where $U(t) \equiv e^{iH_0 t} e^{-iHt}$ is the time-evolution operator relating the two pictures.

Now consider the limit $t \to -\infty$. In the interaction-picture approach, the adiabatic prescription identifies the Fock states of $H_0$ with the asymptotic particle states, after the pole position and residue have been matched by renormalization. In the LSZ approach, the weak asymptotic condition gives instead $\varphi(x) \to \sqrt{Z}\,\varphi_{\text{in}}(x)$ inside suitably smeared matrix elements. Comparing these asymptotic matrix elements motivates the schematic perturbative identification

$$\varphi_{\text{in}}(x) \sim \frac{1}{\sqrt{Z}}\varphi_I(x) \quad \text{inside asymptotic, renormalized matrix elements}\,. \tag{8.120}$$

The two fields differ in two ways: (i) $\varphi_I$ satisfies the free equation associated with the chosen perturbative mass parameter, while $\varphi_{\text{in}}$ uses the physical pole mass $m$; (ii) the relative normalisation is fixed by the pole residue. Both differences are handled by renormalization. After mass and field-strength renormalization, the perturbative Green's functions built from interaction-picture fields have the same external one-particle poles and residues required by LSZ, so the perturbative $S$-matrix from the Dyson expansion agrees with the LSZ $S$-matrix order by order.

## 8.6 The spectral representation

Before deriving the reduction formula, we study the vacuum expectation value of the field commutator. This fixes the normalization constant $Z$.

### Derivation of the Källén–Lehmann representation

We now derive the spectral representation for the vacuum expectation value of the field commutator, showing all intermediate steps.

**Step 1: Definition and completeness.** Consider the vacuum expectation value of the commutator:

$$i\Delta'(x,x') \equiv \langle 0|[\varphi(x),\varphi(x')]|0\rangle = \langle 0|\varphi(x)\varphi(x')|0\rangle - \langle 0|\varphi(x')\varphi(x)|0\rangle\,. \tag{8.121}$$

Insert a complete set of energy-momentum eigenstates $\sum_n |n\rangle\langle n| = 1$ between the field operators. For the first term:

$$\langle 0|\varphi(x)\varphi(x')|0\rangle = \sum_n \langle 0|\varphi(x)|n\rangle\langle n|\varphi(x')|0\rangle\,. \tag{8.122}$$

**Step 2: Use of displacement invariance.** The field operator at point $x$ is related to the field at the origin by the displacement operator:

$$\varphi(x) = e^{iP\cdot x}\varphi(0)e^{-iP\cdot x}\,. \tag{8.123}$$

For a matrix element between eigenstates $|n\rangle$ and $|m\rangle$ with $P^\mu|n\rangle = p_n^\mu|n\rangle$:

$$\begin{aligned}\langle n|\varphi(x)|m\rangle &= \langle n|e^{iP\cdot x}\varphi(0)e^{-iP\cdot x}|m\rangle \\ &= e^{ip_n\cdot x}\langle n|\varphi(0)|m\rangle e^{-ip_m\cdot x} \\ &= e^{i(p_n-p_m)\cdot x}\langle n|\varphi(0)|m\rangle\,. \end{aligned} \tag{8.124}$$

In particular, since $P^\mu|0\rangle = 0$:

$$\langle 0|\varphi(x)|n\rangle = e^{-ip_n\cdot x}\langle 0|\varphi(0)|n\rangle, \qquad \langle n|\varphi(x')|0\rangle = e^{ip_n\cdot x'}\langle n|\varphi(0)|0\rangle\,. \tag{8.125}$$

**Step 3: Evaluation of the two terms.** Substituting (8.125) into (8.122):

$$\begin{aligned}\langle 0|\varphi(x)\varphi(x')|0\rangle &= \sum_n e^{-ip_n\cdot x}\langle 0|\varphi(0)|n\rangle \cdot e^{ip_n\cdot x'}\langle n|\varphi(0)|0\rangle \\ &= \sum_n |\langle 0|\varphi(0)|n\rangle|^2 e^{-ip_n\cdot(x-x')}\,. \end{aligned} \tag{8.126}$$

Similarly, for the second term:

$$\begin{aligned}\langle 0|\varphi(x')\varphi(x)|0\rangle &= \sum_n \langle 0|\varphi(x')|n\rangle\langle n|\varphi(x)|0\rangle \\ &= \sum_n |\langle 0|\varphi(0)|n\rangle|^2 e^{ip_n\cdot(x-x')}\,. \end{aligned} \tag{8.127}$$

**Step 4: The commutator.** Combining (8.126) and (8.127):

$$\begin{aligned} i\Delta'(x-x') &= \langle 0|[\varphi(x),\varphi(x')]|0\rangle \\ &= \sum_n |\langle 0|\varphi(0)|n\rangle|^2 \left(e^{-ip_n\cdot(x-x')} - e^{ip_n\cdot(x-x')}\right)\,. \end{aligned} \tag{8.128}$$

Note that the result depends only on the difference $x-x'$, as required by translation invariance.

**Step 5: Definition of the spectral amplitude.** To convert the discrete sum into a more useful form, we group together all states with the same four-momentum $q$. Introduce the identity:

$$1 = \int d^4q\,\delta^{(4)}(p_n-q)\,. \tag{8.129}$$

Define the **spectral amplitude**:

$$\rho(q) \equiv (2\pi)^3 \sum_n \delta^{(4)}(p_n-q)\,|\langle 0|\varphi(0)|n\rangle|^2\,. \tag{8.130}$$

The factor $(2\pi)^3$ is conventional and matches the normalization of momentum states. Then:

$$\sum_n |\langle 0|\varphi(0)|n\rangle|^2 e^{-ip_n\cdot y} = \frac{1}{(2\pi)^3}\int d^4q\,\rho(q)\,e^{-iq\cdot y}\,, \tag{8.131}$$

where $y = x - x'$.

**Step 6: Lorentz invariance of $\rho(q)$.** We now prove that $\rho(q)$ depends only on $q^2$. Under a Lorentz transformation $\Lambda$:

- The vacuum is invariant: $U(\Lambda)|0\rangle = |0\rangle$.

- The scalar field transforms as $U(\Lambda)\varphi(x)U^{-1}(\Lambda) = \varphi(\Lambda x)$;[3] at the origin this reduces to $U(\Lambda)\varphi(0)U^{-1}(\Lambda) = \varphi(0)$.
- The states transform as: $|n\rangle \longrightarrow |\Lambda n\rangle$ with $p^\mu_{\Lambda n} = \Lambda^\mu{}_\nu p^\nu_n$.

[3]This is the defining transformation rule of a scalar field in the active convention adopted here, where $U(\Lambda)|n\rangle = |\Lambda n\rangle$ with $p_{\Lambda n} = \Lambda p_n$ (third bullet).

*Derivation.* Starting from the translation property $\varphi(x) = e^{iP\cdot x}\varphi(0)e^{-iP\cdot x}$ and the four-vector rule for the momentum operator, written in terms of $\Lambda$ with one index lowered and one raised,

$$U(\Lambda)\, P^\mu\, U^{-1}(\Lambda) \;=\; \Lambda_\nu{}^\mu\, P^\nu\,,$$

where $\Lambda_\nu{}^\mu \equiv \eta_{\nu\rho}\,\Lambda^\rho{}_\sigma\,\eta^{\sigma\mu}$ is the matrix $\Lambda$ with indices "shifted across" (lower–upper instead of upper–lower). For a Lorentz transformation, this coincides with $(\Lambda^{-1})^\mu{}_\nu$, so the inverse the reader might expect from the four-vector rule is hidden in the position of the indices on $\Lambda$. The rule itself is enforced by the requirement $P^\mu|\Lambda n\rangle = (\Lambda p_n)^\mu|\Lambda n\rangle$.

*Action on the exponential.* Conjugation by $U(\Lambda)$ acts term by term in the Taylor expansion of an exponential of operators, because for any $A$, $UA^nU^{-1} = (UAU^{-1})^n$ (insert $\mathbb{1} = U^{-1}U$ between each factor). Since $x_\mu$ is a c-number that commutes through $U$, only $P^\mu$ is conjugated:

$$\begin{aligned} U(\Lambda)\, e^{iP\cdot x}\, U^{-1}(\Lambda) \;&=\; \sum_{n=0}^{\infty}\frac{1}{n!}\big[i\,U(\Lambda)\,(P\cdot x)\,U^{-1}(\Lambda)\big]^n \\ &=\; \exp\!\big[i\,U(\Lambda)\,P^\mu\,U^{-1}(\Lambda)\,x_\mu\big] \;=\; \exp\!\big(i\,\Lambda_\nu{}^\mu\,P^\nu\,x_\mu\big)\,. \end{aligned}$$

The same applies to $e^{-iP\cdot x}$ with the opposite sign in the exponent. Therefore

$$\begin{aligned} U(\Lambda)\,\varphi(x)\,U^{-1}(\Lambda) \;&=\; U(\Lambda)\,e^{iP\cdot x}\,U^{-1}(\Lambda)\;\,U(\Lambda)\,\varphi(0)\,U^{-1}(\Lambda)\;\,U(\Lambda)\,e^{-iP\cdot x}\,U^{-1}(\Lambda) \\ &=\; e^{i\Lambda_\nu{}^\mu P^\nu x_\mu}\,\varphi(0)\,e^{-i\Lambda_\nu{}^\mu P^\nu x_\mu}\,, \end{aligned}$$

where in the last step we used $U(\Lambda)\varphi(0)U^{-1}(\Lambda) = \varphi(0)$ (scalar at the origin) and inserted three copies of $\mathbb{1} = U^{-1}(\Lambda)U(\Lambda)$ between the three factors.

*Index manipulation.* The key step is now the contraction $\Lambda_\nu{}^\mu x_\mu = (\Lambda x)_\nu$: the upper index $\mu$ of $\Lambda$ contracts with $x_\mu$, and the lower index $\nu$ becomes the index of a covector identified with the boosted $\Lambda x$. Indeed, raising and lowering with $\eta$,

$$\Lambda_\nu{}^\mu\, x_\mu \;=\; \eta_{\nu\rho}\,\Lambda^\rho{}_\sigma\,\eta^{\sigma\mu}\,x_\mu \;=\; \eta_{\nu\rho}\,\Lambda^\rho{}_\sigma\,x^\sigma \;=\; \eta_{\nu\rho}\,(\Lambda x)^\rho \;=\; (\Lambda x)_\nu\,.$$

Substituting into the exponent, $\Lambda_\nu{}^\mu P^\nu x_\mu = P^\nu(\Lambda x)_\nu = P\cdot(\Lambda x)$, and finally

$$U(\Lambda)\,\varphi(x)\,U^{-1}(\Lambda) \;=\; e^{iP\cdot(\Lambda x)}\,\varphi(0)\,e^{-iP\cdot(\Lambda x)} \;=\; \varphi(\Lambda x)\,.$$

*Alternative passive convention.* The same physical content can be written with $\Lambda^{-1}$ in the argument: defining $\varphi'(x) \equiv U^{-1}(\Lambda)\,\varphi(x)\,U(\Lambda)$ gives $\varphi'(x) = \varphi(\Lambda^{-1}x)$, obtained from the active form by replacing $\Lambda$ with $\Lambda^{-1}$. This is the convention used in Chapter 1 for the general field transformation $\phi'_r(x) = \phi_r(\Lambda^{-1}x) + \frac{1}{2}\omega_{\mu\nu}(S^{\mu\nu})_r{}^s\phi_s(\Lambda^{-1}x)$, with $S^{\mu\nu} = 0$ in the scalar case.

Therefore:

$$\begin{aligned}\rho(q) &= (2\pi)^3 \sum_n \delta^{(4)}(p_n - q)\,|\langle 0|\varphi(0)|n\rangle|^2 \\ &= (2\pi)^3 \sum_n \delta^{(4)}(p_n - q)\,|\langle 0|U(\Lambda)\varphi(0)U^{-1}(\Lambda)|\Lambda n\rangle|^2 \end{aligned} \tag{8.132}$$

In the second equality we used $\langle 0|U(\Lambda) = \langle 0|$ and $U^{-1}(\Lambda)|\Lambda n\rangle = |n\rangle$, so the matrix element is unchanged. We now apply the scalar transformation $U(\Lambda)\varphi(0)U^{-1}(\Lambda) = \varphi(0)$ and relabel the dummy index $m := \Lambda n$ (a bijection over the spectrum). Under this relabel $p_m = \Lambda p_n$, hence $p_n = \Lambda^{-1} p_m$, and

$$\delta^{(4)}(p_n - q) = \delta^{(4)}(\Lambda^{-1} p_m - q) = \delta^{(4)}\big(\Lambda^{-1}(p_m - \Lambda q)\big) = \delta^{(4)}(p_m - \Lambda q)\,,$$

where the last step uses the Lorentz invariance of the four-dimensional delta, $\delta^{(4)}(\Lambda y) = |\det\Lambda|^{-1}\delta^{(4)}(y) = \delta^{(4)}(y)$ for proper Lorentz transformations. Therefore:

$$\begin{aligned}\rho(q) &= (2\pi)^3 \sum_m \delta^{(4)}(p_m - \Lambda q)\,|\langle 0|\varphi(0)|m\rangle|^2 \\ &= \rho(\Lambda q)\,. \end{aligned} \tag{8.133}$$

Since $\rho(q) = \rho(\Lambda q)$ for all Lorentz transformations, and the only Lorentz-invariant function of a four-vector is a function of its square, we have:

$$\rho(q) = \rho(q^2)\,\theta(q^0)\,, \tag{8.134}$$

where $\theta(q^0)$ restricts to the forward light cone (since all physical states have $p^0 > 0$).

**Step 7: Properties of $\rho(q^2)$.** From the definition (8.130):

- $\rho(q^2) \geq 0$ (it is a sum of absolute squares).
- $\rho(q^2) = 0$ for $q^2 < 0$ (no physical states have spacelike momentum).
- $\rho(q^2)$ is real (obvious from the definition).

**Step 8: Conversion to the spectral integral.** Using (8.134), we can write:

$$\int d^4q\,\rho(q)\,e^{-iq\cdot y} = \int_0^\infty d\sigma^2\,\rho(\sigma^2) \int d^4q\,\delta(q^2 - \sigma^2)\,\theta(q^0)\,e^{-iq\cdot y}\,. \tag{8.135}$$

The inner integral is recognized as the free-field positive-frequency Wightman function:

$$\Delta^{(+)}(y;\sigma) = \int \frac{d^4q}{(2\pi)^3}\,\delta(q^2 - \sigma^2)\,\theta(q^0)\,e^{-iq\cdot y} = \int \frac{d^3q}{(2\pi)^3 2\omega_q}\,e^{-iq\cdot y}\,, \tag{8.136}$$

where $\omega_q = \sqrt{\vec{q}^2 + \sigma^2}$.

**Step 9: Final result.** Substituting back into (8.128):

$$
\begin{aligned}
i\Delta'(x-x') &= \int_0^\infty d\sigma^2 \rho(\sigma^2) \left[\Delta^{(+)}(x-x';\sigma) - \Delta^{(+)}(x'-x;\sigma)\right] \\
&= \int_0^\infty d\sigma^2 \rho(\sigma^2)\, i\Delta(x-x';\sigma)\,,
\end{aligned}
\tag{8.137}
$$

Here $\Delta^{(\pm)}(y;\sigma)$ are the positive- and negative-frequency Wightman two-point functions of a *free* scalar field $\varphi^{(\sigma)}$ of mass $\sigma$,

$$
\begin{aligned}
\Delta^{(+)}(y;\sigma) &= \langle 0|\varphi^{(\sigma)}(y)\varphi^{(\sigma)}(0)|0\rangle\,, \\
\Delta^{(-)}(y;\sigma) &= \langle 0|\varphi^{(\sigma)}(0)\varphi^{(\sigma)}(y)|0\rangle = \Delta^{(+)}(-y;\sigma)\,,
\end{aligned}
\tag{8.138}
$$

each given by the on-shell mode integral $\int d^3q/((2\pi)^3 2\omega_q)\, e^{\mp iq\cdot y}$ with $\omega_q = \sqrt{\vec{q}^2 + \sigma^2}$ (cf. (8.136)). The combination

$$
\Delta(y;\sigma) \;=\; -i\left[\Delta^{(+)}(y;\sigma) - \Delta^{(-)}(y;\sigma)\right]
\tag{8.139}
$$

is the *free-field commutator function* (Pauli–Jordan function) with mass $\sigma$, characterized by the c-number identity

$$
[\varphi^{(\sigma)}(x), \varphi^{(\sigma)}(x')] \;=\; i\,\Delta(x-x';\sigma)\,\mathbb{1}\,.
\tag{8.140}
$$

It is real, Lorentz-invariant, antisymmetric ($\Delta(-y;\sigma) = -\Delta(y;\sigma)$), and vanishes for spacelike separations (microcausality). See Chapter 1 for the explicit mode-expansion derivation, where $\Delta(y;\sigma)$ emerges directly from the canonical commutation relations applied to the free-field expansion.

The **Källén–Lehmann spectral representation** then takes the form:

$$
\boxed{\Delta'(x-x') = \int_0^\infty d\sigma^2\, \rho(\sigma^2)\, \Delta(x-x';\sigma)}
\tag{8.141}
$$

where $\Delta(x-x';\sigma)$ is the free-field commutator function with mass parameter $\sigma$.

### Spectral representation for the propagator

The same derivation applies to the Feynman propagator, defined as the time-ordered two-point function (cf. Chapter 1, where $i\Delta_F = \langle 0|T(\varphi\varphi)|0\rangle$). Writing the $T$-product explicitly and inserting the complete set $\mathbb{1} = |0\rangle\langle 0| + \sum_n |n\rangle\langle n|$ (the vacuum drops out since $\langle 0|\varphi(x)|0\rangle = 0$):

$$\begin{aligned} i\Delta'_F(x-x') &\equiv \langle 0|T\{\varphi(x)\varphi(x')\}|0\rangle \\ &= \theta(t-t')\langle 0|\varphi(x)\varphi(x')|0\rangle + \theta(t'-t)\langle 0|\varphi(x')\varphi(x)|0\rangle \\ &= \theta(t-t')\sum_n e^{-ip_n\cdot(x-x')}|\langle 0|\varphi(0)|n\rangle|^2 \\ &\quad + \theta(t'-t)\sum_n e^{+ip_n\cdot(x-x')}|\langle 0|\varphi(0)|n\rangle|^2 , \end{aligned} \tag{8.142}$$

where in the last line we used $\varphi(x) = e^{iP\cdot x}\varphi(0)e^{-iP\cdot x}$ and $P^\mu|n\rangle = p_n^\mu|n\rangle$. Both Wightman functions are precisely those evaluated in Steps 5–6. Substituting the spectral form (8.131) and the identification of $\Delta^{(+)}(y;\sigma)$ from (8.136), the sums become the spectral integrals

$$\sum_n e^{\mp ip_n\cdot y}|\langle 0|\varphi(0)|n\rangle|^2 = \int_0^\infty d\sigma^2\,\rho(\sigma^2)\,\Delta^{(+)}(\pm y;\sigma)\,, \tag{8.143}$$

hence

$$\begin{aligned} i\Delta'_F(x-x') &= \int_0^\infty d\sigma^2\,\rho(\sigma^2)\,\big[\theta(t-t')\,\Delta^{(+)}(x-x';\sigma) \\ &\qquad\qquad + \theta(t'-t)\,\Delta^{(+)}(x'-x;\sigma)\big] \\ &= \int_0^\infty d\sigma^2\,\rho(\sigma^2)\,i\Delta_F(x-x';\sigma)\,, \end{aligned} \tag{8.144}$$

where in the last step we recognized the standard expression for the free Feynman propagator of mass $\sigma$ in terms of the positive-frequency Wightman function,

$$i\Delta_F(y;\sigma) \;=\; \theta(y^0)\,\Delta^{(+)}(y;\sigma) + \theta(-y^0)\,\Delta^{(+)}(-y;\sigma)\,. \tag{8.145}$$

**Momentum space.** To pass to momentum space, take the Fourier transform of both sides of (8.144). *By definition*, the momentum-space propagator $i\Delta'_F(p)$ is the Fourier transform of the position-space propagator $i\Delta'_F(y)$ (where $y = x - x'$):

$$i\Delta'_F(p) \;\equiv\; \int d^4y\, e^{ip\cdot y}\, i\Delta'_F(y)\,. \tag{8.146}$$

This is just notation/convention — not an identity to be proven. Now substitute the spectral form $i\Delta'_F(y) = \int_0^\infty d\sigma^2\,\rho(\sigma^2)\,i\Delta_F(y;\sigma)$ from (8.144); the only $y$-dependence sits inside $\Delta_F(y;\sigma)$, so the $\sigma^2$-integral commutes with the $y$-integral and we may swap their order:

$$\begin{aligned} i\Delta'_F(p) &= \int d^4y\, e^{ip\cdot y} \int_0^\infty d\sigma^2\, \rho(\sigma^2)\, i\Delta_F(y;\sigma) \\ &= \int_0^\infty d\sigma^2\, \rho(\sigma^2) \underbrace{\int d^4y\, e^{ip\cdot y}\, i\Delta_F(y;\sigma)}_{=\ \dfrac{i}{p^2-\sigma^2+i\epsilon}} \\ &= \int_0^\infty d\sigma^2\, \rho(\sigma^2)\, \frac{i}{p^2-\sigma^2+i\epsilon}, \end{aligned} \tag{8.147}$$

where in the second line we recognized the Fourier transform of the free Feynman propagator of mass $\sigma$ — the standard result derived in Chapter 1:

$$\int d^4y\, e^{ip\cdot y}\, i\Delta_F(y;\sigma) \;=\; \frac{i}{p^2-\sigma^2+i\epsilon}. \tag{8.148}$$

### Spectral density and the physical pole

The spectral representation makes explicit what, in the Peskin–Schroeder language ([6, Section 7.1, Eqs. (7.4)–(7.9)]), is often stated as a pole condition on the full propagator. The structure of $\rho(\sigma^2)$ follows directly from the decomposition of the eigenstates of $P^\mu$ used in Step 4: the vacuum $|0\rangle$ drops out because $\langle 0|\varphi(0)|0\rangle = 0$, and the remaining intermediate states split into

- the **stable one-particle states** $|p\rangle$ of mass $m$, whose four-momentum lies on the hyperboloid $p^2 = m^2$. Their contribution to $\rho(q)$ is concentrated on a single mass shell, giving a $\delta(\sigma^2 - m^2)$ in the spectral integral. The weight is fixed by the field-particle overlap $|\langle 0|\varphi(0)|p\rangle|^2$ and is conventionally called $Z$ (the *wave-function renormalization constant*);
- the **multi-particle continuum**, made of states with $p^2 \geq m_1^2$, where $m_1$ is the threshold for the lowest multi-particle channel into which $\varphi(0)|0\rangle$ has nonzero overlap (e.g. $m_1 = 3m$ in a $\mathbb{Z}_2$-symmetric $\varphi^4$ theory, $m_1 = m + m_\pi$ in nucleon–pion theory, etc.). These states span a continuum of invariant masses, producing a smooth $\rho_{\text{cont.}}(\sigma^2)$ supported on $[m_1^2, \infty)$.

Assuming the one-particle state is *separated from the continuum by a mass gap*, i.e. $m_1 > m$, the spectral density therefore has the form

$$\rho(\sigma^2) = Z\,\delta(\sigma^2 - m^2) + \rho_{\text{cont.}}(\sigma^2)\,\theta(\sigma^2 - m_1^2), \qquad m_1^2 > m^2. \tag{8.149}$$

**Derivation of the coefficient $Z$.** The factor $Z$ in front of $\delta(\sigma^2 - m^2)$ is not a free parameter — it is fixed by the field–particle matrix element. Restricting the sum in (8.130) to one-particle states $|p\rangle$ with covariant normalization $\langle p|p'\rangle = (2\pi)^3\, 2E_p\, \delta^{(3)}(\vec{p}-\vec{p}\,')$,

$$\sum_n^{(1\text{-part})} |n\rangle\langle n| = \int \frac{d^3p}{(2\pi)^3\, 2E_p}\, |p\rangle\langle p|\,, \tag{8.150}$$

and using Lorentz invariance of $\varphi(0)$ together with $U(\Lambda)|p\rangle = |\Lambda p\rangle$, the matrix element $\langle 0|\varphi(0)|p\rangle$ is independent of $p$ on the mass shell. We *define*

$$\langle 0|\varphi(0)|p\rangle = \sqrt{Z}\,, \qquad |\langle 0|\varphi(0)|p\rangle|^2 = Z\,, \tag{8.151}$$

which is the standard definition of the wave-function renormalization constant.

The one-particle contribution to the spectral function is now obtained by substituting these ingredients into the definition (8.130),

$$\rho(q) \;\equiv\; (2\pi)^3 \sum_n \delta^{(4)}(p_n - q)\, |\langle 0|\varphi(0)|n\rangle|^2\,.$$

Restricting the sum to the one-particle sector: $\sum_n \to \int d^3p/[(2\pi)^3\, 2E_p]$, the eigenvalue $p_n^\mu$ becomes the on-shell four-momentum $p^\mu = (E_p, \vec{p})$ with $E_p = \sqrt{\vec{p}^{\,2}+m^2}$, and the matrix-element factor reduces to the constant $|\langle 0|\varphi(0)|p\rangle|^2 = Z$. Therefore

$$\begin{aligned}
\rho^{(1\text{-part})}(q) &= (2\pi)^3 \int \frac{d^3p}{(2\pi)^3\, 2E_p}\, \delta^{(4)}(p-q)\, Z \\
&= Z \int d^4p\, \delta(p^2-m^2)\, \theta(p^0)\, \delta^{(4)}(p-q) \\
&= Z\, \delta(q^2-m^2)\, \theta(q^0)\,,
\end{aligned} \tag{8.152}$$

where in the second line we used the Lorentz-invariant form of the on-shell measure $d^3p/(2\pi)^3\, 2E_p = (2\pi)^{-3}\, d^4p\, \delta(p^2-m^2)\theta(p^0)$ (this rewrites the three-momentum integral over the mass-shell as a covariant four-momentum integral restricted to the positive-energy hyperboloid). Comparing with $\rho(q) = \rho(\sigma^2 = q^2)\, \theta(q^0)$ identifies the $\delta$-coefficient as $Z$.

This structure is illustrated in Fig. 8.1.

Thus the exact Feynman propagator is

$$i\Delta_F'(p) = \frac{iZ}{p^2-m^2+i\epsilon} + \int_{m_1^2}^{\infty} d\sigma^2\, \rho_{\text{cont.}}(\sigma^2)\, \frac{i}{p^2-\sigma^2+i\epsilon}\,. \tag{8.153}$$

Near the one-particle shell the continuum term is regular, so

$$i\Delta_F'(p) \xrightarrow[p^2\to m^2]{} \frac{iZ}{p^2-m^2+i\epsilon} + \text{terms regular at } p^2 = m^2,$$

$$\lim_{p^2\to m^2} (p^2-m^2)\, i\Delta_F'(p) = iZ\,. \tag{8.154}$$

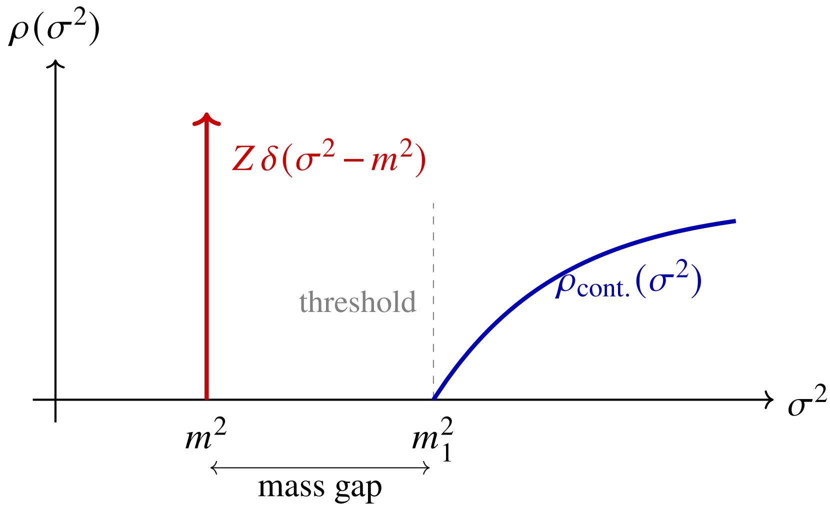


**Figure 8.1.** Schematic structure of the spectral density $\rho(\sigma^2)$ for a theory with a stable one-particle state of mass $m$ separated from the multi-particle continuum by a mass gap. The isolated $\delta$-function at $\sigma^2 = m^2$ carries weight equal to the wave-function renormalization constant $Z$; the smooth distribution starting at the threshold $\sigma^2 = m_1^2$ encodes the multi-particle intermediate states.

This is precisely the form of the propagator that enters the LSZ reduction formula, derived later in this chapter. There, the procedure of amputating external legs and taking the on-shell limit $p^2 \to m^2$ for each external particle picks out exactly the residue of the isolated one-particle pole: the multi-particle continuum and any other analytic background are less singular on the mass shell and are killed by the $(p^2 - m^2)$ amputation factor. As we shall see, each external line of an $n$-particle $S$-matrix element therefore acquires a wave-function renormalization factor $\sqrt{Z}$, which is the practical role played by $Z$ in scattering amplitudes.

**Perturbative (1PI) characterization.** In perturbation theory the same propagator is more conveniently re-expressed in terms of the *one-particle-irreducible* (1PI) self-energy $\Pi(p^2)$, defined as the sum of all 1PI diagrams contributing to the two-point function (the systematic treatment for QED is given in Chapter 6, where the analogous object is $\Pi^{\mu\nu}(k)$ for the photon and $\Sigma(\not{p})$ for the electron). Resumming the chain of 1PI insertions via the Dyson equation yields

$$i\Delta'_F(p) \;=\; \frac{i}{p^2 - m_0^2 - \Pi(p^2) + i\epsilon}, \tag{8.155}$$

where $m_0$ is the bare mass appearing in the Lagrangian (8.1). Requiring that this expression match the spectral form near the physical pole gives two on-shell renormalization conditions:

$$\underbrace{m^2 - m_0^2 - \Pi(m^2) = 0,}_{\text{mass renormalization}} \qquad \underbrace{Z^{-1} = 1 - \Pi'(m^2)}_{\text{field-strength renormalization}} \quad . \tag{8.156}$$

*Small derivation.* Define the inverse propagator $D(p^2) \equiv p^2 - m_0^2 - \Pi(p^2)$, so that $i\Delta'_F(p) = i/D(p^2)$. For the propagator to have a simple pole at the physical mass $p^2 = m^2$, the function $D(p^2)$ must vanish at $p^2 = m^2$:

$$D(m^2) = m^2 - m_0^2 - \Pi(m^2) = 0, \tag{8.157}$$

which is the mass renormalization condition. Taylor-expanding $D(p^2)$ around the physical pole,

$$\begin{aligned} D(p^2) &= (p^2 - m^2)\, D'(m^2) + \mathcal{O}\big((p^2 - m^2)^2\big) \\ &= (p^2 - m^2)\left[1 - \Pi'(m^2)\right] + \mathcal{O}\big((p^2 - m^2)^2\big), \end{aligned} \tag{8.158}$$

the propagator near the pole reads

$$i\Delta'_F(p) \xrightarrow[p^2 \to m^2]{} \frac{i}{(p^2 - m^2)\,[1 - \Pi'(m^2)]} = \frac{i/[1 - \Pi'(m^2)]}{p^2 - m^2}. \tag{8.159}$$

Comparing with the spectral form $i\Delta'_F(p) \to iZ/(p^2 - m^2)$ from (8.154) identifies the residue as $Z = 1/[1 - \Pi'(m^2)]$, i.e. the field-strength renormalization condition $Z^{-1} = 1 - \Pi'(m^2)$.

The first condition states that the radiative correction $\Pi(m^2)$ shifts the bare mass $m_0$ to the physical mass $m$; the second states that the slope of the self-energy at the on-shell point rescales the field–particle overlap, determining $Z$. Different sign conventions for $\Pi$ (e.g. $\Pi \leftrightarrow -\Pi$, or absorbing factors of $i$) change the signs in (8.155)–(8.156), but not the invariant content: the exact propagator has a simple pole at $p^2 = m^2$ and $Z$ is its residue.

In an on-shell renormalization scheme one chooses the renormalized field $\varphi_R = Z^{-1/2}\varphi$ so that the renormalized propagator has unit residue at $p^2 = m^2$; in a generic (e.g. minimal-subtraction) scheme the finite residue remains and reappears as an external-leg factor in scattering amplitudes. The perturbative constant $Z$ defined by (8.156) and the non-perturbative spectral weight in (8.149) should not be identified automatically: they coincide only once a normalization convention for the field has been specified.

### The wave function renormalization constant $Z$

By the assumed structure of the spectrum, $\rho(\sigma^2)$ at $\sigma^2 = m^2$ receives contributions only from the one-particle states. Using (8.66):

$$(2\pi)^3 \int d^3p\, \delta^{(4)}(p - q)\, \frac{Z}{(2\pi)^3 2\omega_p} = Z\,\delta(q^2 - m^2)\,\theta(q^0). \tag{8.160}$$

Separating this contribution:

$$\Delta'(x-x') = Z\,\Delta(x-x';m) \\ + \int_{m_1^2}^{\infty} d\sigma^2\,\rho(\sigma^2)\,\Delta(x-x';\sigma)\,, \tag{8.161}$$

where $m_1^2$ is the threshold of the continuum in the quantum-number sector selected by the field. If a two-particle state in that sector is allowed, then $m_1^2 = 4m^2$; if symmetries forbid that channel, the first threshold is higher.

Taking a time derivative and setting $t' = t$, we use the canonical commutation relations:

$$\lim_{t'\to t} i\frac{\partial}{\partial t}\Delta'(x-x') = \langle 0|[\dot\varphi(\vec{x},t),\varphi(\vec{x}',t)]|0\rangle \\ = -i\delta^{(3)}(\vec{x}-\vec{x}')\,. \tag{8.162}$$

Since the same relation holds for the free $\Delta$ function:

$$\boxed{1 = Z + \int_{m_1^2}^{\infty} \rho(\sigma^2)\,d\sigma^2}\,. \tag{8.163}$$

Together with the positivity of $\rho(\sigma^2)$, this proves:

$$\boxed{0 < Z \le 1}\,. \tag{8.164}$$

The equality $Z = 1$ occurs only when the continuum integral in (8.163) vanishes in this channel. If the field has nonzero overlap with continuum states, then $Z < 1$. Physically, the amplitude for $\varphi(x)$ to produce the isolated one-particle state is reduced because some spectral weight belongs to multiparticle states.

### Analytic structure and dispersion relation

The spectral representation (8.153) reveals the analytic structure of the exact propagator as a function of the complex variable $p^2$, a viewpoint emphasized by Peskin–Schroeder [6]:

- An **isolated simple pole** at $p^2 = m^2$, with residue $Z$, corresponding to the stable one-particle state.
- A **branch cut** along the real axis for $p^2 \ge m_1^2$, arising from the continuous integral of poles in the second term of (8.153). The threshold is set by the lightest multiparticle state that has the same conserved quantum numbers as the *field insertion*, i.e. the operator placed inside the correlation function (here $\varphi(x)$ in $\langle 0|T(\varphi(x)\varphi(x'))|0\rangle$). Only states that $\varphi(0)|0\rangle$ can actually produce

contribute to the spectral sum; more generally, replacing $\varphi$ with a composite local operator $\mathcal{O}(x)$ would change the threshold according to the quantum numbers of $\mathcal{O}$ (for example, an even operator $\mathcal{O} = \varphi^2$ in a $\mathbb{Z}_2$-symmetric theory has a two-particle threshold at $2m$, while $\varphi$ itself has a three-particle threshold at $3m$).

**The role of the $i\epsilon$ prescription.** There are two equivalent ways to view the propagator as a function of $p^2$, and it is useful to be explicit about both since they share the symbol "$i\epsilon$" in different roles.

*View (a): singularities just below the real axis.* Take the spectral form (8.153) literally, with $\epsilon > 0$ small but finite in each denominator $1/(p^2 - \sigma^2 + i\epsilon)$. The singularities of this expression sit at $p^2 = \sigma^2 - i\epsilon$, i.e. infinitesimally *below* the real $p^2$ axis. For real $p^2$ the propagator is then a well-defined distribution; no further shift is needed.

*View (b): boundary value of an analytic function with singularities on the axis.* Treat $\Delta'_F$ as an analytic function of a complex variable $z = p^2$ defined by

$$\Delta'_F(z) = \int_0^\infty d\sigma^2 \, \frac{\rho(\sigma^2)}{z - \sigma^2}, \tag{8.165}$$

*without* any $i\epsilon$ in the denominator. Its singularities now sit *on* the real axis: the pole at $z = m^2$ and the branch cut $z \in [m_1^2, \infty)$. The function is analytic in the upper half-plane and in the lower half-plane separately, but not on the real axis at the singularities. The physical Feynman propagator is then defined as the boundary value of $\Delta'_F(z)$ on the real axis approached *from above*,

$$\Delta'_F(p^2)\big|_{\text{physical}} = \lim_{\epsilon\to 0^+} \Delta'_F(p^2 + i\epsilon), \tag{8.166}$$

where the $+i\epsilon$ now appears in the *argument* of $\Delta'_F$ rather than in the denominator: it is the prescription "evaluate at a point slightly in the upper half-plane and shrink towards the axis".

*Equivalence of the two views.* Plugging $z = p^2 + i\epsilon$ into the formula of view (b) immediately reproduces view (a):

$$\Delta'_F(p^2 + i\epsilon) = \int_0^\infty d\sigma^2 \, \frac{\rho(\sigma^2)}{(p^2 + i\epsilon) - \sigma^2} = \int_0^\infty d\sigma^2 \, \frac{\rho(\sigma^2)}{p^2 - \sigma^2 + i\epsilon}. \tag{8.167}$$

So the two "$+i\epsilon$"s denote the same physical content — in view (a) it sits inside the denominator regulating each pole, in view (b) it parametrizes the direction of approach to the real axis from above. Throughout we shall use the same symbol $\epsilon$ for both. Approaching the real axis from *below* ($-i\epsilon$ in the argument, equivalently in the denominators) would instead produce the complex conjugate of the Feynman propagator, called the *Dyson* or *anti-time-ordered* propagator,[4] named so because its time-ordering convention is reversed. The two boundary values differ on the cut by the *discontinuity*

$$\mathrm{Disc}\,\Delta'_F(p^2) \;\equiv\; \Delta'_F(p^2+i\epsilon)-\Delta'_F(p^2-i\epsilon)\,. \tag{8.168}$$

**Recovering $\rho$ from the discontinuity.** The key analytic identity is the *Sokhotski–Plemelj formula* applied to a real variable $x$:

$$\frac{1}{x\pm i\epsilon} \;=\; \mathcal{P}\frac{1}{x} \;\mp\; i\pi\,\delta(x)\,, \tag{8.169}$$

where $\mathcal{P}$ denotes the Cauchy principal value.[5] Subtracting the two prescriptions cancels the principal value and leaves only the $\delta$:

$$\frac{1}{x+i\epsilon} - \frac{1}{x-i\epsilon} \;=\; -2\pi i\,\delta(x)\,. \tag{8.170}$$

[4] The Feynman propagator is the vacuum expectation value of the *time-ordered* product

$$i\Delta_F(x-y)=\langle 0|T[\varphi(x)\varphi(y)]|0\rangle\,,\qquad T[\varphi(x)\varphi(y)]=\theta(t_x-t_y)\,\varphi(x)\varphi(y)+\theta(t_y-t_x)\,\varphi(y)\varphi(x)\,,$$

which orders the operators with the latest time on the left. The *anti-time-ordered* propagator is defined dually, with the *earliest* time on the left:

$$i\Delta_{\bar F}(x-y)=\langle 0|\bar T[\varphi(x)\varphi(y)]|0\rangle\,,\qquad \bar T[\varphi(x)\varphi(y)]=\theta(t_y-t_x)\,\varphi(x)\varphi(y)+\theta(t_x-t_y)\,\varphi(y)\varphi(x)\,.$$

Physically, $T$ encodes forward propagation of positive-frequency modes (particles) and backward propagation of negative-frequency modes (antiparticles); $\bar T$ does the opposite. In momentum space this swaps the $+i\epsilon$ in each Feynman propagator for $-i\epsilon$, which is why the anti-time-ordered propagator is the complex conjugate of the Feynman one along the real $p^2$ axis. Both propagators appear naturally in the closed-time-path (Schwinger–Keldysh) formalism for non-equilibrium QFT, where the time contour runs forward and then backward in time, attaching $T$ to the forward branch and $\bar T$ to the backward one. The name "Dyson" refers to F. J. Dyson's early use of this object in covariant perturbation theory.

[5] The principal-value distribution $\mathcal{P}(1/x)$ is *not* a function (the function $1/x$ diverges at $x=0$) but a tempered distribution, defined by its action on a test function $f$ as

$$\langle \mathcal{P}\tfrac{1}{x}, f\rangle \;\equiv\; \lim_{\delta\to 0^+}\left[\int_{-\infty}^{-\delta}\frac{f(x)}{x}\,dx+\int_{\delta}^{\infty}\frac{f(x)}{x}\,dx\right] \;=\; \int_{-\infty}^{\infty}\frac{f(x)-f(0)}{x}\,dx\,.$$

The two separate integrals diverge as $-\ln\delta$ but with opposite signs, so their sum is finite: the $1/x$ singularity is removed by the symmetric subtraction. Likewise, the Sokhotski–Plemelj identity (8.169) is an equality between *distributions*, meaning that for every test function $f$,

$$\lim_{\epsilon\to 0^+}\int\frac{f(x)}{x\pm i\epsilon}\,dx \;=\; \mathcal{P}\int\frac{f(x)}{x}\,dx \;\mp\; i\pi\, f(0)\,.$$

Heuristically, writing $1/(x\pm i\epsilon)=(x\mp i\epsilon)/(x^2+\epsilon^2)$ splits the limit into a real part $x/(x^2+\epsilon^2)\to\mathcal{P}(1/x)$ and an imaginary part $\mp\epsilon/(x^2+\epsilon^2)\to\mp\pi\,\delta(x)$ (a Lorentzian of width $\epsilon$ collapsing to a $\pi$-normalized delta).

Applying this to every denominator in the spectral integral (8.153), and noting that for $p^2$ on the cut ($p^2 \geq m_1^2 > m^2$) the isolated pole term contributes $\delta(p^2 - m^2) = 0$,

$$\begin{aligned} \operatorname{Disc}\Delta'_F(p^2) &= \int_{m_1^2}^{\infty} d\sigma^2\, \rho_{\text{cont.}}(\sigma^2) \left[\frac{1}{p^2-\sigma^2+i\epsilon} - \frac{1}{p^2-\sigma^2-i\epsilon}\right] \\ &= \int_{m_1^2}^{\infty} d\sigma^2\, \rho_{\text{cont.}}(\sigma^2)\,(-2\pi i)\,\delta(p^2-\sigma^2) \\ &= -2\pi i\,\rho(p^2)\,, \qquad p^2 \geq m_1^2\,. \end{aligned} \tag{8.171}$$

Furthermore, because $\rho$ is real, the propagator satisfies the Schwarz reflection $\Delta'_F(p^2 - i\epsilon) = [\Delta'_F(p^2 + i\epsilon)]^*$,[6,7] so the discontinuity is purely imaginary:

$$\operatorname{Disc}\Delta'_F(p^2) = \Delta'_F(p^2+i\epsilon) - [\Delta'_F(p^2+i\epsilon)]^* = 2i\,\operatorname{Im}\Delta'_F(p^2+i\epsilon)\,. \tag{8.172}$$

Equating the two expressions for $\operatorname{Disc}\Delta'_F$ gives the announced inversion formula:

$$\boxed{\rho(p^2) \;=\; -\frac{1}{\pi}\operatorname{Im}\Delta'_F(p^2+i\epsilon)\,, \qquad p^2 \geq m_1^2}\,. \tag{8.173}$$

**Dispersion-relation reading.** The Källén–Lehmann formula (8.153) can now be read as a *dispersion relation*. Factoring out the conventional overall $i$ in the time-ordered two-point function, the analytic function $\Delta'_F$ is expressed at arbitrary complex $p^2$ as an integral over its own imaginary part along the cut, plus the isolated pole,

$$\Delta'_F(p^2) \;=\; \frac{Z}{p^2-m^2+i\epsilon} + \int_{m_1^2}^{\infty}\frac{d\sigma^2}{\pi}\,\frac{-\operatorname{Im}\Delta'_F(\sigma^2+i\epsilon)}{p^2-\sigma^2+i\epsilon}\,. \tag{8.174}$$

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[6]The *Schwarz reflection principle* is a theorem of complex analysis: if a function $f(z)$ is analytic in a domain $D$ containing a segment $I$ of the real axis, and is *real on* $I$, then $f$ extends analytically to the mirror image of $D$ across the real axis via $f(\bar{z}) = \overline{f(z)}$. Applied to our propagator: from the spectral form $\Delta'_F(z) = \int_0^\infty d\sigma^2\,\rho(\sigma^2)/(z-\sigma^2)$ with $\rho$ real and supported on $\sigma^2 \geq m^2$, $\Delta'_F(p^2)$ is manifestly *real* for real $p^2 < m^2$ (every denominator $p^2 - \sigma^2$ is real and negative there). Hence the reflection principle applies on the segment $(-\infty, m^2)$, and analytic continuation yields $\Delta'_F(\bar{z}) = \overline{\Delta'_F(z)}$ throughout the analyticity domain. Specializing to $z = p^2 + i\epsilon$ gives $\Delta'_F(p^2 - i\epsilon) = [\Delta'_F(p^2+i\epsilon)]^*$.

[7]*Uniqueness of the analytic continuation.* The reflection above relies on the *identity theorem* for analytic functions: if two functions are analytic on a connected domain $D$ and agree on a set with an accumulation point in $D$ (e.g. a segment of $\mathbb{R}$), they coincide on all of $D$. In our setup, $D = \mathbb{C}\setminus\big(\{m^2\}\cup[m_1^2,\infty)\big)$ is connected, and the candidate reflection $\tilde{f}(z) := \overline{\Delta'_F(\bar{z})}$ is analytic on $D$ and agrees with $\Delta'_F$ on the real segment $(-\infty, m^2) \subset D$ where the latter is real; the identity theorem then forces $\tilde{f} \equiv \Delta'_F$ throughout $D$, which is the Schwarz reflection. The same theorem also implies an important *negative* statement: analytic continuation is unique *only as long as one stays inside* $D$. The cut $[m_1^2, \infty)$ is precisely where analyticity fails, so one cannot continue *across* the cut as a single-valued function: going from $p^2 + i\epsilon$ to $p^2 - i\epsilon$ *around* the cut (through $D$) gives the complex conjugate value, while going *straight through* the cut would land on a different Riemann sheet. The mismatch between the two routes is exactly the discontinuity $\operatorname{Disc}\Delta'_F = -2\pi i\rho$.

This is the same conclusion that Cauchy's integral formula would give by deforming the contour around the analytic structure of $\Delta'_F$ in the complex $p^2$ plane: the full function is determined by its singularities — the pole position and residue $(m^2, Z)$ and the discontinuity (i.e. the imaginary part) along the cut.

**Physical interpretation: cuts come from real intermediate states.** The inversion formula $\rho(p^2) = -\pi^{-1}\,\mathrm{Im}\,\Delta'_F(p^2 + i\epsilon)$ has a very transparent physical meaning, which we now unpack.

**In words.** The imaginary part of the exact propagator is non-zero *only* at invariant masses $\sqrt{p^2}$ for which physical, on-shell, multi-particle intermediate states actually exist. Every time we cross a new kinematical threshold $\sqrt{p^2} = m_a + m_b + \cdots$ at which a multi-particle channel $|X\rangle = |a, b, \ldots\rangle$ becomes accessible, a new piece switches on in $\mathrm{Im}\,\Delta'_F(p^2)$ and therefore in $\rho(p^2)$. Equivalently: **branch cuts of the propagator are built out of real intermediate states**.

**Where this comes from: unitarity.** This is not coincidence — it follows directly from $S$-matrix unitarity. Let us derive it in three short steps.

*Step 1: from unitarity to an operator identity for $T$.* Write the $S$ matrix as $S = \mathbb{1} + iT$, where $T$ encodes all genuine scattering. Unitarity $S^\dagger S = \mathbb{1}$ becomes

$$\mathbb{1} = (\mathbb{1} - iT^\dagger)(\mathbb{1} + iT) = \mathbb{1} + i(T - T^\dagger) + T^\dagger T\,,$$

that is, after cancelling the $\mathbb{1}$,

$$-i\big(T - T^\dagger\big) = T^\dagger T \qquad \Longleftrightarrow \qquad 2\,\mathrm{Im}\,T = T^\dagger T\,. \tag{8.175}$$

*Step 2: take matrix elements between identical states.* Sandwiching (8.175) between the same asymptotic state $|i\rangle$ on both sides, and inserting a complete set $\sum_X |X\rangle\langle X|$ in the right-hand side:

$$2\,\mathrm{Im}\langle i|T|i\rangle \;=\; \langle i|T^\dagger T|i\rangle \;=\; \sum_X |\langle X|T|i\rangle|^2\,.$$

Recognizing $\langle f|T|i\rangle$ as the invariant scattering amplitude $\mathcal{M}(i \to f)$ (up to an overall $(2\pi)^4\delta^{(4)}(\cdot)$ enforcing energy–momentum conservation), this is the *optical theorem*:

$$\boxed{2\,\mathrm{Im}\,\mathcal{M}(\mathrm{i} \to \mathrm{i}) \;=\; \sum_X \int d\Pi_X \big|\mathcal{M}(\mathrm{i} \to X)\big|^2\,,} \tag{8.176}$$

the imaginary part of any *forward* amplitude equals the sum over *all* final states $X$ into which the system can transition, integrated over their Lorentz-invariant phase space $d\Pi_X$.

*Step 3: specialize to the two-point function.* The same unitarity argument applied to a single field operator $\varphi(0)$ acting on the vacuum (rather than to a full scattering amplitude) gives precisely the relation (8.130): inserting $\sum_X |X\rangle\langle X|$ between $\varphi(0)$ and $\varphi(0)^\dagger$ in the spectral integrand produces the spectral density $\rho(p^2)$. One thus arrives at the equivalence

$$\operatorname{Im}\Delta_F'(p^2) \neq 0 \quad \Longleftrightarrow \quad \text{there exists a physical state } |X\rangle \text{ with } p_X^2 = p^2\,. \tag{8.177}$$

**Putting it together.** The "analytic structure" picture of the propagator (poles + cuts on the real $p^2$ axis) is simply unitarity expressed in the language of complex analysis:

- The isolated pole at $p^2 = m^2$ comes from the one-particle intermediate state $|p\rangle$ — a $\delta(\sigma^2 - m^2)$ in $\rho$.
- The branch cut starting at $p^2 = m_1^2$ comes from multi-particle intermediate states. Below $m_1^2$ no on-shell multi-particle state exists in this channel, so $\rho_{\text{cont.}}(\sigma^2) = 0$ and the propagator is analytic there.
- Each higher threshold $(m_a + m_b)^2$, $(m_a + m_b + m_c)^2$, ... where a heavier channel opens adds another onset to $\operatorname{Im}\Delta_F'$, and hence to $\rho(\sigma^2)$.

**Perturbative version.** In perturbation theory the same content is recovered diagrammatically: the imaginary part of the exact propagator is controlled by the imaginary part of the 1PI self-energy via the Dyson form (8.155), and $\operatorname{Im}\Pi(p^2)$ at a given order is computed by "cutting" the contributing Feynman diagrams along all sets of internal propagators that can be simultaneously put on shell. The systematic statement is provided by the *Cutkosky cutting rules*, the diagrammatic incarnation of the optical theorem; their full statement, mathematical justification, and worked examples (bubble and sunset diagrams) are developed in Appendix H.

**Box 8.1 The propagator encodes the spectrum**

The exact propagator, viewed as a function of complex $p^2$, is an analytic function whose singularities encode the spectrum of the theory in the relevant quantum-number sector. The pole gives the particle mass and the overlap of the field with the one-particle state (through $Z$); the branch cut gives the multiparticle thresholds and, through $\rho(\sigma^2)$, the density of accessible intermediate states weighted by their coupling to the field. This "pole-plus-cut" picture, central to the Peskin–Schroeder presentation, provides a powerful organizing principle: statements about propagation and scattering can often be traced back to the analytic structure of two-point functions.

### Spectral representation for the Dirac field

The Källén–Lehmann analysis extends to the Dirac field [1, 6]. Consider the exact fermion two-point function

$$iS_F'(x-y) \equiv \langle 0|T\big(\psi(x)\bar{\psi}(y)\big)|0\rangle\,. \tag{8.178}$$

Inserting a complete set of states and using translation invariance as in the scalar case, Lorentz covariance constrains the momentum-space propagator of a parity-conserving theory to the general form

$$iS_F'(p) = i\int_0^\infty d\sigma^2\,\frac{\rho_V(\sigma^2)\,\not{p}+\rho_S(\sigma^2)\,\sigma}{p^2-\sigma^2+i\epsilon}\,, \tag{8.179}$$

where $\rho_V$ and $\rho_S$ obey positivity constraints inherited from Hilbert-space positivity, but they should not be treated as two arbitrary independent positive scalar densities. The two Lorentz structures—$\not{p}$ and $\sigma\cdot\mathbb{1}$—are the only ones available for the two-point function of a parity-conserving Dirac field. In a parity-violating theory, additional chiral structures can appear.

Just as in the scalar case, the one-particle state contributes a delta-function to both spectral densities, producing an isolated pole:

$$iS_F'(p) \xrightarrow[p^2\to m^2]{} \frac{iZ_2(\not{p}+m)}{p^2-m^2+i\epsilon} + \text{terms regular at } p^2=m^2\,, \tag{8.180}$$

where $m$ is the physical fermion mass and $Z_2$ is the residue of the isolated fermion pole. In a massive theory with canonical normalization one has $0<Z_2\le 1$, with strict inequality only when continuum states carry spectral weight in the same channel.

The perturbative expression for the exact fermion propagator is most usefully written in terms of the one-particle-irreducible self-energy $\Sigma(p)$:

$$iS_F'(p) = \frac{i}{\not{p}-m_0-\Sigma(p)+i\epsilon}\,. \tag{8.181}$$

Lorentz covariance and the discrete symmetries of a parity-conserving theory (such as massive QED with an infrared regulator) restrict $\Sigma$ to the form $\Sigma(p) = A(p^2)\not{p}+B(p^2)$. It is useful to write the inverse propagator as

$$\begin{aligned} S_F'^{-1}(p) &= \not{p}\,C(p^2)-D(p^2),\\ C(p^2) &= 1-A(p^2), \qquad D(p^2) = m_0+B(p^2). \end{aligned} \tag{8.182}$$

The physical mass and the residue are then fixed by

$$D(m^2) = m\,C(m^2), \qquad Z_2^{-1} = C(m^2)+2m^2C'(m^2)-2mD'(m^2), \tag{8.183}$$

where primes denote derivatives with respect to $p^2$. The first condition fixes the physical mass as the pole of the full propagator. To see the second, write $\not{p}=m+\delta$ near the positive-energy pole, so that $p^2-m^2=(\not{p}-m)(\not{p}+m)=2m\,\delta+\mathcal{O}(\delta^2)$ on the pole projector. Expanding

$$\not{p}\,C(p^2)-D(p^2) = \left[C(m^2)+2m^2C'(m^2)-2mD'(m^2)\right](\not{p}-m)+\mathcal{O}\big((\not{p}-m)^2\big)$$