5.5 Scattering by an external field

So far in this chapter, the electromagnetic field has always been a *quantized* field, described by photon creation and annihilation operators. However, in many physical situations the quantum fluctuations of the electromagnetic field are negligible, and it is adequate to treat part of the field as a *classical*, prescribed function of the spacetime coordinates. A prime example is the scattering of electrons by the Coulomb field of a heavy nucleus: the nucleus is so massive that its recoil can be neglected, and its electric field can be treated as a fixed, "external" background.

In this section we develop the formalism for scattering by external fields and apply it to derive the **Mott scattering formula** for relativistic electrons in a Coulomb potential. We will see how the familiar Rutherford formula emerges in the non-relativistic limit, and we will discuss the fascinating spin and helicity properties of the scattered electrons.

The external field in the S-matrix

Combining quantized and classical fields

When an external classical field $A^\mu_{\text{ext}}(x)$ is present in addition to the quantized photon field $A^\mu(x)$, the total electromagnetic potential is:

$$A^\mu_{\text{tot}}(x) = A^\mu(x) + A^\mu_{\text{ext}}(x) \tag{5.353}$$

Here $A^\mu(x)$ is the usual quantized field (a sum of creation and annihilation operators), while $A^\mu_{\text{ext}}(x)$ is a prescribed classical function—it has no operator character. More precisely, in the Fock-space sense, the external field acts as $A^\mu_{\text{ext}}(x)\,\mathbb{1}$, where $\mathbb{1}$ is the identity operator on the photon Hilbert space: it is a c-number that does not create or annihilate any photon. Unlike $A^\mu(x)$, which changes the photon content of a state, A^μ_{ext} simply provides a fixed background that the charged particles "feel" but cannot modify.

The QED interaction Hamiltonian becomes:

$$\mathcal{H}_{\text{int}} = -e\bar{\psi}\gamma^\mu\psi(A_\mu + A_{\text{ext},\mu}) = -e\bar{\psi}\not{A}\psi - e\bar{\psi}\not{A}_{\text{ext}}\psi \tag{5.354}$$

The first term describes the usual interaction with quantized photons (emission, absorption, virtual exchange). The second term describes the interaction with the external field—this is what causes the scattering we want to study.

The S-matrix expansion

The S-matrix expansion now contains both types of interaction:

$$S = \sum_{n=0}^{\infty} \frac{(ie)^n}{n!} \int d^4x_1 \cdots d^4x_n T\left\{N[\bar{\psi}(\not{A} + \not{A}_{\text{ext}})\psi]_{x_1} \cdots N[\bar{\psi}(\not{A} + \not{A}_{\text{ext}})\psi]_{x_n}\right\}$$

(5.355)

For scattering by the external field alone (no photon emission or absorption), the lowest-order contribution comes from the first-order term with only the external field:

$$S^{(1)}_{\text{ext}} = ie \int d^4x\, \bar{\psi}(x) \not{A}_{\text{ext}}(x) \psi(x) \tag{5.356}$$

This describes a single interaction of the electron with the external field—the electron comes in, "feels" the field at some point x, and goes out with changed momentum.

Why is this term non-zero? Recall that in pure QED, the analogous first-order term $S^{(1)} = ie \int d^4x\, \bar{\psi} \not{A} \psi$ gives zero for physical processes: the quantized photon field A^μ carries a factor $e^{\pm ik\cdot x}$, and after integrating over x one obtains $\delta^{(4)}(p' - p \pm k)$, which cannot be satisfied by three on-shell particles (an electron cannot emit or absorb a real photon while remaining on-shell). Here the situation is fundamentally different. The external field $A^\mu_{\text{ext}}(x)$ is *not* a quantized plane wave with a fixed on-shell 4-momentum—it is a prescribed classical function. For a static field, the time integration in $S^{(1)}_{\text{ext}}$ yields $\delta(E' - E)$: **energy is conserved**, and the scattering is elastic ($|\vec{p}\,'| = |\vec{p}|$). However, the spatial Fourier transform $\tilde{A}^\mu_{\text{ext}}(\vec{q})$ can supply *any* 3-momentum transfer $\vec{q} = \vec{p}\,' - \vec{p}$ to the electron, so $\vec{p}\,' \neq \vec{p}$ is allowed. This is the crucial difference with pure QED, where the on-shell condition for the photon constrains all four components of the momentum and leaves no solution. Physically, the "missing" 3-momentum is absorbed by the source of the external field (e.g. a heavy nucleus), which we treat as infinitely massive—it can absorb arbitrary recoil with negligible energy cost. The full system (electron + source) conserves 4-momentum exactly; the external-field approximation simply means that we do not track the source's recoil.

Specialization to static fields

For a **static** external field (one that does not depend on time), we can write:

$$A^\mu_{\text{ext}}(x) = A^\mu_{\text{ext}}(\vec{x}) \tag{5.357}$$

It is convenient to introduce the three-dimensional Fourier transform:

$$A^\mu_{\text{ext}}(\vec{x}) = \frac{1}{(2\pi)^3} \int d^3q\, e^{i\vec{q}\cdot\vec{x}}\, \tilde{A}^\mu_{\text{ext}}(\vec{q})\,, \tag{5.358}$$

with the inverse transform:

$$\tilde{A}^\mu_{\text{ext}}(\vec{q}) = \int d^3x\, e^{-i\vec{q}\cdot\vec{x}}\, A^\mu_{\text{ext}}(\vec{x})\,. \tag{5.359}$$

A note on conventions. The placement of the factor $(2\pi)^3$ in the Fourier transform is a matter of convention. We use the **asymmetric convention**, in which the full $(2\pi)^{-3}$ appears in the transform from momentum to position space (5.358), and no factor of (2π) appears in the inverse (5.359). Some textbooks instead adopt the **symmetric convention**, placing a factor of $(2\pi)^{-3/2}$ in each direction. The two conventions yield Fourier transforms that differ by a factor of $(2\pi)^{3/2}$:

$$\tilde{A}^{\mu}_{\text{ext}}(\vec{q})\Big|_{\text{here}} = (2\pi)^{3/2}\,\tilde{A}^{\mu}_{\text{ext}}(\vec{q})\Big|_{\text{symmetric}}. \tag{5.360}$$

Note that this asymmetric convention is *not* the same as the one used in the Bjorken–Drell field mode expansions adopted throughout these notes, where the symmetric factor $(2\pi)^{-3/2}$ appears in each direction (see Section 1.1). The choice made here for $\tilde{A}^{\mu}_{\text{ext}}$ follows Mandl–Shaw and is convenient because the inverse transform (5.359) takes a clean form without prefactors—this is the quantity that appears directly in the Feynman amplitude. When comparing with other references, always check which convention is adopted: the physics is the same, but numerical prefactors in intermediate expressions will differ.

Matrix element and Feynman rules

Evaluation of the matrix element

Consider an electron scattering from an initial state $|i\rangle$ with momentum $p = (E, \vec{p})$ and spin r, to a final state $|f\rangle$ with momentum $p' = (E', \vec{p}')$ and spin s. We need to evaluate:

$$\langle f|S^{(1)}_{\text{ext}}|i\rangle = ie\int d^4x\,\langle f|\bar{\psi}(x)\not{A}_{\text{ext}}(x)\psi(x)|i\rangle \tag{5.361}$$

Using the field expansions (with box normalization volume V):

$$\psi(x) = \sum_{\vec{k},s}\sqrt{\frac{m}{VE_{\vec{k}}}}\left[b(\vec{k},s)\,u(\vec{k},s)\,e^{-ik\cdot x} + d^{\dagger}(\vec{k},s)\,v(\vec{k},s)\,e^{ik\cdot x}\right] \tag{5.362}$$

$$\bar{\psi}(x) = \sum_{\vec{k},s}\sqrt{\frac{m}{VE_{\vec{k}}}}\left[b^{\dagger}(\vec{k},s)\,\bar{u}(\vec{k},s)\,e^{ik\cdot x} + d(\vec{k},s)\,\bar{v}(\vec{k},s)\,e^{-ik\cdot x}\right] \tag{5.363}$$

For an incoming electron with momentum $\vec{p}$ and spin r, and an outgoing electron with momentum $\vec{p}'$ and spin s:

$$|i\rangle = b^{\dagger}(\vec{p},r)|0\rangle, \qquad |f\rangle = b^{\dagger}(\vec{p}',s)|0\rangle \tag{5.364}$$

The only non-vanishing contribution comes from:

$$\langle 0|b(\vec{p}',s)\,\bar{\psi}(x)\cdots\psi(x)\,b^{\dagger}(\vec{p},r)|0\rangle \tag{5.365}$$

After applying the anticommutation relations:

$$\langle f|S^{(1)}_{\text{ext}}|i\rangle = ie\sqrt{\frac{m}{VE}}\sqrt{\frac{m}{VE'}}\int d^4x\,e^{i(p'-p)\cdot x}\,\bar{u}(\vec{p}',s)\,\not{A}_{\text{ext}}(\vec{x})\,u(\vec{p},r) \tag{5.366}$$

Performing the integrals

The time integral gives:

$$\int dt\, e^{i(E'-E)t} = 2\pi\delta(E'-E) \tag{5.367}$$

This delta function enforces **energy conservation**: $E' = E$. For a free electron with $E^2 = |\vec{p}|^2 + m^2$, this implies $|\vec{p}'| = |\vec{p}|$—the scattering is **elastic**.

Physical interpretation: Energy is conserved because the external field is static (time-independent). A time-varying field could transfer energy to the electron. The momentum, however, is *not* conserved: the electron transfers momentum $\vec{q} = \vec{p}' - \vec{p}$ to the source of the field. Since we treat the source as infinitely heavy, this momentum transfer does not appear as kinetic energy—hence the elastic scattering.

The spatial integral gives:

$$\int d^3x\, e^{-i(\vec{p}'-\vec{p})\cdot\vec{x}} A^{\mu}_{\text{ext}}(\vec{x}) = \tilde{A}^{\mu}_{\text{ext}}(\vec{q}) \tag{5.368}$$

where $\vec{q} = \vec{p}' - \vec{p}$ is the **momentum transfer**.

The final result

Combining everything:

$$\boxed{\langle f|S^{(1)}_{\text{ext}}|i\rangle = (2\pi)\delta(E'-E)\sqrt{\frac{m}{VE}}\sqrt{\frac{m}{VE'}}\mathcal{M}} \tag{5.369}$$

where the **Feynman amplitude** is

$$\boxed{\mathcal{M} = ie\,\bar{u}(\vec{p}',s)\,\tilde{\not{A}}_{\text{ext}}(\vec{q})\,u(\vec{p},r)} \tag{5.370}$$

Comparison with photon scattering

Compare eq. (5.369) with the matrix element for scattering by a real photon (e.g., eq. (7.31) in Mandl-Shaw):

$$\langle f|S|i\rangle_{\text{photon}} = (2\pi)^4\delta^{(4)}(p'+k'-p-k)\times(\text{normalization})\times\mathcal{M} \tag{5.371}$$

The key differences are:

1. **Energy-momentum conservation:** For real photons, we have full 4-momentum conservation $(2\pi)^4\delta^{(4)}(p_f - p_i)$. For an external field, we have only energy conservation $(2\pi)\delta(E'-E)$, because the infinitely heavy source absorbs the momentum.
2. **Photon factors:** For a real photon, the amplitude contains a polarization and normalization factor $\sqrt{1/(2V\omega)}\,\varepsilon_\mu(\vec{k})$. For an external field, this is replaced by the Fourier transform $\tilde{A}^{\mu}_{\text{ext}}(\vec{q})$.

Modified Feynman rules

The standard QED Feynman rules are modified as follows when an external static field is present:

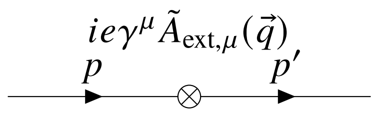
(i) In the relation between the S-matrix element and the amplitude, replace:

$$(2\pi)^4\delta^{(4)}(P_f - P_i) \quad \longrightarrow \quad (2\pi)\delta(E_f - E_i) \tag{5.372}$$

(ii) For each interaction vertex with the external field, write a factor:

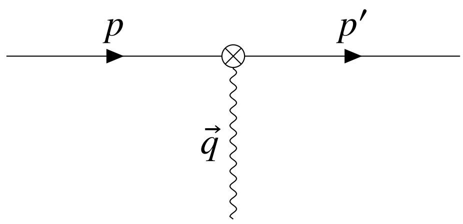
$$ie\gamma^\mu \tilde{A}_{\text{ext},\mu}(\vec{q}) \tag{5.373}$$

where $\vec{q}$ is the momentum transferred *from* the field *to* the particle. The vertex is marked with a cross ($\times$) in Feynman diagrams to distinguish it from a quantized-photon vertex:



Cross section for external field scattering

We consider the elastic scattering of an electron off a static external field generated by a heavy nucleus (charge Ze). At lowest order, the process is described by a single diagram:



Ze (nucleus)

The electron interacts with the Coulomb field of the nucleus via the external-field vertex ($\otimes$), exchanging 3-momentum $\vec{q} = \vec{p}' - \vec{p}$. The wavy line ending on the nucleus represents the classical field, not a quantized photon—no photon appears in the initial or final state.

Derivation of the cross section formula

Starting from eq. (5.369), we derive the differential cross section. The transition probability per unit time is:

$$w = \frac{|\langle f|S^{(1)}_{\text{ext}}|i\rangle|^2}{T} \tag{5.374}$$

Using $[2\pi\delta(E'-E)]^2 = 2\pi T\delta(E'-E)$ (where T is the total time), we get:

$$w = 2\pi\delta(E'-E)\frac{m^2}{V^2 EE'}|\mathcal{M}|^2 \tag{5.375}$$

The number of final states in a momentum-space volume d^3p' is:

$$\frac{Vd^3p'}{(2\pi)^3} = \frac{V|\vec{p}'|^2 d|\vec{p}'|d\Omega'}{(2\pi)^3} = \frac{V|\vec{p}'|E'dE'd\Omega'}{(2\pi)^3} \tag{5.376}$$

where we used $E'dE' = |\vec{p}'|d|\vec{p}'|$.

The incident flux (particles per unit area per unit time) is:

$$\text{flux} = \frac{v}{V} = \frac{|\vec{p}|}{VE} \tag{5.377}$$

where $v = |\vec{p}|/E$ is the electron velocity.

The cross section is:

$$d\sigma = \frac{w \times (\text{density of states})}{\text{flux}} \tag{5.378}$$

Putting it all together and using $\delta(E'-E)$ to set $E' = E$ and $|\vec{p}'| = |\vec{p}|$:

$$\boxed{\frac{d\sigma}{d\Omega'} = \left(\frac{m}{2\pi}\right)^2 |\mathcal{M}|^2} \tag{5.379}$$

This is a remarkably simple result: the cross section is just $(m/2\pi)^2$ times the squared amplitude.

Coulomb scattering: the Mott formula

The Coulomb potential

We now apply the general formalism to the most important case: scattering of electrons by the Coulomb field of a nucleus with charge Ze. This is called **Mott scattering** (after Nevill Mott, who derived it in 1929).

In the Coulomb gauge, the potential of a point nucleus at the origin is:

$$A^\mu_{\text{ext}}(\vec{x}) = \left(\frac{Ze}{4\pi|\vec{x}|}, 0, 0, 0\right) = \left(\phi(\vec{x}), \vec{0}\right) \tag{5.380}$$

Note: only the time component (the scalar potential ϕ) is non-zero. There is no magnetic field—the nucleus is at rest.

Fourier transform of the Coulomb potential

We need to compute $\tilde{A}^{\mu}_{\text{ext}}(\vec{q})$. Since only A^0 is non-zero:

$$\tilde{A}^{0}_{\text{ext}}(\vec{q}) = \int d^3x\, e^{-i\vec{q}\cdot\vec{x}} \frac{Ze}{4\pi|\vec{x}|} \tag{5.381}$$

This integral is a standard result, but let us derive it carefully. It is easier to first compute the Fourier transform of the *Yukawa potential* $V(r) = e^{-\mu r}/r$, and then take the limit $\mu \to 0$.

Using spherical coordinates with $\vec{q}$ along the z-axis:

$$\tilde{V}(\vec{q}) = \int_0^{\infty} r^2 dr \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \frac{e^{-\mu r}}{r} e^{-iqr\cos\theta} \tag{5.382}$$

The ϕ integral gives 2π. For the θ integral, let $u = \cos\theta$:

$$\int_0^{\pi} \sin\theta\, d\theta\, e^{-iqr\cos\theta} = \int_{-1}^{1} du\, e^{-iqru} = \frac{e^{iqr} - e^{-iqr}}{iqr} = \frac{2\sin(qr)}{qr} \tag{5.383}$$

where $q = |\vec{q}|$. Therefore:

$$\tilde{V}(\vec{q}) = 2\pi \int_0^{\infty} r^2 dr \frac{e^{-\mu r}}{r} \cdot \frac{2\sin(qr)}{qr} = \frac{4\pi}{q} \int_0^{\infty} dr\, e^{-\mu r} \sin(qr) \tag{5.384}$$

The remaining integral is evaluated by writing $\sin(qr) = \text{Im}(e^{iqr})$:

$$\begin{aligned}\int_0^{\infty} dr\, e^{-\mu r} \sin(qr) &= \text{Im} \int_0^{\infty} dr\, e^{-\mu r} e^{iqr} = \text{Im} \int_0^{\infty} dr\, e^{-(\mu - iq)r} \\ &= \text{Im}\left[\frac{1}{\mu - iq}\right] = \text{Im}\left[\frac{\mu + iq}{\mu^2 + q^2}\right] = \frac{q}{\mu^2 + q^2}.\end{aligned} \tag{5.385}$$

Thus:

$$\tilde{V}(\vec{q}) = \frac{4\pi}{\mu^2 + q^2} \tag{5.386}$$

Taking $\mu \to 0$ (the Coulomb limit):

$$\int d^3x\, e^{-i\vec{q}\cdot\vec{x}} \frac{1}{|\vec{x}|} = \frac{4\pi}{|\vec{q}|^2} \tag{5.387}$$

Therefore, the momentum-space Coulomb potential is:

$$\boxed{\tilde{A}^{\mu}_{\text{ext}}(\vec{q}) = \left(\frac{Ze}{|\vec{q}|^2}, 0, 0, 0\right)} \tag{5.388}$$

The scattering amplitude

From eq. (5.370):

$$\mathcal{M} = ie\,\bar{u}(\vec{p}\,', s)\,\tilde{\not{A}}_{\text{ext}}(\vec{q})\,u(\vec{p}, r) = ie\,\bar{u}(\vec{p}\,', s)\,\gamma^\mu \tilde{A}_{\text{ext},\mu}(\vec{q})\,u(\vec{p}, r) \tag{5.389}$$

Since only $\tilde{A}^0_{\text{ext}} \neq 0$:

$$\tilde{\not{A}}_{\text{ext}} = \gamma^0 \tilde{A}^0_{\text{ext}} = \gamma^0 \frac{Ze}{|\vec{q}|^2} \tag{5.390}$$

Therefore:

$$\mathcal{M} = \frac{iZe^2}{|\vec{q}|^2}\,\bar{u}(\vec{p}\,', s)\,\gamma^0\,u(\vec{p}, r) \tag{5.391}$$

The squared amplitude (for specific spins r, s) is:

$$|\mathcal{M}|^2 = \frac{Z^2 e^4}{|\vec{q}|^4}\left|\bar{u}(\vec{p}\,', s)\,\gamma^0\,u(\vec{p}, r)\right|^2 \tag{5.392}$$

Unpolarized cross section: the trace calculation

For unpolarized electrons, we average over initial spins and sum over final spins:

$$\overline{|\mathcal{M}|^2} = \frac{1}{2}\sum_{r,s}|\mathcal{M}|^2 = \frac{Z^2 e^4}{2|\vec{q}|^4}\sum_{r,s}\left|\bar{u}(\vec{p}\,', s)\,\gamma^0\,u(\vec{p}, r)\right|^2 \tag{5.393}$$

Using the BD completeness relation $\sum_s u(\vec{p}, s)\,\bar{u}(\vec{p}, s) = (\not{p} + m)/(2m)$:

$$\sum_{r,s}\left|\bar{u}(\vec{p}\,', s)\,\gamma^0\,u(\vec{p}, r)\right|^2 = \text{Tr}\left[\frac{\not{p}' + m}{2m}\gamma^0 \frac{\not{p} + m}{2m}\gamma^0\right] = \frac{1}{4m^2}\text{Tr}\left[(\not{p}' + m)\gamma^0(\not{p} + m)\gamma^0\right] \tag{5.394}$$

Evaluating the trace.

We expand:

$$\text{Tr}[(\not{p}' + m)\gamma^0(\not{p} + m)\gamma^0] = \text{Tr}[\not{p}'\gamma^0\not{p}\gamma^0] + m^2\text{Tr}[\gamma^0\gamma^0] \tag{5.395}$$

The traces with an odd number of gamma matrices vanish. Using $(\gamma^0)^2 = \mathbb{1}_4$ and $\text{Tr}[\mathbb{1}_4] = 4$:

$$= \text{Tr}[\not{p}'\gamma^0\not{p}\gamma^0] + m^2 \cdot 4 \tag{5.396}$$

For the first trace, we use the standard formula. Writing $\not{p}' = \gamma^\alpha p'_\alpha$ and $\not{p} = \gamma^\beta p_\beta$:

$$\text{Tr}[\gamma^\alpha\gamma^0\gamma^\beta\gamma^0] = 4(g^{\alpha 0}g^{\beta 0} - g^{\alpha\beta}g^{00} + g^{\alpha 0}g^{0\beta}) = 4(g^{\alpha 0}g^{\beta 0} - g^{\alpha\beta} + g^{\alpha 0}g^{\beta 0}) \tag{5.397}$$

Therefore:

$$\begin{aligned}\mathrm{Tr}[\not{p}'\gamma^0\not{p}\gamma^0] &= 4p'_\alpha p_\beta(2g^{\alpha 0}g^{\beta 0} - g^{\alpha\beta}) \\ &= 4(2p'^0 p^0 - p'\cdot p) \\ &= 4(2E^2 - p'\cdot p)\end{aligned} \tag{5.398}$$

where we used $p'^0 = p^0 = E$ (energy conservation).
For the dot product $p'\cdot p$:

$$p'\cdot p = E^2 - \vec{p}'\cdot\vec{p} = E^2 - |\vec{p}|^2\cos\theta \tag{5.399}$$

where θ is the scattering angle.
Combining:

$$\begin{aligned}\mathrm{Tr}[(\not{p}'+m)\gamma^0(\not{p}+m)\gamma^0] &= 4(2E^2 - E^2 + |\vec{p}|^2\cos\theta) + 4m^2 \\ &= 4(E^2 + |\vec{p}|^2\cos\theta + m^2)\end{aligned} \tag{5.400}$$

Using $E^2 = |\vec{p}|^2 + m^2$:

$$= 4(|\vec{p}|^2 + m^2 + |\vec{p}|^2\cos\theta + m^2) = 4(2m^2 + |\vec{p}|^2(1+\cos\theta)) \tag{5.401}$$

Or, equivalently:

$$\boxed{\mathrm{Tr}[(\not{p}'+m)\gamma^0(\not{p}+m)\gamma^0] = 8\left(E^2 - |\vec{p}|^2\sin^2\frac{\theta}{2}\right)} \tag{5.402}$$

where we used $1+\cos\theta = 2\cos^2(\theta/2)$ and $E^2 + m^2 = 2E^2 - |\vec{p}|^2$.

The momentum transfer

We need $|\vec{q}|^2$. Since $\vec{q} = \vec{p}' - \vec{p}$ and $|\vec{p}'| = |\vec{p}|$ (elastic scattering):

$$\begin{aligned}|\vec{q}|^2 &= |\vec{p}' - \vec{p}|^2 = |\vec{p}'|^2 + |\vec{p}|^2 - 2\vec{p}'\cdot\vec{p} \\ &= 2|\vec{p}|^2(1-\cos\theta) = 4|\vec{p}|^2\sin^2\frac{\theta}{2}\end{aligned} \tag{5.403}$$

The Mott formula

Substituting eqs. (5.394), (5.402), and (5.403) into eq. (5.379):

$$\begin{aligned}\frac{d\sigma}{d\Omega'} &= \left(\frac{m}{2\pi}\right)^2 \frac{Z^2e^4}{2|\vec{q}|^4} \times \frac{1}{4m^2} \times 8\left(E^2 - |\vec{p}|^2\sin^2\frac{\theta}{2}\right) \\ &= \frac{Z^2e^4}{(2\pi)^2}\cdot\frac{E^2 - |\vec{p}|^2\sin^2(\theta/2)}{16|\vec{p}|^4\sin^4(\theta/2)}\end{aligned} \tag{5.404}$$

where the m^2 from $(m/2\pi)^2$ has cancelled the $1/(4m^2)$ from the BD spin sum. Using $e^2 = 4\pi\alpha$ and $|\vec{p}| = E\beta$ (where $\beta = |\vec{v}|/c$ is the velocity in units of c, not to be confused with the macro `\vspin` for the Dirac spinor v):

$$= \frac{Z^2\alpha^2}{4|\vec{p}|^4 \sin^4(\theta/2)} \cdot \left(E^2 - E^2\beta^2 \sin^2(\theta/2)\right) = \frac{Z^2\alpha^2}{4E^2\beta^4 \sin^4(\theta/2)} \cdot \left(1 - \beta^2 \sin^2(\theta/2)\right) \tag{5.405}$$

This gives the **Mott scattering formula**:

$$\boxed{\frac{d\sigma}{d\Omega'} = \frac{(Z\alpha)^2}{4E^2\beta^4 \sin^4(\theta/2)} \left[1 - \beta^2 \sin^2 \frac{\theta}{2}\right]} \tag{5.406}$$

This is the relativistic cross section for electron scattering by a Coulomb field. The factor in square brackets is the relativistic correction to the Rutherford formula.

The Rutherford limit

In the **non-relativistic limit**, $v \ll 1$, so $E \approx m$ and the correction factor becomes:

$$1 - v^2 \sin^2 \frac{\theta}{2} \approx 1 \tag{5.407}$$

The Mott formula reduces to the **Rutherford scattering formula**:

$$\boxed{\frac{d\sigma}{d\Omega'} = \frac{(Z\alpha)^2}{4m^2 v^4 \sin^4(\theta/2)}} \tag{5.408}$$

This is the famous result derived by Rutherford in 1911 to explain alpha-particle scattering by gold nuclei, which led to the discovery of the atomic nucleus.

Physical features of the Rutherford formula:

- The $\sin^{-4}(\theta/2)$ dependence means that most scattering is at small angles (forward scattering), with a long tail extending to large angles
- The v^{-4} dependence means slow particles are scattered much more strongly
- The Z^2 dependence means heavy nuclei scatter more strongly
- The cross section diverges as $\theta \to 0$; this is because the Coulomb potential has infinite range. In reality, atomic electrons screen the nuclear charge at large distances.

Spin and helicity in Coulomb scattering

The spin properties of Coulomb scattering reveal a beautiful interplay between relativity and quantum mechanics. The behavior is strikingly different in the non-relativistic and ultra-relativistic regimes.

Non-relativistic limit: spin conservation

At non-relativistic energies, the interaction is essentially spin-independent. To see this, consider the matrix element:

$$\bar{u}(\vec{p}\,', s)\,\gamma^0\,u(\vec{p}, r) \tag{5.409}$$

Recall from eq. (1.174) the explicit form of the Dirac spinors in the BD normalization:

$$u(\vec{p}, r) = \sqrt{\frac{E_p + m}{2m}} \begin{pmatrix} \xi^{(r)} \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{E_p + m}\,\xi^{(r)} \end{pmatrix}, \tag{5.410}$$

where $\xi^{(r)}$ is a two-component Pauli spinor.[1] In the non-relativistic limit $|\vec{p}| \ll m$, we have $E_p \to m$ and $\vec{\sigma}\cdot\vec{p}/(E_p + m) \to 0$. The prefactor $\sqrt{(E_p + m)/(2m)} \to \sqrt{2m/(2m)} = 1$, so:

$$u(\vec{p}, r) \xrightarrow{\text{NR}} \begin{pmatrix} \xi^{(r)} \\ 0 \end{pmatrix}, \qquad \bar{u}(\vec{p}\,', s) \xrightarrow{\text{NR}} (\xi^{(s)\dagger}, 0) \tag{5.411}$$

Since $\gamma^0 = \mathrm{diag}(1, 1, -1, -1)$ in the Dirac representation:

$$\bar{u}(\vec{p}\,', s)\,\gamma^0\,u(\vec{p}, r) \xrightarrow{\text{NR}} \xi^{(s)\dagger}\xi^{(r)} = \delta_{sr} \tag{5.412}$$

Spin is conserved in non-relativistic Coulomb scattering: the matrix element vanishes unless $s = r$. This is expected because the Coulomb interaction depends only on position, not on spin—there is no magnetic field in the rest frame of the nucleus.

What about helicity? Helicity is defined as the projection of spin along the momentum direction. If spin is conserved but the momentum direction changes by

[1] Let us verify that this spinor satisfies the BD normalization $\bar{u}u = 1$. We compute $\bar{u}u = u^\dagger\gamma^0 u$ step by step:

$$\begin{aligned} \bar{u}(\vec{p}, r)\,u(\vec{p}, r) &= \frac{E_p + m}{2m}\left(\xi^{(r)\dagger},\, \xi^{(r)\dagger}\tfrac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\right) \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} \begin{pmatrix} \xi^{(r)} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\,\xi^{(r)} \end{pmatrix} \\ &= \frac{E_p + m}{2m}\left[\xi^{(r)\dagger}\xi^{(r)} - \frac{(\vec{\sigma}\cdot\vec{p})^2}{(E_p + m)^2}\,\xi^{(r)\dagger}\xi^{(r)}\right]. \end{aligned}$$

Using $(\vec{\sigma}\cdot\vec{p})^2 = |\vec{p}|^2 = E_p^2 - m^2 = (E_p - m)(E_p + m)$ and $\xi^{(r)\dagger}\xi^{(r)} = 1$:

$$\bar{u}\,u = \frac{E_p + m}{2m}\left[1 - \frac{E_p - m}{E_p + m}\right] = \frac{E_p + m}{2m}\cdot\frac{2m}{E_p + m} = 1\,. \quad \checkmark$$

angle θ, then helicity must change. To see this quantitatively, we use the Wigner d-matrix for spin-$\frac{1}{2}$ (cf. the box in Section 5.2). Setting $j = \frac{1}{2}$ in the general formula $d^j_{m,m'}(\theta) = \langle j,m|e^{-i\theta J_y}|j,m'\rangle$ and using $J_y = \sigma_y/2$:

$$d^{(1/2)}_{m,m'}(\theta) = \langle \tfrac{1}{2},m|e^{-i\theta\sigma_y/2}|\tfrac{1}{2},m'\rangle = \begin{pmatrix} \cos\dfrac{\theta}{2} & -\sin\dfrac{\theta}{2} \\ \sin\dfrac{\theta}{2} & \cos\dfrac{\theta}{2} \end{pmatrix}_{m,m'}, \tag{5.413}$$

where the rows and columns are labelled by $m,m' = +\frac{1}{2},-\frac{1}{2}$. If the initial helicity state is $|+\rangle$ (spin along $\vec{p}$), the probability of finding helicity $|\pm\rangle$ along the new direction $\vec{p}'$ is $|d^{(1/2)}_{m'm}(\theta)|^2$:

$$P(\text{no helicity flip}) = \left|d^{(1/2)}_{++}(\theta)\right|^2 = \cos^2\frac{\theta}{2} \tag{5.414}$$

$$P(\text{helicity flip}) = \left|d^{(1/2)}_{-+}(\theta)\right|^2 = \sin^2\frac{\theta}{2} \tag{5.415}$$

Note the key point: because spin is conserved ($s = r$), the spin state $|\uparrow\rangle$ does not change, but the momentum rotates by θ. The helicity eigenstates *do* change, because they are defined with respect to the momentum direction. The d-matrix above connects the old helicity basis (along $\vec{p}$) to the new one (along $\vec{p}'$), so even though the spin quantum number is unchanged, the helicity can flip. For forward scattering ($\theta = 0$), helicity is trivially preserved. For backward scattering ($\theta = \pi$), $\cos(\theta/2) = 0$ and $\sin(\theta/2) = 1$, so the helicity *always* flips: the momentum reverses but the spin does not, so the projection of spin along the new momentum direction is opposite to the original one.

Ultra-relativistic limit: helicity conservation

In the **ultra-relativistic limit** $E \gg m$, the situation is reversed: *helicity* is conserved, not spin!

To see this, we use the helicity projection operators. For $E \gg m$:

$$\Pi^{\pm}(\vec{p}) \approx \frac{1}{2}(1 \pm \gamma^5) \tag{5.416}$$

Consider a transition where the initial electron has positive helicity ($s = 1$, spin aligned with momentum). The probability of finding the final electron with helicity label s is proportional to:

$$X_s = \left|\bar{u}(\vec{p}',s)\,\gamma^0\,u(\vec{p},1)\right|^2 \tag{5.417}$$

Writing $|X|^2 = X \cdot X^*$ and using $(\bar{u}'\gamma^0 u)^* = \bar{u}\gamma^0 u'$ (since γ^0 is Hermitian), this becomes a trace over the outer products $u\bar{u}$:

$$X_s = \mathrm{Tr}\left[\big(u(\vec{p}',s)\bar{u}(\vec{p}',s)\big)\gamma^0\big(u(\vec{p},1)\bar{u}(\vec{p},1)\big)\gamma^0\right] \tag{5.418}$$

For a *single* helicity state λ, the outer product can be expressed in terms of projection operators (see Appendix E for the proof):

$$u(\vec{p},\lambda)\,\bar{u}(\vec{p},\lambda) = \frac{\not{p}+m}{2m}\cdot\frac{1}{2}(1+\lambda\gamma^5\not{n}_p), \tag{5.419}$$

where $n_p^\mu = (|\vec{p}|/m,\, E_p\hat{p}/m)$ is the covariant spin four-vector. In the **ultra-relativistic limit** $E \gg m$, $n_p^\mu \approx p^\mu/m$ and $\gamma^5\not{n}_p\, u \approx \gamma^5 u$, so the helicity projector reduces to the chirality projector $\frac{1}{2}(1+\lambda\gamma^5)$. The expression for X_s simplifies to:

$$X_s \approx \frac{1}{16m^2}\mathrm{Tr}\left[(\not{p}'+m)\gamma^0(1+\gamma^5)(\not{p}+m)\gamma^0(1\pm\gamma^5)\right] \tag{5.420}$$

Explicit evaluation. We expand $(\not{p}'+m)$ and $(\not{p}+m)$, writing $\Gamma \equiv \gamma^0(1+\gamma^5)$ for brevity:

$$(\not{p}'+m)\,\Gamma\,(\not{p}+m) = \not{p}'\,\Gamma\,\not{p} + m\,\Gamma\,\not{p} + m\,\not{p}'\,\Gamma + m^2\,\Gamma \tag{5.421}$$

The two cross terms linear in m contain three γ-matrices (from Γ and one $\not{p}$), giving traces of an odd number of γ's, which vanish. The two surviving contributions are:

$$X_s = \frac{1}{16m^2}\Big\{\underbrace{\mathrm{Tr}\big[\not{p}'\gamma^0(1+\gamma^5)\not{p}\gamma^0(1\pm\gamma^5)\big]}_{\text{leading, }\sim E^2} + m^2\underbrace{\mathrm{Tr}\big[\gamma^0(1+\gamma^5)\gamma^0(1\pm\gamma^5)\big]}_{\text{subleading, }\sim m^0}\Big\} \tag{5.422}$$

The first comes from $\not{p}'\,\Gamma\,\not{p}$ (two powers of momentum, $\sim E^2$); the second from $m^2\,\Gamma$ (no momenta, $\sim m^0$).
Leading term. Expanding the product of $(1+\gamma^5)$ and $(1\pm\gamma^5)$, we get four traces. Two of them contain γ^5 linearly:

$$\mathrm{Tr}[\not{p}'\gamma^0\not{p}\gamma^0\gamma^5] = p'_\alpha p_\beta\,\mathrm{Tr}[\gamma^\alpha\gamma^0\gamma^\beta\gamma^0\gamma^5] = -4i\,p'_\alpha p_\beta\,\epsilon^{\alpha 0\beta 0} = 0\,, \tag{5.423}$$

since $\epsilon^{\alpha 0\beta 0}$ vanishes (two repeated indices). By the same reasoning, $\mathrm{Tr}[\not{p}'\gamma^0\gamma^5\not{p}\gamma^0] = 0$. The leading term therefore reduces to:

$$\mathrm{Tr}[\not{p}'\gamma^0\not{p}\gamma^0] \pm \mathrm{Tr}[\not{p}'\gamma^0\gamma^5\not{p}\gamma^0\gamma^5] \equiv T_1 \pm T_2\,. \tag{5.424}$$

The key step is to show that $T_2 = T_1$. Anticommute γ^5 past $\not{p}$ using $\{\gamma^5,\gamma^\mu\} = 0$, i.e. $\gamma^5\not{p} = -\not{p}\gamma^5$:

$$\begin{aligned} T_2 &= \mathrm{Tr}[\not{p}'\gamma^0\gamma^5\not{p}\gamma^0\gamma^5] \\ &= \mathrm{Tr}[\not{p}'\gamma^0(-\not{p}\gamma^5)\gamma^0\gamma^5] = -\mathrm{Tr}[\not{p}'\gamma^0\not{p}\,\gamma^5\gamma^0\gamma^5] \\ &= -\mathrm{Tr}[\not{p}'\gamma^0\not{p}\,(-\gamma^0\gamma^5)\,\gamma^5] = +\mathrm{Tr}[\not{p}'\gamma^0\not{p}\,\gamma^0\,(\gamma^5)^2] \\ &= \mathrm{Tr}[\not{p}'\gamma^0\not{p}\gamma^0] = T_1\,. \quad \checkmark \end{aligned} \tag{5.425}$$

(We used $\gamma^5\gamma^0 = -\gamma^0\gamma^5$ in the third line and $(\gamma^5)^2 = \mathbb{1}$ in the last.)

Therefore:

- **No helicity flip** ($s=1$, taking +): leading $= T_1+T_1 = 2T_1 = 8(2E^2 - p'\cdot p) \sim E^2$.
- **Helicity flip** ($s=2$, taking −): leading $= T_1 - T_1 = 0$.

Subleading m^2 term. Using $(\gamma^0)^2 = \mathbb{1}$ and $\gamma^0\gamma^5\gamma^0 = -\gamma^5$:

$$\mathrm{Tr}[\gamma^0(1+\gamma^5)\gamma^0(1\pm\gamma^5)] = \mathrm{Tr}[(1-\gamma^5)(1\pm\gamma^5)]\,. \tag{5.426}$$

For +: $(1-\gamma^5)(1+\gamma^5) = 1-(\gamma^5)^2 = 0$. For −: $(1-\gamma^5)^2 = 2(1-\gamma^5)$, so Tr $= 2\,\mathrm{Tr}[\mathbb{1}_4] = 8$.
Collecting results:

$$X_1 \propto 2T_1 + 0 \sim E^2\,, \qquad X_2 \propto 0 + 8m^2 \sim m^2\,. \tag{5.427}$$

Therefore, in the ultra-relativistic limit:

$$\frac{X_2}{X_1} \sim \left(\frac{m}{E}\right)^2 \to 0 \tag{5.428}$$

Helicity is conserved in ultra-relativistic Coulomb scattering. The probability of helicity flip vanishes as $(m/E)^2$.

Physical interpretation

- **Non-relativistic regime:** The electron moves slowly, so its spin has time to "follow" the momentum direction changes caused by the electric field. The interaction is purely electric, with no magnetic component.
- **Ultra-relativistic regime:** In the electron's instantaneous rest frame, the purely electric field of the nucleus appears as both electric and magnetic fields (Lorentz transformation). The magnetic field couples to the electron's spin, and this spin-orbit coupling tends to preserve helicity. Mathematically, this is connected to chiral symmetry: for massless fermions, left- and right-handed components decouple completely.

The relativistic correction factor $[1 - v^2\sin^2(\theta/2)]$ in the Mott formula (5.406) is a direct consequence of this magnetic scattering. It suppresses large-angle scattering for relativistic electrons.

Summary: two limiting behaviors

	Non-relat. ($|\vec{p}| \ll m$)	Ultra-relat. ($E \gg m$)
Conserved quantity	Spin	Helicity
Helicity flip probability	$\sin^2(\theta/2)$	$\sim (m/E)^2 \to 0$
Physical origin	Electric interaction only	Magnetic spin-orbit coupling

This transition from spin conservation to helicity conservation as the energy increases is a beautiful illustration of how relativistic effects modify the quantum mechanical behavior of spin-1/2 particles.