8 General Methods for Interacting Fields

This chapter develops the general formalism connecting the S-matrix to vacuum expectation values of time-ordered products of field operators. The central result is the *LSZ reduction formula* [25], which provides a rigorous foundation for the Feynman diagram technique and illuminates the physical content of scattering amplitudes. A key tool is the Källén–Lehmann spectral representation [26, 27], which encodes the general structure of two-point functions. We follow primarily the treatment of Bjorken and Drell [10], supplementing it with the perturbative Green-function discussion of Mandl–Shaw [1], the pole-and-residue viewpoint emphasized by Peskin–Schroeder [6], and the compact LSZ presentation of Schwartz [12].

8.1 Introduction: from free to interacting fields

A theory of free fields alone has no physical content. The nature of the physical world is revealed to us only through the interactions between fields, to which we now turn.

In constructing general interactions, we are motivated by analogy with the electromagnetic field. The "minimal" substitution $p_\mu \to p_\mu + eA_\mu$ (in our convention with $D_\mu = \partial_\mu - ieA_\mu$ and $q_e = -e$; cf. Section 5.0 of Chapter 5), corresponding to the classical interaction of a point charge, provides the standard prescription for introducing electrodynamic couplings. This leads to the QED Lagrangian density:

$$\mathcal{L}(x) = \bar{\psi}(x)(i\partial\!\!\!/ + e_0 A\!\!\!/(x) - m_0)\psi(x) - \frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x)\,, \tag{8.1}$$

which describes the local interaction of the electron and photon fields at the same spacetime point x. The coupled field equations are:

$$(i\partial\!\!\!/ - m_0)\psi(x) = -e_0 A\!\!\!/(x)\psi(x), \tag{8.2}$$

$$\partial_\nu F^{\mu\nu}(x) = -e_0\bar{\psi}(x)\gamma^\mu\psi(x)\,. \tag{8.3}$$

It is evident from these equations that in discussing the coupling between fields we face a nonlinear problem of vast complexity—even classically, as found in the study of radiation damping and runaway solutions for classical charges under the influence of their own and each other's fields.

The subscripts on m_0 and e_0 indicate that these are the *bare* mass and charge, which differ from the physically observable values m and e due to renormalization effects. Indeed, perturbation theory calculations show that the corrections to mass and charge are divergent, requiring careful renormalization procedures to extract finite physical predictions.

Despite these complications, a systematic approach is possible. The physical predictions of any quantum field theory are contained in the S-matrix elements, whose squared moduli give the probabilities for physical processes to occur. Alternatively, one can focus on closely related functions called **Green functions**. Although Green functions are less directly related to observables than S-matrix elements, they are easier to calculate, and S-matrix elements can be readily obtained from them. Moreover, S-matrix elements for several different processes can be derived from a single Green function, allowing relations between these processes to be inferred.

8.2 Properties of physical states

The goal of quantum field theory as a physical theory is to describe the dynamics of interacting particles observed in nature. We have seen how the properties of free particles emerge from the quantization of classical fields and how symmetries constrain the form of interaction Lagrangians. The task remaining is to construct and evaluate the general matrix elements that describe the dynamical behavior of interacting particles.

We are interested both in the amplitudes for single particles to propagate in spacetime—the one-particle Green's functions—and in the transition amplitudes between different initial and final states, that is, the S-matrix.

Assumptions on the spectrum

To study the properties of exact states and propagators, we rely on invariance arguments. In particular, displacement and Lorentz invariance play fundamental roles.

We work in the *Heisenberg picture*, where operators carry the full time dependence and states are time-independent. This is the natural framework for an exact, non-perturbative treatment of interacting fields[1]: we need to discuss the behaviour

[1] A word of caution on the interaction picture used in earlier chapters. Haag's theorem [28] states that, for a relativistic quantum field theory with infinitely many degrees of freedom, the Fock space of the free theory and the Hilbert space of the interacting theory are *unitarily inequivalent*: no unitary operator $U(t)$ can map one into the other. This means that the interaction-picture splitting $H = H_0 + H_I$, which relies precisely on such a unitary map, cannot be defined rigorously in the exact theory. In practice, however, this is not a problem: perturbative calculations are always performed within a regularised theory (with ultraviolet cutoff and finite volume), where the number of degrees of freedom is finite and Haag's theorem does not apply. The interaction picture and the Dyson

of the field operators $\varphi(x)$ at asymptotic times $t \longrightarrow \pm\infty$, and this requires operators that evolve in time. Moreover, in the Heisenberg picture states do not evolve, so they can be classified once and for all by the eigenvalues of conserved quantities.

We choose the states Φ to be eigenstates of energy and momentum. This is possible because a conserved energy-momentum four-vector P^μ is guaranteed by displacement invariance. We label each state by the eigenvalues of P^μ together with those of all other mutually commuting constants of the motion (spin, internal charges, etc.), forming a complete set of commuting observables.

We assume the following physically motivated conditions on the eigenspectrum of P^μ:

1. **Forward light cone condition.** The eigenvalues of energy and momentum lie within the forward light cone:
$$P^2 = P_\mu P^\mu \geq 0, \qquad P^0 \geq 0. \tag{8.4}$$
2. **Unique vacuum.** There exists a nondegenerate Lorentz-invariant ground state of lowest energy, the vacuum state
$$\Phi_0 \equiv |0\rangle\,, \tag{8.5}$$
with the zero of energy chosen so that
$$P^0|0\rangle = 0\,. \tag{8.6}$$
From the forward light cone condition (8.4), it follows that $\mathbf{P}|0\rangle = 0$ as well. Lorentz invariance of the vacuum ensures that $|0\rangle$ appears as the vacuum state to all inertial observers.
3. **Stable one-particle states.** There exist stable single-particle states
$$\Phi_{1(i)} \equiv |P^{(i)}\rangle \tag{8.7}$$
with $P^{(i)}_\mu P^{(i)\mu} = m_i^2$ for each stable particle of physical mass m_i. These states transform as irreducible unitary representations of the Poincaré group, labelled by mass m_i and spin s_i.
4. **Mass gap / isolated mass shell.** The vacuum and single-particle states form a discrete part of the spectrum of P^μ. More precisely, in the sector of quantum numbers carried by the field $\varphi(x)$, the one-particle mass hyperboloid $p^2 = m^2$ is *isolated* from the rest of the spectrum: there exists a lowest continuum threshold $m_1^2 > m^2$ such that the continuous part starts only at $p^2 \geq m_1^2$. The

expansion remain perfectly valid tools within this regularised, order-by-order framework. The point here is that for an *exact*, non-perturbative formulation—such as the LSZ reduction formula—one should work directly in the Heisenberg picture, which makes no reference to a free-field Fock space.

threshold m_1 is not fixed universally by the mass of the lightest particle: it is the invariant mass of the lightest multiparticle state with the same conserved quantum numbers as the field insertion. For example, a Z_2-odd scalar field in a ϕ^4 theory couples to odd-particle states, so the first continuum threshold in that channel is typically $3m$, whereas an even composite operator can have a two-particle threshold at $2m$. This separation is crucial: as we shall see, it allows the one-particle contributions to be cleanly extracted in the asymptotic limit $t \to \pm\infty$, while the continuum contributions vanish.

5. **Vanishing vacuum expectation value.** The field $\varphi(x)$ is assumed to satisfy

$$\langle 0|\varphi(x)|0\rangle = 0\,. \tag{8.8}$$

This is not a restriction: if $\langle 0|\varphi(x)|0\rangle = v \neq 0$, one redefines $\varphi(x) \to \varphi(x) - v$. The shifted field satisfies the same equations of motion with modified couplings. This condition is necessary for the correct identification of the asymptotic fields in the LSZ formalism.

6. **Asymptotic completeness.** Intuitively, an *in-state* describes what a scattering experiment looks like in the far past ($t \to -\infty$), when the particles are widely separated and propagate freely; an *out-state* describes the same system in the far future ($t \to +\infty$), after all interactions have ceased. These states will be constructed precisely in the next sections; for now, the key assumption is that they span the full Hilbert space of the interacting theory:

$$\mathcal{H} = \mathcal{H}_{\text{in}} = \mathcal{H}_{\text{out}}\,. \tag{8.9}$$

This ensures that every physical state can be described as a scattering state, and guarantees the unitarity of the S-matrix. Asymptotic completeness is assumed in all standard treatments, although a rigorous proof remains an open problem in axiomatic quantum field theory.

Taken together, these hypotheses define the standard textbook regime in which the LSZ construction works cleanly: stable massive particles with an isolated one-particle shell. Stable bound states are not excluded in principle, provided they appear as isolated poles and admit suitable interpolating fields. The main caveat is that theories with long-range interactions, such as QED, require a more careful treatment of asymptotic states: charged sectors are accompanied by arbitrarily soft photons, so the sharp one-particle pole assumed here is replaced by infrared-sensitive dressed states or inclusive observables.

These assumptions reflect our physical expectation that the interaction does not violently alter the spectrum of states from the free-field case. This is the spirit of perturbation theory: having introduced fields for stable particles into a Lagrangian, we assume that their mutual interactions do not destroy the basic structure of the spectrum.

Notation for the vacuum. In the non-perturbative LSZ construction below, $|0\rangle$ denotes the exact physical vacuum of the interacting theory. In the later perturbative discussion of the Gell-Mann–Low formula, we switch to the standard interaction-picture notation: $|\Omega\rangle$ denotes the exact interacting vacuum and $|0\rangle$ denotes the free vacuum of H_0. This switch is made explicitly when the perturbative formula is introduced.

8.3 In-fields and the asymptotic condition

Since we are primarily concerned with scattering problems, we first construct the states that give a simple description of the physical system at the initial time $t \to -\infty$. At this time the incoming wave packets are widely separated, so no finite-range interaction can exchange momentum between them. Each object is already a physical, dressed particle; its self-energy effects are part of its mass and field normalization.

Construction of in-fields for a scalar field

Consider a Hermitian scalar field $\varphi(x)$ satisfying the wave equation

$$(\Box + m_0^2)\varphi(x) = j(x)\,, \tag{8.10}$$

where $j(x)$ is a scalar operator constructed from the fields interacting with $\varphi(x)$ locally at x. We assume that the full interacting field satisfies the same equal-time commutation relations that we derived for free fields in the canonical quantisation procedure:

$$[\varphi(\vec{x},t),\varphi(\vec{y},t)] = [\pi(\vec{x},t),\pi(\vec{y},t)] = 0, \qquad [\pi(\vec{x},t),\varphi(\vec{y},t)] = -i\delta^{(3)}(\vec{x}-\vec{y})\,, \tag{8.11}$$

where $\pi(x) = \dot{\varphi}(x)$ for theories without derivative couplings. These relations are the quantum counterpart of the classical Poisson brackets and constitute a basic postulate of canonical quantisation: we retain them as an axiom of the interacting theory.

We denote by $\varphi_{\text{in}}(x)$ the operator that creates states of single physical particles. To ensure the correct interpretation, we require:

1. **Covariance.** In the Heisenberg picture, the full interacting field transforms under spacetime translations as

$$e^{iP\cdot a}\,\varphi(x)\,e^{-iP\cdot a} = \varphi(x+a)\,, \tag{8.12}$$

where P^μ is the *exact* four-momentum operator (including interactions).
Let us recall how this comes about. The full Lagrangian $\mathcal{L}$ is invariant under spacetime translations $x^\mu \to x^\mu + a^\mu$. By Noether's theorem, this invariance yields a conserved four-momentum P^μ—the *total* four-momentum of the interacting theory, not just the free part. In Chapter 1, Eq. (1.201), we proved by explicit computation with mode expansions that the free four-momentum satisfies $[\psi(x), P^\mu] = +i\partial^\mu\psi(x)$, or equivalently (by antisymmetry of the commutator) $[P^\mu, \psi(x)] = -i\partial^\mu\psi(x)$. We adopt the latter form here. For the interacting theory, the same relation holds:

$$[P^\mu, \varphi(x)] = -i\partial^\mu\varphi(x)\,. \tag{8.13}$$

This can be seen directly from the canonical formalism. For the temporal component, recall that in the Heisenberg picture every operator evolves according to

$$i\frac{\partial\varphi}{\partial t} = [\varphi, H]\,, \tag{8.14}$$

where H is the *full* Hamiltonian, including interactions. Rearranging:

$$[P^0, \varphi(x)] = [H, \varphi(x)] = -i\,\partial_0\varphi(x)\,. \tag{8.15}$$

For the spatial components, using the equal-time commutation relations (8.11) and the expression $P^i = \int d^3x\,\pi(\vec{x},t)\,\partial^i\varphi(\vec{x},t)$ from Noether's theorem:

$$[P^i, \varphi(\vec{y},t)] = \int d^3x\,[\pi(\vec{x},t), \varphi(\vec{y},t)]\,\partial^i\varphi(\vec{x},t) = -i\partial^i\varphi(\vec{y},t)\,. \tag{8.16}$$

The finite translation (8.12) then follows by exponentiation of the infinitesimal result (8.13). The key identity is the *Hadamard lemma*: for any operator A and any Hermitian generator G,

$$e^{i\alpha G}\,A\,e^{-i\alpha G} = \sum_{n=0}^{\infty}\frac{(i\alpha)^n}{n!}\,\underbrace{[G,[G,\cdots[G,A]\cdots]]}_{n}\,. \tag{8.17}$$

To prove this, define $f(\alpha) = e^{i\alpha G}\,A\,e^{-i\alpha G}$ and Taylor-expand around $\alpha = 0$. Differentiating gives $f'(\alpha) = i\,[G, f(\alpha)]$, so $f^{(n)}(0) = (i)^n\,[G,[G,\cdots[G,A]\cdots]]$ with n nested commutators.
In our case, the generator is $G = a_\mu P^\mu$ (with the role of α absorbed into a_μ), and $A = \varphi(x)$. The first few nested commutators are:

$$\begin{aligned}
[a_\mu P^\mu, \varphi(x)] &= -ia^\mu\partial_\mu\varphi(x)\,,\\
[a_\mu P^\mu, [a_\nu P^\nu, \varphi(x)]] &= [a_\mu P^\mu, -ia^\nu\partial_\nu\varphi(x)]\\
&= (-i)^2 a^\mu a^\nu\partial_\mu\partial_\nu\varphi(x)\,,
\end{aligned} \tag{8.18}$$

where in the second line we used the fact that a^ν is a c-number and that the commutation relation (8.13) holds for $\partial_\nu\varphi$ as well as for φ. Let us spell out why. Since P^μ is an integral over all space and does not depend on the spacetime point x, the derivative $\partial_\nu^{(x)}$ commutes with P^μ inside a commutator. Taking $\partial_\nu^{(x)}$ of both sides of (8.13):

$$\partial_\nu^{(x)}[P^\mu,\varphi(x)] = [P^\mu,\partial_\nu\varphi(x)] = -i\partial^\mu\partial_\nu\varphi(x)\,. \tag{8.19}$$

More generally, P^μ generates translations of *any* local operator $O(x)$ constructed from products of fields and their derivatives at x. This follows by repeated application of the identity $[A,BC] = [A,B]C + B[A,C]$: for a product $O = O_1O_2$,

$$\begin{aligned}[P^\mu,O_1O_2] &= [P^\mu,O_1]O_2 + O_1[P^\mu,O_2]\\ &= (-i\partial^\mu O_1)O_2 + O_1(-i\partial^\mu O_2)\\ &= -i\partial^\mu(O_1O_2)\,,\end{aligned} \tag{8.20}$$

which is just the Leibniz rule. By induction, $[P^\mu,O(x)] = -i\partial^\mu O(x)$ for any local composite operator.
The pattern is clear: the n-th nested commutator gives

$$(-i)^n a^{\mu_1}\cdots a^{\mu_n}\partial_{\mu_1}\cdots\partial_{\mu_n}\varphi(x)\,.$$

Substituting into (8.17):

$$\begin{aligned}e^{iP\cdot a}\,\varphi(x)\,e^{-iP\cdot a} &= \sum_{n=0}^{\infty}\frac{(i)^n(-i)^n}{n!}a^{\mu_1}\cdots a^{\mu_n}\partial_{\mu_1}\cdots\partial_{\mu_n}\varphi(x)\\ &= \sum_{n=0}^{\infty}\frac{1}{n!}(a\cdot\partial)^n\varphi(x) = \varphi(x+a)\,,\end{aligned} \tag{8.21}$$

where we used $(i)^n(-i)^n = 1$ and recognised the Taylor expansion of φ around x. We require that $\varphi_{\rm in}(x)$ inherits this property:

$$[P^\mu,\varphi_{\rm in}(x)] = -i\frac{\partial\varphi_{\rm in}(x)}{\partial x_\mu}\,. \tag{8.22}$$

This is indeed the case, as we shall verify explicitly after constructing $\varphi_{\rm in}(x)$.

2. **Free-field equation.** The spacetime development is described by a free-particle Klein–Gordon equation with the *physical* mass m:

$$(\Box + m^2)\varphi_{\rm in}(x) = 0\,. \tag{8.23}$$

Proof that $\varphi_{\text{in}}(x)$ creates only one-particle states. Consider an arbitrary eigenstate $|n\rangle$ with $P^\mu|n\rangle = p_n^\mu|n\rangle$, and form the matrix element with the vacuum:

$$-i\frac{\partial}{\partial x_\mu}\langle n|\varphi_{\text{in}}(x)|0\rangle = \langle n|[P^\mu,\varphi_{\text{in}}(x)]|0\rangle = p_n^\mu\langle n|\varphi_{\text{in}}(x)|0\rangle\,. \tag{8.24}$$

Iterating this operation and using (8.23):

$$(\Box+m^2)\langle n|\varphi_{\text{in}}(x)|0\rangle = (m^2-p_n^2)\langle n|\varphi_{\text{in}}(x)|0\rangle = 0\,. \tag{8.25}$$

Therefore, the only states produced from the vacuum by $\varphi_{\text{in}}(x)$ are those with $p_n^2 = m^2$, that is, the one-particle states of mass m. □

Fourier expansion of the in-field

Since $\varphi_{\text{in}}(x)$ satisfies the free Klein–Gordon equation, its Fourier expansion takes the standard form:

$$\varphi_{\text{in}}(x) = \int d^3k\left[a_{\text{in}}(k)f_k(x) + a_{\text{in}}^\dagger(k)f_k^*(x)\right], \tag{8.26}$$

with

$$f_k(x) = \frac{1}{\sqrt{(2\pi)^3 2\omega_k}}e^{-ik\cdot x}, \qquad \omega_k = \sqrt{\vec{k}^2+m^2} = k^0\,, \tag{8.27}$$

and the inverse relation

$$a_{\text{in}}(k) = i\int d^3x\, f_k^*(x)\overleftrightarrow{\partial_0}\,\varphi_{\text{in}}(x)\,, \tag{8.28}$$

where $A\overleftrightarrow{\partial_0}B \equiv A(\partial_0 B) - (\partial_0 A)B$.

The operator coefficients satisfy

$$[P^\mu, a_{\text{in}}(k)] = -k^\mu a_{\text{in}}(k), \qquad [P^\mu, a_{\text{in}}^\dagger(k)] = +k^\mu a_{\text{in}}^\dagger(k)\,. \tag{8.29}$$

Repeated application of $a_{\text{in}}^\dagger(k)$ to the vacuum builds up general n-particle eigenstates. To see that these are eigenstates of P^μ, we commute P^μ past each creation operator one at a time using (8.29). For the one-particle state:

$$\begin{aligned}P^\mu a_{\text{in}}^\dagger(k)|0\rangle &= \left([P^\mu, a_{\text{in}}^\dagger(k)] + a_{\text{in}}^\dagger(k)\,P^\mu\right)|0\rangle \\ &= k^\mu a_{\text{in}}^\dagger(k)|0\rangle + a_{\text{in}}^\dagger(k)\,\underbrace{P^\mu|0\rangle}_{=0} = k^\mu|k \text{ in}\rangle\,.\end{aligned} \tag{8.30}$$

For two particles, P^μ first commutes past $a_{\text{in}}^\dagger(k_1)$, picking up k_1^μ, and the remainder $a_{\text{in}}^\dagger(k_1)P^\mu$ acts on $a_{\text{in}}^\dagger(k_2)|0\rangle$, which by (8.30) gives k_2^μ. The pattern repeats for N particles:

$$P^\mu|k_1\cdots k_N \text{ in}\rangle = P^\mu a_{\text{in}}^\dagger(k_1)\cdots a_{\text{in}}^\dagger(k_N)|0\rangle = \sum_{i=1}^N k_i^\mu\,|k_1\cdots k_N \text{ in}\rangle\,, \tag{8.31}$$

with $a_{\text{in}}(k)|0\rangle = 0$.

Explicit construction of $\varphi_{\text{in}}(x)$

To express $\varphi_{\text{in}}(x)$ in terms of the interacting field $\varphi(x)$, we rewrite the field equation (8.10) using the physical mass m by adding a mass counterterm:

$$(\Box + m^2)\varphi(x) = j(x) + \delta m^2 \varphi(x) \equiv \tilde{j}(x)\,, \tag{8.32}$$

where $\delta m^2 = m^2 - m_0^2$. Written in this form, the equation $(\Box + m^2)\varphi = \tilde{j}$ has exactly the structure of a *scattering problem*: $\varphi(x)$ plays the role of the total wave, and $\tilde{j}(x)$ acts as the source that generates scattered waves. The strategy for isolating the free part is the same as in non-relativistic quantum mechanics, where the **Lippmann–Schwinger equation** provides the decomposition of a scattering state into incident and scattered waves. Let us briefly recall the derivation. The full Hamiltonian is $H = H_0 + V$, and we seek an eigenstate $|\psi\rangle$ with energy E:

$$(E - H_0 - V)|\psi\rangle = 0\,. \tag{8.33}$$

Let $|\psi_0\rangle$ be the free state satisfying $(E - H_0)|\psi_0\rangle = 0$ with the same energy. Then $|\psi\rangle - |\psi_0\rangle$ satisfies

$$(E - H_0)(|\psi\rangle - |\psi_0\rangle) = V|\psi\rangle\,. \tag{8.34}$$

Inverting $(E - H_0)$ with a retarded boundary condition ($+i\epsilon$ prescription, selecting outgoing scattered waves) gives

$$|\psi\rangle = |\psi_0\rangle + G_0^{(+)}(E)\,V\,|\psi\rangle\,, \qquad G_0^{(+)}(E) = \frac{1}{E - H_0 + i\epsilon}\,. \tag{8.35}$$

The first term is the incident free wave; the second is the scattered wave, generated by the potential V acting on the full state and propagated by the free retarded Green's function. Subtracting the scattered wave from the total state recovers the free part: $|\psi_0\rangle = |\psi\rangle - G_0^{(+)}V|\psi\rangle$.

We do exactly the same here: subtracting from $\varphi(x)$ the scattered waves generated by $\tilde{j}$ via the retarded Green's function leaves just the free waves propagating with mass m. This gives:

$$\boxed{\sqrt{Z}\,\varphi_{\text{in}}(x) = \varphi(x) - \int d^4y\,\Delta_{\text{ret}}(x-y;m)\,\tilde{j}(y)}\,, \tag{8.36}$$

where $\Delta_{\text{ret}}(x-y;m)$ is the retarded Green's function satisfying

$$(\Box_x + m^2)\Delta_{\text{ret}}(x-y;m) = \delta^{(4)}(x-y), \qquad \Delta_{\text{ret}}(x-y;m) = 0 \text{ for } x^0 < y^0\,. \tag{8.37}$$

The constant $\sqrt{Z}$ normalizes $\varphi_{\text{in}}(x)$ to unit amplitude for creating one-particle states from the vacuum.

Verification of the defining properties. As defined in (8.36), $\varphi_{\text{in}}(x)$ satisfies both conditions (8.22) and (8.23). For the covariance property, we substitute $x \to x+a$ in (8.36):

$$\sqrt{Z}\,\varphi_{\text{in}}(x+a) = \varphi(x+a) - \int d^4y\,\Delta_{\text{ret}}(x+a-y)\,\tilde{j}(y)\,. \tag{8.38}$$

In the first term we use the translation property $\varphi(x+a) = e^{iP\cdot a}\varphi(x)e^{-iP\cdot a}$. In the integral we change variable $y' = y-a$, so that $\Delta_{\text{ret}}(x+a-y) = \Delta_{\text{ret}}(x-y')$ and $\tilde{j}(y) = \tilde{j}(y'+a)$:

$$= e^{iP\cdot a}\varphi(x)e^{-iP\cdot a} - \int d^4y'\,\Delta_{\text{ret}}(x-y')\,\tilde{j}(y'+a)\,. \tag{8.39}$$

Since $\tilde{j}$ is a scalar operator, $\tilde{j}(y'+a) = e^{iP\cdot a}\,\tilde{j}(y')\,e^{-iP\cdot a}$. The retarded Green's function $\Delta_{\text{ret}}(x-y')$ is a c-number, so it commutes with $e^{\pm iP\cdot a}$ and we can factor the unitary conjugation out of the integral:

$$\begin{aligned}
&= e^{iP\cdot a}\varphi(x)e^{-iP\cdot a} - e^{iP\cdot a}\left[\int d^4y'\,\Delta_{\text{ret}}(x-y')\,\tilde{j}(y')\right]e^{-iP\cdot a}\\
&= e^{iP\cdot a}\underbrace{\left[\varphi(x) - \int d^4y'\,\Delta_{\text{ret}}(x-y')\,\tilde{j}(y')\right]}_{=\ \sqrt{Z}\,\varphi_{\text{in}}(x)\ \text{by (8.36)}}e^{-iP\cdot a} = e^{iP\cdot a}\sqrt{Z}\,\varphi_{\text{in}}(x)\,e^{-iP\cdot a}\,,
\end{aligned} \tag{8.40}$$

as required. For the free-field equation, apply $(\Box_x + m^2)$ to both sides of (8.36):

$$\begin{aligned}
\sqrt{Z}\,(\Box_x+m^2)\varphi_{\text{in}}(x) &= (\Box_x+m^2)\varphi(x) - \int d^4y\,\underbrace{(\Box_x+m^2)\Delta_{\text{ret}}(x-y;m)}_{=\ \delta^{(4)}(x-y)}\,\tilde{j}(y)\\
&= \tilde{j}(x) - \int d^4y\,\delta^{(4)}(x-y)\,\tilde{j}(y)\\
&= \tilde{j}(x) - \tilde{j}(x) = 0\,,
\end{aligned} \tag{8.41}$$

where in the first term we used the field equation (8.32), and in the second we used the defining property (8.37) of the retarded Green's function. Since $\sqrt{Z} \neq 0$, we conclude $(\Box + m^2)\varphi_{\text{in}}(x) = 0$.

The weak asymptotic condition

The form of (8.36) might suggest that as $x^0 \to -\infty$, the retarded Green's function term vanishes as it would for a source with finite time support, so that

$$\varphi(x) \overset{?}{\longrightarrow} \sqrt{Z}\,\varphi_{\text{in}}(x) \quad \text{as } x^0 \to -\infty. \tag{8.42}$$

However, this *cannot* hold as an operator identity. The source $\tilde{j}(x) = j(x) + \delta m^2 \varphi(x)$ includes the self-interaction of the field with itself, which persists at all times—even a single isolated particle is always surrounded by its own virtual cloud. There is no time at which $\varphi(x)$ literally becomes a free field.

To see the problem more concretely, consider a matrix element of $\varphi(x)$ between the vacuum and a multiparticle state carrying the same quantum numbers as φ. If (8.42) held as an operator statement, this matrix element would have to vanish for $x^0 \to -\infty$ (since $\varphi_{\rm in}$ creates only one-particle states). In an interacting theory the local field can also have overlap with continuum states in the same quantum-number sector, so the strong limit cannot hold.

The correct asymptotic condition, due to Lehmann, Symanzik, and Zimmermann [25], takes a *weak* form: the convergence holds only inside matrix elements between normalizable (wave-packet) states, and only after a spatial smearing that projects out the one-particle contribution.

Definition of the smeared field. Let $f(\vec{x}, t)$ be a normalizable solution of the free Klein–Gordon equation $(\Box + m^2) f = 0$, for example a smooth wave packet:

$$f(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \tilde{f}(\vec{k})\, e^{-ik\cdot x}, \tag{8.43}$$

where $\tilde{f}(\vec{k})$ is chosen smooth and sufficiently rapidly decreasing, for instance a Schwartz function, so that the particle is localized in momentum space and the oscillatory integrals below are well defined. Define the **smeared field operator**:

$$\varphi^f(t) \equiv i \int d^3x\, f^*(\vec{x}, t)\, \overleftrightarrow{\partial_0}\, \varphi(\vec{x}, t), \tag{8.44}$$

The purpose of this construction is to project the field onto the one-particle sector. To see why, suppose φ were a free field with mode expansion (8.26). Comparing (8.43) with (8.27), we see that $f^*(x) = \int d^3k\, \tilde{f}^*(\vec{k})\, f_k^*(x)$, so by linearity of the spatial integral and (8.44):

$$\varphi^f = \int d^3k\, \tilde{f}^*(\vec{k})\, i \int d^3x\, f_k^*(x)\, \overleftrightarrow{\partial_0}\, \varphi(x). \tag{8.45}$$

Now substitute the mode expansion $\varphi(x) = \int d^3k' \left[a(k') f_{k'}(x) + a^\dagger(k') f_{k'}^*(x)\right]$ and use the orthonormality of the modes in the Klein–Gordon inner product:

$$i \int d^3x\, f_k^*\, \overleftrightarrow{\partial_0}\, f_{k'} = \delta^3(\vec{k} - \vec{k}'), \qquad i \int d^3x\, f_k^*\, \overleftrightarrow{\partial_0}\, f_{k'}^* = 0. \tag{8.46}$$

The first relation picks out $a(\vec{k})$; the second kills $a^\dagger(\vec{k}')$, because the Klein–Gordon inner product gives zero for this pairing of two negative-frequency modes. Therefore:

$$\varphi^f = \int d^3k\, \tilde{f}^*(\vec{k})\, a(\vec{k}) \equiv a(\tilde{f}^*), \tag{8.47}$$

the annihilation operator smeared with the wave packet $\tilde{f}^*$. Since each $a(\vec{k})$ is time-independent, so is $a(\tilde{f}^*)$—the result does not depend on the time at which we evaluate the spatial integral. The key question is what happens for the interacting field.

Time independence of φ^f_{in}. Since $\varphi_{\text{in}}(x)$ satisfies the free Klein–Gordon equation, the analogously defined

$$\varphi^f_{\text{in}} \equiv i \int d^3x\, f^*(\vec{x},t)\, \overleftrightarrow{\partial_0}\, \varphi_{\text{in}}(\vec{x},t) \tag{8.48}$$

is indeed *time-independent*. To see this, compute its time derivative:

$$\begin{aligned} \frac{d}{dt}\varphi^f_{\text{in}} &= i \int d^3x \left[\dot{f}^* \overleftrightarrow{\partial_0}\, \varphi_{\text{in}} + f^* \overleftrightarrow{\partial_0}\, \dot{\varphi}_{\text{in}} \right] \\ &= i \int d^3x \left[\dot{f}^* \dot{\varphi}_{\text{in}} - \ddot{f}^* \varphi_{\text{in}} + f^* \ddot{\varphi}_{\text{in}} - \dot{f}^* \dot{\varphi}_{\text{in}} \right] \\ &= i \int d^3x \left[f^* \ddot{\varphi}_{\text{in}} - \ddot{f}^* \varphi_{\text{in}} \right]. \end{aligned} \tag{8.49}$$

Now use the fact that both f and φ_{in} satisfy the Klein–Gordon equation: $\ddot{f}^* = (\nabla^2 - m^2) f^*$ and $\ddot{\varphi}_{\text{in}} = (\nabla^2 - m^2)\varphi_{\text{in}}$. Substituting:

$$\begin{aligned} \frac{d}{dt}\varphi^f_{\text{in}} &= i \int d^3x \left[f^* (\nabla^2 - m^2)\varphi_{\text{in}} - (\nabla^2 - m^2) f^* \cdot \varphi_{\text{in}} \right] \\ &= i \int d^3x \left[f^* \nabla^2 \varphi_{\text{in}} - (\nabla^2 f^*)\varphi_{\text{in}} \right]. \end{aligned} \tag{8.50}$$

The integrand is $\nabla \cdot (f^* \nabla \varphi_{\text{in}} - \varphi_{\text{in}} \nabla f^*)$, a total divergence. For normalizable wave packets, the fields vanish at spatial infinity, so the integral is zero:

$$\frac{d}{dt}\varphi^f_{\text{in}} = 0. \tag{8.51}$$

This means φ^f_{in} is a time-independent operator. Since φ_{in} is a free field, the same decomposition as in (8.45) applies:

$$\varphi^f_{\text{in}} = \int d^3k\, \tilde{f}^*(\vec{k})\, a_{\text{in}}(\vec{k}) = a_{\text{in}}(\tilde{f}^*), \tag{8.52}$$

the annihilation operator for a particle with wave-packet profile $\tilde{f}^*$, whose value does not depend on the time t at which we evaluate the spatial integral. The Hermitian conjugate $(\varphi^f_{\text{in}})^\dagger = \int d^3k\, \tilde{f}(\vec{k})\, a^\dagger_{\text{in}}(\vec{k}) = a^\dagger_{\text{in}}(\tilde{f})$ is the corresponding creation operator.

Why the continuum contributions vanish. For the interacting field, $\varphi^f(t)$ defined in (8.44) *is* time-dependent: the field $\varphi(x)$ does not satisfy the free Klein–Gordon equation, so the argument above does not go through. The crucial point is what happens to the matrix element $\langle\alpha|\varphi^f(t)|\beta\rangle$ as $t \to -\infty$.

Insert a complete set of exact energy-momentum eigenstates between the operators:

$$\langle\alpha|\varphi^f(t)|\beta\rangle = i\int d^3x \sum_n f^*(\vec{x},t)\overleftrightarrow{\partial_0}\langle\alpha|\varphi(\vec{x},t)|n\rangle\langle n|\beta\rangle\,. \tag{8.53}$$

To illustrate the mechanism most transparently, take $\langle\alpha| = \langle 0|$ (the vacuum). Since $P^\mu|0\rangle = 0$, the translation property gives

$$\langle 0|\varphi(x)|n\rangle = \langle 0|e^{iP\cdot x}\varphi(0)e^{-iP\cdot x}|n\rangle = e^{-ip_n\cdot x}\langle 0|\varphi(0)|n\rangle\,. \tag{8.54}$$

Substituting $f^*(\vec{x},t) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}}\,\tilde{f}^*(\vec{k})\,e^{i\omega_k t - i\vec{k}\cdot\vec{x}}$ and computing the $\overleftrightarrow{\partial_0}$ derivative:

$$\begin{aligned} f^*(\vec{x},t)\overleftrightarrow{\partial_0}\,e^{-ip_n\cdot x} &= f^*\cdot(-iE_n)e^{-ip_n\cdot x} - (i\omega_k)\,f^*\cdot e^{-ip_n\cdot x} \\ &= -i(E_n+\omega_k)\,f^*\,e^{-ip_n\cdot x}\,, \end{aligned} \tag{8.55}$$

where in the second term $\partial_0 f^* = i\omega_k f^*$ for each Fourier mode. The spatial integral of the product $f_k^*\cdot e^{-ip_n\cdot x}$ gives

$$\int d^3x\, e^{-i\vec{k}\cdot\vec{x}}\,e^{i\vec{p}_n\cdot\vec{x}} = (2\pi)^3\,\delta^3(\vec{p}_n - \vec{k})\,, \tag{8.56}$$

which enforces $\vec{p}_n = \vec{k}$. After performing the d^3k integral using this delta function, the time-dependent factors combine into $e^{i\omega_k t}\cdot e^{-iE_n t}\big|_{\vec{p}_n=\vec{k}} = e^{i(\omega_k - E_n)t}$. Including the normalization and prefactors, the result is

$$\langle 0|\varphi^f(t)|\beta\rangle = \sum_\lambda \int d^3k\,\tilde{f}^*(\vec{k})\,G_\lambda(\vec{k})\,e^{i(\omega_k - E_\lambda(\vec{k}))t}\,, \tag{8.57}$$

where the sum runs over all intermediate-state types λ (one-particle, two-particle, etc.) with total momentum $\vec{k}$ and energy $E_\lambda(\vec{k}) = \sqrt{\vec{k}^2 + M_\lambda^2}$, and $G_\lambda(\vec{k})$ collects the matrix elements $\langle 0|\varphi(0)|\lambda,\vec{k}\rangle$, $\langle\lambda,\vec{k}|\beta\rangle$, and kinematic factors.

Now the mass gap assumption plays its role. For **one-particle intermediate states**, $M_\lambda = m$ and $E_\lambda(\vec{k}) = \omega_k$, so the phase vanishes identically:

$$e^{i(\omega_k - E_\lambda)t}\Big|_{\text{1-particle}} = e^{i(\omega_k-\omega_k)t} = 1 \qquad \text{(time-independent)}. \tag{8.58}$$

The one-particle contribution to $\langle 0|\varphi^f(t)|\beta\rangle$ is therefore time-independent and gives the free-field result $\sqrt{Z}\,\langle 0|\varphi^f_{\text{in}}|\beta\rangle$.

For **continuum intermediate states**, the invariant mass satisfies $M_\lambda \geq m_1 > m$, so $E_\lambda(\vec{k}) = \sqrt{\vec{k}^2 + M_\lambda^2} > \omega_k$ for every $\vec{k}$. The phase $\omega_k - E_\lambda(\vec{k})$ is strictly negative, and the exponential $e^{i(\omega_k - E_\lambda)t}$ oscillates *rapidly* as $t \to -\infty$. It is essential that we integrate over a smooth wave packet $\tilde{f}^*(\vec{k})$: for sharp-momentum plane waves, the rapidly oscillating phase would simply keep oscillating forever without decaying, and the argument would fail. With a smooth $\tilde{f}^*(\vec{k})$, however, the Riemann–Lebesgue lemma[2] guarantees that these contributions vanish:

$$\int d^3k\, \tilde{f}^*(\vec{k})\, G_\lambda(\vec{k})\, e^{i(\omega_k - E_\lambda(\vec{k}))t} \xrightarrow{t\to-\infty} 0\,, \tag{8.59}$$

provided $\omega_k - E_\lambda(\vec{k}) \neq 0$ throughout the integration region and the product $\tilde{f}^*(\vec{k}) G_\lambda(\vec{k})$ is integrable. The mass gap guarantees the first condition. If the continuum started at $M_\lambda = m$, there would be states with $E_\lambda = \omega_k$, and the oscillating phase could not eliminate them.

The argument generalises to matrix elements $\langle\alpha|\varphi^f(t)|\beta\rangle$ with arbitrary normalizable $\langle\alpha|$. One inserts completeness and uses translation invariance; the resulting phase $\omega_k + E_\alpha - E_n(\vec{k} + \vec{p}_\alpha)$ again separates one-particle from multiparticle contributions thanks to the mass gap.

Statement of the asymptotic condition. The result is the **weak asymptotic condition** of LSZ:

$$\boxed{\lim_{t\to-\infty} \langle\alpha|\varphi^f(t)|\beta\rangle = \sqrt{Z}\,\langle\alpha|\varphi^f_{\text{in}}|\beta\rangle}\,, \tag{8.60}$$

for any normalizable states $|\alpha\rangle$, $|\beta\rangle$. The factor $\sqrt{Z}$ accounts for the fact that the interacting field has amplitude $\sqrt{Z}$ to create a one-particle state from the vacuum. The word "weak" refers to the fact that the convergence holds only inside matrix elements between normalizable states—it is *not* an operator identity. The smearing with a wave packet f is essential: it simultaneously projects onto the one-particle sector and provides the smooth momentum-space profile needed for the Riemann–Lebesgue lemma to suppress the continuum.

8.4 Out-fields and the S-matrix

Just as we have reduced the dynamics at $t \to -\infty$ to that of free particles using the in-fields, we can do the same at $t \to +\infty$ by defining **out-fields** $\varphi_{\text{out}}(x)$.

[2] The Riemann–Lebesgue lemma states that when a smooth, integrable function is multiplied by a rapidly oscillating phase $e^{i\Omega t}$ and integrated, the result vanishes as $\Omega \to \infty$ (or, equivalently, as $t \to \infty$). The physical intuition is simple: when the oscillation is fast enough, the integrand spends equal amounts of time being positive and negative across the smooth envelope, and these contributions cancel out. In our case, $t \to -\infty$ plays the role of the large parameter, making the phase oscillate faster and faster over the integration region in $\vec{k}$.

Construction of out-fields

The out-fields are defined to satisfy

$$[P^\mu, \varphi_{\text{out}}(x)] = -i\frac{\partial \varphi_{\text{out}}(x)}{\partial x_\mu}, \qquad (\Box + m^2)\varphi_{\text{out}}(x) = 0, \tag{8.61}$$

and therefore $\varphi_{\text{out}}(x)$ produces only one-particle states from the vacuum, just as $\varphi_{\text{in}}(x)$ does. The Fourier expansion is

$$\varphi_{\text{out}}(x) = \int d^3k \left[a_{\text{out}}(k) f_k(x) + a^\dagger_{\text{out}}(k) f^*_k(x) \right]. \tag{8.62}$$

The asymptotic condition for the out-fields takes the form

$$\lim_{t\to+\infty} \langle\alpha|\varphi^f(t)|\beta\rangle = \sqrt{Z}\,\langle\alpha|\varphi^f_{\text{out}}|\beta\rangle, \tag{8.63}$$

or simply $\varphi(x) \to \sqrt{Z}\,\varphi_{\text{out}}(x)$ as $t \to +\infty$ (in the weak sense).

The explicit construction uses the advanced Green's function:

$$\sqrt{Z}\,\varphi_{\text{out}}(x) = \varphi(x) - \int d^4y\,\Delta_{\text{adv}}(x-y;m)\,\tilde{j}(y), \tag{8.64}$$

where

$$(\Box_x + m^2)\Delta_{\text{adv}}(x-y;m) = \delta^{(4)}(x-y), \qquad \Delta_{\text{adv}}(x-y;m) = 0 \text{ for } x^0 > y^0. \tag{8.65}$$

The same normalizing constant $\sqrt{Z}$ appears because the amplitude for $\varphi(x)$ to create a one-particle state from the vacuum is the same:

$$\langle 0|\varphi(x)|p\rangle = \sqrt{Z}\,\langle 0|\varphi_{\text{in}}(x)|p\rangle = \sqrt{Z}\,\langle 0|\varphi_{\text{out}}(x)|p\rangle = \sqrt{Z}\,\frac{e^{-ip\cdot x}}{\sqrt{(2\pi)^3 2\omega_p}}. \tag{8.66}$$

Definition of the S-matrix

The in- and out-states are defined by

$$|p_1\cdots p_n \text{ in}\rangle = a^\dagger_{\text{in}}(p_1)\cdots a^\dagger_{\text{in}}(p_n)|0\rangle, \tag{8.67}$$

$$|p_1\cdots p_m \text{ out}\rangle = a^\dagger_{\text{out}}(p_1)\cdots a^\dagger_{\text{out}}(p_m)|0\rangle. \tag{8.68}$$

The S-matrix element for a transition from an initial state $|\alpha \text{ in}\rangle$ to a final state $|\beta \text{ out}\rangle$ is the probability amplitude:

$$\boxed{S_{\beta\alpha} = \langle\beta \text{ out}|\alpha \text{ in}\rangle} \tag{8.69}$$

By completeness of the in- and out-states, all matrix elements define an operator S on the in-state Hilbert space:

$$\langle\beta \text{ in}|S = \langle\beta \text{ out}|, \qquad S_{\beta\alpha} = \langle\beta \text{ in}|S|\alpha \text{ in}\rangle. \tag{8.70}$$

Properties of the S-matrix

Several important properties of S follow from our initial assumptions. We now prove each one in detail.

Property 1: Stability of the vacuum.

Statement: The vacuum state is stable under scattering, and we can choose $S_{00} = 1$.

Proof. By definition of the S-matrix element:

$$S_{00} = \langle 0 \text{ out} | 0 \text{ in} \rangle . \tag{8.71}$$

We show that the vacuum cannot make a transition to any other state. Consider $\langle n \text{ out}|0 \text{ in}\rangle$ for an arbitrary out-state $|n \text{ out}\rangle$ with four-momentum p_n^μ. Since P^μ is the *same* operator acting on both in- and out-states, we have

$$p_n^\mu \langle n \text{ out}|0 \text{ in}\rangle = \langle n \text{ out}|P^\mu|0 \text{ in}\rangle = 0 , \tag{8.72}$$

where we used $P^\mu|0 \text{ in}\rangle = 0$ (the vacuum has zero energy and momentum). Therefore:

$$\text{if } p_n^\mu \neq 0 \quad \Longrightarrow \quad \langle n \text{ out}|0 \text{ in}\rangle = 0 . \tag{8.73}$$

By our spectral assumptions, the vacuum is the *unique* state with $P^\mu = 0$: every other state (one-particle, multiparticle) has $P^0 > 0$. So the only nonvanishing overlap is with $|n\rangle = |0\rangle$. Using the completeness of out-states, $\sum_n |n \text{ out}\rangle\langle n \text{ out}| = \mathbb{1}$:

$$1 = \langle 0 \text{ in}|0 \text{ in}\rangle = \sum_n |\langle n \text{ out}|0 \text{ in}\rangle|^2 = |S_{00}|^2 , \tag{8.74}$$

since only the $n = 0$ term survives. Hence $|S_{00}| = 1$, and the vacuum state is the same for both the in- and out-descriptions (there is nothing to scatter). The only freedom is an overall phase:

$$\langle 0 \text{ out}| = e^{i\varphi_0}\langle 0 \text{ in}| . \tag{8.75}$$

We *choose* the phase convention $\varphi_0 = 0$, giving:

$$\langle 0 \text{ out}| = \langle 0 \text{ in}| = \langle 0|, \qquad S_{00} = 1 . \tag{8.76}$$

□

Property 2: Stability of one-particle states.

Statement: A single stable particle cannot decay or transform into other particles: $|p \text{ in}\rangle = |p \text{ out}\rangle$.

Proof. The argument parallels that for the vacuum. Consider $\langle n \text{ out}|p \text{ in}\rangle$ for an arbitrary out-state $|n \text{ out}\rangle$ with total four-momentum p_n^μ. Since $P^\mu|p \text{ in}\rangle = p^\mu|p \text{ in}\rangle$:

$$p_n^\mu \langle n \text{ out}|p \text{ in}\rangle = \langle n \text{ out}|P^\mu|p \text{ in}\rangle = p^\mu \langle n \text{ out}|p \text{ in}\rangle\,. \tag{8.77}$$

Therefore:

$$(p_n^\mu - p^\mu)\langle n \text{ out}|p \text{ in}\rangle = 0 \quad\Longrightarrow\quad \text{if } p_n^\mu \neq p^\mu, \text{ then } \langle n \text{ out}|p \text{ in}\rangle = 0\,. \tag{8.78}$$

Now we ask: which out-states can have exactly $p_n^\mu = p^\mu$? Since $p^2 = m^2$ (the particle is on-shell), we need $p_n^2 = m^2$. But by the *mass gap* assumption, the multiparticle continuum starts at $p^2 \geq m_1^2 > m^2$. The only states with invariant mass m^2 are one-particle states with the same conserved quantum numbers. Therefore the sole out-state with $p_n^\mu = p^\mu$ is the one-particle state $|p \text{ out}\rangle$ itself, up to possible degeneracies such as spin or internal labels, which are understood to be included in the state label.

For normalized wave-packet states, completeness of the out-basis gives

$$1 = \langle p \text{ in}|p \text{ in}\rangle = \sum_n |\langle n \text{ out}|p \text{ in}\rangle|^2 = |\langle p \text{ out}|p \text{ in}\rangle|^2\,, \tag{8.79}$$

since only the $n = |p \text{ out}\rangle$ term survives. Hence $|\langle p \text{ out}|p \text{ in}\rangle| = 1$, and $|p \text{ in}\rangle$ and $|p \text{ out}\rangle$ differ at most by a phase. In the sharp-momentum notation used below the same statement is understood distributionally, with Kronecker deltas replaced by Dirac delta functions. Choosing this phase to be zero (a convention compatible with the vacuum phase choice):

$$|p \text{ in}\rangle = |p \text{ out}\rangle \equiv |p\rangle\,. \tag{8.80}$$

Consequently, using the definition (8.70):

$$\langle p \text{ in}|S|p' \text{ in}\rangle = \langle p \text{ out}|p' \text{ in}\rangle = \langle p|p'\rangle = \delta^{(3)}(\vec{p} - \vec{p}')\,. \tag{8.81}$$

The S-matrix acts as the identity on one-particle states: a single isolated particle with momentum p at $t = -\infty$ remains a particle with the same momentum at $t = +\infty$. Note that this result depends crucially on the mass gap—without it, the one-particle state could "leak" into the multiparticle continuum. □

Property 3: Relation between in- and out-fields.

Statement: The S-matrix transforms in-fields to out-fields according to $\varphi_{\text{in}}(x) = S\,\varphi_{\text{out}}(x)\,S^{-1}$.

Proof. We must show the operator identity $\varphi_{\text{in}}(x)\,S = S\,\varphi_{\text{out}}(x)$, or equivalently

$$\langle \beta \text{ in}|\varphi_{\text{in}}(x)\,S|\alpha \text{ in}\rangle = \langle \beta \text{ in}|S\,\varphi_{\text{out}}(x)|\alpha \text{ in}\rangle \tag{8.82}$$

for all in-states $|\alpha\ \text{in}\rangle$, $|\beta\ \text{in}\rangle$ (which form a complete set). Using $\langle\beta\ \text{in}|S = \langle\beta\ \text{out}|$ from (8.70), the right-hand side becomes

$$\text{RHS} = \langle\beta\ \text{out}|\varphi_{\text{out}}(x)|\alpha\ \text{in}\rangle\,. \tag{8.83}$$

For the left-hand side, we insert the completeness relation $\sum_\gamma |\gamma\ \text{in}\rangle\langle\gamma\ \text{in}| = \mathbb{1}$ between $\varphi_{\text{in}}(x)$ and S:

$$\text{LHS} = \sum_\gamma \langle\beta\ \text{in}|\varphi_{\text{in}}(x)|\gamma\ \text{in}\rangle\ \underbrace{\langle\gamma\ \text{in}|S|\alpha\ \text{in}\rangle}_{=\,\langle\gamma\ \text{out}|\alpha\ \text{in}\rangle} = \sum_\gamma \langle\beta\ \text{in}|\varphi_{\text{in}}(x)|\gamma\ \text{in}\rangle\,\langle\gamma\ \text{out}|\alpha\ \text{in}\rangle\,. \tag{8.84}$$

Similarly, inserting $\sum_\gamma |\gamma\ \text{out}\rangle\langle\gamma\ \text{out}| = \mathbb{1}$ on the right-hand side:

$$\text{RHS} = \sum_\gamma \langle\beta\ \text{out}|\varphi_{\text{out}}(x)|\gamma\ \text{out}\rangle\,\langle\gamma\ \text{out}|\alpha\ \text{in}\rangle\,. \tag{8.85}$$

Comparing (8.84) and (8.85), the identity holds if and only if

$$\langle\beta\ \text{in}|\varphi_{\text{in}}(x)|\gamma\ \text{in}\rangle = \langle\beta\ \text{out}|\varphi_{\text{out}}(x)|\gamma\ \text{out}\rangle \qquad \text{for all } \beta,\gamma\,. \tag{8.86}$$

This is the key identity, and it holds because φ_{in} and φ_{out} have *identical algebraic structure*. Both are free fields of mass m with the same mode functions $f_k(x)$ [cf. (8.26) and (8.62)]:

$$\varphi_{\text{in/out}}(x) = \int d^3k\,\Big[a_{\text{in/out}}(k)\,f_k(x) + a^\dagger_{\text{in/out}}(k)\,f^*_k(x)\Big]\,. \tag{8.87}$$

The in- and out-creation operators satisfy the same canonical commutation relations,

$$[a_{\text{in}}(k), a^\dagger_{\text{in}}(k')] = \delta^{(3)}(\vec{k}-\vec{k}') = [a_{\text{out}}(k), a^\dagger_{\text{out}}(k')]\,, \tag{8.88}$$

and both annihilate the same vacuum: $a_{\text{in}}(k)|0\rangle = 0 = a_{\text{out}}(k)|0\rangle$. Since the in-states (8.67) and out-states (8.68) are built from $|0\rangle$ by the same algebraic operations (applying $a^\dagger$'s), and the matrix elements of a, $a^\dagger$ between Fock states are fully determined by the CCR and the vacuum condition, we have for instance

$$a_{\text{in}}(k)|k_1,\dots,k_n\ \text{in}\rangle = \sum_{i=1}^n \delta^{(3)}(\vec{k}-\vec{k}_i)\,|k_1,\dots,\hat{k}_i,\dots,k_n\ \text{in}\rangle\,, \tag{8.89}$$

and exactly the same formula with in $\to$ out. Thus (8.86) follows, and since the in-states are complete:

$$\boxed{\varphi_{\text{in}}(x) = S\,\varphi_{\text{out}}(x)\,S^{-1}} \tag{8.90}$$

□

Property 4: Unitarity of the S-matrix.

Statement: $SS^\dagger = S^\dagger S = 1$.

Proof. From (8.70), we have $\langle\beta\ \text{in}|S = \langle\beta\ \text{out}|$. Taking the Hermitian conjugate:

$$S^\dagger|\beta\ \text{in}\rangle = |\beta\ \text{out}\rangle. \tag{8.91}$$

Now consider:

$$\langle\beta\ \text{in}|SS^\dagger|\alpha\ \text{in}\rangle = \langle\beta\ \text{out}|S^\dagger|\alpha\ \text{in}\rangle. \tag{8.92}$$

Using (8.91) with $|\alpha\ \text{in}\rangle$:

$$S^\dagger|\alpha\ \text{in}\rangle = |\alpha\ \text{out}\rangle, \tag{8.93}$$

so:

$$\langle\beta\ \text{in}|SS^\dagger|\alpha\ \text{in}\rangle = \langle\beta\ \text{out}|\alpha\ \text{out}\rangle. \tag{8.94}$$

Both in-states and out-states form complete orthonormal sets (they describe the same Hilbert space, just with different labels). Therefore:

$$\langle\beta\ \text{out}|\alpha\ \text{out}\rangle = \delta_{\beta\alpha} = \langle\beta\ \text{in}|\alpha\ \text{in}\rangle. \tag{8.95}$$

Hence:

$$\langle\beta\ \text{in}|SS^\dagger|\alpha\ \text{in}\rangle = \delta_{\beta\alpha} = \langle\beta\ \text{in}|\alpha\ \text{in}\rangle. \tag{8.96}$$

Since this holds for all α, β:

$$SS^\dagger = 1\,. \tag{8.97}$$

Repeating the same matrix-element argument with the completeness of the out-states gives:

$$\boxed{SS^\dagger = S^\dagger S = 1}\,. \tag{8.98}$$

Physically, unitarity expresses probability conservation: the total probability for any initial state to evolve into *some* final state is unity. □

Property 5: Lorentz invariance of the S-matrix.

Statement: $U(a,b)\,S\,U^{-1}(a,b) = S$, where $U(a,b)$ is the unitary operator generating the Poincaré transformation $x'_\mu = b_\mu + a_\mu{}^\nu x_\nu$.

Proof. Under a Poincaré transformation, field operators transform as:

$$U(a,b)\,\varphi(x)\,U^{-1}(a,b) = \varphi(ax+b)\,. \tag{8.99}$$

Since $\varphi_{\text{in}}(x)$ and $\varphi_{\text{out}}(x)$ are constructed from $\varphi(x)$ through (8.36) and (8.64), and since the Green's functions $\Delta_{\text{ret/adv}}$ are Lorentz covariant, the in- and out-fields transform the same way:

$$U(a,b)\,\varphi_{\text{in}}(x)\,U^{-1}(a,b) = \varphi_{\text{in}}(ax+b)\,. \tag{8.100}$$

Now use Property 3, equation (8.90):

$$\begin{aligned}\varphi_{\text{in}}(ax+b) &= U(a,b)\,\varphi_{\text{in}}(x)\,U^{-1}(a,b)\\ &= U(a,b)\,S\,\varphi_{\text{out}}(x)\,S^{-1}\,U^{-1}(a,b).\end{aligned} \tag{8.101}$$

But we can also write, using (8.90) directly at the transformed point:

$$\varphi_{\text{in}}(ax+b) = S\,\varphi_{\text{out}}(ax+b)\,S^{-1}. \tag{8.102}$$

Using the transformation law for φ_{out}:

$$\varphi_{\text{out}}(ax+b) = U(a,b)\,\varphi_{\text{out}}(x)\,U^{-1}(a,b). \tag{8.103}$$

Substituting (8.103) into (8.102):

$$\varphi_{\text{in}}(ax+b) = S\,U(a,b)\,\varphi_{\text{out}}(x)\,U^{-1}(a,b)\,S^{-1}. \tag{8.104}$$

Comparing (8.101) and (8.104):

$$U(a,b)\,S\,\varphi_{\text{out}}(x)\,S^{-1}\,U^{-1}(a,b) = S\,U(a,b)\,\varphi_{\text{out}}(x)\,U^{-1}(a,b)\,S^{-1}. \tag{8.105}$$

This can be rewritten as:

$$\left[U(a,b)\,S\,U^{-1}(a,b)\right]\varphi_{\text{out}}(x)\left[U(a,b)\,S\,U^{-1}(a,b)\right]^{-1} = S\,\varphi_{\text{out}}(x)\,S^{-1}. \tag{8.106}$$

Since $\varphi_{\text{out}}(x)$ is an arbitrary field and the out-fields generate the full Hilbert space, we conclude:

$$\boxed{U(a,b)\,S\,U^{-1}(a,b) = S}\,. \tag{8.107}$$

Equivalently, $[S, U(a,b)] = 0$: the S-matrix commutes with all Poincaré transformations. This means that if we Lorentz-transform both the initial and final states, the S-matrix element remains unchanged—as required by the principle of relativity. □

8.5 Comparison with the interaction-picture approach

In Chapter 3, we developed the perturbative approach to scattering using the interaction picture and defined the S-matrix through the Dyson expansion. The present chapter takes a different, more fundamental approach. It is instructive to compare the two viewpoints and clarify their relationship.

Two definitions of the S-matrix

Interaction-picture definition (Chapter 3). In the interaction picture, the S-matrix is defined as the time-evolution operator that connects states at $t = -\infty$ to states at $t = +\infty$:

$$|\Phi(\infty)\rangle = S\,|\Phi(-\infty)\rangle, \qquad S = U_I(+\infty, -\infty)\,. \tag{8.108}$$

The states $|\Phi(t)\rangle$ are interaction-picture states, which evolve in time according to

$$i\,\frac{d}{dt}|\Phi(t)\rangle = V_I(t)\,|\Phi(t)\rangle\,, \tag{8.109}$$

where $V_I(t) = e^{iH_0t}V\,e^{-iH_0t}$ is the interaction Hamiltonian in the interaction picture. In natural units, the Dyson expansion provides a perturbative solution:

$$\begin{aligned} S &= T\exp\left[-i\int_{-\infty}^{+\infty} dt\,V_I(t)\right] \\ &= \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int d^4x_1\cdots d^4x_n\,T\{\mathscr{H}_I(x_1)\cdots\mathscr{H}_I(x_n)\}. \end{aligned} \tag{8.110}$$

LSZ definition (this chapter). In the Heisenberg picture, we define in- and out-states using the asymptotic free fields $\varphi_{\text{in}}(x)$ and $\varphi_{\text{out}}(x)$:

$$|\alpha\ \text{in}\rangle = a^\dagger_{\text{in}}(p_1)\cdots a^\dagger_{\text{in}}(p_n)|0\rangle, \qquad |\beta\ \text{out}\rangle = a^\dagger_{\text{out}}(q_1)\cdots a^\dagger_{\text{out}}(q_m)|0\rangle\,, \tag{8.111}$$

and the S-matrix element is the overlap:

$$S_{\beta\alpha} = \langle\beta\ \text{out}|\alpha\ \text{in}\rangle\,. \tag{8.112}$$

Relationship between the two approaches

The two definitions are *equivalent* when properly interpreted, but they emphasize different aspects:

- **The interaction-picture approach** is *computational*: it provides an explicit perturbation expansion (Dyson series) that can be evaluated order by order using Wick's theorem and Feynman diagrams.
- **The LSZ approach** is *structural*: it relates S-matrix elements to Green's functions without relying on perturbation theory, providing a rigorous definition that holds to all orders.

To see the connection, note that in the interaction picture:

$$S_{fi} = \langle f|S|i\rangle = \langle f, t=+\infty|\Phi(\infty)\rangle = \langle f|U_I(+\infty,-\infty)|i\rangle\,, \tag{8.113}$$

where $|i\rangle$ and $|f\rangle$ are free-particle states (eigenstates of H_0). In the LSZ formalism:

$$S_{fi} = \langle f \text{ out}|i \text{ in}\rangle\,, \tag{8.114}$$

where the in- and out-states are constructed from the exact Heisenberg-picture vacuum using asymptotic fields.

The key insight is that the *free-particle states* $|i\rangle$, $|f\rangle$ of the interaction picture correspond to the *asymptotic states* $|i \text{ in}\rangle$, $|f \text{ out}\rangle$ of the LSZ formalism. The adiabatic hypothesis in the interaction-picture approach plays the role of the weak asymptotic condition in the LSZ formalism—both ensure that interactions can be "turned off" asymptotically.

Comparison of in/out fields with interaction-picture fields

The in-fields and out-fields defined in this chapter are *fundamentally different* from the interaction-picture fields of Chapter 2. This distinction is crucial for understanding the structure of scattering theory.

Interaction-picture fields (Chapter 2). The interaction-picture field is defined by

$$\varphi_I(x) = e^{iH_0 t}\,\varphi_S(\vec{x})\,e^{-iH_0 t} = U_0^\dagger(t,t_0)\,\varphi_S(\vec{x})\,U_0(t,t_0)\,, \tag{8.115}$$

where $\varphi_S(x)$ is the Schrödinger-picture field (time-independent) and H_0 is the free Hamiltonian. Key properties:

- $\varphi_I(x)$ satisfies the *free* field equation determined by the quadratic part chosen for H_0. In a bare split this mass is m_0; in renormalized perturbation theory one usually chooses a renormalized or physical mass and treats the difference as a counterterm.
- The time evolution is governed by H_0 alone.
- $\varphi_I(x)$ is related to φ_S by a unitary transformation at each time t.
- The mass appearing in the field equation is therefore a bookkeeping choice tied to the perturbative split $H = H_0 + V$.

In-fields and out-fields (this chapter). The in-field $\varphi_{\text{in}}(x)$ is defined through the asymptotic condition (8.36):

$$\sqrt{Z}\,\varphi_{\text{in}}(x) = \varphi(x) - \int d^4y\,\Delta_{\text{ret}}(x-y;m)\,\tilde{j}(y), \tag{8.116}$$

where $\varphi(x)$ is the full *Heisenberg-picture* interacting field. Key properties:

- $\varphi_{\text{in}}(x)$ satisfies the free field equation with the *physical* mass: $(\Box + m^2)\varphi_{\text{in}} = 0$.

- $\varphi_{\text{in}}(x)$ is the asymptotic limit of the interacting field as $t \to -\infty$ (in the weak sense).
- The normalization involves the wave function renormalization constant $\sqrt{Z}$.
- The physical mass m (not the bare mass m_0) appears in the field equation.

Summary of differences.

Property	**Int. picture φ_I**	**In/out fields $\varphi_{\text{in/out}}$**
Picture	Interaction picture	Heisenberg picture
Defined via	Unitary transform of φ_S	Asymptotic limit of $\varphi(x)$
Evolution gen.	Free Hamiltonian H_0	Free equation, mass m
Mass parameter	Mass chosen in H_0	Physical mass m
Normalization	Same as φ_S	Involves $\sqrt{Z}$
States created	Fock states of H_0	Physical particle states

Physical interpretation. The interaction-picture field $\varphi_I(x)$ is a mathematical tool for perturbation theory. It satisfies the free equations associated with the chosen H_0 and creates the corresponding Fock states. In contrast, $\varphi_{\text{in}}(x)$ and $\varphi_{\text{out}}(x)$ are the physical asymptotic fields that create actual particles with the observed mass m. The factor $\sqrt{Z}$ accounts for the overlap between the local interacting field $\varphi(x)$ and the isolated one-particle state; the remaining spectral weight, when present, belongs to continuum states.

The precise relation between φ_{in} and φ_I. The two fields live in different pictures and are *not* equal, but they are related by a precise chain of identities. Recall from Chapter 2, Eq. (2.10), that a Schrödinger-picture operator is promoted to a Heisenberg-picture operator by

$$A_H(t) = U^\dagger(t,t_0)\, A_S\, U(t,t_0).$$

Applying this to the field operator $\varphi_S(\vec{x})$ with $t_0 = 0$ and $U(t,0) = e^{-iHt}$ gives

$$\varphi(x) = e^{iHt}\, \varphi_S(\vec{x})\, e^{-iHt}, \tag{8.117}$$

where $H = H_0 + V$ is the full Hamiltonian. The interaction-picture field is obtained by evolving φ_S with the free part H_0 only:

$$\varphi_I(x) = e^{iH_0 t}\, \varphi_S(\vec{x})\, e^{-iH_0 t}. \tag{8.118}$$

Combining these, we can express the Heisenberg field in terms of the interaction-picture field:

$$\varphi(x) = e^{iHt}\, e^{-iH_0 t}\, \varphi_I(x)\, e^{iH_0 t}\, e^{-iHt} = U^\dagger(t)\, \varphi_I(x)\, U(t), \tag{8.119}$$

where $U(t) \equiv e^{iH_0 t} e^{-iHt}$ is the time-evolution operator relating the two pictures.

Now consider the limit $t \to -\infty$. In the interaction-picture approach, the adiabatic prescription identifies the Fock states of H_0 with the asymptotic particle states, after the pole position and residue have been matched by renormalization. In the LSZ approach, the weak asymptotic condition gives instead $\varphi(x) \to \sqrt{Z}\,\varphi_{\rm in}(x)$ inside suitably smeared matrix elements. Comparing these asymptotic matrix elements motivates the schematic perturbative identification

$$\varphi_{\rm in}(x) \sim \frac{1}{\sqrt{Z}}\varphi_I(x) \quad \text{inside asymptotic, renormalized matrix elements}\,. \tag{8.120}$$

The two fields differ in two ways: (i) φ_I satisfies the free equation associated with the chosen perturbative mass parameter, while $\varphi_{\rm in}$ uses the physical pole mass m; (ii) the relative normalisation is fixed by the pole residue. Both differences are handled by renormalization. After mass and field-strength renormalization, the perturbative Green's functions built from interaction-picture fields have the same external one-particle poles and residues required by LSZ, so the perturbative S-matrix from the Dyson expansion agrees with the LSZ S-matrix order by order.

8.6 The spectral representation

Before deriving the reduction formula, we study the vacuum expectation value of the field commutator. This fixes the normalization constant Z.

Derivation of the Källén–Lehmann representation

We now derive the spectral representation for the vacuum expectation value of the field commutator, showing all intermediate steps.

Step 1: Definition and completeness. Consider the vacuum expectation value of the commutator:

$$i\Delta'(x,x') \equiv \langle 0|[\varphi(x),\varphi(x')]|0\rangle = \langle 0|\varphi(x)\varphi(x')|0\rangle - \langle 0|\varphi(x')\varphi(x)|0\rangle\,. \tag{8.121}$$

Insert a complete set of energy-momentum eigenstates $\sum_n |n\rangle\langle n| = 1$ between the field operators. For the first term:

$$\langle 0|\varphi(x)\varphi(x')|0\rangle = \sum_n \langle 0|\varphi(x)|n\rangle\langle n|\varphi(x')|0\rangle\,. \tag{8.122}$$

Step 2: Use of displacement invariance. The field operator at point x is related to the field at the origin by the displacement operator:

$$\varphi(x) = e^{iP\cdot x}\varphi(0)e^{-iP\cdot x}\,. \tag{8.123}$$