with the $\prod(2m_i) = (2m_e)^2(2m_\mu)^2$ in the BD cross section formula (3.55) and the $1/(2m)^2$ from the BD spin sums (3.58), and all mass factors cancel in the final cross section. In the spin sums below we work with the BD completeness relations; the $1/(2m)^2$ prefactors are carried explicitly through the leptonic tensors until the cancellation is shown at the level of the cross section.

Spin-averaged amplitude squared. For unpolarized beams, we average over initial spins (factor $\frac{1}{4}$) and sum over final spins:

$$\overline{|\mathcal{M}|^2} \equiv \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2\,. \tag{5.17}$$

From eq. (5.15), we have:

$$\mathcal{M} = \frac{e^2}{s}\left[\bar{u}_\mu(p_1')\gamma_\nu v_\mu(p_2')\right]\left[\bar{v}_e(p_2)\gamma^\nu u_e(p_1)\right], \tag{5.18}$$

where we suppress spin labels for clarity and used $k^2 = s$. The complex conjugate is:

$$\mathcal{M}^* = \frac{e^2}{s}\left[\bar{v}_\mu(p_2')\gamma_\rho u_\mu(p_1')\right]\left[\bar{u}_e(p_1)\gamma^\rho v_e(p_2)\right], \tag{5.19}$$

where we used $(\bar{\psi}_1\gamma_\nu\psi_2)^* = \bar{\psi}_2\gamma_\nu\psi_1$.

The amplitude squared is therefore:

$$\begin{aligned}|\mathcal{M}|^2 &= \mathcal{M}\mathcal{M}^* \\ &= \frac{e^4}{s^2}\left[\bar{u}_\mu(p_1')\gamma_\nu v_\mu(p_2')\right]\left[\bar{v}_e(p_2)\gamma^\nu u_e(p_1)\right] \\ &\quad\times\left[\bar{v}_\mu(p_2')\gamma_\rho u_\mu(p_1')\right]\left[\bar{u}_e(p_1)\gamma^\rho v_e(p_2)\right].\end{aligned} \tag{5.20}$$

From spinor products to traces. To perform the spin sums, we use the **complete-**

ness relations appropriate to the BD normalization:

$$\boxed{\sum_s u_\sigma(p,s)\bar{u}_\tau(p,s) = \frac{(\not{p}+m)_{\sigma\tau}}{2m}} \tag{5.21}$$

$$\boxed{\sum_s v_\sigma(p,s)\bar{v}_\tau(p,s) = \frac{(\not{p}-m)_{\sigma\tau}}{2m}}, \tag{5.22}$$

where σ, τ are spinor indices. These $1/(2m)$ factors can be carried explicitly or absorbed into external-state normalizations; below we keep them explicit.

We now show in detail how the spin sum converts the product of spinor bilinears into traces. Consider first the muon sector. We can write the spinor bilinears with explicit spinor indices:

$$\left[\bar{u}_\mu(p_1')\gamma_\nu v_\mu(p_2')\right] = \bar{u}_{\mu,\sigma}(p_1')(\gamma_\nu)_\sigma{}^\tau\, v_{\mu,\tau}(p_2') \tag{5.23}$$

$$\left[\bar{v}_\mu(p_2')\gamma_\rho u_\mu(p_1')\right] = \bar{v}_{\mu,\xi}(p_2')(\gamma_\rho)_\xi{}^\zeta\, u_{\mu,\zeta}(p_1')\,. \tag{5.24}$$

Their product, summed over muon spins s_1' and s_2', is:

$$\begin{aligned}
&\sum_{s_1',s_2'}\left[\bar{u}_\mu(p_1')\gamma_\nu v_\mu(p_2')\right]\left[\bar{v}_\mu(p_2')\gamma_\rho u_\mu(p_1')\right] \\
&= \sum_{s_1',s_2'} \bar{u}_{\mu,\sigma}(p_1')(\gamma_\nu)_\sigma{}^\tau\, v_{\mu,\tau}(p_2')\,\bar{v}_{\mu,\xi}(p_2')(\gamma_\rho)_\xi{}^\zeta\, u_{\mu,\zeta}(p_1') \\
&= (\gamma_\nu)_\sigma{}^\tau\,(\gamma_\rho)_\xi{}^\zeta \sum_{s_2'} v_{\mu,\tau}(p_2')\bar{v}_{\mu,\xi}(p_2') \sum_{s_1'} u_{\mu,\zeta}(p_1')\bar{u}_{\mu,\sigma}(p_1')\,.
\end{aligned} \tag{5.25}$$

Using the completeness relations:

$$\sum_{s_2'} v_{\mu,\tau}(p_2')\bar{v}_{\mu,\xi}(p_2') = \frac{(\not{p}_2'-m_\mu)_{\tau\xi}}{2m_\mu} \tag{5.26}$$

$$\sum_{s_1'} u_{\mu,\zeta}(p_1')\bar{u}_{\mu,\sigma}(p_1') = \frac{(\not{p}_1'+m_\mu)_{\zeta\sigma}}{2m_\mu}\,. \tag{5.27}$$

Substituting:

$$\begin{aligned}
&\sum_{s_1',s_2'}\left[\bar{u}_\mu(p_1')\gamma_\nu v_\mu(p_2')\right]\left[\bar{v}_\mu(p_2')\gamma_\rho u_\mu(p_1')\right] \\
&= \frac{1}{(2m_\mu)^2}(\gamma_\nu)_\sigma{}^\tau\,(\not{p}_2'-m_\mu)_{\tau\xi}\,(\gamma_\rho)_\xi{}^\zeta\,(\not{p}_1'+m_\mu)_{\zeta\sigma} \\
&= \frac{1}{(2m_\mu)^2}\left[(\not{p}_1'+m_\mu)\gamma_\nu(\not{p}_2'-m_\mu)\gamma_\rho\right]_{\sigma\sigma} \\
&= \frac{1}{(2m_\mu)^2}\mathrm{Tr}\left[(\not{p}_1'+m_\mu)\gamma_\nu(\not{p}_2'-m_\mu)\gamma_\rho\right],
\end{aligned} \tag{5.28}$$

where in the last step we recognized the sum over the repeated index σ as a trace.

By exactly the same procedure for the electron sector:

$$\sum_{s_1,s_2}\left[\bar{v}_e(p_2)\gamma^\nu u_e(p_1)\right]\left[\bar{u}_e(p_1)\gamma^\rho v_e(p_2)\right]$$
$$= \frac{1}{(2m_e)^2}\mathrm{Tr}\left[(\not{p}_2 - m_e)\gamma^\nu(\not{p}_1 + m_e)\gamma^\rho\right]. \tag{5.29}$$

Therefore, the spin-averaged amplitude squared is:

$$\boxed{\overline{|\mathcal{M}|^2} = \frac{e^4}{4s^2}L^{(\mu)}_{\nu\rho}\cdot L^{(e)\nu\rho}}, \tag{5.30}$$

where we define the **leptonic tensors** (retaining the BD normalization factors):

$$L^{(\mu)}_{\nu\rho} \equiv \frac{1}{(2m_\mu)^2}\mathrm{Tr}\left[(\not{p}'_1 + m_\mu)\gamma_\nu(\not{p}'_2 - m_\mu)\gamma_\rho\right], \tag{5.31}$$
$$L^{(e)\nu\rho} \equiv \frac{1}{(2m_e)^2}\mathrm{Tr}\left[(\not{p}_2 - m_e)\gamma^\nu(\not{p}_1 + m_e)\gamma^\rho\right]. \tag{5.32}$$

The $1/(2m)^2$ prefactors originate from the BD completeness relations (5.21)–(5.22). They will cancel exactly against the external-state normalization factors when computing the cross section (see below, eq. (5.71)).

Evaluation of the muon trace. Expanding the product $(\not{p}'_1 + m_\mu)\gamma_\nu(\not{p}'_2 - m_\mu)\gamma_\rho$ inside the trace in eq. (5.31):

$$L^{(\mu)}_{\nu\rho} = \frac{1}{(2m_\mu)^2}\Big[\mathrm{Tr}(\not{p}'_1\gamma_\nu\not{p}'_2\gamma_\rho) - m_\mu\mathrm{Tr}(\not{p}'_1\gamma_\nu\gamma_\rho)$$
$$+ m_\mu\mathrm{Tr}(\gamma_\nu\not{p}'_2\gamma_\rho) - m_\mu^2\mathrm{Tr}(\gamma_\nu\gamma_\rho)\Big]. \tag{5.33}$$

The second and third terms vanish because they contain an odd number of gamma matrices (three each). Thus:

$$L^{(\mu)}_{\nu\rho} = \frac{1}{(2m_\mu)^2}\left[\mathrm{Tr}(\not{p}'_1\gamma_\nu\not{p}'_2\gamma_\rho) - m_\mu^2\mathrm{Tr}(\gamma_\nu\gamma_\rho)\right]. \tag{5.34}$$

For the first term, we write $\not{p}'_1 = p'_{1\alpha}\gamma^\alpha$ and $\not{p}'_2 = p'_{2\beta}\gamma^\beta$:

$$\mathrm{Tr}(\not{p}'_1\gamma_\nu\not{p}'_2\gamma_\rho) = p'_{1\alpha}p'_{2\beta}\mathrm{Tr}(\gamma^\alpha\gamma_\nu\gamma^\beta\gamma_\rho). \tag{5.35}$$

Using the trace formula:

$$\mathrm{Tr}(\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta) = 4(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\delta}g^{\beta\gamma}), \tag{5.36}$$

we have:

$$\begin{aligned}\mathrm{Tr}(\gamma^\alpha\gamma_\nu\gamma^\beta\gamma_\rho) &= 4(g^\alpha{}_\nu g^\beta{}_\rho - g^{\alpha\beta}g_{\nu\rho} + g^\alpha{}_\rho g^\beta{}_\nu) \\ &= 4(\delta^\alpha_\nu\delta^\beta_\rho - g^{\alpha\beta}g_{\nu\rho} + \delta^\alpha_\rho\delta^\beta_\nu)\,. \end{aligned} \tag{5.37}$$

Contracting with $p'_{1\alpha}p'_{2\beta}$:

$$p'_{1\alpha}p'_{2\beta}\mathrm{Tr}(\gamma^\alpha\gamma_\nu\gamma^\beta\gamma_\rho) = 4\big(p'_{1\nu}p'_{2\rho} - (p'_1\cdot p'_2)g_{\nu\rho} + p'_{1\rho}p'_{2\nu}\big)\,. \tag{5.38}$$

For the second term: $\mathrm{Tr}(\gamma_\nu\gamma_\rho) = 4g_{\nu\rho}$.
Combining:

$$\boxed{L^{(\mu)}_{\nu\rho} = \frac{1}{m_\mu^2}\big[p'_{1\nu}p'_{2\rho} + p'_{1\rho}p'_{2\nu} - (p'_1\cdot p'_2 + m_\mu^2)g_{\nu\rho}\big]}\,. \tag{5.39}$$

Evaluation of the electron trace. By the same calculation for the electron tensor $L^{(e)\nu\rho}$:

$$\boxed{L^{(e)\nu\rho} = \frac{1}{m_e^2}\big[p_2^\nu p_1^\rho + p_2^\rho p_1^\nu - (p_1\cdot p_2 + m_e^2)g^{\nu\rho}\big]}\,. \tag{5.40}$$

Contraction of the leptonic tensors. The spin-averaged amplitude squared is:

$$\overline{|\mathcal{M}|^2} = \frac{e^4}{4s^2}L^{(\mu)}_{\nu\rho}L^{(e)\nu\rho} = \frac{e^4}{4s^2}\cdot\frac{1}{(2m_\mu)^2(2m_e)^2}\mathrm{Tr}[\mu]\,\mathrm{Tr}[e]\,. \tag{5.41}$$

The contraction of the two traces (without the $1/(2m)^2$ prefactors) proceeds as follows. Writing:

$$\mathrm{Tr}[\mu]_{\nu\rho} = 4\big[p'_{1\nu}p'_{2\rho} + p'_{1\rho}p'_{2\nu} - A\,g_{\nu\rho}\big] \quad \text{where } A = p'_1\cdot p'_2 + m_\mu^2 \tag{5.42}$$

$$\mathrm{Tr}[e]^{\nu\rho} = 4\big[p_2^\nu p_1^\rho + p_2^\rho p_1^\nu - B\,g^{\nu\rho}\big] \quad \text{where } B = p_1\cdot p_2 + m_e^2\,. \tag{5.43}$$

The contraction gives:

$$\begin{aligned} L^{(\mu)}_{\nu\rho}L^{(e)\nu\rho} = 16\Big[&(p'_{1\nu}p'_{2\rho})(p_2^\nu p_1^\rho) + (p'_{1\nu}p'_{2\rho})(p_2^\rho p_1^\nu) \\ &+ (p'_{1\rho}p'_{2\nu})(p_2^\nu p_1^\rho) + (p'_{1\rho}p'_{2\nu})(p_2^\rho p_1^\nu) \\ &- A(p_2^\nu p_1^\rho g_{\nu\rho}) - A(p_2^\rho p_1^\nu g_{\nu\rho}) \\ &- B(p'_{1\nu}p'_{2\rho}g^{\nu\rho}) - B(p'_{1\rho}p'_{2\nu}g^{\nu\rho}) + 4AB\Big]\,. \end{aligned} \tag{5.44}$$

Evaluating each term using $p'_{1\nu}p_2^\nu = p'_1 \cdot p_2$, etc.:

$$(p'_{1\nu}p'_{2\rho})(p_2^\nu p_1^\rho) = (p'_1 \cdot p_2)(p'_2 \cdot p_1) \tag{5.45}$$
$$(p'_{1\nu}p'_{2\rho})(p_2^\rho p_1^\nu) = (p'_1 \cdot p_1)(p'_2 \cdot p_2) \tag{5.46}$$
$$(p'_{1\rho}p'_{2\nu})(p_2^\nu p_1^\rho) = (p'_2 \cdot p_2)(p'_1 \cdot p_1) \tag{5.47}$$
$$(p'_{1\rho}p'_{2\nu})(p_2^\rho p_1^\nu) = (p'_1 \cdot p_2)(p'_2 \cdot p_1) \tag{5.48}$$
$$p_2^\nu p_1^\rho g_{\nu\rho} = p_1 \cdot p_2 \tag{5.49}$$
$$p'_{1\nu}p'_{2\rho}g^{\nu\rho} = p'_1 \cdot p'_2 . \tag{5.50}$$

Collecting terms:

$$L^{(\mu)}_{\nu\rho}L^{(e)\nu\rho} = 16\Big[2(p'_1 \cdot p_2)(p'_2 \cdot p_1) + 2(p'_1 \cdot p_1)(p'_2 \cdot p_2) - 2A(p_1 \cdot p_2) - 2B(p'_1 \cdot p'_2) + 4AB\Big]. \tag{5.51}$$

Substituting $A = p'_1 \cdot p'_2 + m_\mu^2$ and $B = p_1 \cdot p_2 + m_e^2$, and simplifying:

$$\mathrm{Tr}[\mu]_{\nu\rho}\,\mathrm{Tr}[e]^{\nu\rho} = 32\Big[(p_1 \cdot p'_1)(p_2 \cdot p'_2) + (p_1 \cdot p'_2)(p_2 \cdot p'_1) + m_\mu^2(p_1 \cdot p_2) + m_e^2(p'_1 \cdot p'_2) + 2m_e^2 m_\mu^2\Big]. \tag{5.52}$$

Including the BD normalization prefactors:

$$\boxed{\overline{|\mathcal{M}|^2} = \frac{e^4}{4s^2} \cdot \frac{1}{(2m_\mu)^2(2m_e)^2}\,\mathrm{Tr}[\mu]_{\nu\rho}\,\mathrm{Tr}[e]^{\nu\rho}}\,. \tag{5.53}$$

Center-of-mass kinematics. In the center-of-mass (CM) frame, the momenta are:

$$p_1 = (E, \vec{p}), \qquad p'_1 = (E, \vec{p}\,') \tag{5.54}$$
$$p_2 = (E, -\vec{p}), \qquad p'_2 = (E, -\vec{p}\,')\,, \tag{5.55}$$

where energy conservation requires all particles to have the same energy E. The relevant scalar products are:

$$p_1 \cdot p_2 = E^2 + |\vec{p}|^2 \tag{5.56}$$
$$p'_1 \cdot p'_2 = E^2 + |\vec{p}\,'|^2 \tag{5.57}$$
$$p_1 \cdot p'_1 = E^2 - |\vec{p}||\vec{p}\,'|\cos\theta \tag{5.58}$$
$$p_1 \cdot p'_2 = E^2 + |\vec{p}||\vec{p}\,'|\cos\theta\,, \tag{5.59}$$

where θ is the scattering angle between $\vec{p}$ and $\vec{p}\,'$, $|\vec{p}| = \sqrt{E^2 - m_e^2}$, and $|\vec{p}\,'| = \sqrt{E^2 - m_\mu^2}$.

Mandelstam variables. For a $2 \to 2$ scattering process, the Mandelstam variables are:

$$s \equiv (p_1 + p_2)^2, \qquad t \equiv (p_1 - p_1')^2, \qquad u \equiv (p_1 - p_2')^2. \tag{5.60}$$

These satisfy $s + t + u = 2m_e^2 + 2m_\mu^2$. In the CM frame: $s = 4E^2$.

Ultrarelativistic limit. In the limit $E \gg m_e, m_\mu$, we have $|\vec{p}| \approx |\vec{p}\,'| \approx E$, and the Mandelstam variables simplify to:

$$s = 4E^2, \qquad t = -\frac{s}{2}(1 - \cos\theta), \qquad u = -\frac{s}{2}(1 + \cos\theta). \tag{5.61}$$

The scalar products become:

$$p_1 \cdot p_2 = p_1' \cdot p_2' = \frac{s}{2}, \qquad p_1 \cdot p_1' = -\frac{t}{2}, \qquad p_1 \cdot p_2' = -\frac{u}{2}. \tag{5.62}$$

Substituting into eq. (5.52) with $m_e = m_\mu = 0$:

$$\mathrm{Tr}[\mu]_{\nu\rho}\, \mathrm{Tr}[e]^{\nu\rho} = 32\left[\frac{t^2}{4} + \frac{u^2}{4}\right] = 8(t^2 + u^2). \tag{5.63}$$

Expressing in terms of the scattering angle:

$$t^2 + u^2 = \frac{s^2}{4}\left[(1 - \cos\theta)^2 + (1 + \cos\theta)^2\right] = \frac{s^2}{2}(1 + \cos^2\theta). \tag{5.64}$$

Therefore:

$$\overline{|\mathcal{M}|^2} = \frac{e^4}{4s^2} \cdot \frac{8(t^2 + u^2)}{(2m_\mu)^2(2m_e)^2} = \frac{2e^4(t^2 + u^2)}{(2m_\mu)^2(2m_e)^2\, s^2}. \tag{5.65}$$

Note that $\overline{|\mathcal{M}|^2}$ still carries the BD normalization factors $1/(2m)^2$; these will cancel in the cross section.

Differential cross section and cancellation of BD normalization factors.
We now derive $d\sigma/d\Omega$ in the CM frame from the BD master formula (3.55). The two-body Lorentz-invariant phase space (3.45) reads

$$d\Phi_2 = (2\pi)^4 \delta^{(4)}(p_A + p_B - p_1' - p_2')\, \frac{d^3 p_1'}{(2\pi)^3\, 2E_1'}\, \frac{d^3 p_2'}{(2\pi)^3\, 2E_2'}. \tag{5.66}$$

In the CM frame $\vec{p}_A + \vec{p}_B = 0$, so the three-momentum delta function enforces $\vec{p}_2\,' = -\vec{p}_1\,'$. Writing $|\vec{p}_1\,'| = |\vec{p}_2\,'| \equiv |\vec{p}\,'|$ and $d^3 p_1' = |\vec{p}\,'|^2\, d|\vec{p}\,'|\, d\Omega$, integration over $\vec{p}_2\,'$ yields

$$d\Phi_2 = \frac{1}{(2\pi)^2}\, \frac{|\vec{p}\,'|^2\, d|\vec{p}\,'|\, d\Omega}{4E_1' E_2'}\, \delta(\sqrt{s} - E_1' - E_2'). \tag{5.67}$$

To perform the $|\vec{p}\,'|$ integration we use $\delta(f(x)) = \delta(x - x_0)/|f'(x_0)|$, which requires the Jacobian $d(E_1' + E_2')/d|\vec{p}\,'|$. Since both final-state energies depend on the *same* variable $|\vec{p}\,'|$ through $E_i' = \sqrt{|\vec{p}\,'|^2 + m_i'^2}$, differentiation gives

$$\frac{dE_i'}{d|\vec{p}\,'|} = \frac{|\vec{p}\,'|}{E_i'}, \tag{5.68}$$

so that

$$\frac{d(E_1' + E_2')}{d|\vec{p}\,'|} = \frac{|\vec{p}\,'|}{E_1'} + \frac{|\vec{p}\,'|}{E_2'} = |\vec{p}\,'|\frac{E_2' + E_1'}{E_1' E_2'} = \frac{|\vec{p}\,'|\sqrt{s}}{E_1' E_2'}, \tag{5.69}$$

where in the last step we used $E_1' + E_2' = \sqrt{s}$ on the support of the delta function. The delta function identity then gives

$$d\Phi_2\Big|_{\mathrm{CM}} = \frac{1}{(2\pi)^2}\frac{|\vec{p}\,'|^2}{4E_1'E_2'}\frac{E_1'E_2'}{|\vec{p}\,'|\sqrt{s}}\,d\Omega = \frac{|\vec{p}\,'|}{16\pi^2\sqrt{s}}\,d\Omega. \tag{5.70}$$

Substituting into the master formula (3.55) with the CM-frame flux factor $\mathcal{F} = 4|\vec{p}\,|\sqrt{s}$ (cf. eq. (3.53)):

$$\begin{aligned}\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{CM}} &= \frac{1}{4|\vec{p}\,|\sqrt{s}}\left(\prod_{i\in\text{fermions}} 2m_i\right)\overline{|\mathcal{M}_{\mathrm{BD}}|^2}\,\frac{|\vec{p}\,'|}{16\pi^2\sqrt{s}}\\ &= \frac{1}{64\pi^2 s}\frac{|\vec{p}\,'|}{|\vec{p}\,|}\left(\prod_{i\in\text{fermions}} 2m_i\right)\overline{|\mathcal{M}_{\mathrm{BD}}|^2}.\end{aligned} \tag{5.71}$$

For $e^+e^- \to \mu^+\mu^-$ with four external fermions: $\prod(2m_i) = (2m_e)^2(2m_\mu)^2$.

Using eq. (5.53), the factor $(2m_e)^2(2m_\mu)^2$ from the cross-section formula *exactly cancels* the $1/(2m_e)^2(2m_\mu)^2$ from the BD completeness relations in the leptonic tensors:

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \frac{1}{64\pi^2 s}\frac{|\vec{p}\,'|}{|\vec{p}\,|}\cdot\cancel{(2m_e)^2(2m_\mu)^2}\cdot\frac{e^4}{4s^2}\cdot\frac{\mathrm{Tr}[\mu]\,\mathrm{Tr}[e]}{\cancel{(2m_\mu)^2(2m_e)^2}}\\ &= \frac{1}{64\pi^2 s}\frac{|\vec{p}\,'|}{|\vec{p}\,|}\cdot\frac{e^4}{4s^2}\cdot\mathrm{Tr}[\mu]\,\mathrm{Tr}[e].\end{aligned} \tag{5.72}$$

All mass factors from the BD normalisation drop out, as they must: the cross section is a physical observable independent of the spinor normalisation convention.

In the ultrarelativistic limit ($|\vec{p}\,'| = |\vec{p}\,|$, $m_e = m_\mu = 0$), using eq. (5.63) and $\alpha = e^2/4\pi$:

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}(1 + \cos^2\theta)}. \tag{5.73}$$

The factor $(1+\cos^2\theta)$ reflects the spin-1 nature of the intermediate photon. At $\theta = 0$ (forward) and $\theta = \pi$ (backward), the cross section is maximized, while at $\theta = \pi/2$ it has a minimum.

Total cross section. Integrating over the solid angle:

$$\begin{aligned}\sigma_{\text{tot}} &= \int \frac{d\sigma}{d\Omega}\, d\Omega = 2\pi \cdot \frac{\alpha^2}{4s} \int_{-1}^{+1} (1+x^2)\, dx \\ &= \frac{\pi\alpha^2}{2s} \cdot \frac{8}{3}.\end{aligned} \tag{5.74}$$

Therefore:

$$\boxed{\sigma_{\text{tot}}(e^+e^- \to \mu^+\mu^-) = \frac{4\pi\alpha^2}{3s}}. \tag{5.75}$$

Numerical estimate. Using $\hbar c \simeq 0.197$ GeV·fm and $\alpha \simeq 1/137$, at $\sqrt{s} = 1$ GeV:

$$\sigma \simeq 87 \text{ nb}. \tag{5.76}$$

A useful practical formula is:

$$\boxed{\sigma(e^+e^- \to \mu^+\mu^-) \simeq \frac{87 \text{ nb}}{s/\text{GeV}^2}}. \tag{5.77}$$

Remark. [The "point cross section"] The result $\sigma \propto 1/s$ is characteristic of point-like electromagnetic scattering. Any deviation from this behavior signals the presence of structure (form factors) or new physics. This cross section serves as the normalization standard for the R-ratio:

$$R \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}, \tag{5.78}$$

which provides crucial information about quark charges and the number of colors.

Comparison with experiment. The QED prediction $\sigma = 4\pi\alpha^2/3s$ has been tested at numerous e^+e^- colliders. Figure 5.1 shows a comparison between the theoretical cross section and experimental measurements from various facilities.

The agreement between theory and experiment is remarkable, typically at the percent level or better. This precision test of QED provides strong evidence that the electron and muon are indeed point-like particles with no internal structure down to scales of $\sim 10^{-18}$ m.

Summary of results. We collect the main results of this calculation in the following table:

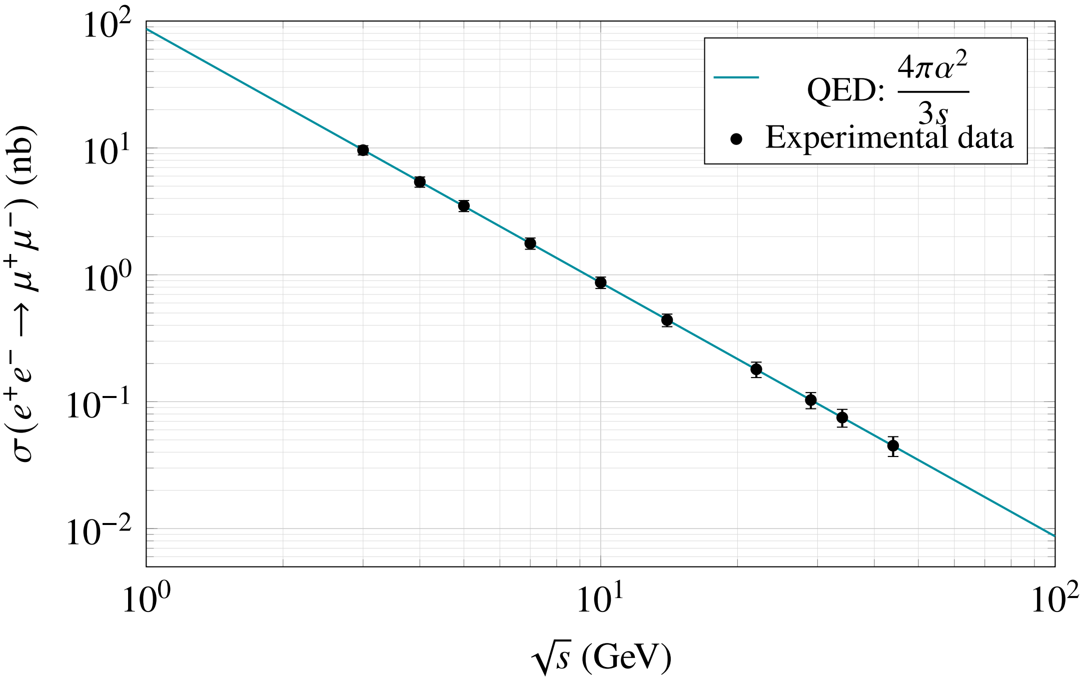


Figure 5.1. Total cross section for $e^+e^- \to \mu^+\mu^-$ as a function of center-of-mass energy $\sqrt{s}$. The solid line shows the QED prediction $\sigma = 4\pi\alpha^2/3s \simeq 86.8$ nb$/(\sqrt{s}/\text{GeV})^2$. The data points represent measurements from various e^+e^- colliders (SPEAR, DORIS, PETRA, PEP, TRISTAN). The excellent agreement over more than two orders of magnitude in energy demonstrates the validity of QED as a description of electromagnetic interactions.

Quantity	**Expression (ultrarelativistic limit)**
Amplitude squared	$\overline{\lvert\mathcal{M}\rvert^2} = \dfrac{2e^4(t^2+u^2)}{s^2} = e^4(1+\cos^2\theta)$ $^{(*)}$
Differential cross section	$\dfrac{d\sigma}{d\Omega} = \dfrac{\alpha^2}{4s}(1+\cos^2\theta)$
Total cross section	$\sigma_{\text{tot}} = \dfrac{4\pi\alpha^2}{3s}$

$^{(*)}$ Here $\overline{\lvert\mathcal{M}\rvert^2}$ denotes the covariant (physical) amplitude squared, i.e. $\prod(2m_i)\times\overline{\lvert\mathcal{M}_{\text{BD}}\rvert^2}$, in which the BD normalization factors have already cancelled (cf. eq. (5.72)).

Helicity considerations

The angular distribution $(1+\cos^2\theta)$ derived above has a deep physical interpretation in terms of helicity conservation. In the ultrarelativistic limit, helicity becomes a powerful organizing principle for understanding scattering amplitudes.

Helicity and chirality. Recall that the **helicity** of a particle is the component of its spin along its direction of motion, $h = \pm\frac{1}{2}$ for spin-$\frac{1}{2}$ particles. We denote right-handed ($h = +1/2$) and left-handed ($h = -1/2$) states as R and L respectively.

The **chirality** operators $P_{L,R} = \frac{1}{2}(1\mp\gamma^5)$ project onto definite chirality states. For a massive fermion, helicity and chirality eigenstates do not coincide. However, in the ultrarelativistic limit $E \gg m$:

$$\text{helicity} \approx \text{chirality} \quad (m \to 0)\,. \tag{5.79}$$

More precisely, the overlap between a helicity eigenstate and the “wrong” chirality state is suppressed by $O(m/E)$.

Chirality conservation at the QED vertex. The electromagnetic interaction $\bar{\psi}\gamma^\mu\psi A_\mu$ can be decomposed as:

$$\bar{\psi}\gamma^\mu\psi = \bar{\psi}_L\gamma^\mu\psi_L + \bar{\psi}_R\gamma^\mu\psi_R\,, \tag{5.80}$$

where $\psi_{L,R} = P_{L,R}\psi$. There are *no* cross-terms $\bar{\psi}_L\gamma^\mu\psi_R$ because:

$$\bar{\psi}_L\gamma^\mu\psi_R = \bar{\psi}P_R\gamma^\mu P_R\psi = \bar{\psi}\gamma^\mu P_L P_R\psi = 0\,, \tag{5.81}$$

using $\gamma^\mu\gamma^5 = -\gamma^5\gamma^\mu$ and $P_L P_R = 0$.

Box 5.2 Helicity conservation in QED

In the massless limit, the QED vertex **conserves chirality**, and hence helicity:

- A left-handed fermion remains left-handed after emitting/absorbing a photon
- A right-handed fermion remains right-handed

This leads to strict **helicity selection rules** for scattering amplitudes.

Helicity vs chirality for antiparticles. For particles in the massless limit, helicity and chirality coincide: a right-handed electron ($h = +1/2$) is right-chiral ($P_R u_R = u_R$), and a left-handed electron ($h = -1/2$) is left-chiral ($P_L u_L = u_L$). This follows directly from the massless Dirac equation: for any solution of $\not{p}\psi = 0$ with $p^\mu = E(1, \hat{p})$, one can show that $\hat{p} \cdot \vec{\Sigma}\,\psi = \gamma^5 \psi$ (see the box below for details).

For antiparticles, the relationship is **inverted**: a right-handed positron ($h = +1/2$) is *left*-chiral. The origin of this inversion is explained in the box below; the key point is that the antiparticle creation operator $b^{s\dagger}$ produces a state with physical helicity $-s$ (not $+s$), while the spinor v^s has the same chirality-helicity relation as u^s. Denoting by the subscripts L, R the *physical helicity*:

$$\text{Particles:} \quad P_R u_R = u_R, \quad P_L u_L = u_L \qquad \text{(chirality = helicity)}, \tag{5.82}$$

$$\text{Antiparticles:} \quad P_L v_R = v_R, \quad P_R v_L = v_L \qquad \text{(chirality = −helicity)}. \tag{5.83}$$

Here $u_{L,R}$ denote the particle spinors with helicity $\mp\frac{1}{2}$, while $v_{L,R}$ denote the v spinors that enter the Feynman amplitude for a physical antiparticle with helicity $\mp\frac{1}{2}$.

Explicit verification. We verify eqs. (5.82)–(5.83) with the ultrarelativistic spinors along $\hat{z}$ in the Dirac representation. From eqs. (1.176)–(1.177), in the limit $m \to 0$ ($p_z/(E+m) \to 1$), dropping the common prefactor:

$$u(\uparrow) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad u(\downarrow) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \quad v(\uparrow) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad v(\downarrow) = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}. \tag{5.84}$$

The helicity operator for motion along z is $\hat{h} = \frac{1}{2}\Sigma^3 = \frac{1}{2}\mathrm{diag}(1, -1, 1, -1)$. Direct multiplication shows that $u(\uparrow)$ and $v(\uparrow)$ have spinor-helicity $+\frac{1}{2}$, while $u(\downarrow)$ and $v(\downarrow)$ have spinor-helicity $-\frac{1}{2}$.

For particles, spinor-helicity = physical helicity, so $u_R = u(\uparrow)$, $u_L = u(\downarrow)$. For antiparticles, the physical helicity is $-s$ (Fact 2 in the box below), so $v_R = v(\downarrow)$, $v_L = v(\uparrow)$.

Using $\gamma^5_{\mathrm{D}} = \left(\begin{smallmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{smallmatrix}\right)$:

$$P_R = \tfrac{1}{2}\begin{pmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & \mathbb{1} \end{pmatrix}, \qquad P_L = \tfrac{1}{2}\begin{pmatrix} \mathbb{1} & -\mathbb{1} \\ -\mathbb{1} & \mathbb{1} \end{pmatrix}. \tag{5.85}$$

Particles (chirality = helicity):

$$P_R u_R = \tfrac{1}{2}\begin{pmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & \mathbb{1} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = u_R, \qquad P_L u_L = \tfrac{1}{2}\begin{pmatrix} \mathbb{1} & -\mathbb{1} \\ -\mathbb{1} & \mathbb{1} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} = u_L. \checkmark \tag{5.86}$$

Antiparticles (chirality = −helicity):

$$P_L v_R = \tfrac{1}{2}\begin{pmatrix} \mathbb{1} & -\mathbb{1} \\ -\mathbb{1} & \mathbb{1} \end{pmatrix}\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} = v_R, \qquad P_R v_L = \tfrac{1}{2}\begin{pmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & \mathbb{1} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = v_L. \checkmark \tag{5.87}$$

The inversion for antiparticles is evident: $v_R = v(\downarrow)$ has its upper and lower two-components with *opposite* signs (unlike u_R where they have the *same* sign), which causes P_L rather than P_R to act as the identity on it.

Box 5.3 Why v_R is left-chiral

The inversion arises from the interplay of two properties of the Dirac field quantization.

Fact 1: Chirality equals spinor helicity (for both u and v). In the massless limit, both u^s and v^s satisfy the same equation $\not{p}\,\psi = 0$. For any such solution with $p^\mu = E(1, \hat{p})$, the Dirac equation implies

$$\hat{p} \cdot \vec{\Sigma}\, \psi = \gamma^5 \psi, \tag{5.88}$$

so the helicity eigenvalue of the *spinor* equals its γ^5 eigenvalue (chirality). This holds for both u^s and v^s.

Fact 2: Physical antiparticle helicity $= -s$. In the Dirac field expansion $\psi \sim a^s u^s e^{-ipx} + b^{s\dagger} v^s e^{+ipx}$, the particle state $a^{s\dagger}|0\rangle$ has physical helicity $+s$, but $b^{s\dagger}|0\rangle$ has physical helicity $-s$.
To see why, consider how ψ transforms under a rotation by θ about the z-axis (at $\vec{p} = 0$ for simplicity):

$$U(R)\,\psi\, U(R)^{-1} = S(R)\,\psi, \qquad S(R) = e^{-i\theta\Sigma^3/2}. \tag{5.89}$$

The spinor matrix $S(R)$ acts on u^s and v^s identically: $S(R)\,u^s = e^{-is\theta}\,u^s$, $S(R)\,v^s = e^{-is\theta}\,v^s$. Matching both sides of the field expansion term by term:

$$U(R)\,a^s\,U(R)^{-1} = e^{-is\theta}\,a^s\,, \qquad U(R)\,b^{s\dagger}\,U(R)^{-1} = e^{-is\theta}\,b^{s\dagger}\,. \tag{5.90}$$

Both a^s and $b^{s\dagger}$ pick up the *same* phase $e^{-is\theta}$, because both multiply a spinor inside ψ. Taking the Hermitian conjugate of the first relation:

$$U(R)\,a^{s\dagger}\,U(R)^{-1} = e^{+is\theta}\,a^{s\dagger}\,. \tag{5.91}$$

For infinitesimal θ, these become:

$$[J_z, a^{s\dagger}] = +s\,a^{s\dagger}\,, \qquad [J_z, b^{s\dagger}] = -s\,b^{s\dagger}\,. \tag{5.92}$$

The sign difference is now clear: $a^{s\dagger}$ is the *conjugate* of a^s (which appears in ψ), picking up the opposite phase $e^{+is\theta}$; while $b^{s\dagger}$ appears *directly* in ψ, retaining the phase $e^{-is\theta}$.

Result. A right-handed positron ($h_{\text{phys}} = +\frac{1}{2}$) is created by $b^{s\dagger}$ with $s = -\frac{1}{2}$. By Fact 1, the spinor $v^{s=-1/2}$ satisfies $\gamma^5 v = -v$, i.e. it is left-chiral: $P_L v_R = v_R$.

Intuitively, the inversion can also be understood from the Dirac hole picture: a right-handed positron is a "hole" in the left-handed negative-energy electron sea; the missing electron was left-chiral (chirality = helicity for particles), and the hole inherits this chirality.

Taking the Dirac adjoint of $P_L v_R = v_R$ and using $P_L^\dagger = P_L$ together with $\gamma^0 P_R = P_L\gamma^0$ (which follows from $\{\gamma^0, \gamma^5\} = 0$), one obtains

$$\bar{v}_R\,P_R = \bar{v}_R\,. \tag{5.93}$$

This is the identity used in the helicity selection rules below.

Helicity selection rules for $e^+e^- \to \mu^+\mu^-$. At the electron vertex, the current is $j_e^\mu = \bar{v}(p_2)\gamma^\mu u(p_1)$, where $u(p_1)$ is the electron spinor and $v(p_2)$ is the positron spinor. Let us check which helicity combinations give non-zero currents.

Case 1: $e_L^- + e_R^+$ (left-handed electron, right-handed positron)

Using $P_L u_L = u_L$ and $P_L v_R = v_R$:

$$\bar{v}_R\gamma^\mu u_L = \bar{v}_R P_R\gamma^\mu P_L u_L = \bar{v}_R\gamma^\mu P_L u_L \neq 0\,, \tag{5.94}$$

where we used $P_R\gamma^\mu = \gamma^\mu P_L$. This is non-zero because $P_L u_L = u_L$.

Case 2: $e_R^- + e_L^+$ (right-handed electron, left-handed positron)

Using $P_R u_R = u_R$ and $P_R v_L = v_L$:

$$\bar{v}_L \gamma^\mu u_R = \bar{v}_L P_L \gamma^\mu P_R u_R = \bar{v}_L \gamma^\mu P_R u_R \neq 0 . \tag{5.95}$$

This is also non-zero.

Case 3: $e^-_L + e^+_L$ (both left-handed)

Using $P_L u_L = u_L$ and $P_R v_L = v_L$:

$$\bar{v}_L \gamma^\mu u_L = \bar{v}_L P_L \gamma^\mu P_L u_L = \bar{v}_L \gamma^\mu \underbrace{P_R P_L}_{=0} u_L = 0 . \tag{5.96}$$

This vanishes because $P_R P_L = 0$.

Case 4: $e^-_R + e^+_R$ (both right-handed)

Similarly:

$$\bar{v}_R \gamma^\mu u_R = \bar{v}_R P_R \gamma^\mu P_R u_R = \bar{v}_R \gamma^\mu \underbrace{P_L P_R}_{=0} u_R = 0 . \tag{5.97}$$

Summary: The virtual photon can only be produced from:

- $e^-_L + e^+_R \to \gamma^*$ (left-handed electron, right-handed positron)
- $e^-_R + e^+_L \to \gamma^*$ (right-handed electron, left-handed positron)

The key insight is that chirality conservation, combined with the inverted helicity-chirality relation for antiparticles, requires the electron and positron to have *opposite* helicities. Similarly, at the muon vertex, the virtual photon can only produce:

- $\gamma^* \to \mu^-_L + \mu^+_R$
- $\gamma^* \to \mu^-_R + \mu^+_L$

Therefore, only **four** helicity amplitudes are non-vanishing (out of the $2^4 = 16$ possible combinations):

$$\begin{aligned} &\mathcal{M}(e^-_L e^+_R \to \mu^-_L \mu^+_R), \qquad &&\mathcal{M}(e^-_L e^+_R \to \mu^-_R \mu^+_L), \\ &\mathcal{M}(e^-_R e^+_L \to \mu^-_L \mu^+_R), \qquad &&\mathcal{M}(e^-_R e^+_L \to \mu^-_R \mu^+_L) . \end{aligned} \tag{5.98}$$

Box 5.4 Wigner d-matrices: group theory and scattering

Group-theoretic origin. The Wigner d-matrices arise as matrix elements of the **irreducible representations of SU(2)**, the universal covering group of the rotation group SO(3). For each spin $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$, there exists a $(2j+1)$-dimensional irreducible representation.

Euler angles. Any rotation in three dimensions can be decomposed into three successive rotations. In the z-y-z **convention** (standard in quantum mechanics), the Euler angles (α, β, γ) parametrize a rotation as:

1. Rotate by angle γ about the original z-axis
2. Rotate by angle β about the (new) y'-axis
3. Rotate by angle α about the (new) z''-axis

The ranges are $\alpha \in [0, 2\pi)$, $\beta \in [0, \pi]$, $\gamma \in [0, 2\pi)$.

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The figure shows the original axes $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ in **blue** and the rotated axes $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in **red**. The **line of nodes** **N** (green) is the intersection of the xy-plane (blue ellipse) and the XY-plane (red ellipse). The three Euler angles are:

- α: angle from **x** to **N**, measured in the xy-plane
- β: angle from **z** to **Z** (the nutation angle)
- γ: angle from **N** to **X**, measured in the XY-plane

Equivalently, using rotations about the *fixed* (original) axes, the same result is achieved by applying rotations in reverse order: first α about z, then β about y, then γ about z:

$$R(\alpha, \beta, \gamma) = R_z(\alpha)\, R_y(\beta)\, R_z(\gamma)\,. \tag{5.99}$$

This is the form used in the quantum mechanical representation below.

Representation of rotations. A general rotation $R(\alpha, \beta, \gamma)$ is represented in the spin-j Hilbert space by:

$$D^j_{m,m'}(\alpha, \beta, \gamma) = \langle j, m | e^{-i\alpha J_z} e^{-i\beta J_y} e^{-i\gamma J_z} | j, m' \rangle = e^{-im\alpha} d^j_{m,m'}(\beta) e^{-im'\gamma}\,. \tag{5.100}$$

The factorization into phases $e^{-im\alpha}$ and $e^{-im'\gamma}$ follows because the basis states $|j, m\rangle$ are **eigenstates of** J_z:

$$J_z |j, m\rangle = m |j, m\rangle \quad \Rightarrow \quad e^{-i\gamma J_z} |j, m'\rangle = e^{-i\gamma m'} |j, m'\rangle\,. \tag{5.101}$$

Thus rotations about the z-axis are diagonal in this basis and contribute only phase factors. In contrast, J_y does not commute with J_z, so $e^{-i\beta J_y}$ mixes different m values—this is what makes $d^j_{m,m'}(\beta)$ a non-trivial matrix.

The **small d-matrix** $d^j_{m,m'}(\theta)$ corresponds to a pure rotation about the y-axis:

$$d^j_{m,m'}(\theta) = \langle j,m|e^{-i\theta J_y}|j,m'\rangle = D^j_{m,m'}(0,\theta,0)\,, \tag{5.102}$$

where the angular momentum generators satisfy the Lie algebra $[J_i, J_j] = i\epsilon_{ijk}J_k$.

Explicit formulas for $j = 1$. The d-matrix for spin-1 is a 3×3 matrix:

$$d^1(\theta) = \begin{pmatrix} \frac{1+\cos\theta}{2} & -\frac{\sin\theta}{\sqrt{2}} & \frac{1-\cos\theta}{2} \\ \frac{\sin\theta}{\sqrt{2}} & \cos\theta & -\frac{\sin\theta}{\sqrt{2}} \\ \frac{1-\cos\theta}{2} & \frac{\sin\theta}{\sqrt{2}} & \frac{1+\cos\theta}{2} \end{pmatrix} \tag{5.103}$$

These are related to spherical harmonics: $d^j_{m,0}(\theta) = \sqrt{\frac{4\pi}{2j+1}}\,Y_{jm}(\theta,0)^*$.

Key properties (following from group representation theory):

- $d^j_{m,m'}(0) = \delta_{m,m'}$ (identity element)
- $d^j_{m,m'}(\theta_1+\theta_2) = \sum_{m''} d^j_{m,m''}(\theta_1)d^j_{m'',m'}(\theta_2)$ (group multiplication)
- $d^j_{m,m'}(\theta)^* = d^j_{m',m}(\theta) = d^j_{m,m'}(-\theta)$ (unitarity)
- $d^j_{-m,-m'}(\theta) = (-1)^{m-m'}d^j_{m,m'}(\theta)$ (reflection symmetry)

Physical meaning in scattering: the sign in $\lambda_B - \lambda_C$. Consider a decay $A \to B + C$ where particle A (spin j, helicity λ_A) decays into B and C moving back-to-back. Let the decay axis make angle θ with the original z-axis.
The helicity of each daughter is defined as the spin projection along *its own direction of motion*:

- Particle B moves along $+\hat{z}'$ with helicity $\lambda_B \Rightarrow$ spin projection along $+\hat{z}'$ is λ_B
- Particle C moves along $-\hat{z}'$ with helicity $\lambda_C \Rightarrow$ spin projection along $-\hat{z}'$ is λ_C

To find C's contribution to the *total* $J_{z'}$ (along $+\hat{z}'$), we must flip the sign: the spin projection of C along $+\hat{z}'$ is $-\lambda_C$. Therefore:

$$J_{z'}^{\text{final}} = \lambda_B + (-\lambda_C) = \lambda_B - \lambda_C\,. \tag{5.104}$$

The amplitude is then proportional to $d^j_{\lambda_A,\lambda_B-\lambda_C}(\theta)$, connecting the initial helicity λ_A to the final net helicity $\lambda_B - \lambda_C$.

Angular dependence from helicity amplitudes. The angular dependence of each helicity amplitude is determined by angular momentum conservation. The initial e^+e^- state with definite helicities has a well-defined z-component of angular momentum $J_z = h_{e^-} - h_{e^+}$ (taking the e^- direction as the z-axis).

For $e^-_L e^+_R$: $J_z = -\frac{1}{2} - (+\frac{1}{2}) = -1$

For $e^-_R e^+_L$: $J_z = +\frac{1}{2} - (-\frac{1}{2}) = +1$

The virtual photon (spin-1) thus carries $J_z = \pm 1$. When it decays to $\mu^+\mu^-$ at angle θ, the amplitude picks up a factor from the rotation matrix $d^1_{m,m'}(\theta)$:

$$\mathcal{M}(e^-_L e^+_R \to \mu^-_L \mu^+_R) \propto d^1_{-1,-1}(\theta) = \frac{1+\cos\theta}{2} \tag{5.105}$$

$$\mathcal{M}(e^-_L e^+_R \to \mu^-_R \mu^+_L) \propto d^1_{-1,+1}(\theta) = \frac{1-\cos\theta}{2} \tag{5.106}$$

$$\mathcal{M}(e^-_R e^+_L \to \mu^-_L \mu^+_R) \propto d^1_{+1,-1}(\theta) = \frac{1-\cos\theta}{2} \tag{5.107}$$

$$\mathcal{M}(e^-_R e^+_L \to \mu^-_R \mu^+_L) \propto d^1_{+1,+1}(\theta) = \frac{1+\cos\theta}{2}\,. \tag{5.108}$$

Squaring and summing (for unpolarized beams, each initial helicity combination has equal weight):

$$\begin{aligned}\sum_{\text{helicities}} |\mathcal{M}|^2 &\propto 2\left(\frac{1+\cos\theta}{2}\right)^2 + 2\left(\frac{1-\cos\theta}{2}\right)^2 \\ &= \frac{1}{2}\left[(1+\cos\theta)^2 + (1-\cos\theta)^2\right] \\ &= 1+\cos^2\theta\,. \end{aligned} \tag{5.109}$$

This provides an elegant derivation of the angular distribution from helicity considerations alone.

Physical interpretation. The $(1+\cos^2\theta)$ distribution can be understood as follows:

- The $\cos^2\theta$ term arises from configurations where the muon helicity matches the electron helicity ($LL \to LL$ or $RR \to RR$ in shorthand notation). These amplitudes are maximal for forward/backward scattering ($\theta = 0, \pi$).
- The constant term (unity) comes from helicity-flip configurations ($LL \to RR$ or $RR \to LL$). These are maximal at $\theta = \pi/2$.

The absence of a $\cos\theta$ term (forward-backward asymmetry) reflects the parity conservation of QED: the process is symmetric under $\theta \to \pi - \theta$.

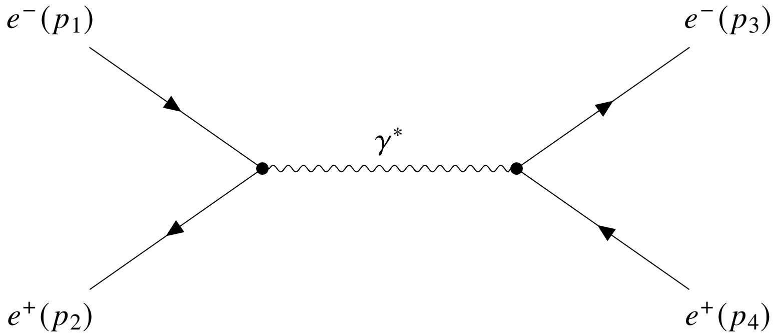
5.3 Bhabha scattering: $e^+e^- \to e^+e^-$

Bhabha scattering is the elastic scattering of electrons and positrons: $e^+e^- \to e^+e^-$. Unlike muon pair production, the same external state can be reached through two distinct tree-level channels (s and t). Their interference, combined with the fermionic sign conventions, produces a rich angular distribution.

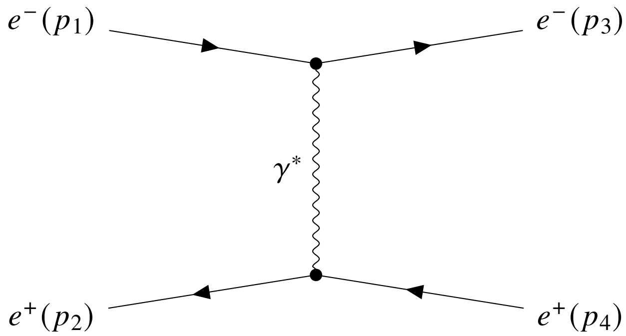
Kinematics and Feynman diagrams. Consider the process:

$$e^-(p_1) + e^+(p_2) \to e^-(p_3) + e^+(p_4)\,. \tag{5.110}$$

At tree level, two diagrams contribute:



(a) s-channel (annihilation)



(b) t-channel (exchange)

In the s-channel, the electron and positron annihilate into a virtual photon with momentum $k = p_1 + p_2$, which then creates a new e^+e^- pair. In the t-channel, a photon is exchanged between the electron and positron lines, with momentum transfer $q = p_1 - p_3$.

Relative sign from Fermi statistics. The two diagrams differ by the assignment of which external particles are handled at each vertex. Recall that at a QED vertex $\bar{\psi}\gamma^\mu\psi$:

- ψ can annihilate an e^- (contributing u) or create an e^+ (contributing v),
- $\bar{\psi}$ can create an e^- (contributing $\bar{u}$) or annihilate an e^+ (contributing $\bar{v}$).

<u>s-channel</u> (annihilation): both incoming particles are absorbed at vertex x, both outgoing particles are emitted at vertex y.

- Vertex x: $\psi(x)$ annihilates $e^-(p_1) \to u(p_1)$; $\bar{\psi}(x)$ annihilates $e^+(p_2) \to \bar{v}(p_2)$.
  Current: $\bar{v}(p_2)\gamma^\mu u(p_1)$.
- Vertex y: $\bar{\psi}(y)$ creates $e^-(p_3) \to \bar{u}(p_3)$; $\psi(y)$ creates $e^+(p_4) \to v(p_4)$.
  Current: $\bar{u}(p_3)\gamma^\nu v(p_4)$.

<u>t-channel</u> (exchange): the electron scatters at vertex x, the positron at vertex y.

- Vertex x: $\psi(x)$ annihilates $e^-(p_1) \to u(p_1)$; $\bar{\psi}(x)$ creates $e^-(p_3) \to \bar{u}(p_3)$.
  Current: $\bar{u}(p_3)\gamma^\mu u(p_1)$.
- Vertex y: $\bar{\psi}(y)$ annihilates $e^+(p_2) \to \bar{v}(p_2)$; $\psi(y)$ creates $e^+(p_4) \to v(p_4)$.
  Current: $\bar{v}(p_2)\gamma^\nu v(p_4)$.

The two channels differ by the exchange of the final-state fermion lines: $v(p_4)$ and $u(p_1)$ swap their vertex assignments. To determine the relative sign, note that the s-channel operator structure (suppressing gamma matrices and spacetime arguments) is

$$\bar{u}_3\, v_4\, \bar{v}_2\, u_1\,, \tag{5.111}$$

while the t-channel has

$$\bar{u}_3\, u_1\, \bar{v}_2\, v_4\,. \tag{5.112}$$

To bring the t-channel ordering into the s-channel ordering, we must move u_1 past $\bar{v}_2$ (one transposition of fermionic operators), and then v_4 past $\bar{v}_2 u_1$ (two more transpositions):

$$\bar{u}_3\, u_1\, \bar{v}_2\, v_4 \xrightarrow{u_1 \leftrightarrow \bar{v}_2} -\bar{u}_3\, \bar{v}_2\, u_1\, v_4 \xrightarrow{v_4 \text{ to front}} +\bar{u}_3\, \bar{v}_2\, v_4\, u_1 \to -\bar{u}_3\, v_4\, \bar{v}_2\, u_1\,. \tag{5.113}$$

Three transpositions give a factor $(-1)^3 = -1$, so the total amplitude is:

$$\boxed{\mathcal{M} = \mathcal{M}_s - \mathcal{M}_t}\,. \tag{5.114}$$

Individual amplitudes. The s-channel amplitude is (with $k^2 = (p_1 + p_2)^2 = s$):

$$\boxed{\mathcal{M}_s = \frac{e^2}{s}\left[\bar{u}(p_3)\gamma^\mu v(p_4)\right]\left[\bar{v}(p_2)\gamma_\mu u(p_1)\right]}\,. \tag{5.115}$$

The t-channel amplitude is (with $q^2 = (p_1 - p_3)^2 = t$):

$$\boxed{\mathcal{M}_t = \frac{e^2}{t}\left[\bar{u}(p_3)\gamma^\nu u(p_1)\right]\left[\bar{v}(p_2)\gamma_\nu v(p_4)\right]}\,. \tag{5.116}$$

Spin-averaged amplitude squared. For unpolarized beams, the spin-averaged amplitude squared is:

$$\begin{aligned}\overline{|\mathcal{M}|^2} &= \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_s - \mathcal{M}_t|^2 \\ &= \frac{1}{4}\sum_{\text{spins}} \left(|\mathcal{M}_s|^2 + |\mathcal{M}_t|^2 - \mathcal{M}_s\mathcal{M}_t^* - \mathcal{M}_s^*\mathcal{M}_t\right) \\ &= X_{ss} + X_{tt} + X_{st}\,,\end{aligned} \tag{5.117}$$

where we define:

$$X_{ss} = \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_s|^2 \tag{5.118}$$

$$X_{tt} = \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_t|^2 \tag{5.119}$$

$$X_{st} = -\frac{1}{4}\sum_{\text{spins}} \left(\mathcal{M}_s\mathcal{M}_t^* + \mathcal{M}_s^*\mathcal{M}_t\right) = -\frac{1}{2}\sum_{\text{spins}} \mathrm{Re}(\mathcal{M}_s\mathcal{M}_t^*)\,. \tag{5.120}$$

Calculation of X_{ss}. This is structurally identical to the $e^+e^- \to \mu^+\mu^-$ calculation (Section 5.2), with the replacement $m_\mu \to m_e$. We have:

$$|\mathcal{M}_s|^2 = \frac{e^4}{s^2}\left[\bar{u}(p_3)\gamma^\mu v(p_4)\right]\left[\bar{v}(p_2)\gamma_\mu u(p_1)\right]\left[\bar{v}(p_4)\gamma^\rho u(p_3)\right]\left[\bar{u}(p_1)\gamma_\rho v(p_2)\right]. \tag{5.121}$$

We now show in detail how to convert this product of spinor bilinears into traces. Writing the spinor indices explicitly:

$$\begin{aligned}|\mathcal{M}_s|^2 = \frac{e^4}{s^2}&\left[\bar{u}_\alpha(p_3)(\gamma^\mu)_{\alpha\beta}v_\beta(p_4)\right]\left[\bar{v}_\gamma(p_2)(\gamma_\mu)_{\gamma\delta}u_\delta(p_1)\right] \\ &\times\left[\bar{v}_\epsilon(p_4)(\gamma^\rho)_{\epsilon\zeta}u_\zeta(p_3)\right]\left[\bar{u}_\eta(p_1)(\gamma_\rho)_{\eta\theta}v_\theta(p_2)\right].\end{aligned} \tag{5.122}$$

When we sum over spins, we use the completeness relations (BD normalization):

$$\sum_s u_\alpha(p,s)\bar{u}_\beta(p,s) = \frac{(\not{p}+m)_{\alpha\beta}}{2m} \tag{5.123}$$

$$\sum_s v_\alpha(p,s)\bar{v}_\beta(p,s) = \frac{(\not{p}-m)_{\alpha\beta}}{2m}\,. \tag{5.124}$$

For the p_3 electron: $\sum_{s_3} u_\zeta(p_3)\bar{u}_\alpha(p_3) = (\not{p}_3+m)_{\zeta\alpha}/(2m)$
For the p_4 positron: $\sum_{s_4} v_\beta(p_4)\bar{v}_\epsilon(p_4) = (\not{p}_4-m)_{\beta\epsilon}/(2m)$

For the p_1 electron: $\sum_{s_1} u_\delta(p_1)\bar{u}_\eta(p_1) = (\not{p}_1 + m)_{\delta\eta}/(2m)$
For the p_2 positron: $\sum_{s_2} v_\theta(p_2)\bar{v}_\gamma(p_2) = (\not{p}_2 - m)_{\theta\gamma}/(2m)$
Substituting these into the spin sum and grouping indices that contract:

$$\sum_{\text{spins}} |\mathcal{M}_s|^2 = \frac{e^4}{s^2}\frac{1}{(2m)^4}(\not{p}_3+m)_{\zeta\alpha}(\gamma^\mu)_{\alpha\beta}(\not{p}_4-m)_{\beta\epsilon}(\gamma^\rho)_{\epsilon\zeta}$$
$$\times(\not{p}_2-m)_{\theta\gamma}(\gamma_\mu)_{\gamma\delta}(\not{p}_1+m)_{\delta\eta}(\gamma_\rho)_{\eta\theta}\,. \tag{5.125}$$

In the first line, the indices form a closed loop: $\zeta \to \alpha \to \beta \to \epsilon \to \zeta$. This is precisely the definition of a trace:

$$(\not{p}_3+m)_{\zeta\alpha}(\gamma^\mu)_{\alpha\beta}(\not{p}_4-m)_{\beta\epsilon}(\gamma^\rho)_{\epsilon\zeta} = \mathrm{Tr}\big[(\not{p}_3+m)\gamma^\mu(\not{p}_4-m)\gamma^\rho\big]\,. \tag{5.126}$$

Similarly, the second line gives:

$$(\not{p}_2-m)_{\theta\gamma}(\gamma_\mu)_{\gamma\delta}(\not{p}_1+m)_{\delta\eta}(\gamma_\rho)_{\eta\theta} = \mathrm{Tr}\big[(\not{p}_2-m)\gamma_\mu(\not{p}_1+m)\gamma_\rho\big]\,. \tag{5.127}$$

Therefore:

$$\boxed{\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_s|^2 = \frac{e^4}{4s^2}\frac{1}{(2m)^4}\mathrm{Tr}\big[(\not{p}_3+m)\gamma^\mu(\not{p}_4-m)\gamma^\rho\big]\,\mathrm{Tr}\big[(\not{p}_2-m)\gamma_\mu(\not{p}_1+m)\gamma_\rho\big]}\,. \tag{5.128}$$

In the ultra-relativistic limit ($m \to 0$), the BD normalization factors $(2m)^{-4}$ cancel against $\prod(2m_i)$ in the cross section formula (cf. eq. (5.72)), so from now on we drop them and use the standard completeness relations $\sum u\bar{u} = \not{p}$, $\sum v\bar{v} = \not{p}$ in the massless limit. Using the trace identity:

$$\mathrm{Tr}(\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta) = 4(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\delta}g^{\beta\gamma})\,, \tag{5.129}$$

we obtain:

$$\mathrm{Tr}(\not{p}_3\gamma^\mu\not{p}_4\gamma^\rho) = 4(p_3^\mu p_4^\rho - (p_3\cdot p_4)g^{\mu\rho} + p_3^\rho p_4^\mu) \tag{5.130}$$
$$\mathrm{Tr}(\not{p}_2\gamma_\mu\not{p}_1\gamma_\rho) = 4(p_{2\mu}p_{1\rho} - (p_2\cdot p_1)g_{\mu\rho} + p_{2\rho}p_{1\mu})\,. \tag{5.131}$$

Contracting these two traces:

$$\begin{aligned}
&\mathrm{Tr}(\not{p}_3\gamma^\mu\not{p}_4\gamma^\rho)\,\mathrm{Tr}(\not{p}_2\gamma_\mu\not{p}_1\gamma_\rho)\\
&= 16\big[(p_3\cdot p_2)(p_4\cdot p_1) - (p_3\cdot p_4)(p_1\cdot p_2) + (p_3\cdot p_1)(p_4\cdot p_2)\\
&\qquad - (p_3\cdot p_4)(p_1\cdot p_2) + 4(p_3\cdot p_4)(p_1\cdot p_2) - (p_3\cdot p_4)(p_1\cdot p_2)\\
&\qquad + (p_4\cdot p_2)(p_3\cdot p_1) - (p_3\cdot p_4)(p_1\cdot p_2) + (p_4\cdot p_1)(p_3\cdot p_2)\big]\\
&= 32\big[(p_1\cdot p_3)(p_2\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3)\big]\,.
\end{aligned} \tag{5.132}$$