5.6 Bremsstrahlung

When an electron is deflected by the Coulomb field of a heavy nucleus, it must emit electromagnetic radiation—this is a fundamental consequence of classical electrodynamics, which states that any accelerated charge radiates. The emitted radiation, and the process itself, are called **bremsstrahlung** (German for "braking radiation"). In this section, we develop the quantum field theoretic treatment of bremsstrahlung in the soft photon limit, deriving the cross section and exploring the famous **infrared divergence** that arises when the photon energy approaches zero.

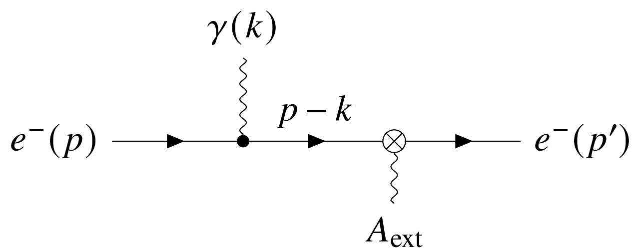
Bremsstrahlung is of considerable practical importance: it is the primary mechanism by which fast electrons lose energy as they pass through matter, and it is widely used in electron accelerators to produce intense photon beams for experiments.

The process and its Feynman diagrams

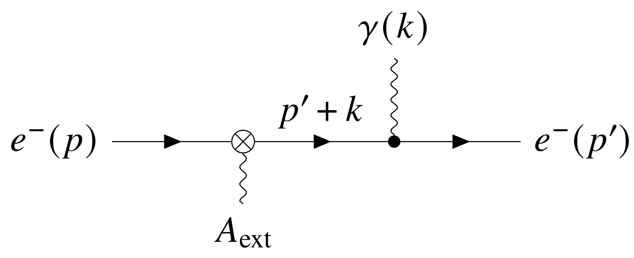
Consider an electron with initial momentum $p = (E, \vec{p})$ scattering in an external Coulomb field and emitting a photon with momentum $k = (\omega, \vec{k})$. The final electron has momentum $p' = (E', \vec{p}')$. Unlike pure Coulomb scattering, this process involves *both* the quantized electromagnetic field (to emit the photon) and the external field (to provide the momentum transfer to the source).

Why do we need both fields? If the electron only interacted with the external field, it could not emit a real photon—energy-momentum conservation would be violated. If it only interacted with the quantized field, there would be no source to absorb the recoil momentum. The external field of the heavy nucleus absorbs momentum without absorbing energy (because the nucleus is infinitely heavy), making the process kinematically allowed.

In lowest order, bremsstrahlung arises from the second-order term ($n = 2$) in the S-matrix expansion. There are two Feynman diagrams, shown below:



(a)



(b)

In diagram (a), the electron first emits the photon, then scatters from the external field. The intermediate electron has momentum $p - k$. In diagram (b), the electron first scatters, then emits the photon. The intermediate electron has momentum $p' + k$. Both diagrams must be included and their amplitudes added coherently.

The S-matrix element

Using the Feynman rules developed in the previous section, we can write down the S-matrix element directly. The structure is similar to Coulomb scattering, but now with an additional photon vertex and propagator.

Writing the amplitude

From the Feynman rules for external fields (Section 5.5) and for photon emission, the S-matrix element is:

$$\langle f|S|i\rangle = 2\pi\delta(E' + \omega - E)\sqrt{\frac{m}{VE}}\sqrt{\frac{m}{VE'}}\sqrt{\frac{1}{2V\omega}}\mathcal{M} \tag{5.429}$$

Note the energy conservation: The delta function enforces $E = E' + \omega$, meaning the initial electron energy equals the final electron energy plus the photon energy. This is different from pure Coulomb scattering where $E = E'$.

The Feynman amplitude $\mathcal{M}$ is the sum of the two diagrams:

$$\mathcal{M} = \mathcal{M}_a + \mathcal{M}_b \tag{5.430}$$

For diagram (a), reading from right to left along the fermion line:

- Final electron: $\bar{u}(\vec{p}')$
- External field vertex: $ie\gamma^\mu \tilde{A}_{\text{ext},\mu}(\vec{q})$
- Electron propagator: $iS_F(p-k) = \frac{i(\not{p}-\not{k}+m)}{(p-k)^2-m^2}$
- Photon emission vertex: $ie\gamma^\nu \varepsilon^*_\nu(k)$
- Initial electron: $u(\vec{p})$

For diagram (b):

- Final electron: $\bar{u}(\vec{p}')$
- Photon emission vertex: $ie\gamma^\nu \varepsilon_\nu^*(k)$
- Electron propagator: $iS_F(p'+k) = \frac{i(\not{p}'+\not{k}+m)}{(p'+k)^2-m^2}$
- External field vertex: $ie\gamma^\mu \tilde{A}_{\text{ext},\mu}(\vec{q})$
- Initial electron: $u(\vec{p})$

Writing $\not{\varepsilon} \equiv \gamma^\nu \varepsilon_\nu^*(k)$ and $\not{A}_{\text{ext}} \equiv \gamma^\mu \tilde{A}_{\text{ext},\mu}(\vec{q})$:

$$\mathcal{M} = -e^2 \bar{u}(\vec{p}') \left[\not{A}_{\text{ext}} iS_F(p-k) \not{\varepsilon} + \not{\varepsilon} iS_F(p'+k) \not{A}_{\text{ext}} \right] u(\vec{p}) \tag{5.431}$$

Simplifying the propagators

The electron propagators can be simplified. For the first propagator:

$$iS_F(p-k) = \frac{i(\not{p}-\not{k}+m)}{(p-k)^2-m^2} \tag{5.432}$$

$$= \frac{i(\not{p}-\not{k}+m)}{p^2-2p\cdot k+k^2-m^2} \tag{5.433}$$

Since the initial electron is on-shell ($p^2 = m^2$) and the photon is massless ($k^2 = 0$):

$$iS_F(p-k) = \frac{i(\not{p}-\not{k}+m)}{-2p\cdot k} \tag{5.434}$$

Similarly, for the second propagator:

$$iS_F(p'+k) = \frac{i(\not{p}'+\not{k}+m)}{(p'+k)^2-m^2} \tag{5.435}$$

$$= \frac{i(\not{p}'+\not{k}+m)}{p'^2+2p'\cdot k+k^2-m^2} = \frac{i(\not{p}'+\not{k}+m)}{2p'\cdot k} \tag{5.436}$$

Substituting back into eq. (5.431):

$$\boxed{\mathcal{M} = -ie^2 \bar{u}(\vec{p}') \left[\not{A}_{\text{ext}} \frac{\not{p}-\not{k}+m}{-2p\cdot k} \not{\varepsilon} + \not{\varepsilon} \frac{\not{p}'+\not{k}+m}{2p'\cdot k} \not{A}_{\text{ext}} \right] u(\vec{p})} \tag{5.437}$$

The momentum transfer to the external field is:

$$\vec{q} = \vec{p}' + \vec{k} - \vec{p} \tag{5.438}$$

Note that three-momentum is not conserved among the electron and photon alone—the difference is absorbed by the external field.

The cross section formula

The derivation of the cross section follows the same pattern as for Coulomb scattering, with the addition of a photon phase space factor.

Transition rate and phase space

The transition probability per unit time is:

$$w = \frac{|\langle f|S|i\rangle|^2}{T} \tag{5.439}$$

Using $[2\pi\delta(E'+\omega-E)]^2 = 2\pi T\delta(E'+\omega-E)$:

$$w = 2\pi\delta(E'+\omega-E)\frac{m^2}{V^2EE'}\frac{1}{2V\omega}|\mathcal{M}|^2 \tag{5.440}$$

The density of final states now includes both the scattered electron and the emitted photon:

$$\frac{V d^3p'}{(2\pi)^3}\times\frac{V d^3k}{(2\pi)^3} \tag{5.441}$$

Dividing by the incident flux $|\vec{p}|/(VE)$ and integrating over $|p'|$ using the energy delta function:

$$\boxed{d\sigma = \frac{m^2}{(2\pi)^5}\frac{1}{2\omega}\frac{|\vec{p}'|}{|\vec{p}|}|\mathcal{M}|^2\, d^3k\, d\Omega'} \tag{5.442}$$

where $d\Omega'$ is the solid angle for the scattered electron direction.

The Bethe-Heitler formula

Evaluating $|\mathcal{M}|^2$ from eq. (5.437), summing over final spins and averaging over initial spins, leads to the **Bethe-Heitler formula** for bremsstrahlung in a Coulomb field. This is a lengthy but straightforward calculation. The full formula is quite complex and we will not derive it here. Instead, we focus on the physically illuminating **soft photon limit**.

The soft photon approximation

When the emitted photon has very low energy ($\omega \to 0$), the calculation simplifies dramatically and reveals deep physical insights.

Simplification of the amplitude

In the soft photon limit $\omega \approx 0$, we have:

- $\vec{q} = \vec{p}' + \vec{k} - \vec{p} \approx \vec{p}' - \vec{p}$ (same as elastic scattering)
- $|\vec{p}'| \approx |\vec{p}|$ (elastic kinematics)
- We can neglect $\not{k}$ in the numerators of the propagators

With these approximations, the amplitude (5.437) becomes:

$$\mathcal{M} \approx -ie^2\bar{u}(\vec{p}')\left[\not{A}_{\text{ext}}\frac{\not{p}+m}{-2p\cdot k}\not{\varepsilon} + \not{\varepsilon}\frac{\not{p}'+m}{2p'\cdot k}\not{A}_{\text{ext}}\right]u(\vec{p}) \tag{5.443}$$

Using the Dirac equation

We can simplify further using the Dirac equation and gamma matrix algebra. Recall that:

$$\not{p}u(\vec{p}) = mu(\vec{p}), \qquad \bar{u}(\vec{p}')\not{p}' = m\bar{u}(\vec{p}') \tag{5.444}$$

For the term $(\not{p}+m)\not{\varepsilon}$, we use the anticommutation relation $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}$:

$$\not{p}\not{\varepsilon} = -\not{\varepsilon}\not{p} + 2(p\cdot\varepsilon) \tag{5.445}$$

$$\therefore \quad (\not{p}+m)\not{\varepsilon} = -\not{\varepsilon}\not{p} + 2(p\cdot\varepsilon) + m\not{\varepsilon} = -\not{\varepsilon}(\not{p}-m) + 2(p\cdot\varepsilon) \tag{5.446}$$

When acting on $u(\vec{p})$:

$$(\not{p}+m)\not{\varepsilon}u(\vec{p}) = -\not{\varepsilon}(\not{p}-m)u(\vec{p}) + 2(p\cdot\varepsilon)u(\vec{p}) = 2(p\cdot\varepsilon)u(\vec{p}) \tag{5.447}$$

since $(\not{p}-m)u(\vec{p}) = 0$.

Similarly, for $\not{\varepsilon}(\not{p}'+m)$:

$$\not{\varepsilon}\not{p}' = -\not{p}'\not{\varepsilon} + 2(\varepsilon\cdot p') \tag{5.448}$$

so

$$\not{\varepsilon}(\not{p}'+m) = -(\not{p}'-m)\not{\varepsilon} + 2(\varepsilon\cdot p') \tag{5.449}$$

When $\bar{u}(\vec{p}')$ acts from the left:

$$\bar{u}(\vec{p}')\not{\varepsilon}(\not{p}'+m) = 2(\varepsilon\cdot p')\bar{u}(\vec{p}') \tag{5.450}$$

since $\bar{u}(\vec{p}')(\not{p}'-m) = 0$.

The factorized amplitude

Substituting eqs. (5.447) and (5.450) into eq. (5.443):

$$\mathcal{M} \approx -ie^2\bar{u}(\vec{p}')\left[\not{A}_{\text{ext}}\frac{2(p\cdot\varepsilon)}{-2p\cdot k} + \frac{2(\varepsilon\cdot p')}{2p'\cdot k}\not{A}_{\text{ext}}\right]u(\vec{p}) \tag{5.451}$$

$$= -ie^2\bar{u}(\vec{p}')\not{A}_{\text{ext}}u(\vec{p})\left[-\frac{p\cdot\varepsilon}{p\cdot k} + \frac{p'\cdot\varepsilon}{p'\cdot k}\right] \tag{5.452}$$

Recognizing that $ie\bar{u}(\vec{p}')\not{A}_{\text{ext}}u(\vec{p}) = \mathcal{M}_0$ is the amplitude for elastic Coulomb scattering (eq. (5.370)), we obtain the beautiful result:

$$\boxed{\mathcal{M} = -e\mathcal{M}_0\left[\frac{p'\cdot\varepsilon}{p'\cdot k} - \frac{p\cdot\varepsilon}{p\cdot k}\right] \qquad (\omega\approx 0)} \tag{5.453}$$

Physical interpretation: The soft bremsstrahlung amplitude **factorizes** into the elastic scattering amplitude times a universal factor that depends only on the initial and final electron momenta and the photon polarization. This factorization is a deep result—it says that soft photon emission is completely determined by the external kinematics, independent of the details of the scattering process.

The soft bremsstrahlung cross section

Substituting eq. (5.453) into the cross section formula (5.442):

$$d\sigma = \frac{m^2}{(2\pi)^5}\frac{1}{2\omega}\frac{|\vec{p}'|}{|\vec{p}|}e^2|\mathcal{M}_0|^2\left|\frac{p'\cdot\varepsilon}{p'\cdot k} - \frac{p\cdot\varepsilon}{p\cdot k}\right|^2 d^3k\, d\Omega' \tag{5.454}$$

In the soft limit, $|\vec{p}'| = |\vec{p}|$, and comparing with the elastic cross section (5.379):

$$\left(\frac{d\sigma}{d\Omega'}\right)_0 = \left(\frac{m}{2\pi}\right)^2|\mathcal{M}_0|^2 \tag{5.455}$$

we obtain:

$$\boxed{\left(\frac{d\sigma}{d\Omega'}\right)_{\text{B}} = \left(\frac{d\sigma}{d\Omega'}\right)_0\frac{\alpha}{(2\pi)^2}\left[\frac{p'\cdot\varepsilon}{p'\cdot k} - \frac{p\cdot\varepsilon}{p\cdot k}\right]^2\frac{d^3k}{\omega} \qquad (\omega\approx 0)} \tag{5.456}$$

where $\alpha = e^2/(4\pi)$ is the fine structure constant.

Unpolarized cross section

If we do not observe the photon polarization, we must sum over the two transverse polarizations. Using the gauge-invariant polarization sum:

$$\sum_{\lambda=1,2} \varepsilon_\mu(k,\lambda)\,\varepsilon^*_\nu(k,\lambda) \to -g_{\mu\nu} \tag{5.457}$$

(the longitudinal terms vanish by current conservation), we obtain:

$$\sum_\lambda \left|\frac{p'\cdot\varepsilon}{p'\cdot k} - \frac{p\cdot\varepsilon}{p\cdot k}\right|^2 = -\left(\frac{p'^\mu}{p'\cdot k} - \frac{p^\mu}{p\cdot k}\right)\left(\frac{p'_\mu}{p'\cdot k} - \frac{p_\mu}{p\cdot k}\right) \tag{5.458}$$

Expanding:

$$= -\frac{p'^2}{(p'\cdot k)^2} + \frac{2p'\cdot p}{(p'\cdot k)(p\cdot k)} - \frac{p^2}{(p\cdot k)^2} \tag{5.459}$$

$$= -\frac{m^2}{(p'\cdot k)^2} + \frac{2p'\cdot p}{(p'\cdot k)(p\cdot k)} - \frac{m^2}{(p\cdot k)^2} \tag{5.460}$$

The unpolarized soft bremsstrahlung cross section is:

$$\boxed{\left(\frac{d\sigma}{d\Omega'}\right)_{\mathrm{B}} = \left(\frac{d\sigma}{d\Omega'}\right)_0 \frac{(-\alpha)}{(2\pi)^2}\left[\frac{p'}{p'\cdot k} - \frac{p}{p\cdot k}\right]^2 \frac{d^3k}{\omega} \qquad (\omega \approx 0)} \tag{5.461}$$

The overall minus sign ensures the cross section is positive (since $p'^2 = p^2 = m^2$ and the dominant term is $2p'\cdot p/(p'\cdot k)(p\cdot k)$ which is positive for small scattering angles).

The infrared divergence

Equation (5.461) reveals a remarkable and troubling feature: the cross section diverges as $\omega \to 0$. This is the famous **infrared catastrophe**.

Origin of the divergence

The divergence arises from the interplay of two factors with opposite ω-dependence.

(i) The soft factor diverges as $1/\omega$. In the soft limit $k \to 0$, the intermediate electron lines approach the mass shell: $(p-k)^2 - m^2 = -2p\cdot k \to 0$ and $(p'+k)^2 - m^2 = 2p'\cdot k \to 0$. Each term in the soft factor individually diverges as $1/\omega$, since $p\cdot k = \omega(E - |\vec{p}|\cos\theta_{pk})$ and $p'\cdot k = \omega(E - |\vec{p}|\cos\theta_{p'k})$, and one must verify that their difference does not cancel this divergence. Factoring out $1/\omega$:

$$\frac{p'\cdot\varepsilon}{p'\cdot k} - \frac{p\cdot\varepsilon}{p\cdot k} = \frac{1}{\omega}\left[\frac{p'\cdot\varepsilon}{E - |\vec{p}|\cos\theta_{p'k}} - \frac{p\cdot\varepsilon}{E - |\vec{p}|\cos\theta_{pk}}\right] \tag{5.462}$$

The expression in brackets is a finite, ω-independent angular factor. Crucially, it does *not* vanish: the two terms have different angular denominators ($\theta_{p'k} \neq \theta_{pk}$ in general, since $\vec{p}' \neq \vec{p}$ after scattering), so their difference is generically non-zero. The soft factor therefore behaves as $\sim 1/\omega$ and its **modulo squared** as $\sim 1/\omega^2$.

Note that the cancellation *would* occur in the forward limit $\vec{p}' = \vec{p}$ (no scattering), which is physically expected: a charge moving at constant velocity does not radiate.

(ii) The photon phase space contributes $\omega\, d\omega$. Writing $d^3k = |\vec{k}|^2 d|\vec{k}| d\Omega_k$ and noting that for massless photons $\omega = |\vec{k}|$:

$$\frac{d^3k}{\omega} = \frac{|\vec{k}|^2 d|\vec{k}|}{\omega} d\Omega_k = \omega\, d\omega\, d\Omega_k \tag{5.463}$$

(The extra $1/\omega$ in d^3k/ω comes from the Lorentz-invariant phase-space measure $d^3k/(2\omega)$.)

(iii) Combining propagators and phase space:

$$d\sigma_{\text{brem}} \sim \underbrace{\frac{1}{\omega^2}}_{\text{soft factor}^2} \times \underbrace{\omega\, d\omega\, d\Omega_k}_{\text{phase space}} = \frac{d\omega}{\omega} d\Omega_k \tag{5.464}$$

The angular integral $\int d\Omega_k$ is finite. But the integral over photon energy:

$$\int_0^{\Delta E} \frac{d\omega}{\omega} = \left[\ln \omega\right]_0^{\Delta E} \longrightarrow +\infty \tag{5.465}$$

is **logarithmically divergent** at the lower limit. The phase-space suppression ($\sim \omega$) is not strong enough to overcome the propagator singularity ($\sim 1/\omega^2$); the net $1/\omega$ behaviour is at the borderline of integrability, and diverges.

Physical interpretation: A zero-energy photon cannot be distinguished from no photon at all, yet the cross section to emit such a photon is infinite—the **infrared catastrophe**.

Regularization with a photon mass

Following Feynman, we can regulate the divergence by giving the photon a small fictitious mass λ. At the end of the calculation, we take $\lambda \to 0$ to recover QED. With $k^2 = \lambda^2$:

$$\omega_\lambda = \sqrt{\lambda^2 + |\vec{k}|^2} \tag{5.466}$$

The propagator denominators become:

$$(p-k)^2 - m^2 = -2p \cdot k + \lambda^2 \tag{5.467}$$

$$(p'+k)^2 - m^2 = 2p' \cdot k + \lambda^2 \tag{5.468}$$

The modified amplitude is:

$$\mathcal{M} = -ie^2\bar{u}(\vec{p}\,')\left[\not{A}_{\text{ext}}\frac{\not{p}-\not{k}+m}{-2p\cdot k+\lambda^2}\not{\epsilon}+\not{\epsilon}\frac{\not{p}'+\not{k}+m}{2p'\cdot k+\lambda^2}\not{A}_{\text{ext}}\right]u(\vec{p}) \tag{5.469}$$

For a massive photon, there are three polarization states. The polarization sum becomes:

$$\sum_{r=1}^{3}\varepsilon_{r\mu}\varepsilon^*_{r\nu} = -g_{\mu\nu}+\frac{k_\mu k_\nu}{\lambda^2} \tag{5.470}$$

The term $k_\mu k_\nu/\lambda^2$ vanishes by current conservation in the limit $\lambda \to 0$, so it does not affect the physical result.

The regularized bremsstrahlung cross section, integrated over photon energies $\lambda \le \omega_\lambda \le \Delta E$, is:

$$\left(\frac{d\sigma}{d\Omega'}\right)_{\text{B}} = \left(\frac{d\sigma}{d\Omega'}\right)_0 \alpha B(\lambda) \tag{5.471}$$

where

$$B(\lambda) = \frac{-1}{(2\pi)^2}\int_{\lambda\le\omega_\lambda\le\Delta E}\frac{d^3k}{\omega_\lambda}\left[\frac{2p'}{2p'\cdot k+\lambda^2}+\frac{2p}{-2p\cdot k+\lambda^2}\right]^2 \tag{5.472}$$

For $\lambda > 0$, the integral $B(\lambda)$ is finite. As $\lambda \to 0$, $B(\lambda) \to +\infty$ logarithmically.

The experimental cross section

In any real experiment, there is a finite energy resolution ΔE. A photon with energy $\omega < \Delta E$ cannot be distinguished from no photon at all. Therefore, what the detector reports as "elastic" is the sum

$$\left(\frac{d\sigma}{d\Omega'}\right)^{\text{elastic-like}}_{\text{Exp}} = \left(\frac{d\sigma}{d\Omega'}\right)_{\text{El}} + \left(\frac{d\sigma}{d\Omega'}\right)^{\text{soft}}_{\text{B}} \tag{5.473}$$

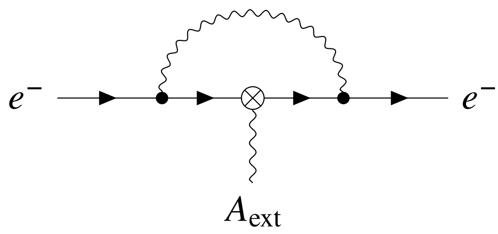
where:

- $(d\sigma/d\Omega')_{\text{El}}$ is the genuine elastic scattering cross section (no photon emitted) including its $O(\alpha)$ virtual radiative corrections (vertex, self-energy, etc.; see below).
- $(d\sigma/d\Omega')^{\text{soft}}_{\text{B}}$ is the *soft* bremsstrahlung contribution (5.471) integrated over $\lambda \le \omega \le \Delta E$ – the region where a photon *is* emitted but is too soft to be resolved by the detector.

Photons with $\omega > \Delta E$ (*hard* bremsstrahlung) *are* detected and constitute a separate, distinguishable inelastic channel, not included in the elastic-like signal.

Why the vertex correction enters at the same order

The soft bremsstrahlung cross section (5.471) is of order α relative to the lowest-order elastic cross section. For consistency, we must include the order-α **radiative corrections** to elastic scattering as well. These come from loop diagrams such as the **vertex correction**:



This correction is of the *same* order in α as bremsstrahlung, despite involving two extra vertices rather than one. To see why, we need to count powers of e carefully in the *cross section*, not in the amplitude.

Bremsstrahlung (one real photon emitted). The emission adds one QED vertex to the Born amplitude, so $\mathcal{M}_{\text{brem}} \sim e\,\mathcal{M}_0$. Since bremsstrahlung has a photon in the final state while elastic scattering does not, the two processes are *distinguishable*—there is no interference between $\mathcal{M}_0$ and $\mathcal{M}_{\text{brem}}$. The bremsstrahlung cross section is the **modulus squared** of the emission amplitude:

$$\sigma_{\text{brem}} \sim |\mathcal{M}_{\text{brem}}|^2 \sim e^2 |\mathcal{M}_0|^2 \sim \alpha\,\sigma_0 \tag{5.474}$$

Vertex correction (one virtual photon exchanged in a loop). The loop adds two extra QED vertices (e^2) but no extra external particles. The corrected amplitude is $\mathcal{M}_0 + \delta\mathcal{M}$ with $\delta\mathcal{M} \sim e^2\,\mathcal{M}_0$. Since the final state is the *same* as the Born process (no photon), the correction contributes through **interference** with $\mathcal{M}_0$:

$$\sigma_{\text{el}} \sim |\mathcal{M}_0 + \delta\mathcal{M}|^2 = |\mathcal{M}_0|^2 + \underbrace{2\,\text{Re}(\mathcal{M}_0^* \delta\mathcal{M})}_{\sim e^2 |\mathcal{M}_0|^2} + \underbrace{|\delta\mathcal{M}|^2}_{\sim e^4 |\mathcal{M}_0|^2} \tag{5.475}$$

The interference term is *linear* in $\delta\mathcal{M}$, so despite $\delta\mathcal{M}$ carrying e^2, the correction to the cross section is:

$$\delta\sigma_{\text{el}} \sim e^2 |\mathcal{M}_0|^2 \sim \alpha\,\sigma_0 \tag{5.476}$$

Summary. The key mechanism is:

	Real photon (bremsstrahlung)	**Virtual photon (vertex)**
Extra powers of e in $\mathcal{M}$	e^1	e^2
Enters σ via	modulus squared ($e^1 \to e^2$)	interference ($e^2 \to e^2$)
Correction to σ	both $\sim \alpha\,\sigma_0$	

One extra power of e in the amplitude, squared, gives e^2. Two extra powers of e, taken linearly (interference), also give e^2. The two mechanisms conspire to produce corrections of the same order—the necessary condition for their infrared divergences to cancel.

Including the vertex correction:

$$\left(\frac{d\sigma}{d\Omega'}\right)_{\text{El}} = \left(\frac{d\sigma}{d\Omega'}\right)_0 [1+\alpha R(\lambda)] \tag{5.477}$$

where $R(\lambda)$ is a function of the photon mass regulator that also diverges as $\lambda \to 0$.

The Bloch-Nordsieck theorem

Combining eqs. (5.473), (5.471), and (5.477):

$$\boxed{\left(\frac{d\sigma}{d\Omega'}\right)_{\text{Exp}} = \left(\frac{d\sigma}{d\Omega'}\right)_0 \left\{1+\alpha\left[B(\lambda)+R(\lambda)\right]+O(\alpha^2)\right\}} \tag{5.478}$$

The Bloch-Nordsieck theorem (1937) states: *The infrared divergences in $B(\lambda)$ and $R(\lambda)$ cancel exactly*, so that

$$\lim_{\lambda\to 0}\left[B(\lambda)+R(\lambda)\right] = \text{finite} \tag{5.479}$$

This remarkable result can be understood as follows:

- $B(\lambda) \to +\infty$ as $\lambda \to 0$ (too many soft photons emitted)
- $R(\lambda) \to -\infty$ as $\lambda \to 0$ (virtual photon loops diverge)
- The two divergences are intimately related and cancel exactly

Physical understanding

The infrared catastrophe arises from treating soft bremsstrahlung and elastic scattering as separate processes. But this separation is artificial: one *always* observes elastic scattering together with some soft bremsstrahlung (photons too soft to detect). The cross section for this combined process is finite.

The Bloch-Nordsieck theorem tells us that the infrared divergence is not a defect of QED, but rather a consequence of asking an unphysical question. The question "What is the probability of scattering with no photon emission?" has no well-defined answer in QED (the probability is zero!). The physically meaningful question is "What is the probability of scattering with photon emission below the detection threshold ΔE?" and this has a finite answer.

Exponentiation and the vanishing of pure elastic scattering

Including all orders of perturbation theory, the Bloch-Nordsieck theorem implies:

$$\left(\frac{d\sigma}{d\Omega'}\right)_{\text{Exp}} = \left(\frac{d\sigma}{d\Omega'}\right)_0 \exp\{\alpha\,[B(\lambda)+R(\lambda)]\} \tag{5.480}$$

The elastic cross section alone would be:

$$\left(\frac{d\sigma}{d\Omega'}\right)_{\text{El}} = \left(\frac{d\sigma}{d\Omega'}\right)_0 e^{\alpha R(\lambda)} \tag{5.481}$$

Since $R(\lambda) \to -\infty$ as $\lambda \to 0$:

$$\boxed{\lim_{\lambda\to 0}\left(\frac{d\sigma}{d\Omega'}\right)_{\text{El}} = 0} \tag{5.482}$$

Remarkable conclusion: The pure elastic cross section, without any photon emission, **vanishes exactly**. There is no truly elastic scattering—all observed scattering is accompanied by emission of radiation. This is the quantum field theoretic manifestation of the classical result that an accelerated charge must radiate.

Physical applications and summary

Radiative corrections in precision experiments

The experimental cross section (5.478) depends on the energy resolution ΔE:

$$\left(\frac{d\sigma}{d\Omega'}\right)_{\text{Exp}} = \left(\frac{d\sigma}{d\Omega'}\right)_0 [1+\alpha f(\Delta E)] \tag{5.483}$$

where $f(\Delta E)$ is the finite radiative correction. In high-energy experiments, these corrections can be as large as 50% in some kinematic regions. They must be carefully calculated and either:

- Applied to theoretical predictions before comparing with data
- Subtracted from experimental data to obtain "radiatively corrected" cross sections

Universality of soft photon emission

The factorization property of eq. (5.453) is universal: it applies to soft photon emission in *any* process involving charged particles, not just Coulomb scattering. The soft photon factor

$$-e\left[\frac{p'\cdot\varepsilon}{p'\cdot k} - \frac{p\cdot\varepsilon}{p\cdot k}\right] \tag{5.484}$$

depends only on the external charged particle momenta and is independent of the details of the hard scattering process.

Summary

Concept	Key result
Bremsstrahlung ampl.	Factorizes: $\mathcal{M} = -e\mathcal{M}_0 \times$ (soft factor)
Infrared divergence	Cross section $\sim \int d\omega/\omega$ diverges as $\omega \to 0$
Physical origin	Zero-mass photon; internal lines go on-shell
Bloch-Nordsieck theorem	Divergences cancel between real and virtual photons
Pure elastic scattering	Vanishes: all scattering is accompanied by radiation

The infrared problem in QED is completely understood and poses no obstacle to making precise predictions. It teaches us that the naive separation of "elastic" and "inelastic" scattering is unphysical when massless particles are involved, and that physically meaningful questions always have finite answers.