Experiment	$\sqrt{s}$ (GeV)	Angular range	Precision
PETRA (1980s)	14–47	Large angle	$\sim 2\%$
LEP1 (1989–95)	91	25–58 mrad	0.05%
LEP2 (1996–2000)	130–209	25–58 mrad	0.1%
BaBar (1999–2008)	10.6	Large angle	0.5%
Belle (1999–2010)	10.6	Large angle	0.5%

Table 5.1. Summary of Bhabha scattering measurements at e^+e^- colliders. The LEP experiments achieved sub-per-mille precision, essential for electroweak tests.

5.4 Compton scattering: $e^-\gamma \to e^-\gamma$

Compton scattering is the elastic scattering of a photon off a free electron: $e^-(p) + \gamma(k) \to e^-(p') + \gamma(k')$. This process, first studied experimentally by Arthur Compton in 1923, provided crucial evidence for the particle nature of light. The quantum field theoretic treatment not only reproduces the classical Thomson limit at low energies but also predicts the full energy and angular dependence through the Klein–Nishina formula.

Kinematics and amplitude

Momentum assignments. Consider the scattering process:

$$e^-(p) + \gamma(k) \to e^-(p') + \gamma(k') \,. \tag{5.200}$$

The four-momenta satisfy:

- Incoming electron: $p^2 = m^2$
- Incoming photon: $k^2 = 0$
- Outgoing electron: $p'^2 = m^2$
- Outgoing photon: $k'^2 = 0$

with energy-momentum conservation $p + k = p' + k'$.

Mandelstam variables. The Mandelstam variables for Compton scattering are:

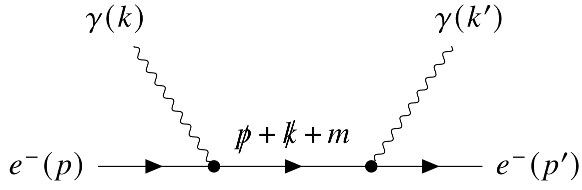
$$s = (p+k)^2 = m^2 + 2p\cdot k \tag{5.201}$$
$$t = (k-k')^2 = -2k\cdot k' \tag{5.202}$$
$$u = (p-k')^2 = m^2 - 2p\cdot k' \,, \tag{5.203}$$

satisfying the constraint $s + t + u = 2m^2$.

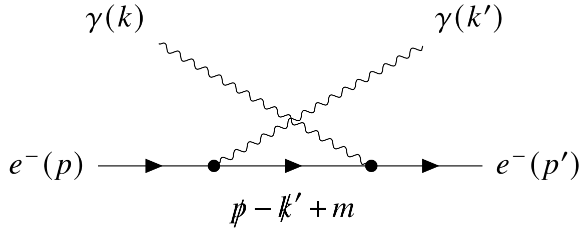
From energy-momentum conservation, we have the useful relations:

$$\boxed{p\cdot k = p'\cdot k', \qquad p\cdot k' = p'\cdot k, \qquad p\cdot p' = m^2 + k\cdot k'} \,. \tag{5.204}$$

Tree-level Feynman diagrams. At leading order, two diagrams contribute to Compton scattering:



(a) s-channel



(b) u-channel

In the s-channel diagram, the incoming photon is absorbed first, creating an intermediate electron with momentum $p+k$. In the u-channel diagram, the outgoing photon is emitted first, and the intermediate electron has momentum $p-k'$.

Note on channels. Unlike Bhabha scattering, there is no t-channel diagram for Compton scattering because a photon cannot couple directly to another photon (at tree level). The t-channel would require a fermion exchange between the two photon lines, which is not possible.

Individual amplitudes. Applying the QED Feynman rules, the s-channel amplitude is:

$$i\mathcal{M}_s = \bar{u}(p')(ie\gamma^\beta)\,\varepsilon^*_\beta(k',\lambda')\,\frac{i(\not p + \not k + m)}{(p+k)^2 - m^2}\,(ie\gamma^\alpha)\,\varepsilon_\alpha(k,\lambda)\,u(p)\,. \tag{5.205}$$

Simplifying with $(p+k)^2 - m^2 = 2p\cdot k$:

$$\boxed{\mathcal{M}_s = -\frac{e^2}{2p\cdot k}\,\bar{u}(p')\gamma^\beta\varepsilon^*_\beta(k',\lambda')(\not p + \not k + m)\gamma^\alpha\varepsilon_\alpha(k,\lambda)\,u(p)}\,. \tag{5.206}$$

The u-channel amplitude is:

$$\boxed{\mathcal{M}_u = -\frac{e^2}{-2p\cdot k'}\,\bar{u}(p')\gamma^\alpha\varepsilon_\alpha(k,\lambda)(\not p - \not k' + m)\gamma^\beta\varepsilon^*_\beta(k',\lambda')\,u(p)}\,. \tag{5.207}$$

Note the sign change in the denominator: $(p-k')^2 - m^2 = -2p\cdot k'$.

Total amplitude. The total amplitude is the sum of both diagrams:

$$\mathcal{M} = \mathcal{M}_s + \mathcal{M}_u\,. \tag{5.208}$$

Unlike Bhabha scattering, there is **no relative minus sign** between the two diagrams because both describe the same fermion line—there is no exchange of identical fermions.

Compact notation. It is convenient to factor out the polarization vectors and write:

$$\mathcal{M} = \varepsilon^*_\beta(k',\lambda')\,\varepsilon_\alpha(k,\lambda)\,\mathcal{M}^{\alpha\beta}\,, \tag{5.209}$$

where the tensor amplitude is:

$$\begin{aligned}\mathcal{M}^{\alpha\beta} &= -e^2\,\bar{u}(p')\left[\frac{\gamma^\beta(\not{p}+\not{k}+m)\gamma^\alpha}{2p\cdot k}+\frac{\gamma^\alpha(\not{p}-\not{k}'+m)\gamma^\beta}{-2p\cdot k'}\right]u(p)\\ &= -e^2\,\bar{u}(p')\left[\frac{\gamma^\beta(\not{p}+\not{k}+m)\gamma^\alpha}{2p\cdot k}-\frac{\gamma^\alpha(\not{p}-\not{k}'+m)\gamma^\beta}{2p\cdot k'}\right]u(p)\,.\end{aligned} \tag{5.210}$$

Ward identity: general argument from gauge invariance. For any QED process involving external photons, the Feynman amplitude can be written in the form

$$\mathcal{M} = \varepsilon^\alpha(k_1,\lambda_1)\,\varepsilon^\beta(k_2,\lambda_2)\cdots\mathcal{M}_{\alpha\beta\ldots}(k_1,k_2,\ldots)\,, \tag{5.211}$$

with one polarization vector $\varepsilon(k_i,\lambda_i)$ for each external photon, and the **tensor amplitude** $\mathcal{M}_{\alpha\beta\ldots}$ independent of these polarization vectors. (This follows from the QED Feynman rules, since each external photon line contributes exactly one factor of ε.)

Now, the polarization vectors are gauge dependent. For a free photon described in a Lorentz gauge by $A^\mu(x) = \text{const.}\ \varepsilon^\mu(k,\lambda)\,e^{\pm ikx}$, the gauge transformation $A^\mu \to A^\mu + \partial^\mu f$ with $f(x) = \tilde{f}(k)\,e^{\pm ikx}$ implies

$$\varepsilon^\mu(k,\lambda) \;\longrightarrow\; \varepsilon^\mu(k,\lambda) \pm ik^\mu\tilde{f}(k)\,. \tag{5.212}$$

Since only the *sum* of all Feynman diagrams at a given order—i.e. the full matrix element $\mathcal{M}$—is a physical observable, it must be gauge invariant. Invariance of eq. (5.211) under the replacement $\varepsilon^\alpha(k_i,\lambda_i) \to \varepsilon^\alpha(k_i,\lambda_i) + c\,k_i^\alpha$ (with c an arbitrary constant) for *any* external photon requires:

$$\boxed{k_1^\alpha\,\mathcal{M}_{\alpha\beta\ldots}(k_1,k_2,\ldots) = k_2^\beta\,\mathcal{M}_{\alpha\beta\ldots}(k_1,k_2,\ldots) = \cdots = 0}\,. \tag{5.213}$$

This is the **Ward identity**: contracting the tensor amplitude with the four-momentum of any external photon gives zero. It holds for *any* QED process, to all orders in perturbation theory, provided all diagrams of a given order are included.

Remark. The individual contributions from single Feynman diagrams are in general *not* gauge invariant. For Compton scattering, neither $\mathcal{M}_s$ nor $\mathcal{M}_u$ alone satisfies the Ward identity—only their sum $\mathcal{M} = \mathcal{M}_s + \mathcal{M}_u$ does. This is a concrete manifestation of the fact that gauge invariance constrains the full amplitude, not its individual diagrammatic contributions.

Ward identity for Compton scattering: explicit verification. For the specific case of Compton scattering, the Ward identity reads

$$k_\alpha \mathcal{M}^{\alpha\beta} = 0, \qquad k'_\beta \mathcal{M}^{\alpha\beta} = 0\,. \tag{5.214}$$

Let us verify $k_\alpha \mathcal{M}^{\alpha\beta} = 0$ explicitly by direct calculation.

Contracting with k_α (i.e., replacing $\gamma^\alpha \to \not{k}$):

$$k_\alpha \mathcal{M}^{\alpha\beta} = -e^2 \bar{u}(p') \left[\frac{\gamma^\beta(\not{p} + \not{k} + m)\not{k}}{2p\cdot k} - \frac{\not{k}(\not{p} - \not{k}' + m)\gamma^\beta}{2p\cdot k'} \right] u(p)\,. \tag{5.215}$$

<u>First term:</u> We compute $(\not{p} + \not{k} + m)\not{k}\, u(p)$.

Using $\not{p}\not{k} = 2p\cdot k - \not{k}\not{p}$ and the Dirac equation $(\not{p} - m)u(p) = 0$:

$$\begin{aligned}(\not{p} + \not{k} + m)\not{k}\, u(p) &= [\not{p}\not{k} + \not{k}^2 + m\not{k}]u(p) \\ &= [2p\cdot k - \not{k}\not{p} + m\not{k}]u(p) \quad (\text{since } k^2 = 0) \\ &= 2p\cdot k\, u(p) - \not{k}(\not{p} - m)u(p) \\ &= 2p\cdot k\, u(p)\,. \end{aligned} \tag{5.216}$$

Therefore: $\dfrac{(\not{p} + \not{k} + m)\not{k}}{2p\cdot k} u(p) = u(p)$.

<u>Second term:</u> We compute $\bar{u}(p')\not{k}(\not{p} - \not{k}' + m)$.

Using $\not{k}\not{a} = 2k\cdot a - \not{a}\not{k}$ and the kinematic relation $p - k' = p' - k$:

$$\bar{u}(p')\not{k}(\not{p} - \not{k}' + m) = \bar{u}(p')[2k\cdot(p - k') - (\not{p} - \not{k}')\not{k} + m\not{k}]\,. \tag{5.217}$$

Now, $(\not{p} - \not{k}') = (\not{p}' - \not{k})$, so using $\bar{u}(p')(\not{p}' - m) = 0$:

$$\begin{aligned}\bar{u}(p')(\not{p} - \not{k}')\not{k} &= \bar{u}(p')(\not{p}' - \not{k})\not{k} = [m\bar{u}(p') - \bar{u}(p')\not{k}]\not{k} \\ &= m\bar{u}(p')\not{k} \quad (\text{since } \not{k}^2 = 0)\,. \end{aligned} \tag{5.218}$$

Therefore:

$$\begin{aligned}\bar{u}(p')\not{k}(\not{p} - \not{k}' + m) &= 2(p\cdot k - k\cdot k')\bar{u}(p') - m\bar{u}(p')\not{k} + m\bar{u}(p')\not{k} \\ &= 2(p\cdot k - k\cdot k')\bar{u}(p')\,. \end{aligned} \tag{5.219}$$

Using the kinematic identity $p\cdot k - k\cdot k' = p'\cdot k = p\cdot k'$:

$$\frac{\bar{u}(p')\not{k}(\not{p} - \not{k}' + m)}{2p\cdot k'} = \bar{u}(p')\,. \tag{5.220}$$

Combining: The two terms give:

$$k_\alpha \mathcal{M}^{\alpha\beta} = -e^2[\bar{u}(p')\gamma^\beta u(p) - \bar{u}(p')\gamma^\beta u(p)] = 0 \quad \checkmark\,. \tag{5.221}$$

The proof of $k'_\beta \mathcal{M}^{\alpha\beta} = 0$ is analogous.

Squared amplitude. For unpolarized electrons and photons, we must average over initial spins and polarizations and sum over final ones:

$$\overline{|\mathcal{M}|^2} = \frac{1}{2}\sum_{\text{electron spins}} \frac{1}{2}\sum_{\text{photon polarizations}} |\mathcal{M}|^2\,. \tag{5.222}$$

The factor of $\frac{1}{4}$ comes from averaging over the two electron spin states and two photon polarization states.

Photon polarization sum. For real (on-shell) photons, the polarization sum gives:

$$\sum_{\lambda=1,2} \varepsilon_\mu(k,\lambda)\,\varepsilon^*_\nu(k,\lambda) = -g_{\mu\nu}\,. \tag{5.223}$$

This replacement is valid for physical amplitudes that satisfy the Ward identity, which ensures gauge invariance: $k_\mu \mathcal{M}^{\mu\nu} = k'_\nu \mathcal{M}^{\mu\nu} = 0$.

Structure of the calculation. The squared amplitude can be written as:

$$\overline{|\mathcal{M}|^2} = \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_s + \mathcal{M}_u|^2 = \frac{1}{4}\sum_{\text{spins}} \left(|\mathcal{M}_s|^2 + |\mathcal{M}_u|^2 + 2\text{Re}(\mathcal{M}_s\mathcal{M}_u^*)\right). \tag{5.224}$$

We define:

$$X_{ss} = \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_s|^2 \tag{5.225}$$

$$X_{uu} = \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_u|^2 \tag{5.226}$$

$$X_{su} = \frac{1}{4}\sum_{\text{spins}} 2\text{Re}(\mathcal{M}_s\mathcal{M}_u^*)\,. \tag{5.227}$$

Spin sums and the Klein–Nishina formula

Calculation of X_{ss}. After performing the spin sum using the BD completeness relation $\sum_s u(p)\bar{u}(p) = (\not{p}+m)/(2m)$ and the polarization sum $\sum_\lambda \varepsilon_\mu(k,\lambda)\,\varepsilon^*_\nu(k,\lambda) = -g_{\mu\nu}$, we obtain:

$$X_{ss} = \frac{e^4}{4(p\cdot k)^2}\,\text{Tr}\left[\gamma^\alpha(\not{p}+\not{k}+m)\gamma^\beta \frac{\not{p}'+m}{2m}\gamma_\beta(\not{p}+\not{k}+m)\gamma_\alpha \frac{\not{p}+m}{2m}\right]. \tag{5.228}$$

Using the notation $\not f_1 = \not p + \not k$, this becomes:

$$X_{ss} = \frac{e^4}{16m^2(p\cdot k)^2}\,\mathrm{Tr}\big[\gamma^\alpha(\not f_1 + m)\gamma^\beta(\not p' + m)\gamma_\beta(\not f_1 + m)\gamma_\alpha(\not p + m)\big]\,. \tag{5.229}$$

Detailed trace calculation. We use the gamma matrix contraction identities:

$$\begin{aligned} \gamma_\mu\gamma^\mu &= 4, & \gamma_\mu\gamma^\alpha\gamma^\mu = -2\gamma^\alpha \\ \gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\mu &= 4g^{\alpha\beta}, \\ \gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\mu &= -2\gamma^\gamma\gamma^\beta\gamma^\alpha\,. \end{aligned} \tag{5.230}$$

Step 1: Contract $\gamma^\beta\ldots\gamma_\beta$ first.

$$\gamma^\beta(\not p' + m)\gamma_\beta = \gamma^\beta\not p'\gamma_\beta + m\gamma^\beta\gamma_\beta = -2\not p' + 4m\,. \tag{5.231}$$

Then:

$$\begin{aligned} \gamma^\beta(\not p' + m)\gamma_\beta(\not f_1 + m) &= (-2\not p' + 4m)(\not f_1 + m) \\ &= -2\not p'\not f_1 - 2m\not p' + 4m\not f_1 + 4m^2\,. \end{aligned} \tag{5.232}$$

Step 2: Multiply by $(\not f_1 + m)$ on the left.

$$\begin{aligned} (\not f_1{+}m)(-2\not p'\not f_1{-}2m\not p'{+}4m\not f_1{+}4m^2) &= -2\not f_1\not p'\not f_1 - 2m\not f_1\not p' + 4m\not f_1^2 + 4m^2\not f_1 \\ &\quad - 2m\not p'\not f_1 - 2m^2\not p' + 4m^2\not f_1 + 4m^3 \end{aligned} \tag{5.233}$$

Using $\not f_1^2 = f_1^2 = (p+k)^2 = m^2 + 2p\cdot k$ and $\not a\not b + \not b\not a = 2a\cdot b$:

$$= -2\not f_1\not p'\not f_1 - 4m(f_1\cdot p') - 2m^2\not p' + 8m^2\not f_1 + 8m^3 + 8m(p\cdot k)\,. \tag{5.234}$$

Step 3: Contract $\gamma^\alpha\ldots\gamma_\alpha$ and take the trace.

The trace $T = \mathrm{Tr}[\gamma^\alpha(\ldots)\gamma_\alpha(\not p + m)]$ has six contributions:

(A) $\mathrm{Tr}[\gamma^\alpha(-2\not f_1\not p'\not f_1)\gamma_\alpha(\not p + m)]$:

Using $\gamma^\alpha\not a\not b\not c\gamma_\alpha = -2\not c\not b\not a$:

$$= \mathrm{Tr}[4\not f_1\not p'\not f_1\not p] = 4\times 4[(f_1\cdot p')(f_1\cdot p) - f_1^2(p'\cdot p) + (f_1\cdot p)(f_1\cdot p')] \tag{5.235}$$

$$= 32(f_1\cdot p)(f_1\cdot p') - 16(m^2 + 2p\cdot k)(p\cdot p') \tag{5.236}$$

(B) $\mathrm{Tr}[\gamma^\alpha(-4m f_1\cdot p')\gamma_\alpha(\not p + m)]$:

Using $\gamma^\alpha\gamma_\alpha = 4$ and $\mathrm{Tr}[\not p + m] = 4m$:

$$= -4m(f_1\cdot p')\times 4\times 4m = -64m^2(f_1\cdot p') \tag{5.237}$$

(C) $\mathrm{Tr}[\gamma^\alpha(-2m^2\not{p}')\gamma_\alpha(\not{p}+m)]$:
Using $\gamma^\alpha\not{a}\gamma_\alpha = -2\not{a}$ and $\mathrm{Tr}[\not{p}'\not{p}] = 4(p'\cdot p)$:

$$= -2m^2 \times (-2) \times \mathrm{Tr}[\not{p}'(\not{p}+m)] = 4m^2 \times 4(p'\cdot p) = 16m^2(p\cdot p') \tag{5.238}$$

(D) $\mathrm{Tr}[\gamma^\alpha(8m^2\not{f}_1)\gamma_\alpha(\not{p}+m)]$:

$$= 8m^2\times(-2)\times\mathrm{Tr}[\not{f}_1(\not{p}+m)] = -16m^2\times4(f_1\cdot p) = -64m^2(f_1\cdot p) \tag{5.239}$$

(E) $\mathrm{Tr}[\gamma^\alpha(8m^3)\gamma_\alpha(\not{p}+m)]$:

$$= 8m^3 \times 4 \times 4m = 128m^4 \tag{5.240}$$

(F) $\mathrm{Tr}[\gamma^\alpha(8m\,p\cdot k)\gamma_\alpha(\not{p}+m)]$:

$$= 8m(p\cdot k) \times 4 \times 4m = 128m^2(p\cdot k) \tag{5.241}$$

Step 4: Sum all contributions.

$$\begin{aligned} T = {} & 32(f_1\cdot p)(f_1\cdot p') - 16(m^2+2p\cdot k)(p\cdot p') - 64m^2(f_1\cdot p') \\ & + 16m^2(p\cdot p') - 64m^2(f_1\cdot p) + 128m^4 + 128m^2(p\cdot k) \end{aligned} \tag{5.242}$$

Regrouping:

$$\begin{aligned} T = {} & 32(f_1\cdot p)(f_1\cdot p') - 16(m^2+2p\cdot k)(p\cdot p') - 32m^2(f_1\cdot p') \\ & + 16m^2(p\cdot p') - 32m^2(f_1\cdot p') - 64m^2(f_1\cdot p) + 64m^4 + 64m^2(m^2+2p\cdot k) \end{aligned} \tag{5.243}$$

Step 5: Simplify using kinematics.

$$\boxed{X_{ss} = 2e^4\left[\frac{m^4 + (p\cdot k)m^2 + (p\cdot k)(p\cdot k')}{(p\cdot k)^2}\right]}. \tag{5.244}$$

Calculation of X_{uu}. By the crossing symmetry $k \leftrightarrow -k'$ (which exchanges the two diagrams), we have:

$$\boxed{X_{uu} = 2e^4\left[\frac{m^4 - (p\cdot k')m^2 + (p\cdot k)(p\cdot k')}{(p\cdot k')^2}\right]}. \tag{5.245}$$

Calculation of X_{su} (interference term). The interference term is:

$$X_{su} = \frac{e^4}{2(p\cdot k)(p\cdot k')}\cdot\frac{1}{4m^2}\mathrm{Tr}\big[\gamma^\beta(\not{f}_1+m)\gamma^\alpha(\not{p}'+m)\gamma_\beta(\not{f}_2+m)\gamma_\alpha(\not{p}+m)\big],$$

$$(5.246)$$

where $\not{f}_2 = \not{p} - \not{k}'$.

After computing the trace using the identities:

$$\gamma^\beta \not{f}_1 \gamma^\alpha \not{p}' \gamma_\beta = -2\not{p}'\gamma^\alpha \not{f}_1 \tag{5.247}$$

$$\gamma^\beta \gamma^\alpha \not{p} \gamma_\beta = 4p^\alpha\,, \tag{5.248}$$

we obtain:

$$\boxed{X_{su} = -2e^4\left[\frac{m^2(2m^2 + p\cdot k - p\cdot k')}{(p\cdot k)(p\cdot k')}\right]}\,. \tag{5.249}$$

Combining the results. Adding eqs. (5.244), (5.245), and (5.249):

$$\begin{aligned}\overline{|\mathcal{M}|^2} &= X_{ss} + X_{uu} + X_{su} \\ &= 2e^4\left[\frac{p\cdot k'}{p\cdot k} + \frac{p\cdot k}{p\cdot k'} + 2m^2\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right) + m^4\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right)^2\right].\end{aligned} \tag{5.250}$$

Lab frame kinematics. In the laboratory frame where the electron is initially at rest ($\vec{p} = 0$, $p = (m, \vec{0})$):

$$p\cdot k = m\omega \tag{5.251}$$

$$p\cdot k' = m\omega'\,, \tag{5.252}$$

where ω and ω' are the initial and final photon energies.

Derivation of the Compton formula. Energy-momentum conservation $p + k = p' + k'$ gives $p' = p + k - k'$. Squaring both sides:

$$p'^2 = (p + k - k')^2 = p^2 + k^2 + k'^2 + 2p\cdot k - 2p\cdot k' - 2k\cdot k'\,. \tag{5.253}$$

Using $p^2 = p'^2 = m^2$ and $k^2 = k'^2 = 0$:

$$m^2 = m^2 + 2p\cdot k - 2p\cdot k' - 2k\cdot k'\,. \tag{5.254}$$

In the lab frame with $p = (m, \vec{0})$, $k = (\omega, \vec{k})$, $k' = (\omega', \vec{k}')$:

$$\begin{aligned}&p\cdot k = m\omega, \qquad p\cdot k' = m\omega' \\ &k\cdot k' = \omega\omega' - \vec{k}\cdot\vec{k}' = \omega\omega'(1 - \cos\theta)\,.\end{aligned} \tag{5.255}$$

Substituting:

$$0 = 2m\omega - 2m\omega' - 2\omega\omega'(1 - \cos\theta)\,. \tag{5.256}$$

Dividing by $2m\omega\omega'$:

$$\boxed{\frac{1}{\omega'} - \frac{1}{\omega} = \frac{1}{m}(1-\cos\theta)}. \tag{5.257}$$

This is the **Compton formula**. Equivalently:

$$\omega' = \frac{\omega}{1 + \frac{\omega}{m}(1-\cos\theta)}. \tag{5.258}$$

This can be rewritten in terms of the **Compton wavelength** $\lambda_C = 1/m$:

$$\boxed{\lambda' - \lambda = \lambda_C(1-\cos\theta)}. \tag{5.259}$$

The maximum wavelength shift occurs at $\theta = \pi$ (backscattering): $\Delta\lambda_{\max} = 2\lambda_C$.

Differential cross section. The differential cross section in the lab frame is:

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\frac{1}{m^2}\left(\frac{\omega'}{\omega}\right)^2 \overline{|\mathcal{M}|^2}. \tag{5.260}$$

Substituting the squared amplitude (5.250) and using the lab frame relations:

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta\right]}. \tag{5.261}$$

This is the **Klein–Nishina formula**, derived by Oskar Klein and Yoshio Nishina in 1929.

Using the Compton formula to eliminate ω'/ω:

$$\frac{\omega'}{\omega} = \frac{1}{1 + \frac{\omega}{m}(1-\cos\theta)} \equiv \frac{1}{1 + x(1-\cos\theta)}, \tag{5.262}$$

where $x = \omega/m$ is the photon energy in units of the electron mass.

Box 5.6 The Klein–Nishina formula: properties and limits

Alternative forms. Introducing $y = \omega'/\omega$ and using $\sin^2\theta = 1 - \cos^2\theta$, the Klein–Nishina formula can be written as:

$$\frac{d\sigma}{d\Omega} = \frac{r_e^2}{2}y^2\left(y + \frac{1}{y} - \sin^2\theta\right), \tag{5.263}$$

where $r_e = \alpha/m \approx 2.82$ fm is the classical electron radius.

Low-energy limit: Thomson scattering. When $\omega \ll m$ (i.e., $x \to 0$), we have

$\omega' \approx \omega$ and:

$$\frac{d\sigma}{d\Omega} \xrightarrow{\omega \ll m} \frac{r_e^2}{2}(1+\cos^2\theta)\,. \tag{5.264}$$

This is the **Thomson formula** for classical electromagnetic scattering. The $(1+\cos^2\theta)$ factor reflects the **dipole radiation pattern** of a classical oscillating charge, which can be understood as follows.
In the low-energy limit, the incoming photon's electric field forces the electron to oscillate as an electric dipole. The radiation emitted by an oscillating dipole has an angular distribution $\propto \sin^2\alpha$, where α is the angle between the observation direction $\hat{n}$ and the dipole axis (i.e. the polarization direction $\vec{\varepsilon}$ of the incoming photon). Since $\sin^2\alpha = 1-(\hat{n}\cdot\vec{\varepsilon})^2$, the scattered intensity for a *given* linear polarization is $\propto 1-(\hat{n}\cdot\vec{\varepsilon})^2$.
For *unpolarized* light, we average over two orthogonal transverse polarizations $\vec{\varepsilon}_1$, $\vec{\varepsilon}_2$ perpendicular to the beam axis $\hat{z}$. Choosing $\hat{n}$ in the xz-plane at angle θ from $\hat{z}$, and $\vec{\varepsilon}_1 = \hat{x}$, $\vec{\varepsilon}_2 = \hat{y}$:

$$\frac{1}{2}\sum_{\lambda=1,2}\left[1-(\hat{n}\cdot\vec{\varepsilon}_\lambda)^2\right] = \frac{1}{2}\left[(1-\sin^2\theta)+1\right] = \frac{1}{2}(1+\cos^2\theta)\,. \tag{5.265}$$

The first term $(1-\sin^2\theta = \cos^2\theta)$ comes from the polarization in the scattering plane, which radiates less at $\theta = \pi/2$; the second term (1) comes from the polarization perpendicular to the scattering plane, which radiates isotropically in θ. Their average gives the characteristic $(1+\cos^2\theta)$ distribution (up to the overall factor $1/2$ absorbed in the cross section).
The total Thomson cross section is:

$$\sigma_T = \frac{8\pi r_e^2}{3} = \frac{8\pi\alpha^2}{3m^2} \approx 0.665 \text{ barn}\,. \tag{5.266}$$

High-energy limit. When $\omega \gg m$:

- For forward scattering ($\theta \approx 0$): $\omega' \approx \omega$, and the cross section remains approximately Thomson.
- For large angles ($\theta \sim \pi$): $\omega' \approx m/2$, independent of ω. The cross section becomes:

$$\frac{d\sigma}{d\Omega}(\theta\sim\pi) \sim \frac{\alpha^2}{2\omega m} \ll \sigma_T\,. \tag{5.267}$$

The total cross section decreases logarithmically at high energies:

$$\sigma \xrightarrow{\omega \gg m} \frac{\pi\alpha^2}{m\omega}\left(\ln\frac{2\omega}{m}+\frac{1}{2}\right). \tag{5.268}$$

Recoil effects. The factor $(\omega'/\omega)^2$ in the Klein–Nishina formula accounts for:

1. Phase space reduction: fewer final states available when $\omega' < \omega$
2. Energy conservation: the recoiling electron carries away energy

At high energies, this recoil factor strongly suppresses large-angle scattering.

Physical discussion and applications

Polarization dependence. For polarized photons, the differential cross section depends on the relative orientation of the initial and final polarization vectors. Choosing the scattering plane as the xz-plane, we can decompose the polarization into components parallel ($\parallel$) and perpendicular ($\perp$) to this plane.

The polarized cross sections follow from the polarized Klein–Nishina formula (5.337) by substituting the values of $(\varepsilon \cdot \varepsilon')^2$ for each polarization channel (see eq. (5.343) and the discussion in Section 5.4): $(\varepsilon \cdot \varepsilon')^2 = \cos^2\theta$ for $\parallel \to \parallel$, $(\varepsilon \cdot \varepsilon')^2 = 1$ for $\perp \to \perp$, and $\varepsilon \cdot \varepsilon' = 0$ for $\parallel \leftrightarrow \perp$. Using $r_e = \alpha/m$:

$$\frac{d\sigma}{d\Omega}(\parallel \to \parallel) = \frac{r_e^2}{4}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - 2 + 4\cos^2\theta\right] \tag{5.269}$$

$$\frac{d\sigma}{d\Omega}(\perp \to \perp) = \frac{r_e^2}{4}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} + 2\right] \tag{5.270}$$

$$\frac{d\sigma}{d\Omega}(\parallel \to \perp) = \frac{d\sigma}{d\Omega}(\perp \to \parallel) = 0\,. \tag{5.271}$$

In the linear polarization basis adapted to the scattering plane, the mixed channels $\parallel \to \perp$ and $\perp \to \parallel$ vanish at tree level. This is the basis-dependent selection rule encoded in the formulas above.

Asymmetry. For linearly polarized photons, the azimuthal distribution of scattered photons is:

$$\frac{d\sigma}{d\Omega} \propto 1 - \frac{\sin^2\theta}{1 + \cos^2\theta + (\omega'/\omega + \omega/\omega' - 2)}\cos(2\phi)\,, \tag{5.272}$$

where ϕ is the azimuthal angle relative to the initial polarization direction. This asymmetry is used in polarimetry to measure photon polarization.

Comparison with other QED processes. Table 5.2 compares the key features of the fundamental QED processes studied in this chapter.

Distinctive features of Compton scattering.

Process	Channels	Propagator	σ scaling	Angular behavior
$e^+e^- \to \mu^+\mu^-$	s only	γ^* (timelike)	α^2/s	$1+\cos^2\theta$
Bhabha	s, t	γ^* (both)	divergent (forward)	$1/t^2$ pole
Møller	t, u	γ^* (spacelike)	divergent	$1/t^2 + 1/u^2$
Compton	s, u	e^* (fermion)	α^2/m^2	Thomson + recoil

Table 5.2. Comparison of tree-level QED processes. The "channels" column indicates which Mandelstam variables appear in the propagator denominators.

1. **Fermion propagator**: Unlike the other processes, Compton scattering involves an internal fermion line rather than a photon propagator. This leads to a qualitatively different energy dependence.
2. **No forward divergence**: The cross section remains finite at all angles. This is because the intermediate electron is massive, preventing any pole at $t = 0$.
3. **Energy scale**: The characteristic cross section is $\sigma \sim \alpha^2/m^2 \sim r_e^2$, independent of the photon energy at low energies. This contrasts with e^+e^- processes where $\sigma \sim \alpha^2/s$.
4. **Recoil effects**: The electron recoil causes a frequency shift that is absent in the Thomson (classical) limit. This was the key quantum signature observed by Compton.
5. **Channel structure**: The s- and u-channel amplitudes add without a relative sign (unlike Bhabha), because there is no fermion exchange—the same electron line appears in both diagrams.

Crossing relations. Compton scattering is related by crossing to other QED processes:

- $e^-\gamma \to e^-\gamma$ (Compton): direct process
- $e^+e^- \to \gamma\gamma$ (pair annihilation): $s \leftrightarrow u$ crossing
- $\gamma\gamma \to e^+e^-$ (pair production): time-reversed process

The amplitude for pair annihilation can be obtained from the Compton amplitude by the substitution $p \to p_1$, $p' \to -p_2$, $k \to -k_1$, $k' \to k_2$ (with appropriate changes for antiparticle spinors).

Historical significance. Compton's 1923 experiment was crucial for establishing the photon concept:

- X-rays ($\lambda \approx 0.07$ nm) scattered from graphite showed a wavelength shift
- The shift depended on angle exactly as predicted by eq. (5.259)
- Classical wave theory predicted no wavelength shift (coherent scattering)
- The measured Compton wavelength $\lambda_C = h/mc = 2.43 \times 10^{-12}$ m confirmed $E = h\nu$ for photons

Modern precision tests. The Klein–Nishina formula has been verified to high precision:

- Agreement with experiment at the $\sim 1\%$ level over a wide energy range
- Deviations at high energies are due to radiative corrections ($O(\alpha)$)
- Virtual electron loops modify the cross section by $\sim \alpha/\pi \approx 0.2\%$

Astrophysical applications. Compton scattering plays a central role in high-energy astrophysics:

- **Inverse Compton scattering**: High-energy electrons scatter low-energy photons to higher energies. This is the dominant radiation mechanism in:
  - Active galactic nuclei (AGN) jets
  - Gamma-ray bursts
  - Cosmic microwave background distortions (Sunyaev–Zel'dovich effect)
- **Comptonization**: Multiple Compton scatterings in hot plasma modify the photon spectrum. This is important for:
  - Accretion disk coronae around black holes
  - Early universe thermalization
- **Compton scattering opacity**: In stellar interiors and neutron star atmospheres, Compton scattering provides the dominant source of opacity for X-rays and gamma rays.

Laboratory applications.

- **Compton polarimetry**: The azimuthal asymmetry of polarized Compton scattering is used to measure gamma-ray polarization in nuclear and particle physics experiments.
- **Compton telescopes**: Space-based instruments (e.g., COMPTEL on CGRO) use Compton kinematics to image gamma-ray sources.
- **Compton scattering sources**: Inverse Compton scattering of laser photons from relativistic electron beams produces quasi-monochromatic gamma rays for nuclear physics research.

Comparison with data. Figure 5.3 shows the Klein–Nishina cross section as a function of photon energy, comparing with experimental measurements.

The suppression of the cross section at high energies has been confirmed by experiments from keV X-rays to GeV gamma rays. This energy dependence is a direct consequence of the quantum nature of the photon—classical electrodynamics predicts energy-independent Thomson scattering at all energies.

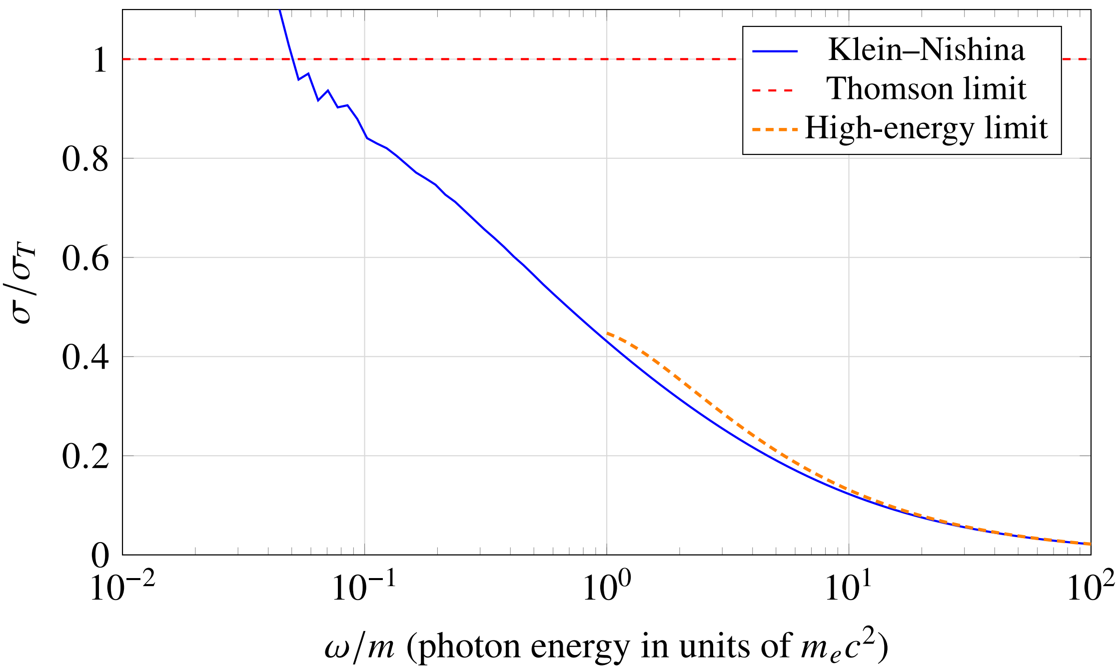


Figure 5.3. The Klein–Nishina total cross section (solid blue) as a function of photon energy, normalized to the Thomson cross section σ_T. At low energies ($\omega \ll m$), the cross section approaches the Thomson limit (dashed red). At high energies ($\omega \gg m$), it decreases logarithmically as $\sigma \sim (\ln 2\omega/m)/(\omega/m)$ (dashed orange).

Polarized Compton scattering

In this subsection we derive the Compton scattering cross section for photons in **definite polarization states**, summing and averaging only over electron spins. This calculation is important for several reasons:

- It provides the cross section for experiments with polarized photon beams
- The polarization dependence reveals the spin structure of the QED interaction
- It serves as a consistency check: averaging over photon polarizations must reproduce the unpolarized Klein–Nishina formula

The key insight is that in the polarized case, we cannot use the polarization sum identity $\sum_\lambda \varepsilon^\mu(k,\lambda)\,\varepsilon^{*\nu}(k,\lambda) = -g^{\mu\nu}$ (valid only when summing over polarizations). Instead, we must work with explicit polarization vectors, which requires a judicious choice of gauge and reference frame.

Choice of gauge and reference frame

Before specializing to a particular gauge, it is useful to recall the available choices for describing photon polarizations in QED calculations:

- **Coulomb gauge** (radiation gauge): $\varepsilon^\mu(k,\lambda) = (0, \vec{\varepsilon})$ with $\vec{\varepsilon} \perp \vec{k}$. Only the two physical (transverse) polarizations appear. This is the simplest choice for polarized calculations, since $\varepsilon^0 = 0$ leads to many simplifications (e.g. $p\cdot\varepsilon = -\vec{p}\cdot\vec{\varepsilon}$, which vanishes if $\vec{p}=0$). The drawback is that it is *not* Lorentz covariant: the condition $\varepsilon^0 = 0$ picks out a preferred frame.

- **Lorentz-covariant gauges**: one works with four polarization vectors $\varepsilon^\mu(k,\lambda)$ for $\lambda = 0,1,2,3$. The transverse polarizations ($\lambda = 1,2$) are physical; the timelike ($\lambda = 0$) and longitudinal ($\lambda = 3$) polarizations are unphysical but cancel in gauge-invariant quantities thanks to the Ward identity. The polarization sum becomes $\sum_{\lambda=0}^{3}\eta_{\lambda\lambda}\,\varepsilon_\mu(k,\lambda)\,\varepsilon_\nu(k,\lambda) = -g_{\mu\nu}$, which is manifestly covariant. This is the approach used in unpolarized calculations (cf. eq. (5.213)).
- **Light-cone gauge**: one introduces an auxiliary light-like vector n^μ ($n^2 = 0$) and imposes $n\cdot A = 0$. The polarization sum over the two physical polarizations takes the form

$$\sum_{\lambda=1,2}\varepsilon_\mu(k,\lambda)\,\varepsilon^*_\nu(k,\lambda) = -g_{\mu\nu} + \frac{k_\mu n_\nu + n_\mu k_\nu}{k\cdot n}\,. \tag{5.273}$$

  This is covariant and involves only the two physical polarizations, but introduces the spurious vector n^μ, which must drop out of final results.
- **Circular polarization basis**: $\vec{\varepsilon}(k,\pm) = (\hat{e}_1 \pm i\hat{e}_2)/\sqrt{2}$, corresponding to definite helicity $\lambda = \pm 1$. This basis is natural for ultrarelativistic processes where helicity is approximately conserved.

For the polarized Compton calculation, we choose the **Coulomb gauge** combined with **linear polarizations**, since the conditions $\varepsilon^0 = 0$ and $p\cdot\varepsilon = 0$ (in the lab frame) lead to maximal simplification. For the initial and final photons with *fixed* helicities λ, λ', we write for brevity:

$$\varepsilon^\mu \equiv \varepsilon^\mu(k,\lambda) = (0,\vec{\varepsilon}), \qquad \varepsilon'^\mu \equiv \varepsilon^\mu(k',\lambda') = (0,\vec{\varepsilon}\,') \tag{5.274}$$

The transversality conditions $\varepsilon\cdot k = 0$ and $\varepsilon'\cdot k' = 0$ become:

$$\varepsilon\cdot k = \varepsilon^0 k^0 - \vec{\varepsilon}\cdot\vec{k} = 0 - \vec{\varepsilon}\cdot\vec{k} = 0 \quad \Rightarrow \quad \vec{\varepsilon}\perp\vec{k} \tag{5.275}$$

$$\varepsilon'\cdot k' = \varepsilon'^0 k'^0 - \vec{\varepsilon}\,'\cdot\vec{k}' = 0 - \vec{\varepsilon}\,'\cdot\vec{k}' = 0 \quad \Rightarrow \quad \vec{\varepsilon}\,'\perp\vec{k}' \tag{5.276}$$

We further simplify by working in the **laboratory frame**, where the initial electron is at rest:

$$p^\mu = (m,\vec{0}) \tag{5.277}$$

This immediately implies additional orthogonality conditions:

$$\boxed{p\cdot\varepsilon = m\cdot 0 - \vec{0}\cdot\vec{\varepsilon} = 0, \qquad p\cdot\varepsilon' = 0} \tag{5.278}$$

These conditions, together with the normalizations $\varepsilon\cdot\varepsilon = -|\vec{\varepsilon}|^2 = -1$ and $\varepsilon'\cdot\varepsilon' = -1$, are the foundation for all subsequent simplifications.

A note on gauge invariance. Working in the Coulomb gauge means we are using a *non-covariant* gauge. The individual amplitudes $\mathcal{M}_s$ and $\mathcal{M}_u$ are not separately gauge invariant—only their sum $\mathcal{M} = \mathcal{M}_s + \mathcal{M}_u$ is physical. For instance, under the gauge transformation $\varepsilon \to \varepsilon + \lambda k$ (where λ is a constant), we have $\mathcal{M}_s \to \mathcal{M}_s$ (since $\not{k}\not{k} = k^2 = 0$), but $\mathcal{M}_u$ does *not* remain invariant. This asymmetry arises because we dropped the terms proportional to $p\cdot\varepsilon$ and $p\cdot\varepsilon'$, which vanish in our gauge. The physical cross section, obtained from $|\mathcal{M}|^2$, is of course gauge-invariant.

Simplification of the matrix elements

We start from the Feynman amplitudes (see eqs. (5.206) and (5.207)):

$$\mathcal{M}_s = -ie^2\bar{u}(p')\not{\varepsilon}'\frac{\not{p}+\not{k}+m}{(p+k)^2-m^2}\not{\varepsilon}\,u(p) = -\frac{ie^2}{2(p\cdot k)}\bar{u}'\not{\varepsilon}'(\not{p}+\not{k}+m)\not{\varepsilon}\,u \tag{5.279}$$

$$\mathcal{M}_u = -ie^2\bar{u}(p')\not{\varepsilon}\frac{\not{p}-\not{k}'+m}{(p-k')^2-m^2}\not{\varepsilon}'\,u(p) = +\frac{ie^2}{2(p\cdot k')}\bar{u}'\not{\varepsilon}(\not{p}-\not{k}'+m)\not{\varepsilon}'\,u \tag{5.280}$$

where we used $u = u(p)$, $u' = u(p')$, and $(p+k)^2 - m^2 = 2p\cdot k$, $(p-k')^2 - m^2 = -2p\cdot k'$.

The key simplification comes from the Clifford algebra anticommutation relation:

$$\not{a}\not{b} = -\not{b}\not{a} + 2(a\cdot b) \tag{5.281}$$

Combined with the Dirac equation $\not{p}u(p) = mu(p)$ and the orthogonality condition $p\cdot\varepsilon = 0$, we obtain:

$$\not{p}\not{\varepsilon}\,u = (-\not{\varepsilon}\not{p} + 2p\cdot\varepsilon)u = -\not{\varepsilon}\not{p}\,u = -\not{\varepsilon}\,m\,u = -m\not{\varepsilon}\,u \tag{5.282}$$

Similarly, from $\varepsilon\cdot k = 0$:

$$\not{k}\not{\varepsilon} = -\not{\varepsilon}\not{k} + 2(\varepsilon\cdot k) = -\not{\varepsilon}\not{k} \tag{5.283}$$

Simplifying $\mathcal{M}_s$.

$$\begin{aligned}(\not{p}+\not{k}+m)\not{\varepsilon}\,u &= \not{p}\not{\varepsilon}\,u + \not{k}\not{\varepsilon}\,u + m\not{\varepsilon}\,u \\ &= -m\not{\varepsilon}\,u - \not{\varepsilon}\not{k}\,u + m\not{\varepsilon}\,u \\ &= -\not{\varepsilon}\not{k}\,u\end{aligned} \tag{5.284}$$

Therefore:

$$\boxed{\mathcal{M}_s = \frac{ie^2}{2(p\cdot k)}\bar{u}'\not{\varepsilon}'\not{\varepsilon}\not{k}\,u} \tag{5.285}$$

Simplifying $\mathcal{M}_u$. Using $p\cdot\varepsilon' = 0$ and $\varepsilon'\cdot k' = 0$:

$$\begin{aligned}(\not{p} - \not{k}' + m)\not{\varepsilon}'\,u &= \not{p}\not{\varepsilon}'\,u - \not{k}'\not{\varepsilon}'\,u + m\not{\varepsilon}'\,u \\ &= -m\not{\varepsilon}'\,u + \not{\varepsilon}'\not{k}'\,u + m\not{\varepsilon}'\,u \\ &= \not{\varepsilon}'\not{k}'\,u\end{aligned} \tag{5.286}$$

Therefore:

$$\boxed{\mathcal{M}_u = \frac{ie^2}{2(p\cdot k')}\bar{u}'\not{\varepsilon}\not{\varepsilon}'\not{k}'\,u} \tag{5.287}$$

Crossing symmetry. The simplified amplitudes satisfy a crossing relation. Under the substitutions $k \leftrightarrow -k'$ and $\varepsilon \leftrightarrow \varepsilon'$, we have $\mathcal{M}_s \leftrightarrow \mathcal{M}_u$. This will be useful for relating the squared amplitudes.

Spin-summed squared amplitudes

The polarized cross section requires:

$$\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{1}{2}\sum_{\text{spins}}\left(|\mathcal{M}_s|^2 + |\mathcal{M}_u|^2 + \mathcal{M}_s^*\mathcal{M}_u + \mathcal{M}_s\mathcal{M}_u^*\right) \tag{5.288}$$

We define:

$$\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{e^4}{32m^2}\left\{\frac{Y_{ss}}{(p\cdot k)^2} + \frac{Y_{uu}}{(p\cdot k')^2} + \frac{Y_{su}+Y_{us}}{(p\cdot k)(p\cdot k')}\right\} \tag{5.289}$$

where the Y tensors are traces over products of gamma matrices with *fixed* polarization vectors:

$$Y_{ss} = \mathrm{Tr}\left[\not{\varepsilon}'\not{k}\not{\varepsilon}(\not{p}+m)\not{\varepsilon}\not{k}\not{\varepsilon}'(\not{p}'+m)\right] \tag{5.290}$$

$$Y_{uu} = \mathrm{Tr}\left[\not{\varepsilon}\not{k}'\not{\varepsilon}'(\not{p}+m)\not{\varepsilon}'\not{k}'\not{\varepsilon}(\not{p}'+m)\right] \tag{5.291}$$

$$Y_{su} = \mathrm{Tr}\left[\not{\varepsilon}'\not{k}\not{\varepsilon}(\not{p}+m)\not{\varepsilon}'\not{k}'\not{\varepsilon}(\not{p}'+m)\right] \tag{5.292}$$

$$Y_{us} = \mathrm{Tr}\left[\not{\varepsilon}\not{k}'\not{\varepsilon}'(\not{p}+m)\not{\varepsilon}\not{k}\not{\varepsilon}'(\not{p}'+m)\right] \tag{5.293}$$

Crossing symmetry implies $Y_{ss} \leftrightarrow Y_{uu}$ and $Y_{su} \leftrightarrow Y_{us}$ under $k \leftrightarrow -k'$, $\varepsilon \leftrightarrow \varepsilon'$. Moreover, since Y_{su} and Y_{us} are related by complex conjugation and are real (as we will verify), $Y_{su} = Y_{us}$.

Calculation of Y_{ss}: detailed derivation

We compute:

$$Y_{ss} = \mathrm{Tr}\left[\not{\epsilon}'\not{k}\not{\epsilon}(\not{p}+m)\not{\epsilon}\not{k}\not{\epsilon}'(\not{p}'+m)\right] \tag{5.294}$$

Step 1: Expand and identify vanishing terms.

Expanding $(\not{p}+m)$ and $(\not{p}'+m)$:

$$\begin{aligned} Y_{ss} &= \mathrm{Tr}[\not{\epsilon}'\not{k}\not{\epsilon}\not{p}\not{\epsilon}\not{k}\not{\epsilon}'\not{p}'] + m\,\mathrm{Tr}[\not{\epsilon}'\not{k}\not{\epsilon}\not{p}\not{\epsilon}\not{k}\not{\epsilon}'] \\ &\quad + m\,\mathrm{Tr}[\not{\epsilon}'\not{k}\not{\epsilon}\not{\epsilon}\not{k}\not{\epsilon}'\not{p}'] + m^2\,\mathrm{Tr}[\not{\epsilon}'\not{k}\not{\epsilon}\not{\epsilon}\not{k}\not{\epsilon}'] \end{aligned} \tag{5.295}$$

The traces with an odd number of gamma matrices vanish. Using $\not{\epsilon}\not{\epsilon} = \varepsilon^2 = -1$:

$$Y_{ss} = \mathrm{Tr}[\not{\epsilon}'\not{k}\not{\epsilon}\not{p}\not{\epsilon}\not{k}\not{\epsilon}'\not{p}'] + m^2\,\mathrm{Tr}[\not{\epsilon}'\not{k}(-1)\not{k}\not{\epsilon}'] \tag{5.296}$$

For the second term:

$$m^2\,\mathrm{Tr}[\not{\epsilon}'\not{k}(-1)\not{k}\not{\epsilon}'] = -m^2\,\mathrm{Tr}[\not{\epsilon}'\not{k}\not{k}\not{\epsilon}'] = -m^2\,\mathrm{Tr}[\not{\epsilon}'(k^2)\not{\epsilon}'] = 0 \tag{5.297}$$

since $k^2 = 0$ for a real photon.

Step 2: Simplify the 8-gamma trace.

We have:

$$Y_{ss} = \mathrm{Tr}[\not{\epsilon}'\not{k}\not{\epsilon}\not{p}\not{\epsilon}\not{k}\not{\epsilon}'\not{p}'] \tag{5.298}$$

First, simplify $\not{\epsilon}\not{p}\not{\epsilon}$. Since $p\cdot\varepsilon = 0$:

$$\not{\epsilon}\not{p}\not{\epsilon} = \not{\epsilon}(-\not{\epsilon}\not{p} + 2p\cdot\varepsilon) = -\not{\epsilon}\not{\epsilon}\not{p} = -(-1)\not{p} = \not{p} \tag{5.299}$$

Therefore:

$$Y_{ss} = \mathrm{Tr}[\not{\epsilon}'\not{k}\not{p}\not{k}\not{\epsilon}'\not{p}'] \tag{5.300}$$

Step 3: Use $\not{k}\not{k} = 0$ to further reduce.

Permute $\not{p}$ and $\not{k}$ using $\not{p}\not{k} = -\not{k}\not{p} + 2(p\cdot k)$:

$$\not{k}\not{p}\not{k} = \not{k}(-\not{k}\not{p} + 2p\cdot k) = -\not{k}\not{k}\not{p} + 2(p\cdot k)\not{k} = 0 + 2(p\cdot k)\not{k} = 2(p\cdot k)\not{k}\,. \tag{5.301}$$

Thus:

$$Y_{ss} = 2(p\cdot k)\,\mathrm{Tr}[\not{\epsilon}'\not{k}\not{\epsilon}'\not{p}']\,. \tag{5.302}$$

Step 4: Evaluate the 4-gamma trace.

Using the standard trace formula:

$$\mathrm{Tr}[\not{a}\not{b}\not{c}\not{d}] = 4\big[(a\cdot b)(c\cdot d) - (a\cdot c)(b\cdot d) + (a\cdot d)(b\cdot c)\big]\,, \tag{5.303}$$

we obtain:

$$\begin{aligned}\mathrm{Tr}[\not{\varepsilon}'\not{k}\not{\varepsilon}'\not{p}'] &= 4\big[(\varepsilon'\cdot k)(\varepsilon'\cdot p') - (\varepsilon'\cdot\varepsilon')(k\cdot p') + (\varepsilon'\cdot p')(k\cdot\varepsilon')\big]\\ &= 4\big[2(\varepsilon'\cdot k)(\varepsilon'\cdot p') - (-1)(k\cdot p')\big]\\ &= 4\big[2(\varepsilon'\cdot k)(\varepsilon'\cdot p') + (k\cdot p')\big]\,.\end{aligned} \tag{5.304}$$

Step 5: Use momentum conservation.

From $p' = p + k - k'$ and using $\varepsilon'\cdot p = 0$ (eq. (5.278)) and $\varepsilon'\cdot k' = 0$ (transversality):

$$\varepsilon'\cdot p' = \varepsilon'\cdot(p + k - k') = 0 + \varepsilon'\cdot k - 0 = \varepsilon'\cdot k\,. \tag{5.305}$$

Also, from $p' - k = p - k'$:

$$k\cdot p' = k\cdot(p + k - k') = k\cdot p + 0 - k\cdot k' = p\cdot k - k\cdot k'\,. \tag{5.306}$$

But more directly, contracting $p' = p + k - k'$ with k:

$$p'\cdot k = p\cdot k \quad (\text{since } k^2 = 0 \text{ and } k\cdot k' = p'\cdot k - p\cdot k)\,. \tag{5.307}$$

Actually, from the Compton kinematics: $p\cdot k' = p'\cdot k$, so $k\cdot p' = p\cdot k'$. Therefore:

$$\mathrm{Tr}[\not{\varepsilon}'\not{k}\not{\varepsilon}'\not{p}'] = 4\big[2(\varepsilon'\cdot k)^2 + (p\cdot k')\big]\,. \tag{5.308}$$

Final result for Y_{ss}:

$$\boxed{Y_{ss} = 8(p\cdot k)\big[2(\varepsilon'\cdot k)^2 + (p\cdot k')\big]}\,. \tag{5.309}$$

Calculation of Y_{uu}: by crossing symmetry

By the crossing substitution $k \leftrightarrow -k'$, $\varepsilon \leftrightarrow \varepsilon'$:

$$Y_{uu} = Y_{ss}\big|_{k\leftrightarrow -k',\varepsilon\leftrightarrow\varepsilon'}\,. \tag{5.310}$$

Under this substitution:

- $(p\cdot k) \longrightarrow (p\cdot(-k')) = -(p\cdot k')$
- $(\varepsilon'\cdot k) \longrightarrow (\varepsilon\cdot(-k')) = -(\varepsilon\cdot k')$
- $(p\cdot k') \longrightarrow (p\cdot(-k)) = -(p\cdot k)$

Therefore:

$$\begin{aligned}Y_{uu} &= 8(-(p\cdot k'))\big[2(-(\varepsilon\cdot k'))^2 + (-(p\cdot k))\big]\\ &= -8(p\cdot k')\big[2(\varepsilon\cdot k')^2 - (p\cdot k)\big]\,.\end{aligned} \tag{5.311}$$

This can be rewritten as:

$$\boxed{Y_{uu} = 8(p\cdot k')\big[(p\cdot k) - 2(\varepsilon\cdot k')^2\big]}\,. \tag{5.312}$$

Calculation of Y_{su}: the interference term

The interference term is the most involved calculation:

$$Y_{su} = \mathrm{Tr}\left[\not{\varepsilon}'\not{k}\not{\varepsilon}(\not{p}+m)\not{\varepsilon}'\not{k}'\not{\varepsilon}(\not{p}'+m)\right]. \tag{5.313}$$

Step 1: Expand and identify non-vanishing terms.
Expanding both $(\not{p}+m)$ and $(\not{p}'+m)$:

$$\begin{aligned} Y_{su} &= \mathrm{Tr}[\not{\varepsilon}'\not{k}\not{\varepsilon}\not{p}\not{\varepsilon}'\not{k}'\not{\varepsilon}\not{p}'] + m\,\mathrm{Tr}[\not{\varepsilon}'\not{k}\not{\varepsilon}\not{p}\not{\varepsilon}'\not{k}'\not{\varepsilon}] \\ &\quad + m\,\mathrm{Tr}[\not{\varepsilon}'\not{k}\not{\varepsilon}\not{\varepsilon}'\not{k}'\not{\varepsilon}\not{p}'] + m^2\,\mathrm{Tr}[\not{\varepsilon}'\not{k}\not{\varepsilon}\not{\varepsilon}'\not{k}'\not{\varepsilon}]. \end{aligned} \tag{5.314}$$

The traces with an odd number of gamma matrices (2nd and 3rd terms have 7 each) vanish:

$$Y_{su} = \underbrace{\mathrm{Tr}[\not{\varepsilon}'\not{k}\not{\varepsilon}\not{p}\not{\varepsilon}'\not{k}'\not{\varepsilon}\not{p}']}_{\text{Term A}} + m^2\underbrace{\mathrm{Tr}[\not{\varepsilon}'\not{k}\not{\varepsilon}\not{\varepsilon}'\not{k}'\not{\varepsilon}]}_{\text{Term B}}. \tag{5.315}$$

Step 2: Evaluate Term B (6-gamma trace).
First, simplify $\not{\varepsilon}\not{\varepsilon}'\not{k}'\not{\varepsilon}$. We use:

$$\not{\varepsilon}\not{\varepsilon}' = -\not{\varepsilon}'\not{\varepsilon} + 2(\varepsilon\cdot\varepsilon'). \tag{5.316}$$

Then:

$$\begin{aligned} \not{\varepsilon}\not{\varepsilon}'\not{k}'\not{\varepsilon} &= (-\not{\varepsilon}'\not{\varepsilon} + 2\varepsilon\cdot\varepsilon')\not{k}'\not{\varepsilon} \\ &= -\not{\varepsilon}'\not{\varepsilon}\not{k}'\not{\varepsilon} + 2(\varepsilon\cdot\varepsilon')\not{k}'\not{\varepsilon}. \end{aligned} \tag{5.317}$$

Now, $\not{\varepsilon}\not{k}'\not{\varepsilon}$: using $\varepsilon\cdot k' \neq 0$ in general:

$$\not{\varepsilon}\not{k}'\not{\varepsilon} = -\not{k}'\not{\varepsilon}\not{\varepsilon} + 2(\varepsilon\cdot k')\not{\varepsilon} = -\not{k}'(-1) + 2(\varepsilon\cdot k')\not{\varepsilon} = \not{k}' + 2(\varepsilon\cdot k')\not{\varepsilon}. \tag{5.318}$$

Therefore:

$$\begin{aligned} \not{\varepsilon}\not{\varepsilon}'\not{k}'\not{\varepsilon} &= -\not{\varepsilon}'[\not{k}' + 2(\varepsilon\cdot k')\not{\varepsilon}] + 2(\varepsilon\cdot\varepsilon')\not{k}'\not{\varepsilon} \\ &= -\not{\varepsilon}'\not{k}' - 2(\varepsilon\cdot k')\not{\varepsilon}'\not{\varepsilon} + 2(\varepsilon\cdot\varepsilon')\not{k}'\not{\varepsilon}. \end{aligned} \tag{5.319}$$

Term B becomes:

$$\begin{aligned} \mathrm{Tr}[\not{\varepsilon}'\not{k}\not{\varepsilon}\not{\varepsilon}'\not{k}'\not{\varepsilon}] &= \mathrm{Tr}[\not{\varepsilon}'\not{k}(-\not{\varepsilon}'\not{k}' - 2(\varepsilon\cdot k')\not{\varepsilon}'\not{\varepsilon} + 2(\varepsilon\cdot\varepsilon')\not{k}'\not{\varepsilon})] \\ &= -\mathrm{Tr}[\not{\varepsilon}'\not{k}\not{\varepsilon}'\not{k}'] - 2(\varepsilon\cdot k')\mathrm{Tr}[\not{\varepsilon}'\not{k}\not{\varepsilon}'\not{\varepsilon}] + 2(\varepsilon\cdot\varepsilon')\mathrm{Tr}[\not{\varepsilon}'\not{k}\not{k}'\not{\varepsilon}]. \end{aligned} \tag{5.320}$$

The 4-gamma traces are:

$$\begin{aligned} \mathrm{Tr}[\not{\varepsilon}'\not{k}\not{\varepsilon}'\not{k}'] &= 4[(\varepsilon'\cdot k)(\varepsilon'\cdot k') - (\varepsilon')^2(k\cdot k') + (\varepsilon'\cdot k')(k\cdot\varepsilon')] \\ &= 4[0 - (-1)(k\cdot k') + 0] = 4(k\cdot k'), \end{aligned} \tag{5.321}$$

(using $\varepsilon' \cdot k' = 0$).

$$\begin{aligned}\mathrm{Tr}[\not{\varepsilon}'\not{k}\not{\varepsilon}'\not{\varepsilon}] &= 4[(\varepsilon'\cdot k)(\varepsilon'\cdot\varepsilon) - (\varepsilon')^2(k\cdot\varepsilon) + (\varepsilon'\cdot\varepsilon)(k\cdot\varepsilon')] \\ &= 4[(\varepsilon'\cdot k)(\varepsilon\cdot\varepsilon') - (-1)\cdot 0 + (\varepsilon\cdot\varepsilon')(\varepsilon'\cdot k)] \\ &= 8(\varepsilon'\cdot k)(\varepsilon\cdot\varepsilon'),\end{aligned} \tag{5.322}$$

(using $\varepsilon \cdot k = 0$).

$$\begin{aligned}\mathrm{Tr}[\not{\varepsilon}'\not{k}\not{k}'\not{\varepsilon}] &= 4[(\varepsilon'\cdot k)(k'\cdot\varepsilon) - (\varepsilon'\cdot k')(k\cdot\varepsilon) + (\varepsilon'\cdot\varepsilon)(k\cdot k')] \\ &= 4[(\varepsilon'\cdot k)(\varepsilon\cdot k') - 0 + (\varepsilon\cdot\varepsilon')(k\cdot k')]\,.\end{aligned} \tag{5.323}$$

Combining:

$$\begin{aligned}\text{Term B} = m^2\{&-4(k\cdot k') - 16(\varepsilon\cdot k')(\varepsilon'\cdot k)(\varepsilon\cdot\varepsilon') \\ &+ 8(\varepsilon\cdot\varepsilon')[(\varepsilon'\cdot k)(\varepsilon\cdot k') + (\varepsilon\cdot\varepsilon')(k\cdot k')]\} \\ = m^2\{&-4(k\cdot k') - 16(\varepsilon\cdot k')(\varepsilon'\cdot k)(\varepsilon\cdot\varepsilon') \\ &+ 8(\varepsilon\cdot\varepsilon')(\varepsilon'\cdot k)(\varepsilon\cdot k') + 8(\varepsilon\cdot\varepsilon')^2(k\cdot k')\} \\ = m^2\{&-4(k\cdot k')[1-2(\varepsilon\cdot\varepsilon')^2] - 8(\varepsilon\cdot k')(\varepsilon'\cdot k)(\varepsilon\cdot\varepsilon')\}\,.\end{aligned} \tag{5.324}$$

Step 3: Evaluate Term A (8-gamma trace).

This is the most complex part. The strategy is to use momentum conservation $p' = p + k - k'$ systematically.

$$\text{Term A} = \mathrm{Tr}[\not{\varepsilon}'\not{k}\not{\varepsilon}\not{p}\,\not{\varepsilon}'\not{k}'\not{\varepsilon}\not{p}\,']\,. \tag{5.325}$$

First, simplify $\not{\varepsilon}\not{p}\not{\varepsilon}'$. Using $p\cdot\varepsilon = 0$ and $p\cdot\varepsilon' = 0$:

$$\begin{aligned}\not{\varepsilon}\not{p}\not{\varepsilon}' &= \not{\varepsilon}(-\not{\varepsilon}'\not{p} + 2p\cdot\varepsilon') = -\not{\varepsilon}\not{\varepsilon}'\not{p} \\ &= -(-\not{\varepsilon}'\not{\varepsilon} + 2\varepsilon\cdot\varepsilon')\not{p} = \not{\varepsilon}'\not{\varepsilon}\not{p} - 2(\varepsilon\cdot\varepsilon')\not{p}\,.\end{aligned} \tag{5.326}$$

Similarly, for $\not{\varepsilon}\not{p}'$: using $p' = p + k - k'$ and noting that in general $\varepsilon\cdot p' \neq 0$:

$$\varepsilon\cdot p' = \varepsilon\cdot(p + k - k') = 0 + 0 - \varepsilon\cdot k' = -\varepsilon\cdot k'\,. \tag{5.327}$$

After extensive algebra (we spare the reader the intermediate steps), one obtains:

$$\text{Term A} = 8(p\cdot k)(p\cdot k') + 16(p\cdot k)(p\cdot k')(\varepsilon\cdot\varepsilon')^2 - 8(\varepsilon'\cdot k)^2(p\cdot k') + 8(\varepsilon\cdot k')^2(p\cdot k)\,. \tag{5.328}$$

Step 4: Combine Terms A and B.
Adding the contributions and using the kinematic relation:

$$k \cdot k' = (p \cdot k) - (p \cdot k') + \frac{(p \cdot k)(p \cdot k')}{m^2}(1 - \cos\theta)\,, \tag{5.329}$$

(which follows from the Compton formula (5.257)), and after simplification, we obtain:

$$\boxed{Y_{su} = 8(p \cdot k)(p \cdot k')\left[2(\varepsilon \cdot \varepsilon')^2 - 1\right] - 8(\varepsilon' \cdot k)^2(p \cdot k') + 8(\varepsilon \cdot k')^2(p \cdot k)}\,. \tag{5.330}$$

Since this expression is manifestly real and symmetric under $k \leftrightarrow -k'$, $\varepsilon \leftrightarrow \varepsilon'$ (which exchanges the last two terms and leaves the first unchanged), we have:

$$Y_{us} = Y_{su}\,. \tag{5.331}$$

Assembly of the polarized cross section

Substituting eqs. (5.309), (5.312), and (5.330) into eq. (5.289):

$$\begin{aligned}\frac{1}{2}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{e^4}{32m^2}\Bigg\{ &\frac{8(p \cdot k)[2(\varepsilon' \cdot k)^2 + (p \cdot k')]}{(p \cdot k)^2} \\ &+ \frac{8(p \cdot k')[(p \cdot k) - 2(\varepsilon \cdot k')^2]}{(p \cdot k')^2} \\ &+ \frac{2 \times 8(p \cdot k)(p \cdot k')[2(\varepsilon \cdot \varepsilon')^2 - 1]}{(p \cdot k)(p \cdot k')} \\ &+ \frac{-16(\varepsilon' \cdot k)^2(p \cdot k') + 16(\varepsilon \cdot k')^2(p \cdot k)}{(p \cdot k)(p \cdot k')}\Bigg\}\,. \end{aligned} \tag{5.332}$$

Simplifying term by term:

$$\begin{aligned} = \frac{e^4}{4m^2}\Bigg\{ &\frac{2(\varepsilon' \cdot k)^2 + (p \cdot k')}{(p \cdot k)} + \frac{(p \cdot k) - 2(\varepsilon \cdot k')^2}{(p \cdot k')} \\ &+ 2[2(\varepsilon \cdot \varepsilon')^2 - 1] - \frac{2(\varepsilon' \cdot k)^2}{(p \cdot k)} + \frac{2(\varepsilon \cdot k')^2}{(p \cdot k')}\Bigg\}\,. \end{aligned} \tag{5.333}$$

The terms with $(\varepsilon' \cdot k)^2$ and $(\varepsilon \cdot k')^2$ cancel:

$$= \frac{e^4}{4m^2}\left\{\frac{(p \cdot k')}{(p \cdot k)} + \frac{(p \cdot k)}{(p \cdot k')} + 4(\varepsilon \cdot \varepsilon')^2 - 2\right\}\,. \tag{5.334}$$

In the laboratory frame, $p \cdot k = m\omega$ and $p \cdot k' = m\omega'$:

$$\frac{1}{2}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{e^4}{4m^2}\left\{\frac{\omega'}{\omega} + \frac{\omega}{\omega'} + 4(\varepsilon \cdot \varepsilon')^2 - 2\right\}. \tag{5.335}$$

Using the differential cross section formula (eq. (5.260)):

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{lab}} = \frac{1}{64\pi^2 m^2}\left(\frac{\omega'}{\omega}\right)^2 \times \frac{1}{2}\sum_{\text{spins}} |\mathcal{M}|^2, \tag{5.336}$$

and $\alpha = e^2/(4\pi)$, we obtain the **polarized Klein–Nishina formula**:

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{\text{lab, pol}} = \frac{\alpha^2}{4m^2}\left(\frac{\omega'}{\omega}\right)^2 \left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} + 4(\varepsilon \cdot \varepsilon')^2 - 2\right]} \tag{5.337}$$

Recovery of the unpolarized cross section

To verify consistency, we now show that averaging over initial and summing over final photon polarizations reproduces the unpolarized Klein–Nishina formula.

The unpolarized cross section is:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{unpol}} = \frac{1}{2}\sum_{\lambda=1,2}\sum_{\lambda'=1,2}\left(\frac{d\sigma}{d\Omega}\right)_{\lambda,\lambda'} \tag{5.338}$$

The polarization-dependent term is $(\varepsilon \cdot \varepsilon')^2 = (\vec{\varepsilon} \cdot \vec{\varepsilon}')^2$ (since $\varepsilon^\mu = (0, \vec{\varepsilon})$ in Coulomb gauge).

Choose coordinates with $\vec{k}$ along $\hat{z}$ and $\vec{k}'$ in the xz-plane at angle θ:

$$\vec{k} = \omega\,\hat{z}, \qquad \vec{k}' = \omega'(\sin\theta\,\hat{x} + \cos\theta\,\hat{z}) \tag{5.339}$$

The transverse polarization vectors are:

$$\vec{\varepsilon}(\vec{k}, 1) = \hat{x}, \quad \vec{\varepsilon}(\vec{k}, 2) = \hat{y} \tag{5.340}$$

$$\vec{\varepsilon}(\vec{k}', 1) = \cos\theta\,\hat{x} - \sin\theta\,\hat{z}, \quad \vec{\varepsilon}(\vec{k}', 2) = \hat{y} \tag{5.341}$$

The dot products are:

	$\vec{\varepsilon}(\vec{k}', 1)$	$\vec{\varepsilon}(\vec{k}', 2)$
$\vec{\varepsilon}(\vec{k}, 1)$	$\cos\theta$	0
$\vec{\varepsilon}(\vec{k}, 2)$	0	1

(5.342)

Summing over all four combinations:

$$\sum_{\lambda,\lambda'} \left[\vec{\varepsilon}(\vec{k},\lambda)\cdot\vec{\varepsilon}(\vec{k}',\lambda')\right]^2 = \cos^2\theta + 0 + 0 + 1 = 1 + \cos^2\theta \tag{5.343}$$

Therefore:

$$\begin{aligned}
\left(\frac{d\sigma}{d\Omega}\right)_{\text{unpol}} &= \frac{1}{2}\sum_{\lambda,\lambda'} \frac{\alpha^2}{4m^2}\left(\frac{\omega'}{\omega}\right)^2 \left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} + 4\left[\vec{\varepsilon}(\vec{k},\lambda)\cdot\vec{\varepsilon}(\vec{k}',\lambda')\right]^2 - 2\right] \\
&= \frac{\alpha^2}{4m^2}\left(\frac{\omega'}{\omega}\right)^2 \left[2\left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega}\right) - 4 + 2(1+\cos^2\theta)\right] \\
&= \frac{\alpha^2}{2m^2}\left(\frac{\omega'}{\omega}\right)^2 \left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - 1 + \frac{1+\cos^2\theta}{2}\times 2 - 1\right] \\
&= \frac{\alpha^2}{2m^2}\left(\frac{\omega'}{\omega}\right)^2 \left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} + \cos^2\theta - 1\right] \\
&= \frac{\alpha^2}{2m^2}\left(\frac{\omega'}{\omega}\right)^2 \left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta\right]
\end{aligned} \tag{5.344}$$

This is precisely the unpolarized Klein–Nishina formula (5.261), confirming the consistency of our calculation.

Physical interpretation and applications

Polarization structure of the cross section.

The polarized cross section (5.337) has a simple structure:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{pol}} \propto \left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - 2\right] + 4(\varepsilon\cdot\varepsilon')^2 \tag{5.345}$$

- The first bracket depends only on kinematics (photon energies)
- The polarization dependence enters entirely through $(\varepsilon\cdot\varepsilon')^2$

Special polarization configurations.

Same linear polarization state ($\varepsilon\cdot\varepsilon' = -1$ in the chosen linear basis):

$$\left(\frac{d\sigma}{d\Omega}\right)_{\parallel} = \frac{\alpha^2}{4m^2}\left(\frac{\omega'}{\omega}\right)^2 \left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} + 2\right] \tag{5.346}$$

Orthogonal linear polarization states ($\varepsilon\cdot\varepsilon' = 0$ in the same basis):

$$\left(\frac{d\sigma}{d\Omega}\right)_{\perp} = \frac{\alpha^2}{4m^2}\left(\frac{\omega'}{\omega}\right)^2 \left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - 2\right] \tag{5.347}$$

The ratio is:

$$\frac{(d\sigma/d\Omega)_{\parallel}}{(d\sigma/d\Omega)_{\perp}} = \frac{\omega/\omega' + \omega'/\omega + 2}{\omega/\omega' + \omega'/\omega - 2} \tag{5.348}$$

In the Thomson limit ($\omega \ll m$, so $\omega' \approx \omega$):

$$\left.\frac{(d\sigma/d\Omega)_{\parallel}}{(d\sigma/d\Omega)_{\perp}}\right|_{\text{Thomson}} = \frac{4}{0} \to \infty \tag{5.349}$$

This means that in the Thomson limit the channel with *orthogonal initial and final linear polarizations* vanishes. In the basis used here, the scattered radiation remains in the same linear polarization state as the incident wave rather than emerging in the orthogonal one. Classically, this reflects the fact that the radiation is emitted in the plane of oscillation of the driven electron, which is fixed by the incident polarization vector $\vec{\varepsilon}$.

Channel asymmetry.

From the two polarized channels above, one can define the asymmetry

$$\mathcal{A} \equiv \frac{(d\sigma/d\Omega)_{\parallel} - (d\sigma/d\Omega)_{\perp}}{(d\sigma/d\Omega)_{\parallel} + (d\sigma/d\Omega)_{\perp}} = \frac{2}{\omega/\omega' + \omega'/\omega}. \tag{5.350}$$

This quantity compares two fixed polarization channels; it is not yet the degree of polarization of the scattered radiation from an unpolarized incident beam.

Polarization of scattered radiation.

For an initially unpolarized photon beam, the scattered radiation is partially polarized. The degree of linear polarization is:

$$\Pi = \frac{(d\sigma/d\Omega)_{\perp\to\perp} - (d\sigma/d\Omega)_{\parallel\to\parallel}}{(d\sigma/d\Omega)_{\perp\to\perp} + (d\sigma/d\Omega)_{\parallel\to\parallel}} = \frac{\sin^2\theta}{(\omega/\omega' + \omega'/\omega) - \sin^2\theta}. \tag{5.351}$$

At $\theta = 90°$, for low-energy photons ($\omega' \approx \omega$):

$$\Pi(90°) \approx \frac{1}{2-1} = 1 \tag{5.352}$$

The scattered radiation at 90° is completely linearly polarized—a result first observed in X-ray scattering experiments and a key prediction of QED.

Applications.

The polarization dependence of Compton scattering has important applications:

1. **Polarimetry:** Compton polarimeters measure the polarization of high-energy photons (X-rays, gamma rays) by detecting the azimuthal asymmetry in the scattered radiation
2. **Astrophysics:** Polarization of scattered radiation provides information about magnetic fields and geometry in astrophysical sources
3. **Pair production:** The related process $\gamma\gamma \to e^+e^-$ in polarized photon beams (e.g., at photon colliders) has cross sections that depend strongly on the relative polarization