## 6.4 One-loop vertex correction in QED and its renormalization

**Physical motivation: what is the vertex correction?**
**The QED vertex at tree level.** In classical (tree-level) QED, the interaction between an electron and a photon is described by a single vertex where an electron absorbs or emits a photon. This interaction is encoded in the Lagrangian term (see Chapter 5 for the sign conventions)

$$\mathcal{L}_{\text{int}} = +e\,\bar{\psi}\gamma^{\mu}\psi\,A_{\mu}, \tag{6.141}$$

which gives the Feynman rule $+ie\gamma^{\mu}$ for the electron-photon vertex. At tree level, this coupling is simply the electric charge multiplying the tree vertex. In renormalized perturbation theory this is written in terms of the renormalized coupling $e$ (with the usual $\mu^{\epsilon/2}$ factor in $d = 4-\epsilon$), while the difference from the bare charge $e_0$ is carried by counterterm constants.
**Quantum corrections modify the vertex.** However, quantum field theory tells us that the vacuum is not empty: virtual particle-antiparticle pairs constantly fluctuate in and out of existence. When an electron interacts with a photon, these quantum fluctuations modify the interaction. In particular, the electron can:

- emit a virtual photon just before interacting with the external photon,
- reabsorb the virtual photon just after,
- creating a "loop" of virtual particles around the vertex.

This process is called the **vertex correction**: quantum loops "dress" the bare vertex and effectively modify the strength and structure of the electron-photon coupling.
**Physical consequences.** The vertex correction has profound physical implications:

1. **Anomalous magnetic moment:** The electron's magnetic moment deviates from the Dirac prediction $g = 2$. The leading correction, computed by Schwinger in 1948, gives $g-2 = \alpha/\pi \approx 0.00116$, in spectacular agreement with experiment.
2. **Charge renormalization:** Together with vacuum polarization, the vertex correction contributes to the running of the electric charge with energy scale.
3. **Form factors:** The corrected vertex introduces momentum-dependent "form factors" that describe how the electron's charge and magnetic moment are distributed.

**Technical aspects.** The vertex correction is the most subtle of the one-loop structures because:

(i) it is a genuine three-point function with nontrivial kinematics (two independent momenta $p$ and $p'$);

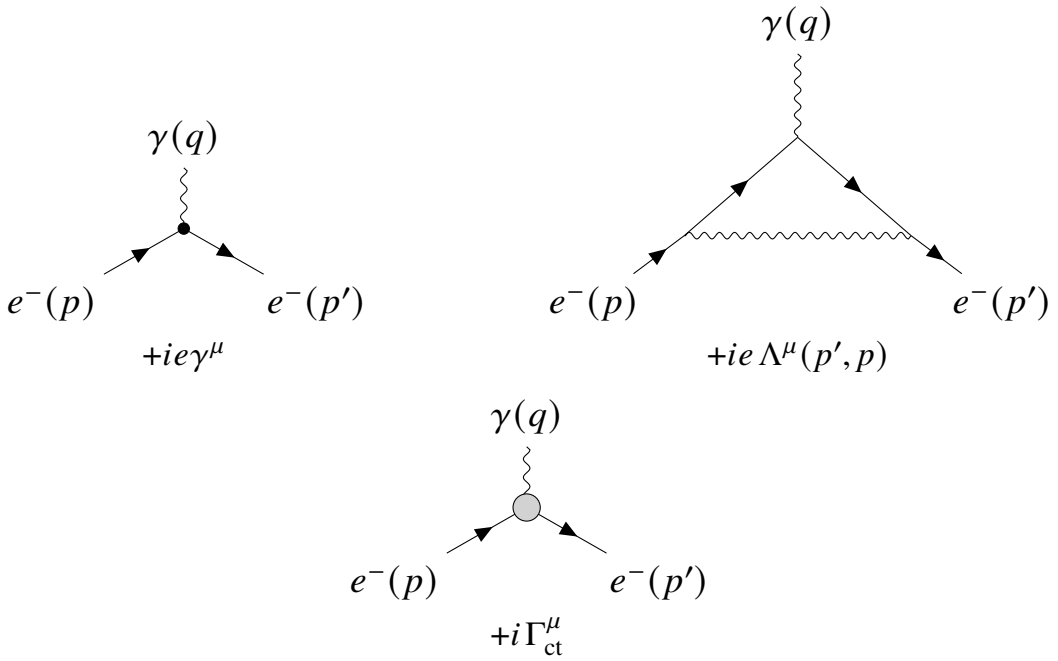
(ii) it is constrained by gauge invariance through the Ward–Takahashi identity, which relates the vertex to the electron self-energy;
(iii) for on-shell external fermions it contains infrared (IR) singularities (separate from UV divergences), which cancel only when including soft photon emission (Bloch–Nordsieck theorem).

Here we focus on the UV renormalization via counterterms; IR issues are noted but not fully treated.

**Diagrams and definitions**

Diagrammatically, the proper vertex is the sum of the tree vertex and the one-loop triangle, plus the counterterm vertex. The renormalized vertex function is:

$$\Gamma^{\mu}_{R}(p',p) = \gamma^{\mu} + \Lambda^{\mu}(p',p) + \Gamma^{\mu}_{\mathrm{ct}}. \tag{6.142}$$




The third diagram is the *counterterm vertex*: it represents a local insertion $\Gamma^{\mu}_{\mathrm{ct}}$, proportional to $\gamma^{\mu}$, whose coefficient is an as-yet-undetermined constant. Its value will be fixed by a renormalization condition (see below, p. 291). The grey blob distinguishes it from the ordinary QED vertex.

We use the kinematic conventions

$$q \equiv p' - p, \qquad p^2 = p'^2 = m^2 \ \text{(on-shell, when stated explicitly)}. \tag{6.143}$$

The proper vertex $\Gamma^{\mu}(p',p)$ is the object that enters physical amplitudes (e.g. electron scattering in an external field).

**Vertex renormalization (Mandl–Shaw approach)**

Just as we derived $\delta m$ and $Z_2$ from the pole structure of the propagator, we now determine the vertex renormalization constant $Z_1$ from the exact vertex at zero

momentum transfer. The key tool is the Ward–Takahashi identity, which we derive first.

*Derivation of the Ward–Takahashi identity.* Start from the elementary algebraic identity: for a free fermion propagator $S_0(k) = i/(\not{k} - m)$,

$$\not{q} = (\not{k} + \not{q} - m) - (\not{k} - m) = S_0^{-1}(k+q) - S_0^{-1}(k)\,. \tag{6.144}$$

Multiplying on the left by $S_0(k+q)$ and on the right by $S_0(k)$:

$$S_0(k+q)\,\not{q}\,S_0(k) = S_0(k) - S_0(k+q)\,. \tag{6.145}$$

In words: inserting $\not{q}$ between two adjacent propagators is equivalent to removing one of them and subtracting the two resulting terms.

*One-loop verification (Mandl–Shaw).* We verify the identity explicitly at one loop in Feynman gauge.

*(Convention: renormalized parameters.)* Throughout this verification we use the **renormalized** mass $m$ and coupling $e$, exactly as in the one-loop self-energy calculation (6.128). This is the natural choice in the counterterm method: the tree-level propagator $S_0(q) = i/(\not{q} - m + i\epsilon)$ and the tree-level vertex $+ie\gamma^\mu$ are read off from $\mathcal{L}_{\text{ren}}$, which is built from the *renormalized* parameters $m$ and $e$ (not the bare $m_0$ and $e_0$). The word "free" would be misleading here: $S_0$ is not the propagator of the original bare theory, but the zeroth-order propagator in the perturbative expansion around $\mathcal{L}_{\text{ren}}$. All deviations of $m_0, e_0$ from $m, e$ are relegated to the counterterms $(Z_2 - 1), \delta m, (Z_1 - 1), (Z_3 - 1)$, treated as separate perturbative insertions. Since the Ward–Takahashi identity in the form $q_\mu \Gamma^\mu = S^{-1}(p') - S^{-1}(p)$ involves the mass only through the *difference* $(\not{p}' - m) - (\not{p} - m) = \not{q}$, the mass choice drops out algebraically and the same identity continues to hold when eventually rewritten in bare or renormalized form.

The full vertex at one loop is $\Gamma^\mu = \gamma^\mu + \Lambda^\mu$, where $\Lambda^\mu$ is given by (same renormalized coupling and $\mu^\epsilon$ as in (6.128), Feynman gauge)

$$\Lambda^\mu(p', p) \;=\; -e^2\mu^\epsilon \int \frac{d^d k}{(2\pi)^d}\, \gamma^\alpha\, S_0(p'-k)\,\gamma^\mu\, S_0(p-k)\,\gamma_\alpha\, D_0(k)\,, \tag{6.146}$$

with $D_0(k) = -i/(k^2 + i\epsilon)$ the (scalar) free photon propagator in Feynman gauge and $S_0(q) = i/(\not{q} - m + i\epsilon)$; the overall $-e^2 = (-ie)^2$ comes from the two virtual-photon–fermion vertices, and the central $\gamma^\mu$ is the amputated external photon leg. Contracting with $q_\mu$ replaces $\gamma^\mu \to \not{q}$ in the integrand. Now apply identity (6.145) with $k \to p - k$ and $k + q \to p' - k$:

$$S_0(p'-k)\,\not{q}\,S_0(p-k) = S_0(p-k) - S_0(p'-k)\,. \tag{6.147}$$

Substituting back, with the *same* overall coefficient $-e^2\mu^\epsilon$:

$$\begin{aligned} q_\mu \Lambda^\mu(p',p) &= -e^2\mu^\epsilon \int \frac{d^dk}{(2\pi)^d}\, \gamma^\alpha \big[S_0(p-k) - S_0(p'-k)\big]\gamma_\alpha\, D_0(k) \\ &= \Sigma(\not{p}) - \Sigma(\not{p}'), \end{aligned} \tag{6.148}$$

where in the last step we recognized each of the two integrals as the one-loop self-energy (6.128), since $\Sigma(\not{q}) = -e^2\mu^\epsilon \int \frac{d^dk}{(2\pi)^d}\gamma^\alpha S_0(q-k)\gamma_\alpha D_0(k)$, evaluated at momentum $p$ and $p'$, respectively. Adding the tree-level contribution (using renormalized $m$, as above) $q_\mu\gamma^\mu = \not{q} = (\not{p}' - m) - (\not{p} - m)$:

$$\begin{aligned} q_\mu \Gamma^\mu &= \not{q} + \Sigma(\not{p}) - \Sigma(\not{p}') \\ &= \big(\not{p}' - m - \Sigma(\not{p}')\big) - \big(\not{p} - m - \Sigma(\not{p})\big) \\ &= S^{-1}(p') - S^{-1}(p), \end{aligned} \tag{6.149}$$

where $S^{-1}(p) = \not{p} - m - \Sigma(\not{p})$ is the inverse one-loop (pre-counterterm) propagator built from the renormalized $\mathcal{L}_{\text{ren}}$; the $m$'s cancel algebraically as anticipated, so the result would be identical if written with $m_0$.

*All-orders argument.* The one-loop check already contains the essential local identity. To extend it to arbitrary order, fix *one* skeleton diagram obtained from a vertex graph by removing the external photon line. What remains is a fermion line, running from the incoming momentum $p$ to the outgoing momentum $p'$, dressed by an arbitrary set of virtual-photon insertions.

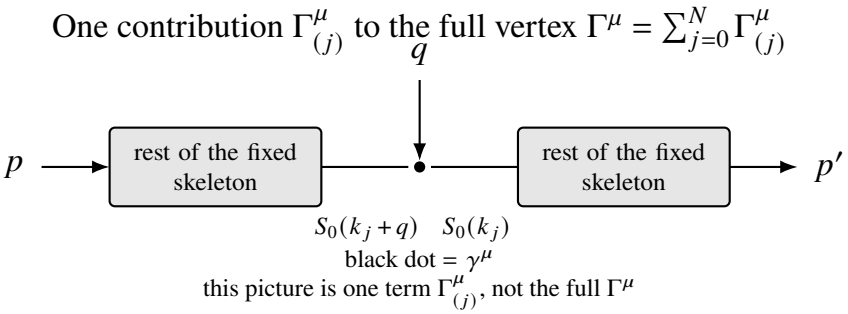
The crucial point is the following distinction:

- $\Gamma^\mu(p',p)$ is the *full proper vertex*: the sum of all amputated diagrams with two external fermions and one external photon;
- $\Gamma^\mu_{(j)}(p',p)$ is *one particular term* in that sum, namely the diagram obtained by inserting the external photon at the $j$-th possible position along the chosen fermion line;
- the black dot in the diagram below is only the *elementary* QED vertex $\gamma^\mu$, not the whole $\Gamma^\mu$.

If the external photon is attached at position $j$, the only local factor that depends on this insertion is

$$S_0(k_j + q)\,\gamma^\mu\, S_0(k_j),$$

while everything else in the skeleton is unchanged. A single contribution $\Gamma^\mu_{(j)}$ therefore has the schematic form shown below:




The full proper vertex is the sum over all possible insertion points:

$$\Gamma^{\mu}(p',p) = \sum_{j=0}^{N} \Gamma^{\mu}_{(j)}(p',p). \tag{6.150}$$

Now contract one fixed term with $q_\mu$. The contraction acts only on the black-dot vertex and turns $\gamma^\mu$ into $\not{q}$. Therefore, inside $\Gamma^{\mu}_{(j)}$ one has the replacement

$$S_0(k_j+q)\,\gamma^{\mu}\,S_0(k_j) \xrightarrow{q_\mu} S_0(k_j+q)\,\not{q}\,S_0(k_j) = S_0(k_j) - S_0(k_j+q), \tag{6.151}$$

where we used the propagator identity (6.145).

This produces two *reduced* diagrams. We denote them by $D_j$ and $D_{j+1}$: they are the same fixed skeleton, but cut at two neighboring positions along the fermion line. With this notation,

$$q_\mu \Gamma^{\mu}_{(j)} = D_j - D_{j+1}. \tag{6.152}$$

Summing over all insertion points gives a telescopic series:

$$q_\mu \Gamma^{\mu} \;=\; \sum_{j=0}^{N} \big(D_j - D_{j+1}\big) \;=\; D_0 - D_{N+1} \;=\; S^{-1}(p') - S^{-1}(p), \tag{6.153}$$

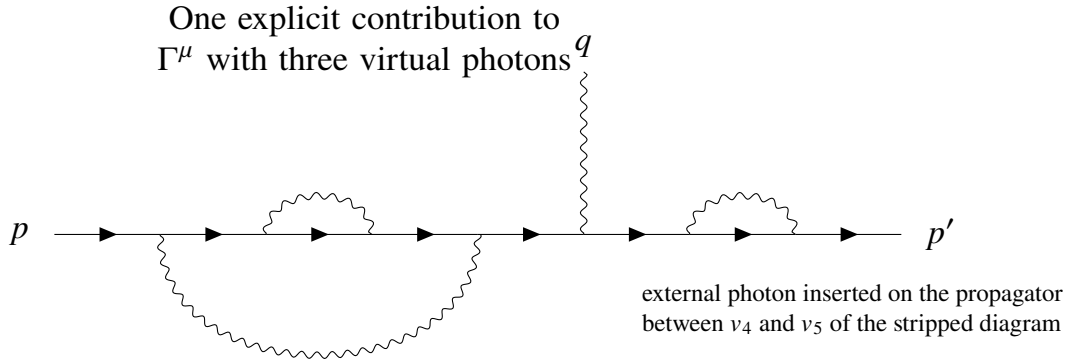
The cancellation is pairwise: every "$-D_{j+1}$" coming from insertion $j$ cancels the "$+D_{j+1}$" coming from insertion $j+1$. Only the two boundary terms survive. By definition of the labeling, $D_0$ is the cut adjacent to the outgoing leg and gives $S^{-1}(p')$, while $D_{N+1}$ is the cut adjacent to the incoming leg and gives $S^{-1}(p)$. Hence

$$\boxed{q_\mu \Gamma^{\mu}(p',p) = S^{-1}(p') - S^{-1}(p),} \tag{6.154}$$

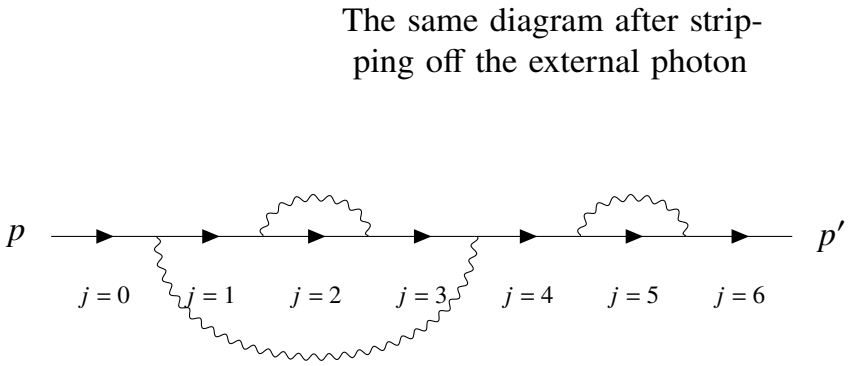
where $S^{-1}(\not{p}) = \not{p} - m - \Sigma_R(\not{p})$ is the (exact) inverse fermion propagator, written here in the renormalized convention; in bare perturbation theory one would replace $m \to m_0$ and $\Sigma_R \to \Sigma$ without changing the identity.

*Concrete example with three actual virtual photons.* The next picture is not the sum over all insertions. It is one single term in that sum. The three curved wavy

lines are genuine virtual photons, while the vertical line labeled $q$ is the external photon. To keep the correspondence with the general schematic completely literal, we choose an insertion point that separates a two-photon subdiagram on the left from a one-photon subdiagram on the right. Reinserting the external photon on that propagator gives the following explicit contribution:




Now erase only the external photon $q$, while keeping the three virtual photons fixed. One obtains the stripped diagram below. Along its fermion line there are seven propagators, so the external photon can be reinserted in seven different places:




For this explicit three-photon pattern the full contribution to the proper vertex is therefore

$$\Gamma^\mu_{3\gamma} = \Gamma^\mu_{(0)} + \Gamma^\mu_{(1)} + \Gamma^\mu_{(2)} + \Gamma^\mu_{(3)} + \Gamma^\mu_{(4)} + \Gamma^\mu_{(5)} + \Gamma^\mu_{(6)}. \tag{6.155}$$

After contraction with $q_\mu$, each insertion gives one difference of reduced diagrams:

$$q_\mu \Gamma^\mu_{(0)} = D_0 - D_1, \qquad q_\mu \Gamma^\mu_{(1)} = D_1 - D_2, \qquad q_\mu \Gamma^\mu_{(2)} = D_2 - D_3,$$
$$q_\mu \Gamma^\mu_{(3)} = D_3 - D_4, \qquad q_\mu \Gamma^\mu_{(4)} = D_4 - D_5, \qquad q_\mu \Gamma^\mu_{(5)} = D_5 - D_6,$$
$$q_\mu \Gamma^\mu_{(6)} = D_6 - D_7.$$

Summing them shows the telescopic cancellation with no abstraction left:

$$\begin{aligned} q_\mu \Gamma^\mu_{3\gamma} &= (D_0 - D_1) + (D_1 - D_2) + (D_2 - D_3) + (D_3 - D_4) \\ &\quad + (D_4 - D_5) + (D_5 - D_6) + (D_6 - D_7) \\ &= D_0 - D_7. \end{aligned}$$

This is the same telescopic mechanism as before, but now displayed on a genuinely more structured diagram rather than on a minimal toy picture.

*Application to vertex renormalization.* Taking the limit $q \to 0$ (i.e. $p' \to p$) with $p$ on shell gives

$$\Gamma^\mu(p,p) = \left.\frac{\partial S^{-1}(\not{p})}{\partial p_\mu}\right|_{\not{p}=m}. \tag{6.156}$$

Since $S^{-1}(\not{p}) = \not{p} - m - \Sigma(\not{p})$, and $\not{p} = \gamma^\nu p_\nu$ depends on $p_\mu$ only through $\partial\not{p}/\partial p_\mu = \gamma^\mu$, the chain rule gives

$$\frac{\partial S^{-1}(\not{p})}{\partial p_\mu} = \gamma^\mu - \frac{\partial \Sigma(\not{p})}{\partial p_\mu} = \gamma^\mu - \Sigma'(\not{p})\,\gamma^\mu = \gamma^\mu\big(1 - \Sigma'(\not{p})\big). \tag{6.157}$$

Evaluating on shell and recalling that $Z_2^{-1} = 1 - \Sigma'(m)$ [eq. (6.114)]:

$$\Gamma^\mu(p,p) = \gamma^\mu\big(1 - \Sigma'(m)\big) = \frac{\gamma^\mu}{Z_2}. \tag{6.158}$$

This is a remarkable result: *gauge invariance completely determines the vertex at zero momentum transfer in terms of the electron self-energy.*

Now consider the physical amplitude for electron scattering in a weak external field. As we will show in Section 6.5, external-leg self-energy corrections reduce on shell to a purely multiplicative factor: each external fermion leg carries a factor $Z_2^{1/2}$ (the square root of the full-propagator pole residue), and each external photon carries $Z_3^{1/2}$. At zero momentum transfer the amplitude is proportional to

$$\underbrace{Z_2^{1/2}\cdot Z_2^{1/2}}_{\text{two fermion legs}}\; Z_3^{1/2}\, e_0\, \bar u(p)\,\Gamma^\mu(p,p)\,u(p) = Z_2\, Z_3^{1/2}\, e_0\, \bar u(p)\,\Gamma^\mu(p,p)\,u(p).$$

Now we use the result (6.158) just derived from the Ward–Takahashi identity: $\Gamma^\mu(p,p) = \gamma^\mu/Z_2$. Substituting:

$$Z_2\, Z_3^{1/2}\, e_0\, \frac{1}{Z_2}\, \bar u(p)\,\gamma^\mu\,u(p) = Z_3^{1/2}\, e_0\, \bar u(p)\,\gamma^\mu\,u(p).$$

The factor $Z_2$ from the two external fermion legs is cancelled exactly by the $1/Z_2$ that the Ward–Takahashi identity forces on the vertex at $q = 0$. The physical charge $e$ is defined by requiring this to equal $e\,\bar u\gamma^\mu u$, so in $d = 4 - \epsilon$

$$\boxed{\mu^{\epsilon/2} e = Z_3^{1/2}\, e_0\,.} \tag{6.159}$$

and in four dimensions this reduces to $e = Z_3^{1/2} e_0$. The charge renormalization is controlled entirely by $Z_3$, the photon wave-function renormalization.

To connect with the standard notation, one introduces the **vertex renormalization constant** $Z_1$. The bare interaction vertex is $e_0\bar{\psi}_0\gamma^\mu\psi_0 A_{0\mu}$. Substituting $\psi_0 = Z_2^{1/2}\psi$ and $A_{0\mu} = Z_3^{1/2}A_\mu$, this becomes $e_0 Z_2 Z_3^{1/2}\bar{\psi}\gamma^\mu\psi A_\mu$. The constant $Z_1$ is defined as the ratio of this coefficient to the renormalized coupling:

$$e_0 Z_2 Z_3^{1/2} \equiv \mu^{\epsilon/2} Z_1 e\,, \tag{6.160}$$

where $\mu$ is the renormalization scale (needed to keep $e$ dimensionless for $d \neq 4$). In other words, $Z_1$ is the overall multiplicative renormalization of the $\bar{\psi}\gamma^\mu\psi A_\mu$ vertex. The Ward identity result (6.156) then implies

$$\boxed{Z_1 = Z_2\,.} \tag{6.161}$$

**Counterterm reformulation**

The same result can be organized via counterterms. Start from the bare interaction term (in $d = 4-\epsilon$)

$$\mathcal{L}_{\rm int}^{(0)} = +e_0\bar{\psi}_0\gamma^\mu\psi_0 A_\mu^{(0)}. \tag{6.162}$$

Substituting $\psi_0 = Z_2^{1/2}\psi$ and $A_\mu^{(0)} = Z_3^{1/2}A_\mu$:

$$\mathcal{L}_{\rm int}^{(0)} = e_0 Z_2 Z_3^{1/2}\,\bar{\psi}\gamma^\mu\psi A_\mu = \mu^{\epsilon/2} Z_1\, e\,\bar{\psi}\gamma^\mu\psi A_\mu\,, \tag{6.163}$$

where $Z_1$ is defined in (6.160). Now split $Z_1 = 1+\delta Z_1$ and write

$$\mathcal{L}_{\rm int}^{(0)} = +\mu^{\epsilon/2} e\,\bar{\psi}\gamma^\mu\psi A_\mu \;+\; \mu^{\epsilon/2}\delta Z_1\, e\,\bar{\psi}\gamma^\mu\psi A_\mu. \tag{6.164}$$

The first term gives the usual tree vertex $+ie\mu^{\epsilon/2}\gamma^\mu$. The second term is the *vertex counterterm*, with Feynman rule

$$\Gamma^\mu_{\rm ct} = +ie\,\mu^{\epsilon/2}\,\delta Z_1\,\gamma^\mu. \tag{6.165}$$

**One-loop 1PI vertex function: explicit integral**

The one-loop 1PI correction $\Lambda^\mu(p',p)$ is defined by

$$\Gamma^\mu(p',p) \equiv \gamma^\mu + \Lambda^\mu(p',p), \tag{6.166}$$

where $\Gamma^\mu$ is the regularized loop vertex before adding the local counterterm insertion. In covariant gauge, the one-loop contribution is

$$(+ie\mu^{\epsilon/2})\Lambda^\mu(p',p) = (+ie\mu^{\epsilon/2})^3 \int \frac{d^dk}{(2\pi)^d}\,\gamma^\alpha\,\frac{i(\not{p}'-\not{k}+m)}{(p'-k)^2-m^2+i\epsilon}\,\gamma^\mu \times \frac{i(\not{p}-\not{k}+m)}{(p-k)^2-m^2+i\epsilon}\,\gamma^\beta D_{\alpha\beta}(k), \tag{6.167}$$

with

$$D_{\alpha\beta}(k) = \frac{-i}{k^2 + i\epsilon}\left(g_{\alpha\beta} - (1-\xi)\frac{k_\alpha k_\beta}{k^2}\right). \tag{6.168}$$

In Feynman gauge $\xi = 1$, $D_{\alpha\beta}(k) = -ig_{\alpha\beta}/(k^2 + i\epsilon)$, so the Dirac structure simplifies to

$$\Lambda^\mu(p', p) = -e^2\mu^\epsilon \int \frac{d^d k}{(2\pi)^d} \frac{\gamma^\alpha(\not{p}' - \not{k} + m)\gamma^\mu(\not{p} - \not{k} + m)\gamma_\alpha}{\big(k^2 + i\epsilon\big)\big((p'-k)^2 - m^2 + i\epsilon\big)\big((p-k)^2 - m^2 + i\epsilon\big)}. \tag{6.169}$$

The integral is logarithmically UV divergent: for large $k$, the denominator scales like $k^2 \cdot k^2 \cdot k^2 \sim k^6$, the numerator scales like $k^2$ (two factors of $k$), so the integrand behaves like $d^d k\, k^2/k^6 = d^d k/k^4$, which is logarithmically divergent in $d = 4$ and becomes a $1/\epsilon$ pole in $d = 4 - \epsilon$.

**Lorentz/Dirac decomposition and what is physically measurable**

Including the counterterm, the **renormalized proper vertex** is

$$\Gamma^\mu_R(p', p) \;\equiv\; \Gamma^\mu(p', p) + \delta Z_1\, \gamma^\mu = \gamma^\mu + \Lambda^\mu(p', p) + \delta Z_1\, \gamma^\mu, \tag{6.170}$$

where $\delta Z_1 \equiv Z_1 - 1$ will be fixed by a renormalization condition. For on-shell electrons, $\bar{u}(p')\,\Gamma^\mu_R\, u(p)$ must be a Lorentz four-vector built from the available ingredients: the Dirac matrices $\gamma^\mu$ and $\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$, and the two independent four-momenta, which we can take to be $P^\mu \equiv p^\mu + p'^\mu$ and $q^\mu \equiv p'^\mu - p^\mu$. Since the vertex carries one free Lorentz index $\mu$, the candidate structures sandwiched between $\bar{u}(p')$ and $u(p)$ are

$$\gamma^\mu, \quad P^\mu, \quad q^\mu, \quad \sigma^{\mu\nu}P_\nu, \quad \sigma^{\mu\nu}q_\nu\,.$$

(Terms like $\gamma^\mu \not{P}$ or $\gamma^\mu \not{q}$ can be reduced using the anticommutation relation $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$; higher products of gamma matrices are not independent in four dimensions.) The on-shell Dirac equation, $\not{p}\, u(p) = m\, u(p)$ and $\bar{u}(p')\not{p}' = m\,\bar{u}(p')$, allows further simplification. In particular, the Gordon identity

$$\bar{u}(p')\,\gamma^\mu\, u(p) = \bar{u}(p')\left[\frac{P^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right] u(p)$$

shows that $P^\mu$ is not independent of $\gamma^\mu$ and $\sigma^{\mu\nu}q_\nu$ between on-shell spinors. Similarly, contracting $\sigma^{\mu\nu}P_\nu$ with the spinors and using the Dirac equation yields a combination of $\gamma^\mu$ and $q^\mu$, so it is also redundant.

After eliminating all dependent structures, three independent form factors remain:

$$\bar{u}(p')\,\Gamma^\mu_R(p', p)\, u(p) = \bar{u}(p')\left[\gamma^\mu\, F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\, F_2(q^2) + q^\mu\, F_3(q^2)\right] u(p),$$

(6.171)

where the scalar coefficients $F_i$ can depend only on the Lorentz invariant $q^2$ (since $p^2 = p'^2 = m^2$ on shell). The $q^\mu F_3$ term does not contribute when the vertex is coupled to a conserved external current ($q_\mu J^\mu = 0$), and is pure-gauge in that sense. The two physically relevant form factors are: $F_1(q^2)$ (charge form factor) and $F_2(q^2)$ (Pauli form factor; $F_2(0)$ gives the anomalous magnetic moment).

**Ward–Takahashi identity and why QED is special**

Gauge invariance implies the Ward–Takahashi identity for the proper vertex:

$$q_\mu \Gamma^\mu(p', p) = S^{-1}(p') - S^{-1}(p), \tag{6.172}$$

where $S^{-1}(\not{p}) = \not{p} - m - \Sigma(\not{p})$ is the regularized inverse propagator. This identity has two crucial consequences:

**(1) Transversality of the UV divergence.** The UV divergent part of $\Lambda^\mu$ must be proportional to $\gamma^\mu$: no new Dirac structures appear in the divergence. The argument is the following. Since $\Sigma(\not{p})$ is at most linearly divergent by power counting, its divergent part has the polynomial form $\Sigma_{\text{div}}(\not{p}) = A\not{p} + Bm$ with $A, B$ constants, so the right-hand side of (6.172) contributes $\left[S^{-1}(p') - S^{-1}(p)\right]_{\text{div}} = -A\not{q} = -A\, q_\mu \gamma^\mu$. On the left-hand side, the WTI must hold separately for the divergent parts; comparing Dirac structures, the only $\Lambda^\mu_{\text{div}}$ that yields $-A\, q_\mu \gamma^\mu$ upon contraction with $q_\mu$ is $\Lambda^\mu_{\text{div}} = -A\,\gamma^\mu$ (other structures such as $\sigma^{\mu\nu} q_\nu$ contribute only to finite form factors like the anomalous magnetic moment). Hence a single counterterm $\delta Z_1\, \gamma^\mu$ suffices to absorb the vertex divergence.

**(2) Equality of renormalization constants: $Z_1 = Z_2$.** The Ward–Takahashi identity (6.172) is an exact consequence of gauge invariance of the bare action $\mathcal{L}_0$. In the counterterm method we split $\mathcal{L}_0 = \mathcal{L}_{\text{ren}} + \delta\mathcal{L}$ and treat $\delta\mathcal{L}$ as a perturbative insertion. The counterterms may look like new interactions, but they are not: $\mathcal{L}_{\text{ren}} + \delta\mathcal{L}$ is *identically* equal to $\mathcal{L}_0$, so the full theory is the same gauge-invariant theory we started from. Now, the renormalized Green's functions $\Gamma^\mu_R$ and $S^{-1}_R$ include *both* the loop corrections computed from $\mathcal{L}_{\text{ren}}$ *and* the counterterm insertions from $\delta\mathcal{L}$. Together, they reproduce the Green's functions of the full bare theory, merely re-expressed in terms of renormalized fields and parameters. Since the bare theory is gauge invariant, the WTI must hold for the renormalized quantities as well:

$$q_\mu \Gamma^\mu_R(p', p) = S^{-1}_R(p') - S^{-1}_R(p)\,.$$

*Explicit check.* The bare gauge invariance of $\mathcal{L}_0$ gives, via the standard path-integral derivation, the bare WTI $q_\mu \Gamma^\mu_0 = S^{-1}_0(p') - S^{-1}_0(p)$. The bare and renormalized 1PI functions are related by external-leg rescalings: from $\psi_0 = Z_2^{1/2}\psi$ and $A_0 = Z_3^{1/2} A$ one obtains $S^{-1}_0(p) = Z_2^{-1} S^{-1}_R(p)$ and, with $e_0 = \mu^{\epsilon/2} Z_e e$, $\Gamma^\mu_0 = Z_1^{-1}\Gamma^\mu_R$, where $Z_1 = Z_e Z_2 Z_3^{1/2}$. Substituting into the bare WTI,

$$q_\mu \Gamma^\mu_R = \frac{Z_1}{Z_2}\left[S^{-1}_R(p') - S^{-1}_R(p)\right],$$

which collapses to the simple form above precisely when $Z_1 = Z_2$. In this sense the renormalized WTI is equivalent to the statement that gauge invariance survives the field rescaling. We now show, working directly in renormalized perturbation theory with counterterms, that this forces $Z_1 = Z_2$.

The counterterms shift the proper vertex and inverse propagator as

$$\Gamma^{\mu}_{R} = \Gamma^{\mu} + \delta Z_1\, \gamma^{\mu}, \qquad S_R^{-1}(\not p) = S^{-1}(\not p) + (Z_2 - 1)\not p - \delta m, \tag{6.173}$$

where $\Gamma^{\mu} = \gamma^{\mu} + \Lambda^{\mu}$ and $S^{-1}(\not p) = \not p - m - \Sigma(\not p)$ are the regularized one-loop quantities before counterterm insertions. Compute both sides of the renormalized identity $q_\mu \Gamma^{\mu}_{R} = S_R^{-1}(p') - S_R^{-1}(p)$ separately. On the left-hand side, using the regularized identity (6.172):

$$q_\mu \Gamma^{\mu}_{R} = q_\mu \Gamma^{\mu} + \delta Z_1\, q_\mu \gamma^{\mu} = \left[S^{-1}(p') - S^{-1}(p)\right] + \delta Z_1\, \not q\,. \tag{6.174}$$

On the right-hand side, the mass counterterms cancel in the difference:

$$\begin{aligned} S_R^{-1}(p') - S_R^{-1}(p) &= \left[S^{-1}(p') - S^{-1}(p)\right] + (Z_2 - 1)(\not p' - \not p) \\ &= \left[S^{-1}(p') - S^{-1}(p)\right] + (Z_2 - 1)\, \not q\,. \end{aligned} \tag{6.175}$$

Equating (6.174) and (6.175), the regularized Ward–Takahashi terms cancel on both sides, leaving

$$\delta Z_1\, \not q = (Z_2 - 1)\, \not q\,,$$

which, since it must hold for arbitrary $q$, requires

$$Z_1 = Z_2 \qquad \Longleftrightarrow \qquad \delta Z_1 = \delta Z_2\,. \tag{6.176}$$

This is the key simplification in QED: once the electron wavefunction renormalization is fixed, the vertex counterterm is fixed automatically.

**Renormalization condition for the charge and determination of $\delta Z_1$**

To fix the *finite* part of $\delta Z_1$ one chooses a renormalization scheme. In the on-shell (Thomson) scheme—the standard QED renormalization prescription, treated in detail in Peskin–Schroeder [6] §10.3, Schwartz [12] Ch. 19, and Itzykson–Zuber [16] §8.4—the renormalized charge is defined by demanding that the electron couples to a soft external photon with strength $e$, i.e. in the limit of vanishing momentum transfer:

$$\Gamma^{\mu}_{R}(p', p)\Big|_{q^2 \to 0,\; p^2 = p'^2 = m^2} = \gamma^{\mu}. \tag{6.177}$$

Using (6.171) and (6.165), this condition becomes

$$\gamma^{\mu} + \Lambda^{\mu}(p', p)\Big|_{\text{Thomson}} + \delta Z_1\, \gamma^{\mu} = \gamma^{\mu}, \tag{6.178}$$

so

$$\delta Z_1 = -\,\Lambda^\mu(p',p)|_{\text{Thomson}} \Big/ \gamma^\mu, \tag{6.179}$$

meaning: $\delta Z_1$ is minus the coefficient of $\gamma^\mu$ in the Thomson limit of $\Lambda^\mu$ (it cancels the UV pole). In practice $\Lambda^\mu$ is decomposed in the basis (6.171),

$$\bar u(p')\,\Lambda^\mu(p',p)\,u(p) = \bar u(p')\Big[\,\gamma^\mu\left(F_1(q^2)-1\right) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\,F_2(q^2)\Big]u(p), \tag{6.180}$$

(the would-be $F_3$ piece vanishes by current conservation), and the bare vertex $\gamma^\mu$ contributes 1 to $F_1$. The renormalized vertex is then

$$\bar u(p')\,\Gamma^\mu_R\,u(p) = \bar u(p')\Big[\,\gamma^\mu\left(F_1(q^2)+\delta Z_1\right) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2)\Big]u(p), \tag{6.181}$$

so the Thomson condition (6.177) reads $F_1(0)+\delta Z_1 = 1$, i.e.

$$\delta Z_1 = 1 - F_1(0), \qquad \text{equivalently} \qquad F_1^{\text{ren}}(0) \equiv F_1(0)+\delta Z_1 = 1. \tag{6.182}$$

Hence $\delta Z_1$ cancels the UV pole of $F_1(0)$ together with its scheme-dependent finite part, while $F_2(q^2)$ is UV finite and unaffected by the counterterm (it carries the anomalous magnetic moment, $F_2(0) = \alpha/2\pi + O(\alpha^2)$).

**UV vs IR: what "the difficult part" really is**

At one loop and on shell, $\Lambda^\mu$ contains *both* UV and IR singularities:

$$\Lambda^\mu(p',p) \sim \left(\frac{c_{\text{UV}}}{\epsilon_{\text{UV}}} + \frac{c_{\text{IR}}}{\epsilon_{\text{IR}}} + \text{finite}\right)\gamma^\mu + (\text{other finite Dirac structures}). \tag{6.183}$$

The counterterm $\delta Z_1$ is fixed to remove the UV pole (and impose the chosen renormalization condition), but *it cannot remove the IR pole*: that singularity has a completely different physical origin. The IR divergence arises because the loop integral receives contributions from virtual photons with arbitrarily small momenta. However, no real detector has infinite energy resolution: a final-state electron with energy $E$ is experimentally indistinguishable from the same electron accompanied by a photon so soft that the total energy remains within the detector resolution $\Delta E$. The physically measurable quantity is therefore the *inclusive* cross section, which sums:

1. the purely virtual process (with the IR-divergent vertex correction), and
2. the real-emission process in which one or more soft photons with total energy below $\Delta E$ are radiated.

The Bloch–Nordsieck theorem guarantees that the IR singularities in these two contributions cancel exactly, leaving a finite inclusive cross section that depends on $\Delta E$ only through a harmless logarithm $\ln(\Delta E/E)$. This is why the vertex correction

is "harder" than the propagator corrections: even after UV renormalization, a purely virtual on-shell vertex amplitude is not IR finite by itself—it only becomes finite once combined with real emission.

**Summary.** The renormalized vertex to one loop is

$$\Gamma^{\mu}_{R}(p',p) = \gamma^{\mu} + \Lambda^{\mu}(p',p) + \delta Z_1\,\gamma^{\mu} + \mathcal{O}(e^4), \tag{6.184}$$

with $\delta Z_1$ chosen to cancel the UV pole and satisfy the charge renormalization condition (e.g. (6.177)). In QED, gauge invariance further enforces $Z_1 = Z_2$ (6.176), tightly linking vertex and electron-propagator renormalization. In the on-shell (Thomson) scheme, $\delta Z_1 = 1 - F_1(0)$ from (6.182), and (6.181) reduces to a manifestly finite (in the UV) on-shell matrix element:

$$\bar{u}(p')\,\Gamma^{\mu}_{R}(p',p)\,u(p) = \bar{u}(p')\Big[\,\gamma^{\mu}\left(1 + F_1(q^2) - F_1(0)\right) + \frac{i\sigma^{\mu\nu}q_{\nu}}{2m}\,F_2(q^2)\,\Big]u(p) + \mathcal{O}(e^4). \tag{6.185}$$

The combination $F_1(q^2) - F_1(0)$ is the physical, UV-subtracted charge form factor (vanishing at $q^2 = 0$ and carrying the IR singularity that will cancel against soft real emission), while $F_2(q^2)$ gives the Pauli form factor and the anomalous magnetic moment $a_e = F_2(0)$.

## 6.5 External line renormalization

In Mandl–Shaw's treatment, the key point is that an $S$-matrix element is obtained from *amputated* Green functions with adiabatic switching and on-shell pole extraction. External self-energy insertions do not introduce new independent on-shell structures; they only modify the pole residues of the exact propagators. We now show this explicitly.

### External fermion lines

Let $\mathcal{M}$ be the rest of an amputated diagram to which an incoming electron spinor is attached. At first order, adding one self-energy insertion on the external leg gives

$$\text{(no insertion)}: \quad \mathcal{M}\,u(p),$$

and

$$\text{(one insertion on the external leg)}: \quad \mathcal{M}\,\frac{i}{\not{p} - m + i\epsilon}\left[-i\Sigma(\not{p}) + i\delta m\right]u(p).$$

Notation choice (rigorous): in this section we are using *renormalized perturbation theory*, i.e.

$$\mathcal{L} = \bar{\psi}\,(i\partial\!\!\!/ - m)\psi + e\,\bar{\psi}\gamma^{\mu}\psi A_{\mu} \;+\; \delta\mathcal{L},$$

with $\delta\mathcal{L} \supset -\delta m\,\bar{\psi}\psi$. Therefore the zeroth-order fermion propagator is

$$S_F^{(0)}(p) = \frac{i}{\not{p} - m + i\epsilon},$$

and the mass shift appears in the insertion $+i\delta m$. Equivalently, in a bare split one writes

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m_0)\psi + e_0\bar{\psi}\gamma^{\mu}\psi A_{\mu} + \cdots$$

and expands around

$$S_{0,\mathrm{bare}}(p) = \frac{i}{\not{p} - m_0 + i\epsilon}.$$

The two organizations are identical order by order once $m_0 = m + \delta m$. Here is the origin of this structure, step by step:

1. The factor $\mathcal{M}$ is the hard amputated subamplitude.
2. The external electron line that connects $\mathcal{M}$ to the insertion contributes the zeroth-order renormalized propagator $S_F^{(0)}(p) = i/(\not{p} - m + i\epsilon)$.
3. The blob on that line is the 1PI two-point insertion plus counterterm: $-i\Sigma(\not{p}) + i\delta m$.
4. The incoming asymptotic state contributes the spinor $u(p)$.

Equivalent Dyson viewpoint (same formula, different language). We denote by $G_{\mathrm{ext}}(p)$ the *dressed external (half-)line*: the object that describes how the asymptotic incoming spinor $u(p)$, prepared at $t \to -\infty$, propagates and accumulates self-energy insertions before reaching the hard subamplitude $\mathcal{M}$ at the scattering vertex. It is *not* a full Green function: it is the truncated correlator that attaches *one* on-shell asymptotic state on one end and is left *open* (un-amputated) on the other end—ready to be inserted into $\mathcal{M}$. Diagrammatically,

$$\underbrace{u(p)}_{\text{in}} \xrightarrow{S_0(p)} \underbrace{\bullet}_{(-i\Sigma + i\delta m)} \xrightarrow{S_0(p)} \cdots \xrightarrow{S_0(p)} \mathcal{M}.$$

Expanding the geometric series of self-energy insertions reads

$$\begin{aligned} G_{\mathrm{ext}}(p) &= S_0(p)\,u(p) \\ &\quad + S_0(p)\left[-i\Sigma(\not{p}) + i\delta m\right] S_0(p)\,u(p) + \cdots, \end{aligned} \tag{6.186}$$

with

$$S_0(p) = S_F^{(0)}(p) = \frac{i}{\not{p} - m + i\epsilon}.$$

Each term has a simple meaning: the zeroth-order term $S_0(p)\,u(p)$ is the bare free propagation of the incoming spinor; the first-order term $S_0\,(-i\Sigma + i\delta m)\,S_0\,u$ inserts one renormalized self-energy blob (1PI loop $-i\Sigma$ together with its mass counterterm $+i\delta m$) between two free propagators; the higher-order terms iterate the same structure. Summing the geometric series gives the exact resummed external line

$$G_{\text{ext}}(p) = S(p)\,u(p), \qquad S(p) = \frac{i}{\not{p} - m - \Sigma(\not{p}) + \delta m + i\epsilon},$$

i.e. the exact propagator $S(p)$ of the renormalized field acting on the asymptotic spinor. Hence (6.186) is just the perturbative expansion of $S(p)\,u(p)$ around $S_0(p)\,u(p)$. For an incoming fermion, removing the external propagator factor means multiplying by $(\not{p} - m)/i$ on that leg:

$$\frac{\not{p} - m}{i} G_{\text{ext}}(p) = \left[1 + \frac{i}{\not{p} - m + i\epsilon}\big(-i\Sigma(\not{p}) + i\delta m\big) + \cdots\right] u(p).$$

Keeping only one insertion, the corrected leg has the form "tree + inserted leg", i.e.

$$\mathcal{M}\,u(p) \;\longrightarrow\; \mathcal{M}\left[1 + \frac{i}{\not{p} - m + i\epsilon}\big(-i\Sigma(\not{p}) + i\delta m\big)\right] u(p). \tag{6.187}$$

Using the on-shell expansion

$$\Sigma(\not{p}) = A + B(\not{p} - m) + \Sigma_c(\not{p}), \qquad \delta m = A,$$

with $\Sigma_c(\not{p}) = O\big((\not{p} - m)^2\big)$ [cf. eqs. (6.109), (6.112), (6.110)], the renormalized self-energy combination $-i\Sigma + i\delta m = -i[B(\not{p} - m) + \Sigma_c]$ enters (6.187) as

$$\mathcal{M}\,u(p) \;\longrightarrow\; \mathcal{M}\left[1 + \frac{B\,(\not{p} - m)}{\not{p} - m + i\epsilon} + \frac{\Sigma_c(\not{p})}{\not{p} - m + i\epsilon}\right] u(p). \tag{6.188}$$

The $\Sigma_c$ piece vanishes on shell ($\Sigma_c = O((\not{p} - m)^2)$ divided by a linear denominator gives $O(\not{p} - m)$, which is killed by $u(p)$). The remaining $B$ piece, however, has the genuinely *indeterminate* form $\frac{(\not{p} - m)}{\not{p} - m + i\epsilon}$: naively cancelling numerator and denominator gives $B$, while using $(\not{p} - m)\,u(p) = 0$ gives 0. The one-insertion calculation alone **cannot** fix the value of this $0/0$, and therefore cannot determine

the external-leg factor: an explicit (infrared, adiabatic) regulator is needed to give it a definite meaning.

The proper resolution is given in the next subsection (Mandl–Shaw viewpoint), where the adiabatic switching of the interaction selects the unique finite answer

$$1+\frac{B\,(\not p-m)}{\not p-m+i\epsilon}\;\longrightarrow\;1+\tfrac{1}{2}B=Z_2^{1/2}+O(B^2).$$

The factor of $\frac{1}{2}$ in front of $B$ is the substantive content of the Mandl–Shaw calculation: it is what distinguishes $Z_2^{1/2}$ (the correct external-leg factor) from $Z_2$ (the residue of the dressed propagator after amputation).

**From one insertion to the exact external-leg factor.**
The exact propagator already resums all such insertions:

$$S(\not p)=\frac{i}{\not p-m-\Sigma(\not p)}\xrightarrow{\not p\to m}\frac{iZ_2}{\not p-m+i\epsilon}\quad\text{[eq. (6.116)]}.$$

When the external pole factor is removed in the $S$-matrix construction, the net external-leg normalization is $Z_2^{1/2}$. Hence

$$\boxed{\begin{aligned}u(p)&\to Z_2^{1/2}u(p), &\qquad \bar u(p)&\to Z_2^{1/2}\bar u(p),\\ v(p)&\to Z_2^{1/2}v(p), &\qquad \bar v(p)&\to Z_2^{1/2}\bar v(p).\end{aligned}}\tag{6.189}$$

Operationally: compute amputated diagrams and attach $Z_2^{1/2}$ to each external electron/positron leg.

### Adiabatic switching (Mandl–Shaw viewpoint)

We now re-derive the external-leg factor $Z_2^{1/2}$ following step by step the treatment of Mandl–Shaw [1], §9.4. The argument is more delicate than the propagator-pole argument above, and the reason to go through it is conceptual: the apparently innocent factor $(\not p-m)/(\not p-m+i\epsilon)$ that controls the external-leg correction is in fact a $0/0$ on shell, and giving it a precise meaning requires an explicit infrared (adiabatic) regulator. The trick is that the result of the regulated calculation is finite and *independent of the choice of regulator*, so it is unambiguous.

**Setup (notation matching Mandl–Shaw).** We work in renormalized perturbation theory. The renormalized free fermion propagator and the renormalized self-energy are

$$S_F^{(0)}(p)=\frac{i}{\not p-m+i\epsilon},\qquad \widetilde\Sigma(\not p)\equiv\Sigma(\not p)-\delta m=B\,(\not p-m)+\Sigma_c(\not p),$$

where the on-shell expansion (6.109) has been used and $\delta m = A = \Sigma(m)$ has been subtracted by the mass counterterm; cf. properties (6.110) ($\Sigma_c$ vanishes quadratically at $\not{p} = m$). The renormalized self-energy insertion contributes a Feynman factor $-i\widetilde{\Sigma}(\not{p})$.

**One self-energy insertion on the incoming line [Mandl–Shaw Eq. (9.38)].** At order $e^2$, summing the tree term and the single insertion on the external incoming electron leg gives

$$\begin{aligned} u(p) &\longrightarrow u(p) + S_F^{(0)}(p)\left[-i\widetilde{\Sigma}(\not{p})\right] u(p) \\ &= u(p) + \frac{i}{\not{p} - m + i\epsilon}\left[-i\right]\left[B(\not{p} - m) + \Sigma_c(\not{p})\right] u(p) \\ &= \left[1 + \frac{B\,(\not{p} - m)}{\not{p} - m + i\epsilon} + \frac{\Sigma_c(\not{p})}{\not{p} - m + i\epsilon}\right] u(p). \end{aligned} \tag{6.190}$$

**The $\Sigma_c$ piece drops out [Mandl–Shaw Eq. (9.39)].** By (6.110) we have $\Sigma_c(p) = O\big((\not{p} - m)^2\big)$ on shell, so $\Sigma_c(p)/(\not{p} - m + i\epsilon)$ vanishes linearly as $\not{p} \to m$. On the on-shell spinor $u(p)$ this term gives zero and we are left with

$$u(p) \longrightarrow \left[1 + \frac{B\,(\not{p} - m)}{\not{p} - m + i\epsilon}\right] u(p). \tag{6.191}$$

**The $0/0$ ambiguity.** The factor $(\not{p} - m)/(\not{p} - m + i\epsilon)$ in (6.191) looks like 1 if one cancels numerator and denominator naively, but the Dirac equation gives $(\not{p} - m)\,u(p) = 0$, so the same expression also looks like 0. As Mandl–Shaw stress, the term *"is indeterminate, as it stands"*: extracting a definite limit requires regulating how $\not{p}$ approaches $m$.

**The adiabatic hypothesis [Mandl–Shaw Eq. (9.40)].** The regulator is the adiabatic switching of the interaction. Introduce a smooth real function $f(t)$ with $f(t) \to 0$ as $t \to \pm\infty$ and $f(t) \simeq 1$ during a long time interval $T$ around the scattering, and replace the (renormalized) QED interaction Hamiltonian density by

$$\mathscr{H}_I(x) \longrightarrow \mathscr{H}_I^{(f)}(x) = -e\, f(t)\, \bar{\psi}(x)\, \not{A}(x)\, \psi(x) - \delta m\, [f(t)]^2\, \bar{\psi}(x)\, \psi(x). \tag{6.192}$$

The original theory is recovered as $f(t) \to 1$. (The mass counterterm carries $[f(t)]^2$ because each occurrence of $\delta m$ accompanies two powers of the coupling at second order.) *Note on conventions.* Mandl–Shaw write $e_0$ rather than $e$ here because they work directly with the bare Lagrangian (not split into renormalized part plus counterterms). At the order $e^2$ at which we are computing, the distinction is immaterial: $e_0 = e + O(e^3)$ via $Z_e = 1 + O(e^2)$, so the second-order self-energy involves $e^2$ or $e_0^2$ interchangeably. We use the renormalized $e$ to keep the notation aligned with the rest of the chapter.

**Fourier representation [Mandl–Shaw Eqs. (9.41)–(9.42)].** Writing the Fourier transform of $f(t)$ as

$$f(t) = \int_{-\infty}^{+\infty} dE\, F(E)\, e^{iEt} = \int_{-\infty}^{+\infty} dE\, F(E)\, e^{iq\cdot x}, \qquad q \equiv (E, \mathbf{0}), \tag{6.193}$$

the normalization conditions become

$$f(0) = \int_{-\infty}^{+\infty} dE\, F(E) = 1, \qquad F(E) \to \delta(E) \text{ as } T \to \infty, \tag{6.194}$$

i.e. $F(E)$ is sharply peaked at $E = 0$ with width $\sim 1/T$. The auxiliary 4-vector $q = (E, \mathbf{0})$ has the structure of an external-field momentum: the modified interaction (6.192) conserves three-momentum at each vertex (because $f$ depends only on $t$) but *not* energy.

**Modified self-energy insertion [Mandl–Shaw Eq. (9.43)].** Each of the two QED vertices in the second-order self-energy carries a factor $f(t)$, hence in momentum space a factor $F(E)$ together with an "energy momentum" $q = (E, \mathbf{0})$ flowing into that vertex. Calling the two such momenta $q$ and $q'$, the internal fermion propagator now carries 4-momentum $\not{p} - \not{q} - \not{q}'$ (instead of $\not{p}$), and the analogue of (6.191) reads

$$u(p) \;\longrightarrow\; \left[ 1 + \int dE\, dE'\, F(E)\, F(E') \, \frac{B\,(\not{p} - \not{q} - m)}{\not{p} - \not{q} - \not{q}' - m + i\epsilon} \right] u(p). \tag{6.195}$$

The original ambiguity of (6.191) is gone: numerator and denominator are now genuinely different polynomials in $E, E'$, and the integral is well defined.

**First replacement: use $(\not{p} - m)\,u(p) = 0$.** Acting on $u(p)$, we may freely add $-\frac{1}{2}(\not{p} - m)$ to the numerator of (6.195):

$$(\not{p} - \not{q} - m)\,u(p) = \big[(\not{p} - \not{q} - m) - \tfrac{1}{2}(\not{p} - m)\big] u(p) = \tfrac{1}{2}\big(\not{p} - 2\not{q} - m\big)\, u(p). \tag{6.196}$$

**Second replacement: symmetrize $q \leftrightarrow q'$.** The remaining factor $F(E)F(E')$ in (6.195) is symmetric under $q \leftrightarrow q'$, so in the numerator we may replace $2\not{q} \to \not{q} + \not{q}'$:

$$\tfrac{1}{2}\big(\not{p} - 2\not{q} - m\big) \;\longrightarrow\; \tfrac{1}{2}\big(\not{p} - \not{q} - \not{q}' - m\big). \tag{6.197}$$

The numerator is now exactly $\frac{1}{2}$ times the denominator: cancellation becomes trivial.

**The integral becomes elementary [Mandl–Shaw Eq. (9.44)].** Substituting (6.196)–(6.197) into (6.195), the integrand reduces to $\frac{1}{2}B$, independent of $E, E'$:

$$u(p) \longrightarrow \left[1 + \tfrac{1}{2}B \int dE\, F(E) \int dE'\, F(E')\right] u(p) = \left(1 + \tfrac{1}{2}B\right) u(p), \tag{6.198}$$

where we used the normalization (6.194) on each $F$. The result is finite, unambiguous, and—as Mandl–Shaw emphasize—*independent* of the precise shape of $F(E)$, hence valid in the limit $F(E) \to \delta(E)$ which restores the original theory.

**Identification with $Z_2^{1/2}$ [Mandl–Shaw Eq. (9.45a)].** Recalling $Z_2 = (1-B)^{-1}$ from (6.114), expand $Z_2^{1/2} = (1-B)^{-1/2} = 1 + \frac{1}{2}B + O(B^2)$. Comparing with (6.198):

$$\boxed{u(p) \longrightarrow Z_2^{1/2}\, u(p) + O(e^4)\,.} \tag{6.199}$$

Although derived only at second order, the result generalizes to all orders (the exact propagator residue $Z_2$ remains the only new structure).

**Other spinors and external photons [Mandl–Shaw Eqs. (9.45b)–(9.45c)].** Identical arguments applied to the other three external fermion polarizations give

$$\bar{u}(p) \to Z_2^{1/2} \bar{u}(p), \quad v(p) \to Z_2^{1/2} v(p), \quad \bar{v}(p) \to Z_2^{1/2} \bar{v}(p). \tag{6.200}$$

For an external physical photon, the same logic with the vacuum-polarization tensor $\Pi^{\mu\nu}(k) = (k^2 g^{\mu\nu} - k^\mu k^\nu)\,\Pi(k^2)$ contracted into a transverse polarization $\varepsilon^\nu(k)$ gives $\Pi^{\mu\nu}\varepsilon_\nu = k^2\,\Pi(k^2)\,\varepsilon^\mu = 0$ on shell; all that remains is the residue $Z_3$ of the exact photon propagator, split as $Z_3^{1/2}$ on the polarization vector:

$$\varepsilon^\mu(k) \longrightarrow Z_3^{1/2}\, \varepsilon^\mu(k). \tag{6.201}$$

**Operational rule.** The Feynman recipe extracted from the above derivation is exactly that of Mandl–Shaw §9.4:

1. compute the *amputated* 1PI Green function for the hard process using renormalized perturbation theory;
2. attach to each external leg the on-shell wavefunction $(u, \bar{u}, v, \bar{v}, \varepsilon)$ of the asymptotic particle;
3. multiply by one factor $Z_2^{1/2}$ for each external fermion line and one factor $Z_3^{1/2}$ for each external photon line.

This reproduces (6.189) and (6.202); the underlying physics is the identification of the asymptotic states (defined via adiabatic switching as bare Fock states) with the physical, dressed in/out states relevant for the $S$-matrix.

### External photon lines

For a physical external photon ($k^2 = 0$, $k\cdot\varepsilon = 0$), an explicit vacuum-polarization insertion contributes

$$\Pi^{\mu\nu}(k)\,\varepsilon_\nu(k) = \big(k^2 g^{\mu\nu} - k^\mu k^\nu\big)\Pi(k^2)\,\varepsilon_\nu(k) = k^2\Pi(k^2)\,\varepsilon^\mu(k) = 0,$$

so again no extra on-shell structure survives.

The exact photon propagator has transverse pole residue $Z_3$:

$$D^{\mu\nu}(k) \xrightarrow{k^2\to 0} \frac{-iZ_3}{k^2 + i\epsilon}\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) + \text{regular}.$$

Removing this pole factor gives one factor $Z_3^{1/2}$ per external photon:

$$\boxed{\varepsilon^\mu(k) \to Z_3^{1/2}\,\varepsilon^\mu(k)\,.} \tag{6.202}$$

### Combining factors and charge renormalization

The bare interaction written in renormalized fields is

$$e_0\bar\psi_0\gamma^\mu\psi_0 A_{0\mu} = e_0 Z_2 Z_3^{1/2}\bar\psi\gamma^\mu\psi A_\mu \equiv \mu^{\epsilon/2} Z_1 e\,\bar\psi\gamma^\mu\psi A_\mu$$

[eq. (6.160)]. Gauge invariance then imposes

$$Z_1 = Z_2 \qquad \text{(Ward–Takahashi identity)}.$$

Hence

$$\boxed{\mu^{\epsilon/2} e = Z_3^{1/2} e_0\,,} \tag{6.203}$$

which in four dimensions is $e = Z_3^{1/2} e_0$.

This is the Mandl–Shaw conclusion: charge renormalization is controlled by photon wave-function renormalization, while fermion external-leg normalization and vertex renormalization are tied by $Z_1 = Z_2$.

## 6.6 Systematic renormalization of QED

Having computed the one-loop corrections to the propagators and vertex, and understood how external legs are renormalized, we now organize the full renormalization programme starting from the bare QED Lagrangian.

**The bare QED Lagrangian**

We start from the bare QED Lagrangian in covariant gauge:

$$\mathcal{L}_0 = -\frac{1}{4} F_{0\mu\nu} F_0^{\mu\nu} + \bar{\psi}_0 (i\partial\!\!\!/ - m_0)\psi_0 + e_0 \bar{\psi}_0 \gamma^\mu \psi_0 A_{0\mu} - \frac{1}{2\xi_0} (\partial \cdot A_0)^2, \quad (6.204)$$

where $F_{0\mu\nu} \equiv \partial_\mu A_{0\nu} - \partial_\nu A_{0\mu}$. The subscript "0" denotes bare quantities. The last term is the *gauge-fixing* term, necessary to define the photon propagator.

**Why the gauge-fixing term is needed.**

The Maxwell Lagrangian $\mathcal{L}_{\text{Maxwell}} = -\frac{1}{4} F_{0\mu\nu} F_0^{\mu\nu}$ is invariant under gauge transformations $A_{0\mu}(x) \mapsto A_{0\mu}(x) + \partial_\mu \alpha(x)$. Because of this invariance, the quadratic operator $O^{\mu\nu} = g^{\mu\nu}\Box - \partial^\mu \partial^\nu$ governing the free photon field is *not invertible*: the equation of motion $O^{\mu\nu} A_\nu = 0$ is satisfied by every pure-gauge configuration $A_\mu^{\text{pg}} = \partial_\mu \alpha$, regardless of the choice of $\alpha(x)$:

$$O^{\mu\nu} A_\nu^{\text{pg}} = \left(g^{\mu\nu}\Box - \partial^\mu \partial^\nu\right)\partial_\nu \alpha = \partial^\mu \Box \alpha - \partial^\mu \Box \alpha = 0.$$

In other words, the operator cannot distinguish between field configurations that differ only by a gauge transformation—it is "blind" along pure-gauge directions. This means we cannot construct a propagator for the photon: a propagator is supposed to tell us how a source $J^\mu$ produces a field response $A_\nu$, but if the kinetic operator annihilates an entire family of field configurations, the inverse simply does not exist. *Physically,* the gauge symmetry says that field configurations related by $A_\mu \to A_\mu + \partial_\mu \alpha$ are physically equivalent (they describe the same electromagnetic field $F_{\mu\nu}$); the path integral overcounts each physical configuration by the infinite volume of the gauge orbit, and this redundancy manifests itself precisely as the non-invertibility of the kinetic operator. The gauge-fixing term removes the redundancy by selecting one representative per orbit, lifting the zero modes and producing an invertible quadratic form. In perturbation theory we need the photon propagator, i.e. the inverse of this quadratic form.

Expanding the Maxwell term to quadratic order and integrating by parts:

$$-\frac{1}{4} F_{0\mu\nu} F_0^{\mu\nu} = \frac{1}{2} A_{0\mu} \left(g^{\mu\nu}\Box - \partial^\mu \partial^\nu\right) A_{0\nu}, \quad (6.205)$$

up to total derivatives. Similarly,

$$-\frac{1}{2\xi_0} (\partial \cdot A_0)^2 = \frac{1}{2} A_{0\mu} \left(-\frac{1}{\xi_0} \partial^\mu \partial^\nu\right) A_{0\nu}. \quad (6.206)$$

The total quadratic photon Lagrangian is therefore

$$\mathcal{L}_{\text{Maxwell}} + \mathcal{L}_{\text{gf}} = \frac{1}{2} A_{0\mu} \left[g^{\mu\nu}\Box - \left(1 - \frac{1}{\xi_0}\right) \partial^\mu \partial^\nu\right] A_{0\nu}. \quad (6.207)$$

The propagator $D_0^{\mu\nu}(x-y)$ is the response of the free photon field at $x$ to a unit point-like source at $y$: it solves

$$\left[g^{\mu\alpha}\Box - \left(1 - \frac{1}{\xi_0}\right) \partial^\mu \partial^\alpha\right] D_{0\alpha\nu}(x-y) = i\,\delta_\nu^\mu\, \delta^{(4)}(x-y)\,.$$

In other words, the propagator is the Green's function—i.e. the inverse—of the quadratic differential operator that governs the free field. Physically, this is why the propagator describes how a disturbance created at one spacetime point propagates to another: it encodes all the information about the free dynamics. In position space finding this inverse requires solving a PDE, but Fourier transforming ($\partial_\mu \to ik_\mu$, so $\Box \to -k^2$ and $\partial^\mu\partial^\nu \to -k^\mu k^\nu$) turns the differential operator into an algebraic one:

$$\widetilde{O}^{\mu\nu}(k) = -k^2 g^{\mu\nu} + \left(1 - \frac{1}{\xi_0}\right) k^\mu k^\nu. \tag{6.208}$$

*Without the gauge-fixing term* ($\xi_0 \to \infty$, equivalently dropping $\mathcal{L}_{\text{gf}}$), this reduces to $\widetilde{O}^{\mu\nu}_{\text{Maxwell}}(k) = -k^2 g^{\mu\nu} + k^\mu k^\nu$, which is manifestly non-invertible: contracting with $k_\nu$ gives

$$\widetilde{O}^{\mu\nu}_{\text{Maxwell}}(k)\, k_\nu = -k^2 k^\mu + k^\mu k^2 = 0,$$

so $k^\mu$ is a null eigenvector for every $k$. (This is the momentum-space image of the position-space pure-gauge mode $A_\mu = \partial_\mu \alpha$, which becomes $\widetilde{A}_\mu(k) = ik_\mu \widetilde{\alpha}(k)$.) Equivalently, the matrix $\widetilde{O}^{\mu\nu}_{\text{Maxwell}}(k) = -k^2(g^{\mu\nu} - k^\mu k^\nu/k^2)$ is $-k^2$ times the transverse projector, which has rank 3 (not 4) and hence vanishing determinant, so its inverse does not exist. The gauge-fixing term restores the longitudinal piece via the $-k^\mu k^\nu/\xi_0$ contribution in (6.208), lifting the zero mode and making the operator invertible. We need $D_{0\alpha\nu}(k)$ such that $\widetilde{O}^{\mu\alpha} D_{0\alpha\nu} = i\,\delta^\mu_\nu$. Since the only available tensors built from $k^\mu$ are $g^{\mu\nu}$ and $k^\mu k^\nu$, we write the ansatz $D_0^{\mu\nu} = A\, g^{\mu\nu} + B\, k^\mu k^\nu$ and substitute:

$$\widetilde{O}^{\mu\alpha} D_{0\alpha\nu} = -k^2 A\ \delta^\mu_\nu + \left[-k^2 B + \left(1 - \frac{1}{\xi_0}\right)(A + Bk^2)\right] k^\mu k_\nu \overset{!}{=} i\,\delta^\mu_\nu.$$

Matching the $\delta^\mu_\nu$ and $k^\mu k_\nu$ structures separately:

$$A = \frac{-i}{k^2}, \qquad B = \frac{i\,(1-\xi_0)}{k^4},$$

which gives the covariant-gauge propagator (adding the Feynman $i\epsilon$ prescription for time ordering):

$$D_0^{\mu\nu}(k) = \frac{-i}{k^2 + i\epsilon}\left[g^{\mu\nu} - (1-\xi_0)\frac{k^\mu k^\nu}{k^2}\right]. \tag{6.209}$$

Special choices: $\xi_0 = 1$ (Feynman gauge), $\xi_0 \to 0$ (Landau gauge). Thus, the gauge-fixing term is not a physical interaction but a device needed to define the propagator and perform perturbation theory in a manifestly Lorentz-covariant way.

**Field and parameter renormalization**

We introduce *renormalized* fields $(\psi, A_\mu)$ and parameters $(m, e, \xi)$ by

$$\psi_0 = \sqrt{Z_2}\,\psi, \qquad A_{0\mu} = \sqrt{Z_3}\,A_\mu, \tag{6.210}$$

$$m_0 = m + \delta m_0, \qquad e_0 = \mu^{\epsilon/2} Z_e\, e, \qquad \xi_0 = Z_3\,\xi, \tag{6.211}$$

where $\mu$ is the renormalization scale (introduced so that $e$ remains dimensionless for $d \neq 4$). The renormalization constants $Z_2, Z_3, Z_e$ and the bare mass shift $\delta m_0$ are determined order by order to cancel UV poles. After splitting the Lagrangian, it is convenient to trade $\delta m_0$ for the coefficient $\delta m$ of the local counterterm $-\delta m\,\bar\psi\psi$.

*Origin of $\xi_0 = Z_3\xi$.* The bare gauge-fixing term, expressed in renormalized fields via $A_{0\mu} = \sqrt{Z_3}\,A_\mu$, becomes

$$-\frac{1}{2\xi_0}(\partial\cdot A_0)^2 = -\frac{Z_3}{2\xi_0}(\partial\cdot A)^2,$$

i.e. a gauge-fixing term for the renormalized field with effective parameter $\xi_{\text{eff}} = \xi_0/Z_3$. Defining the renormalized gauge parameter as $\xi \equiv \xi_{\text{eff}} = \xi_0/Z_3$—equivalently $\xi_0 = Z_3\xi$—ensures that the gauge-fixing term retains its canonical form $-\frac{1}{2\xi}(\partial\cdot A)^2$ in renormalized variables. This convention reflects the fact that $\mathcal{L}_{\text{gf}}$ is not gauge invariant, so it does not receive its own independent renormalization: the only $Z$ factor that appears is the photon wavefunction renormalization $Z_3$, which is fully absorbed by the rescaling $\xi_0 \to \xi$. As a check, the bare and renormalized photon propagators have the same tensor structure, related simply by $D_0^{\mu\nu}(k;\xi_0) = Z_3\,D^{\mu\nu}(k;\xi)$.

It is customary to trade $Z_e$ for the *vertex* renormalization constant $Z_1$, defined by

$$Z_1 \equiv Z_e\,Z_2\,Z_3^{1/2}. \tag{6.212}$$

With this definition, substituting (6.210)–(6.211) into the bare interaction gives the identity

$$\begin{aligned} e_0\,\bar\psi_0\gamma^\mu\psi_0\,A_{0\mu} &= \big(\mu^{\epsilon/2}Z_e e\big)\big(Z_2\bar\psi\gamma^\mu\psi\big)\big(\sqrt{Z_3}\,A_\mu\big) \\ &= \mu^{\epsilon/2}\underbrace{Z_e Z_2 Z_3^{1/2}}_{=Z_1}\, e\,\bar\psi\gamma^\mu\psi\,A_\mu = \mu^{\epsilon/2}\,Z_1\,e\,\bar\psi\gamma^\mu\psi\,A_\mu, \end{aligned} \tag{6.213}$$

so $Z_1$ is precisely the multiplicative factor that converts the bare interaction term into a renormalized one (tree-level vertex plus counterterm: $Z_1 = 1 + \delta Z_1$).

**Why the renormalization scale $\mu$ is needed.**

In $d$ dimensions the action must be dimensionless: $[S] = 0$, hence $[\mathcal{L}] = d$. From the kinetic terms one reads off

$$[\psi] = \frac{d-1}{2}, \qquad [A] = \frac{d-2}{2}. \tag{6.214}$$

Requiring $[\mathcal{L}_{\text{int}}] = [e\bar{\psi}\gamma^\mu\psi A_\mu] = d$ gives

$$[e] = \frac{4-d}{2} = \frac{\epsilon}{2}. \tag{6.215}$$

Thus, in $d \neq 4$ the bare coupling is dimensionful. To keep the *renormalized* coupling $e$ dimensionless, one writes

$$e_0 = \mu^{\epsilon/2} Z_e\, e, \tag{6.216}$$

where $\mu$ is an arbitrary mass scale. This ensures that $[e] = 0$ while $[e_0] = \epsilon/2$ as required.

This has two consequences:

- UV divergences appear as $1/\epsilon$ poles, while finite parts contain logarithms $\ln(\mu^2/m^2)$;
- changing $\mu$ changes $e$ (at fixed $e_0$), which is the origin of the renormalization-group running.

**Splitting the Lagrangian: $\mathcal{L}_{\text{ren}} + \delta\mathcal{L}$**

Substituting the renormalization relations into $\mathcal{L}_0$, one can reorganize it as

$$\mathcal{L}_0 = \mathcal{L}_{\text{ren}} + \delta\mathcal{L}, \tag{6.217}$$

where the *renormalized Lagrangian* has the same form as QED written in terms of renormalized quantities:

$$\mathcal{L}_{\text{ren}} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\,(i\partial\!\!\!/ - m)\psi + \mu^{\epsilon/2} e\,\bar{\psi}\gamma^\mu\psi\, A_\mu - \frac{1}{2\xi}\,(\partial\cdot A)^2, \tag{6.218}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and the *counterterm Lagrangian* collects all remaining terms:

$$\begin{aligned}\delta\mathcal{L} = &-\frac{1}{4}\,(Z_3 - 1)\,F_{\mu\nu}F^{\mu\nu} + (Z_2 - 1)\,\bar{\psi}\, i\partial\!\!\!/\,\psi - \delta m\,\bar{\psi}\psi \\ &+ \mu^{\epsilon/2}(Z_1 - 1)\, e\,\bar{\psi}\gamma^\mu\psi\, A_\mu - \frac{1}{2}\left(\frac{Z_3}{\xi_0} - \frac{1}{\xi}\right)(\partial\cdot A)^2.\end{aligned} \tag{6.219}$$

With the choice $\xi_0 = Z_3\xi$, the gauge-fixing counterterm vanishes (or can be absorbed). The key point is that $\delta\mathcal{L}$ is treated as an *interaction* in perturbation theory: its terms generate additional Feynman rules (counterterm insertions) whose coefficients are fixed order by order to cancel UV poles.

**Detailed derivation of $\mathcal{L}_{\text{ren.}}$ and $\delta\mathcal{L}$**

**Step 1: Gauge kinetic term.** From $A_{0\mu} = \sqrt{Z_3}A_\mu$ we have $F_{0\mu\nu} = \sqrt{Z_3}F_{\mu\nu}$, hence

$$-\frac{1}{4} F_{0\mu\nu}F_0^{\mu\nu} = -\frac{1}{4} Z_3\, F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}\,(Z_3 - 1)\,F_{\mu\nu}F^{\mu\nu}. \tag{6.220}$$

**Step 2: Fermion kinetic and mass terms.** Using $\psi_0 = \sqrt{Z_2}\psi$:

$$\bar{\psi}_0 i\partial\!\!\!/ \psi_0 = Z_2 \bar{\psi} i\partial\!\!\!/ \psi = \bar{\psi} i\partial\!\!\!/ \psi + (Z_2 - 1)\bar{\psi} i\partial\!\!\!/ \psi. \tag{6.221}$$

For the mass term, with $m_0 = m + \delta m_0$:

$$-m_0 \bar{\psi}_0 \psi_0 = -(m + \delta m_0) Z_2 \bar{\psi}\psi. \tag{6.222}$$

Expanding and defining

$$\delta m \equiv m(Z_2 - 1) + Z_2 \delta m_0,$$

so that $\delta m$ is the coefficient of $-\bar{\psi}\psi$ in $\delta\mathcal{L}$, one obtains

$$\bar{\psi}_0(i\partial\!\!\!/ - m_0)\psi_0 = \bar{\psi}(i\partial\!\!\!/ - m)\psi + (Z_2 - 1)\bar{\psi} i\partial\!\!\!/ \psi - \delta m \bar{\psi}\psi. \tag{6.223}$$

**Step 3: Interaction term.** Using (6.212):

$$\begin{aligned} +e_0 \bar{\psi}_0 \gamma^\mu \psi_0 A_{0\mu} &= +\mu^{\epsilon/2} Z_1 e \bar{\psi}\gamma^\mu \psi A_\mu \\ &= +\mu^{\epsilon/2} e \bar{\psi}\gamma^\mu \psi A_\mu + \mu^{\epsilon/2}(Z_1 - 1) e \bar{\psi}\gamma^\mu \psi A_\mu. \end{aligned} \tag{6.224}$$

**Step 4: Gauge-fixing term.** Since $\partial \cdot A_0 = \sqrt{Z_3}\,\partial \cdot A$:

$$-\frac{1}{2\xi_0}(\partial \cdot A_0)^2 = -\frac{Z_3}{2\xi_0}(\partial \cdot A)^2. \tag{6.225}$$

With $\xi_0 = Z_3 \xi$ this equals $-\frac{1}{2\xi}(\partial \cdot A)^2$, so the gauge-fixing term is automatically renormalized. If one keeps $\xi_0 \neq Z_3\xi$, an additional counterterm $-\frac{1}{2}\left(\frac{Z_3}{\xi_0} - \frac{1}{\xi}\right)(\partial \cdot A)^2$ appears.

**Step 5: Collect results.** Summing the "1" pieces gives $\mathcal{L}_{\text{ren}}$ (6.218); summing the "$(Z_i - 1)$" and "$\delta m$" pieces gives $\delta\mathcal{L}$ (6.219).

**Counterterm Feynman rules**

The counterterms in $\delta\mathcal{L}$ generate additional vertices in perturbation theory. In momentum space:

$$\text{Electron 2-point:} \qquad i\Sigma_{\text{ct}}(p\!\!\!/) = i\left[(Z_2 - 1)p\!\!\!/ - \delta m\right], \tag{6.226}$$

$$\text{Photon 2-point:} \qquad i\Pi^{\mu\nu}_{\text{ct}}(q) = -i(Z_3 - 1)\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right), \tag{6.227}$$

$$\text{Vertex 3-point:} \qquad i\Gamma^\mu_{\text{ct}} = +ie\,\mu^{\epsilon/2}(Z_1 - 1)\gamma^\mu. \tag{6.228}$$

These insertions are included in Feynman diagrams exactly like ordinary vertices. Their role is to cancel the UV-divergent parts of loop diagrams.

**Counterterm diagrams: detailed discussion.**

In perturbation theory, one computes Green functions using $\mathcal{L}_{\text{ren}}$ as the "main" Lagrangian and treats $\delta\mathcal{L}$ as additional interaction vertices. To a given loop order, one includes all diagrams built from the usual QED vertices *plus* diagrams where one (or more) vertex is replaced by a counterterm insertion.

**(i) Fermion two-point counterterm.** From

$$\delta\mathcal{L}_{\bar\psi\psi} = (Z_2 - 1)\,\bar\psi\, i\partial\!\!\!/\,\psi - \delta m\,\bar\psi\psi, \tag{6.229}$$

one obtains the 1PI insertion (6.226) on a fermion line. Diagrammatically:


The $(Z_2 - 1)$ term renormalizes the wavefunction (proportional to $\not{p}$), while $\delta m$ renormalizes the mass (proportional to the identity).

**(ii) Photon two-point counterterm.** From

$$\delta\mathcal{L}_{AA} = -\frac{1}{4}\,(Z_3 - 1)\,F_{\mu\nu}F^{\mu\nu}, \tag{6.230}$$

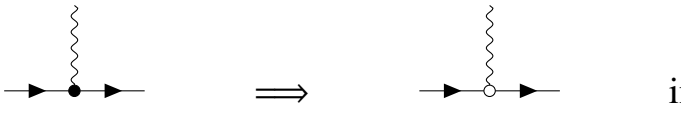
expanding to quadratic order gives (6.227). Diagrammatically:


This counterterm cancels the UV-divergent vacuum polarization and implements the photon renormalization condition.

**(iii) Vertex counterterm.** From

$$\delta\mathcal{L}_{\bar\psi\psi A} = +(Z_1 - 1)\,e\,\bar\psi\gamma^\mu\psi\,A_\mu, \tag{6.231}$$

one obtains the vertex insertion (6.228). Diagrammatically:




This counterterm cancels the UV-divergent one-loop vertex correction. The Ward identity $Z_1 = Z_2$ (see below) relates the vertex and fermion wavefunction counterterms.

**(iv) Gauge-fixing counterterm (optional).** If $\xi_0 \neq Z_3\xi$, the term

$$\delta\mathcal{L}_{\text{gf}} = -\frac{1}{2}\left(\frac{1}{\xi_0} - \frac{1}{\xi}\right)(\partial\cdot A)^2 \tag{6.232}$$

modifies only the longitudinal photon propagator and does not affect physical S-matrix elements. With $\xi_0 = Z_3\xi$ this term vanishes.

**One-loop divergences and renormalization conditions**

At one loop, UV divergences arise from three 1PI Green functions:

$$\Sigma(\not p) \qquad \text{(electron self-energy)}, \tag{6.233}$$
$$\Pi^{\mu\nu}(q) \qquad \text{(photon vacuum polarization)}, \tag{6.234}$$
$$\Gamma^{\mu}(p',p) \qquad \text{(vertex correction)}. \tag{6.235}$$

Renormalization means choosing $(Z_2-1),(Z_3-1),(Z_1-1),\delta m$ such that, order by order in $e$, the sums

$$\Sigma(\not p)+\Sigma_{\rm ct}(\not p), \tag{6.236}$$
$$\Pi^{\mu\nu}(q)+\Pi^{\mu\nu}_{\rm ct}(q), \tag{6.237}$$
$$\Gamma^{\mu}(p',p)+\Gamma^{\mu}_{\rm ct}, \tag{6.238}$$

are UV finite and satisfy the chosen *renormalization conditions*.

**On-shell scheme.**

One imposes that renormalized propagators have poles at the physical mass with unit residues, and that the renormalized charge equals the physical charge at zero momentum transfer.

For the electron propagator

$$S_R(\not p)=\frac{i}{\not p - m - \Sigma_R(\not p)}, \qquad \Sigma_R \equiv \Sigma+\Sigma_{\rm ct}, \tag{6.239}$$

the on-shell conditions are

$$\Sigma_R(\not p)\Big|_{\not p=m}=0, \qquad \frac{\partial}{\partial \not p}\Sigma_R(\not p)\Big|_{\not p=m}=0, \tag{6.240}$$

which determine $\delta m$ and $Z_2$.

For the photon, one requires transversality and that the renormalized vacuum polarization vanishes at $q^2=0$ (so the photon remains massless):

$$\Pi^{\mu\nu}_R(q)=\left(q^2 g^{\mu\nu}-q^\mu q^\nu\right)\Pi_R(q^2), \qquad \Pi_R(0)=0, \tag{6.241}$$

which fixes $Z_3$.

The charge is fixed by a condition on the vertex at $q=p'-p=0$.

**$\overline{\rm MS}$ scheme**

Alternatively, one chooses counterterms to subtract only the UV poles (and the standard $\overline{\rm MS}$ constants) in dimensional regularization. Then $Z_2,Z_3,Z_1,\delta m$ contain only $1/\epsilon$-type terms:

$$Z_i-1=\frac{c_i}{\epsilon}+\cdots, \qquad \delta m=\frac{c_m}{\epsilon}m+\cdots, \tag{6.242}$$

where the coefficients $c_i, c_m$ are determined by the divergent parts of the loop integrals.

**Ward identity and relations among renormalization constants**

Gauge invariance implies the Ward–Takahashi identity, which enforces (to all orders)

$$Z_1 = Z_2 \,. \tag{6.243}$$

This is a powerful constraint: the vertex renormalization is not independent of the fermion wavefunction renormalization. Consequently, the charge renormalization is controlled entirely by the photon field renormalization:

$$Z_e = \frac{Z_1}{Z_2 Z_3^{1/2}} = \frac{1}{Z_3^{1/2}} \,. \tag{6.244}$$

This is one of the main simplifications of QED renormalization.

**Practical workflow**

To renormalize QED at a given loop order:

1. Compute the divergent parts of $\Sigma$, $\Pi^{\mu\nu}$, $\Gamma^{\mu}$ in $d = 4 - \epsilon$.
2. Choose a renormalization scheme (on-shell or $\overline{\text{MS}}$).
3. Solve for $(Z_2 - 1)$, $(Z_3 - 1)$, $(Z_1 - 1)$, $\delta m$ so that the renormalized 1PI functions satisfy the chosen conditions.
4. Include counterterm insertions consistently in higher-loop diagrams.

**Worked example: renormalization of the electron propagator**

**Aim.** We present a complete example of renormalization focusing on the electron two-point function at one loop. We show that the counterterm method follows naturally from rewriting the bare Lagrangian in terms of renormalized quantities.

**1. Setup: bare theory and the split $\mathcal{L}_0 = \mathcal{L}_{\text{ren.}} + \delta\mathcal{L}$**

Start from the bare Lagrangian (6.204) and introduce renormalized fields/parameters via (6.210)–(6.211). The split $\mathcal{L}_0 = \mathcal{L}_{\text{ren}} + \delta\mathcal{L}$ was derived above. The fermion two-point counterterm is

$$\delta\mathcal{L}_{\bar\psi\psi} = (Z_2 - 1)\,\bar\psi\, i\partial\!\!\!/\,\psi - \delta m\,\bar\psi\psi \,. \tag{6.245}$$

**2. The renormalized self-energy.**

The full renormalized electron propagator is

$$S_R(p\!\!\!/) = \frac{i}{p\!\!\!/ - m - \Sigma_R(p\!\!\!/) + i\epsilon} \,. \tag{6.246}$$

At one loop, the renormalized self-energy is

$$\Sigma_R(p\!\!\!/) = \Sigma(p\!\!\!/) + \Sigma_{\text{ct}}(p\!\!\!/) = \Sigma(p\!\!\!/) + (Z_2 - 1)p\!\!\!/ - \delta m, \tag{6.247}$$

where $\Sigma(p\!\!\!/)$ is the loop self-energy (UV divergent in $d = 4 - \epsilon$).

**3. On-shell renormalization conditions.**

In the on-shell scheme we demand:

- the propagator has a pole at the physical mass $m$;
- the residue at the pole is 1.

Write the general Dirac decomposition of the self-energy:

$$\Sigma(\not{p}) = \not{p}\,\Sigma_V(p^2) + m\,\Sigma_S(p^2), \tag{6.248}$$

where $\Sigma_V$ and $\Sigma_S$ are scalar functions containing UV poles. Then

$$\Sigma_R(\not{p}) = \not{p}\Big[\Sigma_V(p^2) + (Z_2 - 1)\Big] + m\,\Sigma_S(p^2) - \delta m. \tag{6.249}$$

**Pole condition.** The inverse propagator vanishes at the pole:

$$S_R^{-1}(\not{p}) = \not{p} - m - \Sigma_R(\not{p}) = 0 \quad \text{at} \quad \not{p} = m\,. \tag{6.250}$$

Evaluating at $\not{p} = m$ gives

$$\delta m = m\Big[\Sigma_V(m^2) + \Sigma_S(m^2)\Big] + m(Z_2 - 1). \tag{6.251}$$

**Residue condition.** Near the pole, the residue-one condition requires

$$\frac{\partial}{\partial \not{p}}\Sigma_R(\not{p})\Big|_{\not{p}=m} = 0\,. \tag{6.252}$$

Using (6.247):

$$\frac{\partial}{\partial \not{p}}\Sigma_R = \frac{\partial}{\partial \not{p}}\Sigma + (Z_2 - 1), \tag{6.253}$$

so

$$Z_2 - 1 = -\frac{\partial}{\partial \not{p}}\Sigma(\not{p})\Big|_{\not{p}=m}\,. \tag{6.254}$$

Inserting (6.254) into (6.251) determines $\delta m$ purely in terms of $\Sigma$ evaluated on-shell.

**4. $\overline{\text{MS}}$ scheme**

In $\overline{\text{MS}}$, one does not impose on-shell conditions. Instead, one chooses $Z_2$ and $\delta m$ to cancel only the $1/\epsilon$ poles. If the one-loop result has the form

$$\Sigma(\not{p}) = \frac{1}{\epsilon}\Big(A\,\not{p} + B\,m\Big) + \text{finite}, \tag{6.255}$$

then one sets

$$Z_2 - 1 = -\frac{A}{\epsilon} + \cdots, \tag{6.256}$$

$$\delta m = \frac{B\,m}{\epsilon} + \cdots, \tag{6.257}$$

so that $\Sigma_R(\not{p})$ is finite.

**5. Final result.**

Combining the loop self-energy and counterterm, the renormalized inverse propagator is

$$S_R^{-1}(\not{p}) = \not{p} - m - \Sigma(\not{p}) - (Z_2 - 1)\not{p} + \delta m\,. \tag{6.258}$$

By construction, the UV divergent $1/\epsilon$ parts cancel between $\Sigma(\not{p})$ and the counterterm contributions. The propagator is finite and satisfies the chosen renormalization conditions.

## 6.7 General renormalization theory: power counting and the classification of divergences

So far in this chapter, we have carried out the renormalization of QED diagram by diagram. It is natural to ask: *how do we know in advance which diagrams are divergent?* And more fundamentally: *what makes QED special, in that only a finite number of counterterms suffice to remove all divergences?*

This section addresses these questions by developing the general theory of ultraviolet divergences, following Weinberg [7], Chapter 12. The central result is that the degree of divergence of any Feynman diagram can be determined by simple power counting, and that the requirement of **renormalizability**—the property that only finitely many types of counterterms are needed—places severe constraints on the allowed interactions. We derive all the key formulas in detail.

### The superficial degree of divergence

Consider a general quantum field theory containing fields of various types, labelled $f$, and interactions of various types, labelled $i$. Each interaction of type $i$ involves $n_{if}$ fields of type $f$ and $d_i$ derivatives. We want to determine when a given Feynman diagram produces a divergent momentum-space integral.

**Definition.** The **superficial degree of divergence** $D$ of a Feynman diagram is defined as the number of powers of loop momentum in the numerator of the integrand minus the number in the denominator, plus four for each independent loop momentum. In the region where all internal momenta go to infinity together with a common factor $\kappa \to \infty$, the integral behaves as:

$$\int^{\infty} \kappa^{D-1}\, d\kappa\,. \tag{6.259}$$

Therefore:

- $D > 0$: the diagram is **power-law divergent** (the integral diverges as $\Lambda^D$);
- $D = 0$: the diagram is **logarithmically divergent** ($\sim \ln \Lambda$);
- $D < 0$: the diagram is **superficially convergent** (at least in the region where all momenta grow together).

**Notation.** For a given connected, one-particle-irreducible (1PI) Feynman diagram, define:

$$\begin{aligned} I_f &\equiv \text{number of internal lines of field type } f\,, \\ E_f &\equiv \text{number of external lines of field type } f\,, \\ N_i &\equiv \text{number of vertices of interaction type } i\,. \end{aligned} \tag{6.260}$$

### Propagator behavior and the spin parameter $s_f$

The asymptotic behavior of the propagator $\Delta_f(k)$ for a field of type $f$ at large momentum $k$ is written as:

$$\Delta_f(k) \sim k^{-2+2s_f} \qquad \text{for } |k| \to \infty\,, \tag{6.261}$$

where $s_f$ is a parameter related to the spin of the field.

**Box 6.3 Values of $s_f$ for common fields**

- **Scalar field** ($\phi$): The propagator is $\Delta(k) = i/(k^2 - m^2)$, which behaves as $k^{-2}$ at large $k$. Hence $s_\phi = 0$.
- **Dirac field** ($\psi$): The propagator is $S(k) = i(\not{k} + m)/(k^2 - m^2)$, which behaves as $k^{-1}$ at large $k$ (the numerator contributes one power of $k$). Hence $s_\psi = 1/2$.
- **Massive vector field** ($A_\mu$, Proca): The propagator tends to a (non-vanishing) *constant* at large $|k|$—i.e. scales as $|k|^0$, the same as $k^{-2+2s_f}$ with $s_f = 1$—because of the $k_\mu k_\nu/m^2$ term. Hence $s_A = 1$ for a massive vector.[a]
- **Photon** ($A_\mu$, massless gauge field): Thanks to gauge invariance, the effective propagator in any gauge is $\sim k^{-2}$. Hence $s_\gamma = 0$, the same as a scalar.

More generally, for a massive field in the Lorentz representation $(A, B)$ one has $s_f = A + B$. Here $(A, B)$ are the two half-integer labels of the standard classification of irreducible representations of the (complexified) Lorentz algebra $\mathfrak{so}(1,3)_{\mathbb{C}} \simeq \mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$, so each of $A, B \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \dots\}$ labels an SU(2) irrep. The familiar fields correspond to:

| | | |
|---|---|---|
| $(0,0)$ | scalar | $s_f = 0$ |
| $(\frac{1}{2},0)$, $(0,\frac{1}{2})$ | Weyl (L, R) spinors; Dirac is their sum | $s_f = \frac{1}{2}$ |
| $(\frac{1}{2},\frac{1}{2})$ | vector $A_\mu$ | $s_f = 1$ |
| $(1,0)\oplus(0,1)$ | antisymmetric tensor $F_{\mu\nu}$ | $s_f = 1$ |

The total spin contained in $(A,B)$ ranges from $|A-B|$ to $A+B$; the *maximum* spin $A+B$ controls the worst high-momentum behaviour of the propagator, which is what enters power counting. The photon is special: despite being a spin-1 particle, gauge invariance ensures that it propagates like a scalar at high momenta. This is crucial for the renormalizability of QED.

[a]The Proca Lagrangian is $\mathcal{L}_{\text{Proca}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu$. The mass term breaks gauge invariance, but it makes the quadratic form invertible without any gauge-fixing term: the equations of motion $\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0$ imply $\partial_\mu A^\mu = 0$ as a consistency condition (taking $\partial_\nu$ on both sides), so the field automatically carries $4-1=3$ physical polarizations of a massive spin-1 particle. The momentum-space propagator is

$$D^{\mu\nu}_{\text{Proca}}(k) = \frac{-i}{k^2 - m^2 + i\epsilon}\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{m^2}\right).$$

The longitudinal piece $k^\mu k^\nu/m^2$ grows like $|k|^2$, which combined with the $1/(k^2-m^2) \sim |k|^{-2}$ prefactor gives a non-decaying contribution: $D^{\mu\nu}_{\text{Proca}}(k) \to (i/m^2)\,\hat{k}^\mu \hat{k}^\nu + O(|k|^{-2})$ as $|k| \to \infty$, i.e. a constant tensor. In the power-counting formula (6.261) this corresponds to $s_A = 1$. This bad UV behaviour is the reason why non-gauge-invariant massive vector theories are generically non-renormalizable; the gauge-invariant resolution (Higgs mechanism) restores the soft $\sim |k|^{-2}$ behaviour.

### Derivation of the formula for $D$

We now compute $D$ by counting powers of momentum in a generic Feynman diagram.

**Step 1: Contribution from propagators.** Each internal line of type $f$ contributes a propagator $\Delta_f(k) \sim k^{-2+2s_f}$. With $I_f$ such lines, the total contribution to $D$ is:

$$\text{propagators:} \qquad \sum_f I_f(-2+2s_f)\,. \tag{6.262}$$

**Step 2: Contribution from vertices.** Each vertex of type $i$ carries $d_i$ derivatives, which in momentum space become factors of momenta. With $N_i$ such vertices, the contribution is:

$$\text{vertices:} \qquad \sum_i N_i\, d_i\,. \tag{6.263}$$

**Step 3: Number of independent loop momenta.** Each of the $\sum_f I_f$ internal lines can be labelled with an independent four-momentum. However, the momentum-conserving delta function at each vertex imposes a constraint on these momenta. With $\sum_i N_i$ vertices, we have $\sum_i N_i$ delta functions, but one of them merely enforces overall momentum conservation and does not constrain internal momenta. The number of independent loop momenta is therefore:

$$L = \sum_f I_f - \sum_i N_i + 1\,, \tag{6.264}$$

which is exactly the number of independent loops in the diagram. Each loop contributes a four-dimensional integration $d^4k$, giving a factor of +4 per loop:

$$\text{loops:} \qquad 4L = 4\left[\sum_f I_f - \sum_i N_i + 1\right]. \tag{6.265}$$

**Step 4: Combining all contributions.** Adding (6.262), (6.263), and (6.265):

$$\begin{aligned} D &= \sum_f I_f(-2+2s_f) + \sum_i N_i d_i + 4\sum_f I_f - 4\sum_i N_i + 4 \\ &= \sum_f I_f(2+2s_f) + \sum_i N_i(d_i-4) + 4\,. \end{aligned} \tag{6.266}$$

This is a correct formula, but it is inconvenient: $D$ depends on the internal structure of the diagram (the numbers $I_f$ of internal lines), which varies from diagram to diagram even for the same physical process. We want a formula that depends only on the *external* content of the diagram.

### The topological identity

To eliminate $I_f$, we use a fundamental identity that relates internal and external lines to vertices.

**Topological identity.** Consider all the lines of type $f$ attached to vertices. Each interaction of type $i$ has $n_{if}$ lines of type $f$, so the total number of line-ends at vertices is $\sum_i N_i n_{if}$. Now, each *internal* line connects two vertices (contributing 2 line-ends), while each *external* line connects to only one vertex (contributing 1 line-end). Therefore:

$$\boxed{2I_f + E_f = \sum_i N_i n_{if}} \tag{6.267}$$

This identity holds for each field type $f$ separately. Solving for $I_f$:

$$I_f = \frac{1}{2}\left(\sum_i N_i n_{if} - E_f\right). \tag{6.268}$$

### The master formula for $D$

Substituting (6.268) into (6.266):

$$\begin{aligned} D &= \sum_f \frac{1}{2}\left(\sum_i N_i n_{if} - E_f\right)(2+2s_f) + \sum_i N_i(d_i-4)+4 \\ &= \sum_f \sum_i N_i n_{if}(1+s_f) - \sum_f E_f(1+s_f) + \sum_i N_i(d_i-4)+4 \\ &= \sum_i N_i\left[d_i + \sum_f n_{if}(1+s_f) - 4\right] - \sum_f E_f(1+s_f)+4\,. \end{aligned} \tag{6.269}$$

Defining the **index of divergence** of an interaction of type $i$:

$$\boxed{\Delta_i \equiv 4 - d_i - \sum_f n_{if}(1+s_f)} \tag{6.270}$$

the formula for $D$ becomes:

$$\boxed{D = 4 - \sum_f E_f(s_f+1) - \sum_i N_i \Delta_i} \tag{6.271}$$

This is the **master formula** for the superficial degree of divergence. Its power lies in the fact that it separates the dependence on the physical process (through $E_f$) from the dependence on the specific diagram (through $N_i$).

**Box 6.4 Dimensional analysis derivation of $\Delta_i$**

The parameter $\Delta_i$ has a direct interpretation in terms of dimensional analysis (in natural units $\hbar = c = 1$, where [mass] = [momentum] = [energy] = [length]$^{-1}$).
**Step 1: Field dimensionality.** The propagator of a field $f$ is the Fourier transform of a vacuum expectation value of a product of two fields:

$$\Delta_f(k) = \int d^4x\, e^{ik\cdot x}\langle 0|T\{\phi_f(x)\phi_f(0)\}|0\rangle\,. \tag{6.272}$$

The integration $d^4x$ has dimension [mass]$^{-4}$, and the propagator behaves as $k^{-2+2s_f}$, so:

$$[\Delta_f] = -4 + 2\mathscr{D}_f = -2 + 2s_f \quad\Longrightarrow\quad \mathscr{D}_f = 1 + s_f\,, \tag{6.273}$$

where $\mathscr{D}_f$ is the mass dimension of the field $\phi_f$.
For the standard fields:

| Field | $s_f$ | $\mathscr{D}_f = 1 + s_f$ |
|---|---|---|
| Scalar $\phi$ | 0 | 1 |
| Dirac $\psi$ | 1/2 | 3/2 |
| Photon $A_\mu$ | 0 | 1 |

**Step 2: Coupling constant dimensionality.** An interaction $i$ with $n_{if}$ fields and $d_i$ derivatives has dimension $d_i + \sum_f n_{if}\mathscr{D}_f$. The action $S = \int d^4x\,\mathcal{L}$ must be dimensionless, so $\mathcal{L}$ has dimension $[\text{mass}]^4$. The coupling constant $g_i$ must therefore have dimension:

$$[g_i] = 4 - d_i - \sum_f n_{if}(1 + s_f) = \Delta_i\,. \tag{6.274}$$

Thus, $\Delta_i$ **is nothing but the mass dimension of the coupling constant** of interaction $i$.

### Classification of interactions: renormalizable, super-renormalizable, and non-renormalizable

The master formula (6.271) immediately leads to a classification of interactions based on $\Delta_i$.

**Case 1:** $\Delta_i \geq 0$ **for all interactions (renormalizable theories).**

If all interactions satisfy $\Delta_i \geq 0$, the term $-\sum_i N_i\Delta_i$ is non-positive, and the formula gives an **upper bound** on $D$ that depends only on the external content of the diagram:

$$D \leq 4 - \sum_f E_f(s_f + 1)\,. \tag{6.275}$$

Only a *finite* number of combinations of external lines ($E_f$) can give $D \geq 0$. This means that only a finite number of types of diagrams are divergent, and therefore only a finite number of counterterms are needed. Such theories are called **renormalizable**.

Within the renormalizable class, one further distinguishes:

- $\Delta_i = 0$: **marginal** (or strictly renormalizable) interactions. The coupling is dimensionless.
- $\Delta_i > 0$: **relevant** (or super-renormalizable) interactions. Adding more vertices *lowers* $D$; these interactions produce divergences only at low orders.

**Case 2:** $\Delta_i < 0$ **for some interactions (non-renormalizable theories).**

If any interaction has $\Delta_i < 0$, then adding more vertices of that type *increases* $D$. No matter how many external lines we have, by including enough such vertices we can make $D \geq 0$. This means that *every* amplitude eventually diverges at sufficiently high order, and infinitely many different counterterms are needed. Such interactions are called **non-renormalizable** (or **irrelevant**), and theories containing them are called non-renormalizable.

| **Type** | $\Delta_i$ | $[g_i]$ | **Effect on** $D$ |
|---|---|---|---|
| Super-renormalizable (relevant) | $> 0$ | positive mass dim. | lowers $D$ |
| Renormalizable (marginal) | $= 0$ | dimensionless | does not change $D$ |
| Non-renormalizable (irrelevant) | $< 0$ | negative mass dim. | raises $D$ |

### Application to QED

Let us verify that QED is renormalizable by computing $\Delta_i$ for each interaction in the QED Lagrangian:

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(i\not{\partial} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + e\bar{\psi}\gamma^\mu\psi A_\mu + \mathcal{L}_{\text{ct}}, \tag{6.276}$$

where $\mathcal{L}_{\text{ct}}$ contains the counterterms.

| **Interaction** | $d_i$ | $n_{i\gamma}$ | $n_{ie}$ | $\Delta_i = 4 - d_i - n_{i\gamma} - \frac{3}{2}n_{ie}$ | **Type** |
|---|---|---|---|---|---|
| $e\bar{\psi}\gamma^\mu\psi A_\mu$ | 0 | 1 | 2 | $4-0-1-3=0$ | marginal |
| $F_{\mu\nu}F^{\mu\nu}$ (kinetic) | 2 | 2 | 0 | $4-2-2-0=0$ | marginal |
| $\bar{\psi}\not{\partial}\psi$ (kinetic) | 1 | 0 | 2 | $4-1-0-3=0$ | marginal |
| $m\bar{\psi}\psi$ (mass) | 0 | 0 | 2 | $4-0-0-3=1$ | super-ren. |

Since all $\Delta_i \geq 0$, **QED is renormalizable**. The bound (6.275) gives, with $s_\gamma = 0$ and $s_e = 1/2$:

$$\boxed{D \leq 4 - E_\gamma - \frac{3}{2}E_e} \tag{6.277}$$

**Box 6.5 Enumeration of superficially divergent amplitudes in QED**

The bound (6.277) lists the 1PI amplitudes with $D \geq 0$. Since $E_e$ is even (fermion lines come in pairs), the possibilities are:

| $E_e$ | $E_\gamma$ | $D \leq$ | **Amplitude** | **Status** |
|---|---|---|---|---|
| 0 | 1 | 3 | Photon tadpole | Vanishes (Furry) |
| 0 | 2 | 2 | Photon self-energy $\Pi^{\mu\nu}$ | Divergent |
| 0 | 3 | 1 | Three-photon vertex | Vanishes (Furry) |
| 0 | 4 | 0 | Light-by-light | Finite (gauge inv.) |
| 2 | 0 | 1 | Electron self-energy $\Sigma$ | Divergent |
| 2 | 1 | 0 | Vertex $\Gamma^\mu$ | Divergent |
| 1 | 0,1,2 | $\frac{5}{2}, \frac{3}{2}, \frac{1}{2}$ | Odd-fermion | Absent (fermion-# cons.) |

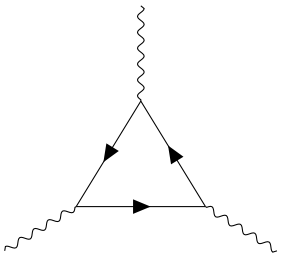
Only **three** 1PI amplitudes are genuinely divergent in QED:

- Photon self-energy $\Pi^{\mu\nu}(q)$, $D \leq 2$ — absorbed by $Z_3$.
- Electron self-energy $\Sigma(p)$, $D \leq 1$ — absorbed by $Z_2$ and $\delta m$.
- Vertex $\Gamma^\mu(p', p)$, $D \leq 0$ — absorbed by $Z_1$ (with $Z_1 = Z_2$ by Ward).

These are exactly the three structures renormalized in the preceding sections: power counting guarantees *no other divergences* can appear, at any order.

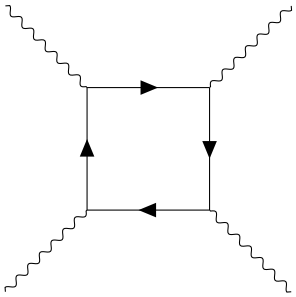
**Remark: the role of symmetries.** Several amplitudes that power counting allows to diverge are actually finite thanks to symmetry:

- The **three-photon vertex** ($E_\gamma = 3$, $D \leq 1$) vanishes by **Furry's theorem** (charge conjugation invariance): any closed fermion loop with an odd number of photon vertices gives zero. The lowest-order diagram is the fermion triangle:



+ (reversed orientation) = 0.

  The two orientations of the closed fermion loop differ by a relative sign (each $C$-conjugation reverses the loop and contributes a $(-1)^{n_\gamma}$ for $n_\gamma$ photon vertices), so for odd $n_\gamma$ they cancel exactly.
- **Light-by-light scattering** ($E_\gamma = 4$, $D = 0$) is rendered finite by **gauge invariance**. The divergent part would have to be a constant $K(\eta_{\mu\nu}\eta_{\rho\sigma} + \text{perms})$, but current conservation $q^\mu M_{\mu\nu\rho\sigma} = 0$ forces $K = 0$. The lowest-order diagram is the fermion box (plus its $3! = 6$ photon-leg permutations):



+ (perms).

Naive power counting gives $D = 0$ (logarithmic divergence: 4 propagators $\sim k^{-4}$ vs. $\int d^4k$). However, the WTI requires that $q_i^\mu M_{\mu\nu\rho\sigma} = 0$ for each external photon, which forces enough powers of external momenta in the numerator to make the integral UV finite.

- **Odd-fermion-number amplitudes** are absent because each QED vertex contains one $\psi$ and one $\bar{\psi}$, so fermion number conservation forces external fermion fields to occur in pairs.

Symmetries thus play a dual role: they reduce the number of independent divergences, and they guarantee the consistency of the renormalization program (the available counterterms match the divergences exactly).

### Why the superficial degree of divergence is not the whole story: subdivergences

The superficial degree of divergence $D$ characterizes the behavior of the integral when *all* internal momenta go to infinity together. However, divergences can also arise when only the momenta of a *subgraph* go to infinity while others remain fixed. These are called **subdivergences**.

**Example.** Consider a two-loop contribution to Compton scattering ($E_e = 2$, $E_\gamma = 2$), for which $D \leq -1$. The overall integral is superficially convergent. However, if the diagram contains a one-loop electron self-energy insertion on an internal fermion line, that subgraph has $D = 1$ and diverges, even though the full diagram is superficially convergent.

A theorem due to Weinberg states that a Feynman integral is *actually* convergent if and only if both the full integral *and* every possible subintegration have $D < 0$. In a renormalizable theory, every divergent subgraph corresponds to one of the finitely many divergent amplitudes enumerated above, and therefore comes accompanied by its counterterm. The counterterms cancel the subdivergences, rendering the full integral convergent.

### The cancellation of divergences and overlapping divergences

The renormalization program works as follows. Any graph or subgraph with $D \geq 0$ produces a divergent contribution that is a **polynomial in external momenta of**

**degree** $D$, plus a finite remainder. Such a polynomial is exactly what would be produced by a local interaction with $E_f$ fields and at most $D$ derivatives. For a renormalizable theory ($\Delta_i \geq 0$ for all $i$), these are precisely the interactions already present in the Lagrangian. Therefore:

- The divergent part of $\Sigma(p)$ ($D = 1$) is a first-order polynomial in $\not{p}$: $\Sigma^{\text{div}} = A\not{p} + Bm$, which is absorbed by $(Z_2 - 1)$ and $\delta m$.
- The divergent part of $\Pi^{\mu\nu}(q)$ ($D = 2$) is a second-order polynomial in $q$, but gauge invariance ($q^\mu \Pi_{\mu\nu} = 0$) reduces it to a single constant $C$ multiplying $(q^2\eta_{\mu\nu} - q_\mu q_\nu)$, absorbed by $(Z_3 - 1)$.
- The divergent part of $\Gamma^\mu(p', p)$ ($D = 0$) is a constant proportional to $\gamma^\mu$: $\Gamma^{\mu,\text{div}} = L\gamma^\mu$, absorbed by $(Z_1 - 1)$.

The Ward identity further relates these constants: $L = A$ (the vertex divergence equals the $\not{p}$-coefficient of the self-energy divergence), i.e., $Z_1 = Z_2$, reducing the number of independent renormalization constants.

**Overlapping divergences.** A subtlety arises when two divergent subgraphs share an internal line, so they cannot be treated as independent divergent integrals. In QED, this occurs starting at two-loop order, for instance when two vertex corrections overlap inside a photon or electron self-energy. Despite this complication, it can be shown (through the BPHZ prescription of Bogoliubov, Parasiuk, Hepp, and Zimmermann) that the counterterms still cancel all divergences. The key idea is to consider all possible "forests" of nested subgraphs and subtract their divergences systematically, starting from the innermost and working outward.

## Beyond renormalizability: effective field theories

**Why non-renormalizable theories are not hopeless.** If we include interactions with $\Delta_i < 0$, infinitely many types of counterterms are needed, one for every interaction allowed by symmetries. This may seem catastrophic, but it is not: as long as we include *all* interactions consistent with the symmetries of the theory, every divergence can still be absorbed into a coupling constant. In this sense, non-renormalizable theories are just as "renormalizable" as renormalizable ones—they simply require infinitely many parameters.

**Dimensional suppression.** The key insight is that the coupling constant $g_i$ of a non-renormalizable interaction has dimension $[\text{mass}]^{\Delta_i}$ with $\Delta_i < 0$. If all such couplings are set by a single high-energy scale $M$, then $g_i \sim M^{\Delta_i}$. At energies $E \ll M$, the contribution of such an interaction to any amplitude is suppressed by a factor:

$$\left(\frac{E}{M}\right)^{|\Delta_i|} \ll 1 \,. \tag{6.278}$$

This is the modern perspective of **effective field theory** (EFT): at low energies, non-renormalizable interactions are negligible, and the physics is well described by the renormalizable part of the Lagrangian alone. The success of QED is not because nature is fundamentally described by a renormalizable theory, but because the non-renormalizable corrections are suppressed by powers of $E/M$, where $M$ is some very high scale (at least $\sim 10^7$ GeV, from the precision of the electron's anomalous magnetic moment).

**Example: the Euler–Heisenberg Lagrangian.** When the electron is "integrated out" from QED at energies $E \ll m_e$, one obtains an effective Lagrangian for photon-photon interactions:

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{2\alpha^2}{45m_e^4}\left[(F_{\mu\nu}F^{\mu\nu})^2 + 7(F_{\mu\nu}\tilde{F}^{\mu\nu})^2\right] + \cdots \tag{6.279}$$

The leading correction is a dimension-8 operator (four field strengths $F_{\mu\nu}$, each of dimension 2, giving total dimension 8), so the coupling has $[g] = 4 - 8 = -4$. Its effects are suppressed by $(E/m_e)^4$ compared to the free-field term.

### Wilson's renormalization group and the floating cutoff

The traditional approach—developed in the 1960s by **Bogoliubov, Parasiuk, Hepp,** and **Zimmermann** (**BPHZ**, already mentioned in passing for overlapping divergences, §6.7)—treats the cutoff $\Lambda$ as an unphysical regulator, sent to infinity at the end of the calculation while counterterms absorb all $\Lambda$-dependence into a finite number of renormalized parameters. Wilson [17] pioneered an alternative viewpoint in which $\Lambda$ is kept *finite and physical*: it is the scale below which the theory is defined. Renormalization then becomes a statement about how the description of the theory must change as we lower the cutoff to focus on physics at progressively longer wavelengths.

**Integrating out high-momentum shells.** Imagine a Euclidean path integral with a sharp UV cutoff $\Lambda_0$:

$$Z = \int_{|k|<\Lambda_0} \mathcal{D}\phi\, e^{-S_{\Lambda_0}[\phi]}, \qquad S_{\Lambda_0}[\phi] = \int d^dx \sum_i g_i(\Lambda_0)\, \mathcal{O}_i[\phi],$$

where $\{\mathcal{O}_i\}$ is a (in principle infinite) basis of local operators compatible with the symmetries of the theory and $g_i(\Lambda_0)$ are the bare couplings. Split the field into a low-momentum and a high-momentum part, $\phi = \phi_< + \phi_>$, with $\phi_<$ supported on $|k| < \Lambda$ and $\phi_>$ on $\Lambda < |k| < \Lambda_0$. Performing the path integral over $\phi_>$ defines the **Wilsonian effective action** at the new scale $\Lambda$,

$$e^{-S_\Lambda[\phi_<]} = \int_{\Lambda<|k|<\Lambda_0} \mathcal{D}\phi_>\, e^{-S_{\Lambda_0}[\phi_<+\phi_>]},$$

which has the same form as the original action but with renormalized, $\Lambda$-dependent couplings $g_i(\Lambda)$. Iterating the procedure for infinitesimal shells $\Lambda \to \Lambda - d\Lambda$ generates a flow $g_i(\Lambda)$ in coupling space: this is the *Wilson renormalization group flow*. The whole construction is physical: the couplings $g_i(\Lambda)$ encode what an observer working at energies $\lesssim \Lambda$ can measure, with the modes $\Lambda < |k| < \Lambda_0$ already absorbed into them.

**The floating cutoff.** The phrase "floating cutoff" refers to the fact that $\Lambda$ is not a fixed regulator but a variable dial. The same physics admits infinitely many descriptions, one for each value of $\Lambda$, related by the RG flow. Demanding that physical observables (Green functions of $\phi_<$ at momenta $\ll \Lambda$) be invariant under a shift of $\Lambda$ gives a non-trivial constraint on how the couplings must evolve.

**Classifying operators by dimension.** In $d$ spacetime dimensions, an operator $\mathcal{O}_i$ has a canonical mass dimension $[\mathcal{O}_i]$, and the corresponding coupling has dimension $\Delta_i = d - [\mathcal{O}_i]$. The natural dimensionless coupling is

$$\mathcal{G}_i(\Lambda) \equiv \Lambda^{-\Delta_i} g_i(\Lambda),$$

which measures the strength of $\mathcal{O}_i$ in units of the cutoff. Cutoff-independence of physical observables, combined with the flow $g_i(\Lambda)$, gives the **Wilson RG equation**

$$\Lambda \frac{d}{d\Lambda} \mathcal{G}_i(\Lambda) = \beta_i\big(\mathcal{G}(\Lambda)\big), \tag{6.280}$$

where the $\beta_i$ are calculable order by order in perturbation theory and have a universal piece coming from the rescaling of $\Lambda$ in the definition of $\mathcal{G}_i$, plus an interaction-induced piece from the loop integrals over the shell $\Lambda < |k| < \Lambda_0$.

**Relevant, marginal, irrelevant.** Linearizing (6.280) near the free (Gaussian) fixed point $\mathcal{G}_i^* = 0$, the leading behaviour is purely dimensional:

$$\Lambda \frac{d\mathcal{G}_i}{d\Lambda} = -\Delta_i \mathcal{G}_i + O(\mathcal{G}^2) \implies \mathcal{G}_i(\Lambda) \simeq \mathcal{G}_i(\Lambda_0) \left(\frac{\Lambda}{\Lambda_0}\right)^{-\Delta_i}.$$

Three cases:

- $\Delta_i > 0$ (*relevant*, super-renormalizable): $\mathcal{G}_i$ *grows* as $\Lambda \to 0$. Examples: the mass term ($\Delta = 2$ in $d = 4$), which is why fine tuning is required to keep scalar masses small (the hierarchy problem).
- $\Delta_i = 0$ (*marginal*, renormalizable): the dimensional flow vanishes; the actual evolution is logarithmic, controlled entirely by the loop-induced piece of $\beta_i$ (running of $\alpha$ in QED, of $\alpha_s$ in QCD).

- $\Delta_i < 0$ (*irrelevant*, non-renormalizable): $\mathcal{G}_i$ *decays* as

$$\mathcal{G}_i(\Lambda) \simeq \mathcal{G}_i(\Lambda_0)\left(\frac{\Lambda}{\Lambda_0}\right)^{|\Delta_i|}, \qquad \Lambda \ll \Lambda_0.$$

  For $|\Delta_i| = 4$ (e.g. the Euler–Heisenberg operator $(F_{\mu\nu}F^{\mu\nu})^2$, $\Delta = -4$ in $d = 4$) and a low-energy probe at $\Lambda/\Lambda_0 \sim 10^{-2}$, the suppression is by a factor $10^{-8}$.

**Why renormalizable theories "work so well."** Because irrelevant couplings die off as a power of $\Lambda/\Lambda_0$, the low-energy effective action is dominated by the finitely many relevant and marginal operators—exactly the ones admitted by traditional renormalization. This is not a tautology but a deep statement: an unknown high-scale theory ($\Lambda_0 \sim M_{\rm Pl}$, say) projects, at experimentally accessible energies, onto a renormalizable Lagrangian. The non-renormalizable operators are still there, but their effects are systematically suppressed by powers of $E/\Lambda_0$ and are usually unobservable. This is the modern *effective field theory* picture: every renormalizable QFT is to be read as the leading approximation of an EFT, valid up to some scale $\Lambda_0$ where new degrees of freedom appear.

**Comparison with BPHZ.** The traditional and Wilsonian viewpoints are equivalent on physical predictions but differ in interpretation:

| | BPHZ / continuum | Wilson |
|---|---|---|
| Cutoff $\Lambda$ | unphysical regulator, $\to \infty$ | finite, physical |
| Bare couplings | infinite, fine-tuned | finite, $\Lambda$-dependent |
| Counterterms | cancel divergences | encode integrated-out modes |
| Non-renorm. ops. | forbidden | present but irrelevant |

The Wilsonian language makes manifest *why* the BPHZ procedure is internally consistent and *why* ignoring non-renormalizable operators is justified at low energies.

## Accidental symmetries

The requirement of renormalizability is so restrictive that it can *force* the Lagrangian to have symmetries that were not assumed from the outset. These are called **accidental symmetries**.

**Example: QED with multiple lepton flavors.** Consider the most general renormalizable, Lorentz- and gauge-invariant Lagrangian for photons and $N$ species of charged spin-1/2 fields $\psi_i$. A priori, the kinetic terms, mass terms, and interaction terms could all involve non-diagonal matrices in flavor space. However, by field redefinitions (non-singular changes of basis $\psi_i \to S_{ij}\psi_j$), the Lagrangian can always

be brought to the diagonal form:

$$\mathcal{L} = -\frac{1}{4} Z_3 F_{\mu\nu} F^{\mu\nu} + \sum_i \bar{\psi}_i \left( i\partial\!\!\!/ + eA\!\!\!/ - m_i \right) \psi_i \,, \tag{6.281}$$

This diagonal form reveals several **accidental symmetries**:

- **Individual lepton flavor conservation**: The number of leptons of each flavor ($e$, $\mu$, $\tau$) minus the number of antileptons is separately conserved. Processes like $\mu \to e + \gamma$ are forbidden, even though they were not explicitly excluded.
- **C, P, and T invariance**: The Lagrangian (6.281) is automatically invariant under charge conjugation, parity, and time reversal. No explicit assumption of these symmetries was needed.

These symmetries are "accidental" because they are not fundamental—they emerge automatically from the constraints of renormalizability. Non-renormalizable interactions (such as a Pauli term $\bar{\psi}[\gamma_\mu, \gamma_\nu]\psi F^{\mu\nu}$, or four-fermion interactions) would violate them, but their effects are suppressed by powers of $E/M$.

**Remark.** Most of the experimentally observed symmetries of elementary particle physics—lepton flavor conservation, baryon number conservation, CP invariance of the strong and electromagnetic interactions—are accidental symmetries of the renormalizable Standard Model Lagrangian. Their violation by non-renormalizable interactions is a subject of active experimental investigation (neutrino oscillations violate lepton flavor; proton decay would violate baryon number).

# Summary: renormalization constants and their physical effects

The entire one-loop renormalization of QED is controlled by four quantities—three multiplicative constants and one additive shift—each absorbing the UV divergence of one 1PI amplitude. The Ward–Takahashi identity $Z_1 = Z_2$ reduces the independent constants to three.

| Const. | 1PI amplitude | Bare → ren. | Physical role |
|---|---|---|---|
| $Z_3$ | Photon s.e. $\Pi^{\mu\nu}$ | $A_{0\mu} = \sqrt{Z_3}\, A_\mu$ | Charge ren.: $\mu^{\epsilon/2} e = Z_3^{1/2} e_0$ |
| $Z_2$ | Electron s.e. $\Sigma$ | $\psi_0 = \sqrt{Z_2}\, \psi$ | Fermion w.f. ren. |
| $Z_1$ | Vertex $\Gamma^\mu$ | $e_0 Z_2 Z_3^{1/2} = \mu^{\epsilon/2} Z_1 e$ | Vertex ren. ($Z_1 = Z_2$, Ward) |
| $\delta m$ | Electron s.e. $\Sigma$ | $m_0 = m + \delta m$ | Mass renormalization |

| **Object** | **Effect of renormalization** |
|---|---|
| Internal photon propagator | $\dfrac{-ig_{\alpha\beta}}{k^2} \longrightarrow \dfrac{-ig_{\alpha\beta}}{k^2}\left[1 - e^2\Pi_c(k^2)\right]$ |
| Internal fermion propagator | $\dfrac{i}{\not{p} - m} \longrightarrow \dfrac{i}{\not{p} - m - \Sigma_c(\not{p})}$ |
| QED vertex | $ie\gamma^\mu \longrightarrow ie\left[\gamma^\mu + e^2\Lambda_c^\mu(p', p)\right]$ |
| External fermion spinor | $u(p) \longrightarrow Z_2^{1/2} u(p)$ (similarly $\bar{u}, v, \bar{v}$) |
| External photon polarization | $\varepsilon^\mu(k) \longrightarrow Z_3^{1/2}\varepsilon^\mu(k)$ |

The finite corrections $\Pi_c(k^2)$, $\Sigma_c(\not{p})$, and $\Lambda_c^\mu(p', p)$ are the physically observable radiative effects (vacuum polarization, anomalous magnetic moment, Lamb shift, running coupling). All UV divergences are absorbed into $Z_2$, $Z_3$, $\delta m$ (with $Z_1 = Z_2$), and the charge renormalization depends *only* on $Z_3$:

$$\boxed{e = Z_3^{1/2} e_0 .} \tag{6.282}$$