6.3 Electron propagator renormalization in QED

Physical interpretation: mass and wavefunction renormalization

The electron self-energy $\Sigma(\not{p})$ encodes how virtual photon emission and reabsorption modify the electron's propagation through the vacuum. Unlike the photon self-energy (which affects charge renormalization), the electron self-energy leads to two distinct physical effects:

- **Mass renormalization:** The "bare" mass m_0 in the Lagrangian is not the physical mass we measure. Quantum corrections shift the pole of the propagator from $\not{p} = m_0$ to $\not{p} = m$, where m is the *physical* (renormalized) mass. The mass counterterm δm (the coefficient of $-\bar{\psi}\psi$ in the counterterm Lagrangian) absorbs the infinite part of this shift.
- **Wavefunction renormalization:** The residue of the propagator pole is modified by quantum corrections. The factor Z_2 (with $Z_2 - 1$ containing a UV divergence) relates the bare field ψ_0 to the renormalized field $\psi = Z_2^{-1/2}\psi_0$. This ensures that the renormalized propagator has unit residue at the physical mass pole.

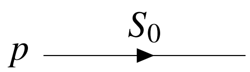
The on-shell renormalization scheme, commonly used in QED, imposes:

1. The propagator pole is at $\not{p} = m$ (physical mass).
2. The residue at the pole is i (canonical normalization).

These conditions uniquely determine δm and Z_2 in terms of the self-energy function.

Diagrammatic overview and the need for renormalization

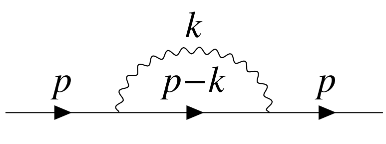
Free propagator. The starting point is the tree-level (free) electron propagator, represented diagrammatically as a single directed line:



$$= \quad S_0(\not{p}) = \frac{i}{\not{p} - m_0 + i\epsilon}$$

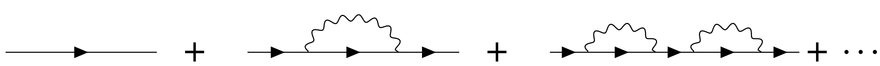
where m_0 is the bare mass parameter in the Lagrangian.

Self-energy insertion. Interactions modify propagation: even in the vacuum, the electron can emit and reabsorb virtual photons. The one-loop 1PI *self-energy* correction $-i\Sigma(\not{p})$ is given by:



$$= \quad -i\Sigma(\not{p})$$

Geometric series (Dyson resummation). Inserting one self-energy correction on the fermion line, then two, then three, etc., produces a geometric series:



Algebraically, this reads:

$$S(\not p) = S_0 + S_0\big(-i\Sigma\big)S_0 + S_0\big(-i\Sigma\big)S_0\big(-i\Sigma\big)S_0 + \cdots = \frac{i}{\not p - m_0 - \Sigma(\not p) + i\epsilon}. \quad (6.104)$$

Strictly speaking, $\Sigma(\not p)$ here denotes the *one-particle irreducible* (1PI) self-energy, i.e. the sum of all self-energy diagrams that cannot be disconnected by cutting a single internal fermion line. Using the 1PI self-energy avoids double counting: a reducible insertion (two 1PI blobs joined by a bare propagator) is already generated by the geometric series itself.

Why renormalization is needed. The loop integral defining $\Sigma(\not p)$ is ultraviolet divergent. At one loop:

$$\Sigma(\not p) \sim e_0^2 \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\mu(\not p - \not k + m_0)\gamma_\mu}{\big(k^2 + i\epsilon\big)\big((p-k)^2 - m_0^2 + i\epsilon\big)}. \quad (6.105)$$

Power counting: the Dirac numerator $\gamma^\mu(\not p - \not k + m)\gamma_\mu$ scales as k for large k, while the two propagators supply $1/k^4$. Including the measure $d^4k \sim k^3\,dk$, the integral scales as $k^3\,dk \cdot k/k^4 = dk$, giving a superficial degree of divergence $D = 1$ (linearly divergent). In dimensional regularization the linear power divergence is absent—all UV divergences manifest as $1/\epsilon$ poles—but the integral is still divergent and requires renormalization.

Lorentz decomposition. By Lorentz covariance, $\Sigma(\not p)$ is a 4×4 matrix in Dirac space that can only depend on $\not p$ and the identity matrix $\mathbb{1}$. The most general form is therefore

$$\Sigma(\not p) = \not p\,\Sigma_V(p^2) + m_0\,\Sigma_S(p^2), \quad (6.106)$$

where Σ_V and Σ_S are Lorentz-scalar functions of p^2. The coefficient of $\mathbb{1}$ is proportional to m_0 because in the chiral limit $m_0 \to 0$ the Lagrangian has a chiral symmetry that forces Σ to be purely proportional to $\not p$. In dimensional regularization ($d = 4 - \epsilon$), each scalar function contains an ultraviolet pole:

$$\Sigma_V(p^2) = \frac{c_V}{\epsilon} + \Sigma_V^{\text{fin}}(p^2), \qquad \Sigma_S(p^2) = \frac{c_S}{\epsilon} + \Sigma_S^{\text{fin}}(p^2), \quad (6.107)$$

where c_V, c_S are constants (independent of p^2) and the superscript "fin" denotes the finite remainder.

Mass renormalization and wave-function renormalization (Mandl–Shaw approach). We now show that the divergences in $\Sigma(\not p)$ can be absorbed into a redefinition of the mass and an overall rescaling of the field, without introducing counterterms explicitly.

Start from the Dyson-resummed propagator (6.104):

$$S(\not p) = \frac{i}{\not p - m_0 - \Sigma(\not p) + i\epsilon}. \quad (6.108)$$

The key idea is to expand $\Sigma(\not p)$ around the point $\not p = m$, where m is the *physical* (observed) mass, still to be determined:

$$\Sigma(\not p) = \underbrace{\Sigma(m)}_{\equiv\ A} + \underbrace{\left.\frac{\partial\Sigma}{\partial\not p}\right|_{\not p=m}}_{\equiv\ B}(\not p - m) + \Sigma_c(\not p)\,, \tag{6.109}$$

where $\Sigma_c(\not p)$ collects all higher-order terms in $(\not p - m)$ and satisfies

$$\Sigma_c(m) = 0\,, \qquad \left.\frac{\partial\Sigma_c}{\partial\not p}\right|_{\not p=m} = 0\,. \tag{6.110}$$

The constants A and B are divergent (they contain $1/\epsilon$ poles). In contrast, $\Sigma_c(\not p)$ is *finite*: the superficial degree of divergence of the self-energy is $D = 1$, so its UV-divergent part is a polynomial of degree ≤ 1 in $\not p$. The subtraction of A (degree 0) and $B(\not p - m)$ (degree 1) therefore removes all divergent pieces, leaving a finite remainder.

Substituting (6.109) into (6.108):

$$\begin{aligned} S(\not p) &= \frac{i}{\not p - m_0 - A - B(\not p - m) - \Sigma_c(\not p)} \\ &= \frac{i}{(1-B)(\not p - m) - (m_0 + A - m) - \Sigma_c(\not p)}\,. \end{aligned} \tag{6.111}$$

Step 1: fix the physical mass. We *define* m by requiring that the propagator has a pole at $\not p = m$. Since $\Sigma_c(m) = 0$, the denominator vanishes at $\not p = m$ if and only if

$$\boxed{m = m_0 + A = m_0 + \Sigma(m)\,.} \tag{6.112}$$

This determines the **mass shift**: the bare mass m_0 differs from the physical mass m by the self-energy evaluated on shell. Defining $\delta m \equiv m - m_0 = \Sigma(m)$, we see that δm is UV-divergent.

Step 2: fix the residue. With condition (6.112) satisfied, the propagator becomes

$$S(\not p) = \frac{i}{(1-B)\big(\not p - m\big) - \Sigma_c(\not p)}\,. \tag{6.113}$$

Factoring out $(1 - B)$ and defining the **wave-function renormalization constant**

$$\boxed{Z_2 \equiv \frac{1}{1-B} = \frac{1}{1-\Sigma'(m)}\,,} \tag{6.114}$$

where $\Sigma'(m) \equiv \partial\Sigma/\partial\not p\big|_{\not p=m}$, we obtain

$$S(\not p) = \frac{iZ_2}{\not p - m - Z_2\,\Sigma_c(\not p)}\,. \tag{6.115}$$

Since Z_2 is a momentum-independent constant and $\Sigma_c(m) = 0$ by construction, $Z_2\,\Sigma_c(\not{p}) \to Z_2 \cdot 0 = 0$ as $\not{p} \to m$, and the propagator (6.115) has a pole at $\not{p} = m$ with residue Z_2:

$$S(\not{p}) \xrightarrow{\not{p}\to m} \frac{iZ_2}{\not{p} - m}\,. \tag{6.116}$$

To restore canonical normalization (unit residue), we define the **renormalized field** $\psi = Z_2^{-1/2}\psi_0$, whose propagator is

$$\begin{aligned} S_R(\not{p}) \equiv Z_2^{-1}\,S(\not{p}) &= \frac{i}{\not{p} - m - Z_2\,\Sigma_c(\not{p})} \\ &\xrightarrow{\not{p}\to m} \frac{i}{\not{p} - m}\,. \end{aligned} \tag{6.117}$$

At the pole, the divergent Z_2 in the numerator of S is cancelled by the Z_2^{-1} prefactor, giving a residue $Z_2/Z_2 = 1$. Away from the pole, the product $Z_2\,\Sigma_c(\not{p})$ appears divergent because Z_2 has $1/\epsilon$ poles while $\Sigma_c \neq 0$. However, $Z_2 = 1 + \mathcal{O}(\alpha)$ and $\Sigma_c = \mathcal{O}(\alpha)$, so $Z_2\,\Sigma_c = \Sigma_c + \mathcal{O}(\alpha^2)$: at one-loop accuracy the product reduces to Σ_c alone, which is finite. The $\mathcal{O}(\alpha^2)$ divergent remainder is absorbed by two-loop counterterms, and so on order by order.

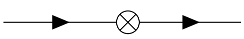
Remark (Källén–Lehmann bound). Independently of perturbation theory, the spectral representation of the exact propagator (see Section 8.6) implies $0 \le Z_2 \le 1$: the exact Z_2 is finite and measures the overlap of ψ_0 with the one-particle state. The $1/\epsilon$ poles in the perturbative expansion of Z_2 are artifacts of computing the exact (finite) quantity order by order; they are systematically removed by renormalization.

To summarize: the two renormalization conditions—pole at $\not{p} = m$ and unit residue—uniquely determine

$$\delta m = \Sigma(m)\,, \qquad Z_2 = \frac{1}{1 - \Sigma'(m)}\,.$$

In perturbation theory both quantities contain $1/\epsilon$ poles; these are absorbed by the counterterms introduced below.

Counterterm reformulation. The same result can be organized systematically via counterterms. Splitting the bare Lagrangian as $\mathcal{L}_0 = \mathcal{L}_{\text{ren}} + \delta\mathcal{L}$, the fermion counterterm $\delta\mathcal{L}_{\bar\psi\psi} = (Z_2 - 1)\,\bar\psi\, i\not{\partial}\psi - \delta m\,\bar\psi\psi$ produces a two-point vertex (denoted by $\times$):

 $$= i\Sigma_{\text{ct}}(\not{p}) \;=\; i\Big[(Z_2 - 1)\not{p} - \delta m\Big]$$

The values of $(Z_2 - 1)$ and δm are precisely those determined above from the pole structure. This vertex is added to each order of perturbation theory so that the renormalized self-energy $\Sigma_R = \Sigma + \Sigma_{\text{ct}}$ is finite.

Renormalized propagator.

The central question is: *why does the counterterm vertex end up inside the denominator of the propagator, next to the loop self-energy?* The answer is Dyson resummation, which we now spell out step by step.

(i) The counterterm is a 1PI two-point insertion. A 1PI (one-particle-irreducible) two-point diagram is an amputated graph with exactly two external fermion legs that cannot be split into two disconnected pieces by cutting a single internal line. Both contributions at order α satisfy this:

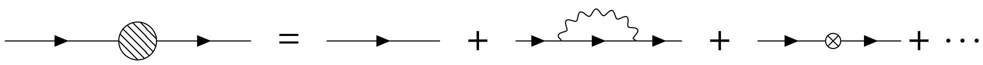
- the loop self-energy (two amputated legs, one photon bridge);
- the counterterm vertex (two amputated legs, a single insertion).

Summing all 1PI contributions defines the total **1PI two-point function**

$$-i\Sigma_R(\not{p}) \;=\; \underbrace{-i\Sigma(\not{p})}_{\text{loop}} \;+\; \underbrace{+i\Sigma_{\rm ct}(\not{p})}_{\text{counterterm}} \quad\Longrightarrow\quad \Sigma_R(\not{p}) \;\equiv\; \Sigma(\not{p}) + \Sigma_{\rm ct}(\not{p}). \quad (6.118)$$

The *opposite* sign of the loop's $-i\Sigma$ and the counterterm's $+i\Sigma_{\rm ct}$ is precisely what allows the counterterm to subtract the UV pole of Σ.

(ii) Dyson resummation puts 1PI insertions in the denominator. The full propagator is obtained by stringing together an arbitrary number of 1PI insertions on the bare propagator line:



Algebraically:

$$S_R \;=\; S_0 \;+\; S_0\,(-i\Sigma_R)\,S_0 \;+\; S_0\,(-i\Sigma_R)\,S_0\,(-i\Sigma_R)\,S_0 \;+\; \cdots \tag{6.119}$$

which is a geometric series. Using $S_0 = i/(\not{p} - m + i\epsilon)$ and summing,

$$\boxed{S_R(\not{p}) \;=\; \frac{i}{\not{p} - m - \Sigma_R(\not{p}) + i\epsilon}\,, \qquad \Sigma_R(\not{p}) \equiv \Sigma(\not{p}) + \Sigma_{\rm ct}(\not{p})} \tag{6.120}$$

This is the reason the counterterm *appears in the denominator*: it is a 1PI two-point insertion, and Dyson resummation stacks every such insertion into the inverse propagator $S_R^{-1} = -i\,(\not{p} - m - \Sigma_R)$.

(iii) Explicit cancellation of the UV divergence. We now check that Σ_R is indeed finite. The one-loop self-energy admits the Dirac decomposition

$$\Sigma(\not{p}) = \not{p}\,\Sigma_V(p^2) + m\,\Sigma_S(p^2),$$

and in dimensional regularization each of Σ_V, Σ_S has the structure

$$\Sigma_V(p^2) = \Sigma_V^{\rm div} + \Sigma_V^{\rm fin}(p^2), \qquad \Sigma_S(p^2) = \Sigma_S^{\rm div} + \Sigma_S^{\rm fin}(p^2), \tag{6.121}$$

where $\Sigma^{\rm div}_{V,S} \propto 1/\epsilon$ are p-independent constants. (They must be constants because the superficial degree of divergence of the fermion self-energy is $D = 1$: the UV-divergent part is a polynomial in $\not{p}$ of degree ≤ 1, so $\not{p}\Sigma_V$ and $m\Sigma_S$ capture all of it.)

The counterterm is, by construction, itself a polynomial of degree 1 in $\not{p}$:

$$\Sigma_{\rm ct}(\not{p}) = (Z_2 - 1)\,\not{p} - \delta m,$$

with $(Z_2 - 1)$ and δm two *free constants* to be chosen. Adding, the two 1PI contributions combine as

$$\Sigma_R(\not{p}) \;=\; \left[\Sigma_V(p^2) + (Z_2 - 1)\right]\not{p} \;+\; \left[m\,\Sigma_S(p^2) - \delta m\right]. \tag{6.122}$$

Matching (6.122) against the divergent pieces of (6.121) shows that the $1/\epsilon$ poles cancel *separately* in the $\not{p}$-coefficient and in the scalar piece iff

$$\boxed{(Z_2 - 1)\big|_{\rm div} = -\Sigma^{\rm div}_V\,, \qquad \delta m\big|_{\rm div} = m\,\Sigma^{\rm div}_S} \tag{6.123}$$

With these choices,

$$\Sigma_R(\not{p}) = \left[\Sigma^{\rm fin}_V(p^2) + \Delta Z_2\right]\not{p} + \left[m\,\Sigma^{\rm fin}_S(p^2) - \Delta m\right] \qquad \text{(finite)},$$

where ΔZ_2 and Δm denote the finite remainders of $(Z_2 - 1)$ and δm; these finite pieces are fixed by the chosen scheme (on-shell, $\overline{\rm MS}$, ...). The full denominator of (6.120) therefore becomes

$$\not{p} - m - \Sigma_R(\not{p}) \;=\; (\not{p} - m) - \Sigma(\not{p}) - (Z_2 - 1)\not{p} + \delta m,$$

in which every $1/\epsilon$ pole on the right-hand side has been explicitly absorbed, term by term, by the counterterm coefficients (6.123).

One-loop calculation (counterterm method)

Counterterm Feynman rule. We derive the fermion counterterm from the bare Lagrangian. The free fermion part reads

$$\mathcal{L}_0^{\rm free} = \bar\psi_0\big(i\not{\partial} - m_0\big)\psi_0\,.$$

We now express everything in terms of renormalized quantities. Introduce the renormalized field ψ and the physical mass m via

$$\psi_0 = \sqrt{Z_2}\,\psi\,, \qquad m_0 = m + \delta m_0\,,$$

where Z_2 is the wave-function renormalization constant and δm_0 is the bare mass shift. Substituting:

$$\mathcal{L}_0^{\rm free} = Z_2\,\bar\psi\big(i\not{\partial} - m - \delta m_0\big)\psi\,.$$

We split this into $\mathcal{L}_{\text{ren}} + \delta\mathcal{L}$ by writing $Z_2 = 1 + (Z_2 - 1)$ and expanding:

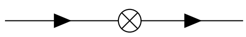
$$\mathcal{L}_0^{\text{free}} = \bar{\psi}\, i\partial\!\!\!/\, \psi - m\bar{\psi}\psi + (Z_2 - 1)\,\bar{\psi}\, i\partial\!\!\!/\, \psi - \big[\underbrace{m(Z_2 - 1) + Z_2\,\delta m_0}_{\equiv\ \delta m}\big]\,\bar{\psi}\psi\,. \tag{6.124}$$

The first two terms are the renormalized Lagrangian $\mathcal{L}_{\text{ren}} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$; the remaining terms form the counterterm

$$\delta\mathcal{L}_{\bar{\psi}\psi} = (Z_2 - 1)\,\bar{\psi}\, i\partial\!\!\!/\psi - \delta m\,\bar{\psi}\psi\,, \tag{6.125}$$

with $\delta m \equiv m(Z_2 - 1) + Z_2\,\delta m_0 = Z_2 m_0 - m$. Note that δm combines the mass shift and the wave-function renormalization into a single coefficient.

In momentum space, $i\partial\!\!\!/ \to p\!\!\!/$, so the counterterm vertex gives the Feynman rule

 $= i\Big[(Z_2 - 1)\,p\!\!\!/ - \delta m\Big]$

and the renormalized self-energy becomes

$$\Sigma_R(p\!\!\!/) = \Sigma(p\!\!\!/) + (Z_2 - 1)\,p\!\!\!/ - \delta m\,. \tag{6.126}$$

One-loop self-energy integral (Feynman gauge). In dimensional regularization $d = 4 - \epsilon$ and Feynman gauge $\xi = 1$:

$$D_{\mu\nu}(k) = \frac{-ig_{\mu\nu}}{k^2 + i\epsilon}, \qquad S_0(p) = \frac{i}{p\!\!\!/ - m + i\epsilon}\,. \tag{6.127}$$

Which m and e enter the loop? In the counterterm method the bare Lagrangian has been split as $\mathcal{L}_0 = \mathcal{L}_{\text{ren}} + \delta\mathcal{L}$; propagators and vertices of $\mathcal{L}_{\text{ren}}$ are written in terms of the **renormalized** quantities m and e (the physical mass and coupling). The bare $m_0 = m + \delta m_0$ and $e_0 = \mu^{\epsilon/2} Z_1 (Z_2\sqrt{Z_3})^{-1} e$ do *not* appear in (6.127): their deviations from m, e are absorbed into the counterterm coefficients $(Z_2 - 1)$, δm, $(Z_1 - 1)$, $(Z_3 - 1)$, treated as separate perturbative insertions. *Throughout this subsection m and e always mean the renormalized quantities.*

The one-loop self-energy is

$$\Sigma(p\!\!\!/) = -e^2\mu^{\epsilon} \int \frac{d^d k}{(2\pi)^d}\, \frac{\gamma^{\mu}(p\!\!\!/ - k\!\!\!/ + m)\gamma_{\mu}}{\big((p-k)^2 - m^2 + i\epsilon\big)\,(k^2 + i\epsilon)}\,. \tag{6.128}$$

The factor μ^{ϵ} keeps the renormalized coupling e dimensionless in $d = 4 - \epsilon$ dimensions.

Dirac algebra and Feynman parametrization. Using the d-dimensional identities $\gamma^{\mu}\gamma^{\alpha}\gamma_{\mu} = (2 - d)\gamma^{\alpha}$ and $\gamma^{\mu}\gamma_{\mu} = d$:

$$\gamma^{\mu}(p\!\!\!/ - k\!\!\!/ + m)\gamma_{\mu} = (2 - d)(p\!\!\!/ - k\!\!\!/) + dm\,. \tag{6.129}$$

Combining denominators with Feynman parameter x and shifting $\ell = k-(1-x)p$:

$$\Sigma(\not{p}) = -e^2\mu^\epsilon \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \frac{(2-d)(x\not{p}-\not{\ell})+dm}{(\ell^2-\Delta+i\epsilon)^2}, \tag{6.130}$$

where $\Delta = (1-x)m^2 - x(1-x)p^2$. The $\not{\ell}$ term vanishes by symmetry.

Scalar master integral — origin of $\Gamma(\epsilon/2)$.

After dropping $\not{\ell}$, the only loop integral left in (6.130) is the *standard d-dimensional scalar bubble* with a double pole at $\ell^2 = \Delta$. Wick rotating $\ell^0 \to i\ell_E^0$ (so that $d^d\ell \to i\, d^d\ell_E$ and $\ell^2-\Delta \to -(\ell_E^2+\Delta)$) reduces it to an ordinary Euclidean integral:

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{1}{(\ell^2-\Delta+i\epsilon)^2} = \frac{i}{(4\pi)^{d/2}}\,\Gamma\left(2-\frac{d}{2}\right)\Delta^{d/2-2}. \tag{6.131}$$

The Euclidean integral has been done using the well-known spherical formula $\int d^d\ell_E\,(\ell_E^2+\Delta)^{-n} = (4\pi)^{d/2}\Delta^{d/2-n}\Gamma(n-d/2)/\Gamma(n)$. With $d=4-\epsilon$, $2-d/2=\epsilon/2$, so the master integral acquires the factor $\Gamma(\epsilon/2)$, whose Laurent expansion contains a simple pole:

$$\Gamma\left(\frac{\epsilon}{2}\right) = \frac{2}{\epsilon} - \gamma_E + O(\epsilon). \tag{6.132}$$

This is where $\Gamma(\epsilon/2)$ comes from: the logarithmic UV divergence that a cutoff regulator would present as $\ln(\Lambda^2/\Delta)$ is replaced, in dimensional regularization, by $\Gamma(\epsilon/2)\,\Delta^{-\epsilon/2} \sim 2/\epsilon + \ldots$.

Explicit form of $\Sigma(\not{p})$ and its $1/\epsilon$ pole. Substituting (6.131) into (6.130) (and using $\not{\ell} \to 0$):

$$\Sigma(\not{p}) = -\frac{i\,e^2\mu^\epsilon}{(4\pi)^{d/2}}\,\Gamma\left(\frac{\epsilon}{2}\right)\int_0^1 dx\,\left[(2-d)\,x\,\not{p} + d\,m\right]\Delta^{-\epsilon/2}. \tag{6.133}$$

Expanding in ϵ ($2-d=-2+\epsilon$, $d=4-\epsilon$, $(4\pi)^{-d/2} = (4\pi)^{-2}+O(\epsilon)$, $\Delta^{-\epsilon/2} = 1+O(\epsilon)$), the leading $1/\epsilon$ pole is

$$\Sigma^{\rm div}(\not{p}) = -\frac{i\,e^2}{(4\pi)^2}\,\frac{2}{\epsilon}\int_0^1 dx\,\left[-2x\,\not{p}+4m\right] = -\frac{i\,e^2}{16\pi^2}\,\frac{2}{\epsilon}\left[-\not{p}+4m\right], \tag{6.134}$$

where we used $\int_0^1 x\,dx = \frac{1}{2}$, $\int_0^1 dx = 1$. The crucial structural point is: *the divergent part is a polynomial of degree 1 in $\not{p}$*, with p-independent coefficients, exactly matching the polynomial form of the counterterm $(Z_2-1)\not{p} - \delta m$. Writing $\Sigma^{\rm div}$ in the Dirac decomposition $\Sigma^{\rm div}(\not{p}) = \Sigma_V^{\rm div}\,\not{p} + m\,\Sigma_S^{\rm div}$, eq. (6.134) identifies

$$\Sigma_V^{\rm div} = \frac{i\,e^2}{16\pi^2}\,\frac{2}{\epsilon}, \qquad m\,\Sigma_S^{\rm div} = -\frac{i\,e^2}{16\pi^2}\,\frac{2}{\epsilon}\,(4m) \implies \Sigma_S^{\rm div} = -\frac{4\,i\,e^2}{16\pi^2}\,\frac{2}{\epsilon}.$$

Cancellation of the pole in (6.122) is the polynomial equality

$$(Z_2-1)\big|_{\text{div}}\not{p}-\delta m\big|_{\text{div}} \;=\; -\Sigma^{\text{div}}(\not{p}), \tag{6.135}$$

which is *one equation between two polynomials of degree 1 in* $\not{p}$. Two such polynomials are equal iff they agree coefficient by coefficient, so (6.135) splits into *two independent scalar equations*:

$$\underbrace{(Z_2-1)\big|_{\text{div}}=-\Sigma_V^{\text{div}},}_{\text{matching the }\not{p}\text{ coefficient}} \qquad \underbrace{\delta m\big|_{\text{div}}=m\,\Sigma_S^{\text{div}}}_{\text{matching the }\not{p}\text{-independent part}} \quad .$$

Each determines exactly one counterterm constant, as expected.
Dirac decomposition. The one-loop self-energy has the structure

$$\Sigma(\not{p}) = \not{p}\,\Sigma_V(p^2)+m\,\Sigma_S(p^2), \tag{6.136}$$

so the renormalized self-energy becomes

$$\Sigma_R(\not{p}) = \not{p}\left[\Sigma_V(p^2)+(Z_2-1)\right]+m\,\Sigma_S(p^2)-\delta m\,. \tag{6.137}$$

Thus (Z_2-1) adjusts the $\not{p}$ coefficient, while δm adjusts the scalar part.

On-shell renormalization conditions

Pole condition. On-shell renormalization imposes a pole at the physical mass:

$$S_R^{-1}(\not{p})\big|_{\not{p}=m}=0 \quad\Rightarrow\quad \delta m = m\left[\Sigma_V(m^2)+\Sigma_S(m^2)\right]+m(Z_2-1). \tag{6.138}$$

Residue condition. Unit residue at the pole requires:

$$\frac{\partial}{\partial\not{p}}S_R^{-1}(\not{p})\bigg|_{\not{p}=m}=1 \quad\Rightarrow\quad Z_2-1=-\frac{\partial}{\partial\not{p}}\Sigma(\not{p})\bigg|_{\not{p}=m}. \tag{6.139}$$

Cancellation of divergences. Combining the inverse propagator:

$$S_R^{-1}(\not{p}) = \not{p}-m-\Sigma(\not{p})-(Z_2-1)\not{p}+\delta m, \tag{6.140}$$

the divergent parts of $\Sigma(\not{p})$ are cancelled by those of $(Z_2-1)\not{p}-\delta m$. The finite pieces are fixed by the chosen scheme (on-shell as above, or $\overline{\text{MS}}$ where one subtracts only the pole).