after using $D(m^2) = mC(m^2)$ gives the coefficient that inverts to the pole residue Z_2. Together with a charge-renormalization condition, these equations define the on-shell renormalization scheme for a massive theory (see Chapter 6).

In a massive theory with a mass gap, the analytic structure mirrors the scalar case: the exact fermion propagator has an isolated pole at $p^2 = m^2$ and a branch cut starting at the lowest multiparticle threshold in the same charge sector. In QED the photon is massless, so the charged fermion is accompanied by arbitrarily soft photons and the branch point reaches the would-be pole at $p^2 = m^2$. This is the infrared reason why the simple isolated-pole LSZ formalism must be modified for charged external states in QED.

8.7 The LSZ reduction formula for scalar fields

We now develop the central result of this chapter: the **reduction formula** of Lehmann, Symanzik, and Zimmermann, which expresses S-matrix elements in terms of vacuum expectation values of time-ordered products of field operators.

Removing a particle from the in-state

Consider the S-matrix element

$$S_{\beta,\alpha p} = \langle \beta \text{ out} | \alpha p \text{ in} \rangle , \tag{8.184}$$

where $|\alpha p \text{ in}\rangle$ is the in-state with an assemblage α of particles plus an additional particle with momentum p. Using (8.67):

$$\begin{aligned} \langle \beta \text{ out} | \alpha p \text{ in} \rangle &= \langle \beta \text{ out} | a^\dagger_{\text{in}}(p) | \alpha \text{ in} \rangle \\ &= \langle \beta \text{ out} | a^\dagger_{\text{out}}(p) | \alpha \text{ in} \rangle \\ &\quad + \langle \beta \text{ out} | a^\dagger_{\text{in}}(p) - a^\dagger_{\text{out}}(p) | \alpha \text{ in} \rangle . \end{aligned} \tag{8.185}$$

The first term equals $\langle \beta - p \text{ out} | \alpha \text{ in} \rangle$, representing an out-state with particle p removed (if present in β); otherwise this term vanishes.

For the second term, we use (8.26)–(8.28) and the asymptotic conditions. Recall from (8.28) that the annihilation operator is extracted via

$$a_{\text{in}}(k) = i \int d^3x \, f_k^*(x) \overleftrightarrow{\partial_0} \varphi_{\text{in}}(x) . \tag{8.186}$$

Taking the Hermitian conjugate (and using $f_k^{**} = f_k$):

$$a^\dagger_{\text{in}}(k) = -i \int d^3x \, f_k(x) \overleftrightarrow{\partial_0} \varphi_{\text{in}}(x) , \tag{8.187}$$

with the same relation holding for the out-operators with $\varphi_{\text{in}} \to \varphi_{\text{out}}$. Subtracting:

$$\begin{aligned}&\langle\beta \text{ out}|a^\dagger_{\text{in}}(p) - a^\dagger_{\text{out}}(p)|\alpha \text{ in}\rangle \\ &\quad = -i\int d^3x\, f_p(x) \overleftrightarrow{\partial_0} \langle\beta \text{ out}|[\varphi_{\text{in}}(x) - \varphi_{\text{out}}(x)]|\alpha \text{ in}\rangle\,. \end{aligned} \tag{8.188}$$

Applying the weak asymptotic condition, we replace $\varphi_{\text{in}}(\vec{x},x^0)$ by $(1/\sqrt{Z})\varphi(\vec{x},x^0)$ in the limit $x^0 \to -\infty$, and $\varphi_{\text{out}}(\vec{x},x^0)$ by $(1/\sqrt{Z})\varphi(\vec{x},x^0)$ as $x^0 \to +\infty$:

$$\begin{aligned}\langle\beta \text{ out}|\alpha p \text{ in}\rangle &= \langle\beta - p \text{ out}|\alpha \text{ in}\rangle \\ &\quad + \frac{i}{\sqrt{Z}}\Big(\lim_{x^0\to+\infty} - \lim_{x^0\to-\infty}\Big) \\ &\qquad \times \int d^3x\, f_p(\vec{x},x^0) \overleftrightarrow{\partial_0} \langle\beta \text{ out}|\varphi(\vec{x},x^0)|\alpha \text{ in}\rangle\,. \end{aligned} \tag{8.189}$$

Converting to a four-dimensional integral

The time limits can be incorporated into a four-dimensional volume integral using the identity:

$$\begin{aligned}&\Big(\lim_{x^0\to+\infty} - \lim_{x^0\to-\infty}\Big)\int d^3x\, g_1(x) \overleftrightarrow{\partial_0} g_2(x) \\ &\quad = \int_{-\infty}^{+\infty} d^4x \frac{\partial}{\partial x^0}\Big[g_1(x) \overleftrightarrow{\partial_0} g_2(x)\Big] \\ &\quad = \int d^4x \left[g_1(x)\frac{\partial^2}{\partial(x^0)^2}g_2(x) - \frac{\partial^2 g_1(x)}{\partial(x^0)^2}g_2(x)\right]. \end{aligned} \tag{8.190}$$

Detailed derivation of (8.190). The first equality is the fundamental theorem of calculus applied to the time integration:

$$\Big[\lim_{x^0\to+\infty} - \lim_{x^0\to-\infty}\Big] F(x^0) = \int_{-\infty}^{+\infty} dx^0 \frac{\partial}{\partial x^0} F(x^0)\,, \tag{8.191}$$

with $F(x^0) = \int d^3x\, g_1(x) \overleftrightarrow{\partial_0} g_2(x)$. The second equality is the *temporal integration by parts*, obtained by expanding the bidirectional derivative $g_1 \overleftrightarrow{\partial_0} g_2 = g_1\,\partial_0 g_2 - (\partial_0 g_1)\, g_2$ and applying Leibniz:

$$\begin{aligned}\frac{\partial}{\partial x^0}\big[g_1 \overleftrightarrow{\partial_0} g_2\big] &= \frac{\partial}{\partial x^0}\big[g_1\,\partial_0 g_2 - (\partial_0 g_1)\, g_2\big] \\ &= \underbrace{(\partial_0 g_1)(\partial_0 g_2)}_{\text{cross term}} + g_1\,\partial_0^2 g_2 \\ &\quad - (\partial_0^2 g_1)\, g_2 - \underbrace{(\partial_0 g_1)(\partial_0 g_2)}_{\text{cross term}} \\ &= g_1\,\partial_0^2 g_2 - (\partial_0^2 g_1)\, g_2\,. \end{aligned} \tag{8.192}$$

The cross terms $(\partial_0 g_1)(\partial_0 g_2)$ cancel: this is precisely the Wronskian structure encoded by $\overleftrightarrow{\partial_0}$, and it is what makes the resulting volume integral amenable to the Klein–Gordon equation in the next step. Observe that, after this manipulation, both time derivatives have been concentrated on a single factor at a time, which can then be traded for spatial derivatives via $\partial_0^2 = \nabla^2 - m^2$ (for a free wave function) or kept on the matrix element.

Applying (8.190) to (8.189), with $g_1 = f_p$ and $g_2 = \langle \beta \text{ out}|\varphi(x)|\alpha \text{ in}\rangle$, we obtain:

$$\begin{aligned}\langle \beta \text{ out}|\alpha p \text{ in}\rangle = \langle \beta - p \text{ out}|\alpha \text{ in}\rangle \\ + \frac{i}{\sqrt{Z}} \int d^4x \left[f_p(x) \frac{\partial^2}{\partial (x^0)^2} \langle \beta \text{ out}|\varphi(x)|\alpha \text{ in}\rangle \right. \\ \left. - \frac{\partial^2 f_p(x)}{\partial (x^0)^2} \langle \beta \text{ out}|\varphi(x)|\alpha \text{ in}\rangle \right].\end{aligned} \tag{8.193}$$

Since $f_p(x)$ satisfies the Klein–Gordon equation:

$$\frac{\partial^2 f_p(x)}{\partial (x^0)^2} = (\nabla^2 - m^2) f_p(x)\,, \tag{8.194}$$

we substitute (8.194) into the second term of (8.193):

$$\begin{aligned}= \langle \beta - p \text{ out}|\alpha \text{ in}\rangle \\ + \frac{i}{\sqrt{Z}} \int d^4x \left[f_p(x) \frac{\partial^2}{\partial (x^0)^2} \langle \beta \text{ out}|\varphi(x)|\alpha \text{ in}\rangle \right. \\ \left. - (\nabla^2 - m^2) f_p(x) \cdot \langle \beta \text{ out}|\varphi(x)|\alpha \text{ in}\rangle \right].\end{aligned} \tag{8.195}$$

Now we integrate the $\nabla^2 f_p$ term by parts twice in $\vec{x}$ (boundary terms vanish since the fields fall off at spatial infinity):

$$-\int d^3x\, (\nabla^2 f_p)\, \langle \beta \text{ out}|\varphi|\alpha \text{ in}\rangle = -\int d^3x\, f_p\, \nabla^2 \langle \beta \text{ out}|\varphi|\alpha \text{ in}\rangle\,. \tag{8.196}$$

Substituting back into (8.195):

$$= \langle \beta - p \text{ out}|\alpha \text{ in}\rangle + \frac{i}{\sqrt{Z}} \int d^4x\, f_p(x) \left[\frac{\partial^2}{\partial (x^0)^2} - \nabla^2 + m^2 \right] \langle \beta \text{ out}|\varphi(x)|\alpha \text{ in}\rangle\,. \tag{8.197}$$

Recognising the d'Alembertian $\Box = \partial^2/\partial (x^0)^2 - \nabla^2$, we arrive at:

$$\boxed{\langle \beta \text{ out}|\alpha p \text{ in}\rangle = \langle \beta - p \text{ out}|\alpha \text{ in}\rangle + \frac{i}{\sqrt{Z}} \int d^4x\, f_p(x) (\Box + m^2) \langle \beta \text{ out}|\varphi(x)|\alpha \text{ in}\rangle\,.} \tag{8.198}$$

Removing a particle from the out-state

Let us now remove a particle p' from the assemblage $\beta = \gamma p'$. Since $|\gamma p' \text{ out}\rangle = a^\dagger_{\text{out}}(p')|\gamma \text{ out}\rangle$, taking the Hermitian conjugate gives $\langle \gamma p' \text{ out}| = \langle \gamma \text{ out}|a_{\text{out}}(p')$. Therefore:

$$\langle \gamma p' \text{ out}|\varphi(x)|\alpha \text{ in}\rangle = \langle \gamma \text{ out}|a_{\text{out}}(p')\,\varphi(x)|\alpha \text{ in}\rangle\,. \tag{8.199}$$

We now add and subtract a term with $a_{\text{in}}(p')$ acting to the right:

$$\begin{aligned} &= \langle \gamma \text{ out}|a_{\text{out}}(p')\,\varphi(x)|\alpha \text{ in}\rangle - \langle \gamma \text{ out}|\varphi(x)\,a_{\text{in}}(p')|\alpha \text{ in}\rangle \\ &\quad + \langle \gamma \text{ out}|\varphi(x)\,a_{\text{in}}(p')|\alpha \text{ in}\rangle\,. \end{aligned} \tag{8.200}$$

In the last term, $a_{\text{in}}(p')$ acts on $|\alpha \text{ in}\rangle$. If particle p' is present in the assemblage α, then $a_{\text{in}}(p')|\alpha \text{ in}\rangle = |\alpha - p' \text{ in}\rangle$ (up to normalisation); this gives the *disconnected* contribution $\langle \gamma \text{ out}|\varphi(x)|\alpha - p' \text{ in}\rangle$. Hence:

$$\begin{aligned} &\langle \gamma p' \text{ out}|\varphi(x)|\alpha \text{ in}\rangle \\ &= \langle \gamma \text{ out}|\varphi(x)|\alpha - p' \text{ in}\rangle + \langle \gamma \text{ out}|\big[a_{\text{out}}(p')\,\varphi(x) - \varphi(x)\,a_{\text{in}}(p')\big]|\alpha \text{ in}\rangle\,. \end{aligned} \tag{8.201}$$

Expressing the second term via the asymptotic condition. Using the analogue of (8.187) for the annihilation operator:

$$a_{\text{out}}(p') = i\int d^3y\, f^*_{p'}(y) \overleftrightarrow{\partial_0}\, \varphi_{\text{out}}(y)\,, \qquad a_{\text{in}}(p') = i\int d^3y\, f^*_{p'}(y) \overleftrightarrow{\partial_0}\, \varphi_{\text{in}}(y)\,, \tag{8.202}$$

the second term in (8.201) becomes:

$$\begin{aligned} &\langle \gamma \text{ out}|a_{\text{out}}(p')\,\varphi(x) - \varphi(x)\,a_{\text{in}}(p')|\alpha \text{ in}\rangle \\ &\quad = i\int d^3y\, f^*_{p'}(y) \overleftrightarrow{\partial_0}\, \langle \gamma \text{ out}|\big[\varphi_{\text{out}}(y)\,\varphi(x) - \varphi(x)\,\varphi_{\text{in}}(y)\big]|\alpha \text{ in}\rangle\,. \end{aligned} \tag{8.203}$$

Now we apply the weak asymptotic condition: $\varphi_{\text{out}}(y) \to \varphi(y)/\sqrt{Z}$ as $y^0 \to +\infty$ and $\varphi_{\text{in}}(y) \to \varphi(y)/\sqrt{Z}$ as $y^0 \to -\infty$, inside matrix elements. Consider the ordering, with x held *fixed* (it is an arbitrary but finite spacetime point in the Green function): when $y^0 \to +\infty$, $\varphi_{\text{out}}(y)\,\varphi(x) = T(\varphi(y)\varphi(x))$ since $y^0 > x^0$ (with x fixed); when $y^0 \to -\infty$, $\varphi(x)\,\varphi_{\text{in}}(y) = T(\varphi(y)\varphi(x))$ since $x^0 > y^0$ (again with x fixed). Therefore:

$$\begin{aligned} &\langle \gamma \text{ out}|\big[\varphi_{\text{out}}(y)\,\varphi(x) - \varphi(x)\,\varphi_{\text{in}}(y)\big]|\alpha \text{ in}\rangle \\ &\quad = \frac{1}{\sqrt{Z}}\left(\lim_{y^0\to+\infty} - \lim_{y^0\to-\infty}\right)\langle \gamma \text{ out}|T(\varphi(y)\varphi(x))|\alpha \text{ in}\rangle\,. \end{aligned} \tag{8.204}$$

Converting to a four-dimensional integral. Combining (8.203) and (8.204), and using the same identity (8.190), now with

$$g_1 = f^*_{p'}, \qquad g_2 = Z^{-1/2}\langle \gamma \text{ out}|T(\varphi(y)\varphi(x))|\alpha \text{ in}\rangle \ :$$

$$= \frac{i}{\sqrt{Z}}\int d^4y \left[f^*_{p'}(y)\frac{\partial^2}{\partial(y^0)^2}\langle \gamma \text{ out}|T(\varphi(y)\varphi(x))|\alpha \text{ in}\rangle \right.$$
$$\left. - \frac{\partial^2 f^*_{p'}(y)}{\partial(y^0)^2}\langle \gamma \text{ out}|T(\varphi(y)\varphi(x))|\alpha \text{ in}\rangle \right]. \tag{8.205}$$

Since $f^*_{p'}(y)$ also satisfies the Klein–Gordon equation:

$$\frac{\partial^2 f^*_{p'}(y)}{\partial(y^0)^2} = (\nabla_y^2 - m^2) f^*_{p'}(y)\,, \tag{8.206}$$

we substitute into the second term of (8.205):

$$= \frac{i}{\sqrt{Z}}\int d^4y \left[f^*_{p'}(y)\frac{\partial^2}{\partial(y^0)^2}\langle \gamma \text{ out}|T(\varphi(y)\varphi(x))|\alpha \text{ in}\rangle \right.$$
$$\left. - (\nabla_y^2 - m^2) f^*_{p'}(y)\cdot\langle \gamma \text{ out}|T(\varphi(y)\varphi(x))|\alpha \text{ in}\rangle \right]. \tag{8.207}$$

The $\nabla_y^2 f^*_{p'}$ piece is now *integrated by parts twice* in $\vec{y}$, transferring the Laplacian from the wave function onto the matrix element. Denoting $G(y,x) \equiv \langle \gamma \text{ out}|T(\varphi(y)\varphi(x))|\alpha \text{ in}\rangle$ for brevity:

$$\int d^3y\,(\nabla_y^2 f^*_{p'})\,G = \underbrace{\oint_{\partial V} d\mathbf{S}\cdot(\nabla_y f^*_{p'})\,G}_{=0} - \int d^3y\,(\nabla_y f^*_{p'})\cdot(\nabla_y G)$$
$$= -\underbrace{\oint_{\partial V} d\mathbf{S}\cdot f^*_{p'}\,(\nabla_y G)}_{=0} + \int d^3y\, f^*_{p'}(\nabla_y^2 G)$$
$$= \int d^3y\, f^*_{p'}(y)\,\nabla_y^2 G(y,x)\,. \tag{8.208}$$

Both surface terms at $|\vec{y}| \to \infty$ vanish: for normalizable wave packets this is automatic, and for the sharp plane-wave choice it is understood distributionally, with the usual adiabatic/$i\epsilon$ damping. Substituting (8.208) back into (8.207):

$$= \frac{i}{\sqrt{Z}}\int d^4y\, f^*_{p'}(y)\left[\frac{\partial^2}{\partial(y^0)^2} - \nabla_y^2 + m^2\right]\langle \gamma \text{ out}|T(\varphi(y)\varphi(x))|\alpha \text{ in}\rangle$$
$$= \frac{i}{\sqrt{Z}}\int d^4y\, f^*_{p'}(y)\,(\Box_y + m^2)\,\langle \gamma \text{ out}|T(\varphi(y)\varphi(x))|\alpha \text{ in}\rangle\,. \tag{8.209}$$

It is useful to keep the arrow notation explicit: in the following formula the operator $(\Box_y + m^2)$ acts on the time-ordered matrix element, not on the external wave function $f^*_{p'}(y)$. Only later, when one rewrites the result in momentum space and continues the external momentum off shell before taking the pole residue, does the same differential operator become the usual amputation factor. The final result is:

$$\begin{aligned}\langle \gamma p' \text{ out}|\varphi(x)|\alpha \text{ in}\rangle &= \langle \gamma \text{ out}|\varphi(x)|\alpha - p' \text{ in}\rangle \\ &+ \frac{i}{\sqrt{Z}} \int d^4y\, f^*_{p'}(y)\,(\Box_y + m^2)\,\langle \gamma \text{ out}|T(\varphi(y)\varphi(x))|\alpha \text{ in}\rangle\,. \end{aligned} \tag{8.210}$$

The general reduction formula

Repeatedly applying this technique to remove all particles from the states, we arrive at the vacuum expectation value of a time-ordered product:

$$\begin{aligned}&\langle p_1 \cdots p_n \text{ out}|q_1 \cdots q_m \text{ in}\rangle \\ &= \left(\frac{i}{\sqrt{Z}}\right)^{m+n} \left[\prod_{i=1}^{m} \int d^4x_i\, f_{q_i}(x_i) \overrightarrow{(\Box_{x_i} + m^2)}\right] \\ &\quad \times \langle 0|T(\varphi(y_1)\cdots\varphi(y_n)\varphi(x_1)\cdots\varphi(x_m))|0\rangle \\ &\quad \times \left[\prod_{j=1}^{n} \int d^4y_j\, \overleftarrow{(\Box_{y_j} + m^2)} f^*_{p_j}(y_j)\right], \end{aligned} \tag{8.211}$$

valid when all $p_j \neq q_i$ (forward-scattering terms have been dropped for simplicity).

A note on arrow notation. Throughout this chapter, in LSZ-type formulas with differential operators sandwiched between wave functions and Green functions, the arrows indicate the *direction in which the operator acts*:

- $\overrightarrow{(\Box + m^2)}$ (right-pointing) acts on whatever stands to its *right* — in the formulas above, it acts on the Green function $\langle 0|T(\cdots)|0\rangle$.
- $\overleftarrow{(\Box + m^2)}$ (left-pointing) acts on whatever stands to its *left* — again the Green function.

This convention will be used identically in the Dirac and photon LSZ formulas below, with $(\Box + m^2)$ replaced by the appropriate free wave operator ($i\not{\partial} - m$ for Dirac fields, $\Box$ for the massless photon). The arrows are purely a bookkeeping device: by integration by parts (boundary terms vanish for normalisable wave packets or under the standard $i\epsilon$ prescription), one can freely move the operator onto the external wave function, where it produces the on-shell amputation factor.

The LSZ reduction formula can be written more compactly. The derivation above uses on-shell wave packets, but the momentum-space formula is understood

as a residue prescription: first Fourier transform the Green function with an arbitrary external four-momentum, multiply by the inverse free wave operator, and only then take the limit $p^2 \to m^2$. The normalisation factors are evaluated on the physical mass shell. With this standard convention, inserting the explicit plane waves (8.27) for $f_p(x)$ and $f^*_{p'}(y)$ gives:

$$f_p(x) = \frac{e^{-ip\cdot x}}{\sqrt{(2\pi)^3 2\omega_p}}, \qquad f^*_{p'}(y) = \frac{e^{+ip'\cdot y}}{\sqrt{(2\pi)^3 2\omega_{p'}}}, \tag{8.212}$$

we note the key identity:

$$(\Box + m^2)e^{\pm ip\cdot x} = (-p^2 + m^2)e^{\pm ip\cdot x}, \tag{8.213}$$

which follows from $\partial_\mu e^{\pm ip\cdot x} = \pm ip_\mu e^{\pm ip\cdot x}$ and hence $\Box\, e^{\pm ip\cdot x} = (\pm ip_\mu)(\pm ip^\mu)e^{\pm ip\cdot x} = -p^2\, e^{\pm ip\cdot x}$.

Let us show explicitly how this works. Consider a typical *incoming* factor from (8.211):

$$\int d^4x_i\, f_{q_i}(x_i)\,(\Box_{x_i} + m^2)\,\langle 0|T(\cdots\varphi(x_i)\cdots)|0\rangle\,. \tag{8.214}$$

We integrate by parts *twice*, transferring $\Box_{x_i}$ from the Green's function onto $f_{q_i}(x_i)$. Let us denote $G(x_i) \equiv \langle 0|T(\cdots\varphi(x_i)\cdots)|0\rangle$ for brevity. Writing $\Box = \partial_\mu\partial^\mu$, the first integration by parts gives:

$$\int d^4x_i\, f_{q_i}\,(\partial_\mu\partial^\mu G) = \big[\text{boundary}\big] - \int d^4x_i\,(\partial_\mu f_{q_i})\,(\partial^\mu G)\,, \tag{8.215}$$

and the second:

$$-\int d^4x_i\,(\partial_\mu f_{q_i})\,(\partial^\mu G) = -\big[\text{boundary}\big] + \int d^4x_i\,(\partial_\mu\partial^\mu f_{q_i})\,G = \int d^4x_i\,(\Box f_{q_i})\,G\,. \tag{8.216}$$

For normalizable wave packets the boundary terms vanish; for the sharp plane-wave formula this step is understood distributionally, with the usual adiabatic/$i\epsilon$ damping. Calling the left-hand side of (8.214) I_i, and including the mass term (which requires no integration by parts), we obtain:

$$\begin{aligned} I_i &= \int d^4x_i\,\big[(\Box_{x_i} + m^2)\,f_{q_i}(x_i)\big]\,\langle 0|T(\cdots\varphi(x_i)\cdots)|0\rangle \\ &= \int d^4x_i\,\frac{(\Box_{x_i} + m^2)\,e^{-iq_i\cdot x_i}}{\sqrt{(2\pi)^3 2\omega_{q_i}}}\,\langle 0|T(\cdots\varphi(x_i)\cdots)|0\rangle \\ &= \frac{m^2 - q_i^2}{\sqrt{(2\pi)^3 2\omega_{q_i}}}\int d^4x_i\, e^{-iq_i\cdot x_i}\,\langle 0|T(\cdots\varphi(x_i)\cdots)|0\rangle\,. \end{aligned}$$

The remaining integral over x_i is precisely the Fourier transform of the Green's function at momentum q_i. Similarly, for a typical *outgoing* factor, the arrow notation means that the Klein–Gordon operator initially acts on the Green function to its left; integrating by parts moves it onto the outgoing plane wave:

$$\begin{aligned}&\int d^4y_j\,\langle 0|T(\cdots\varphi(y_j)\cdots)|0\rangle\,(\Box_{y_j}+m^2)\,f^*_{p_j}(y_j)\\&\quad=\frac{m^2-p_j^2}{\sqrt{(2\pi)^3 2\omega_{p_j}}}\int d^4y_j\,e^{+ip_j\cdot y_j}\,\langle 0|T(\cdots\varphi(y_j)\cdots)|0\rangle\,.\end{aligned}\tag{8.217}$$

This gives the same amputation factor $(m^2-p_j^2)$, now associated with the outgoing external pole.

Combining all m incoming and n outgoing factors, the reduction formula (8.211) becomes:

$$\begin{aligned}&\langle p_1\cdots p_n\ \text{out}|q_1\cdots q_m\ \text{in}\rangle\\&=\left(\frac{i}{\sqrt{Z}}\right)^{m+n}\prod_{i=1}^{m}\frac{m^2-q_i^2}{\sqrt{(2\pi)^3 2\omega_{q_i}}}\prod_{j=1}^{n}\frac{m^2-p_j^2}{\sqrt{(2\pi)^3 2\omega_{p_j}}}\\&\qquad\times\widetilde{G}(p_1,\ldots,p_n;q_1,\ldots,q_m)\,,\end{aligned}\tag{8.218}$$

where $\widetilde{G}$ is the Fourier transform of the full $(m+n)$-point Green's function, with the external momenta kept off shell until the final *on-shell limit* $p_j^2,q_i^2\to m^2$ is taken (see below). Each factor (m^2-p^2) *removes one external propagator*: since the Green's function in momentum space contains a pole $\sim i/(p^2-m^2)$ for each external leg, the product $(m^2-p^2)\cdot i/(p^2-m^2)=-i$ yields a finite, non-zero result on shell. The order of operations is essential:

1. keep $p^2\neq m^2$ (off-shell), so that $\widetilde{G}(p,\ldots)$ is a regular function with the one-particle pole isolated;
2. multiply by the amputation factor (m^2-p^2), which cancels the pole and leaves a finite residue;
3. *only now* take the on-shell limit $p^2\to m^2$, i.e.

$$\lim_{p^2\to m^2}(m^2-p^2)\,\widetilde{G}(p,\ldots)\;=\;-i\cdot\big[\text{amputated residue}\big]\,.\tag{8.219}$$

Reversing this order would give either a divergence (setting $p^2=m^2$ before amputating) or an indeterminate $0\cdot\infty$ (amputating with p already on shell). Hence the LSZ formula is properly understood as a residue prescription: amputate first, mass-shell limit last.

Why the connected Green's function gives the scattering amplitude. The full Green's function $\widetilde{G}$ in (8.218) contains connected and disconnected parts. LSZ reduction applied recursively separates the same structures in the S-matrix: disconnected Green-function components reproduce the lower-particle overlaps that appeared explicitly in the reduction steps, while the fully connected component gives the genuine transition amplitude.

What are "lower-cluster terms"? By *lower-cluster terms* (or "disconnected lower-cluster contributions") we mean those pieces of the N-point Green's function ($N = m + n$) that *factorise* into products of correlators on fewer points. They are "lower" because each factor in the product involves strictly less than N external fields. The general decomposition reads

$$\widetilde{G}^{(N)}_{\text{full}} = \widetilde{G}^{(N)}_{\text{conn}} + \sum_{\substack{\text{partitions of} \\ \{x_1,\dots,x_N\} \\ \text{with } \geq 2 \text{ blocks}}} \prod_{\text{blocks } B} \widetilde{G}^{(|B|)}_{\text{conn}}\,, \tag{8.220}$$

where each non-trivial partition groups the N external points into two or more disjoint subsets ("clusters"), each cluster fully connected internally but disconnected from the others. Every block has $|B| < N$ external points, hence the term "lower".

Paradigmatic example: $2 \to 2$. For the four-point function the non-trivial partitions of $\{x_1, x_2, y_1, y_2\}$ into clusters of size at least one are: the three pairings $\{(x_1y_1)(x_2y_2)\}$, $\{(x_1y_2)(x_2y_1)\}$, $\{(x_1x_2)(y_1y_2)\}$, plus partitions involving singletons. Singleton-clusters yield factors of $\langle 0|\varphi|0\rangle = 0$ after the field shift, so the surviving disconnected piece is a sum of products of two two-point functions:

$$\widetilde{G}^{(4)}_{\text{disc}} = \sum_{\text{pairings}} \widetilde{G}^{(2)}_{\text{conn}}(\cdot,\cdot)\,\widetilde{G}^{(2)}_{\text{conn}}(\cdot,\cdot)\,. \tag{8.221}$$

Physically these are the configurations in which the two "in" particles propagate independently to the two "out" particles *without ever interacting* — they are precisely the forward-scattering pieces that build the identity part of the S-matrix, $S = \mathbb{1} + iT$, and that have to be subtracted to isolate the non-trivial scattering amplitude $\mathcal{M}$.

Where they enter the LSZ recursion. The lower-cluster terms are exactly the contributions that appeared explicitly at each step of the reduction in the form of overlaps with one fewer particle, e.g.

$$\langle \gamma p' \text{ out}|\varphi(x)|\alpha \text{ in}\rangle = \underbrace{\langle \gamma \text{ out}|\varphi(x)|\alpha - p' \text{ in}\rangle}_{\text{lower cluster}} + (\text{piece feeding the connected } \widetilde{G}_{\text{conn}})\,. \tag{8.222}$$

Applying LSZ recursively to all particles collects all such lower-cluster pieces, while the genuinely new physics at each step lives in the connected component. *Stripping the disconnected lower-cluster terms* thus means isolating $\widetilde{G}^{(N)}_{\text{conn}}$ from $\widetilde{G}^{(N)}_{\text{full}}$ in (8.220); the result is what the LSZ formula relates to the non-trivial scattering amplitude.

Single isolated external insertion. If an external field insertion is completely disconnected from all the others, its contribution factorises through the one-point function,

$$\begin{aligned}&\langle 0|T(\varphi(x_1)\cdots\varphi(x_i)\cdots\varphi(x_{m+n}))|0\rangle\big|_{x_i \text{ isolated}}\\ &\quad=\langle 0|\varphi(x_i)|0\rangle\,\langle 0|T(\varphi(x_1)\cdots\hat{\varphi}(x_i)\cdots\varphi(x_{m+n}))|0\rangle\,,\end{aligned}\tag{8.223}$$

where the hat over $\hat{\varphi}(x_i)$ is the standard *omission notation*: $T(\varphi(x_1)\cdots\hat{\varphi}(x_i)\cdots\varphi(x_{m+n}))$ denotes the time-ordered product of all the fields *except* $\varphi(x_i)$, i.e. the $(m+n-1)$-point Green's function on the remaining points $\{x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{m+n}\}$. The same convention is used throughout the rest of this chapter.

Our field has been shifted/renormalised so that $\langle 0|\varphi|0\rangle = 0$, hence such terms vanish.

Disconnected groups of external insertions. For $2\to 2$ scattering, the disconnected part of the exact four-point function contains products of exact two-point functions, schematically

$$\begin{aligned}\widetilde{G}_{\text{disc.}} &= (2\pi)^4\delta^{(4)}(q_1-p_1)\,i\Delta'_F(q_1)\,(2\pi)^4\delta^{(4)}(q_2-p_2)\,i\Delta'_F(q_2)\\ &\quad+(p_1\leftrightarrow p_2)+\cdots\,.\end{aligned}\tag{8.224}$$

These terms describe particles that propagate from the in-state to the out-state without participating in a common connected interaction. In the recursive LSZ derivation they are precisely the forward/no-scattering overlaps such as $\langle\beta - p \text{ out}|\alpha \text{ in}\rangle$; in the operator language they belong to the identity part and, more generally, to products of lower-point S-matrix elements.

The fully connected scattering amplitude is therefore defined after subtracting these lower-cluster contributions:

$$S_{\beta\alpha} = S^{\text{disc.}}_{\beta\alpha} + i(2\pi)^4\delta^{(4)}(P_\beta - P_\alpha)\,\mathcal{M}_{\beta\alpha}\,,\tag{8.225}$$

and $\mathcal{M}_{\beta\alpha}$ is obtained from the connected Green's function. This is the precise sense in which the physically non-trivial scattering amplitude is determined by $\widetilde{G}_{\text{conn.}}$.

Box 8.2 The LSZ Reduction Formula

After stripping the disconnected lower-cluster terms, the connected part of the S-matrix element is obtained from the connected Green's function by removing the external propagators and then putting the external momenta on shell. Suppressing the standard external state normalisation factors displayed in (8.218), this is written schematically as

$$\boxed{S_{fi}^{\text{conn.}} = \mathcal{A}_{\text{LSZ}}\,\widetilde{G}_{\text{conn.}}(p_1,\dots,p_r)\Big|_{p_i^2=m^2}} \tag{8.226}$$

where

$$\mathcal{A}_{\text{LSZ}} = \prod_{\text{ext. legs}} \left[\frac{i}{\sqrt{Z}}\,(m^2-p^2)\right]. \tag{8.227}$$

Here $\widetilde{G}_{\text{conn.}}$ is the Fourier transform of the connected time-ordered Green's function, and the momenta are kept off shell until after the factors in $\mathcal{A}_{\text{LSZ}}$ have cancelled the external poles. Disconnected terms give the forward-scattering and lower-cluster delta functions already separated in the reduction steps.

Box 8.3 Renormalisation scheme and the fate of $\sqrt{Z}$

The factor $1/\sqrt{Z}$ for each external leg in (8.226) originates from the LSZ asymptotic condition $\varphi(x) \to \sqrt{Z}\,\varphi_{\text{in/out}}(x)$, with Z the residue of the *exact* propagator at the physical one-particle pole:

$$i\Delta_F'(p^2) \xrightarrow{\;p^2\to m^2_{\text{phys}}\;} \frac{iZ}{p^2-m^2_{\text{phys}}+i\epsilon} + (\text{regular}). \tag{8.228}$$

Whether a finite external-leg factor appears explicitly in scattering amplitudes depends on the *renormalisation scheme*, i.e. on how the finite parts of the counterterms and the normalization of the renormalized field are chosen. It is important to distinguish the pole residue of the propagator from the renormalization constant that relates bare and renormalized fields.

On-shell (OS) scheme. The field renormalisation $\varphi_{\text{bare}} = Z_\phi^{1/2}\varphi_R$ is fixed order by order so that the renormalised propagator has unit residue at the physical pole:

$$\lim_{p^2\to m^2_{\text{phys}}} (p^2-m^2_{\text{phys}})\, i\Delta^{\text{OS}}_{F,R}(p^2) = i\,. \tag{8.229}$$

Consequence: when the Green's functions in (8.226) are computed with the OS-renormalized field φ_R, the LSZ residue factor for that field is 1 and no

separate $\sqrt{Z_{\text{pole}}}$ multiplies the amplitude. The mass parameter is automatically the physical pole mass. The counterterm δZ_ϕ absorbs both the UV divergence *and* the finite part required to enforce unit residue. This is the natural scheme whenever external states are physical on-shell particles.

Minimal subtraction ($\overline{\text{MS}}$ and similar). Counterterms are chosen to remove *only the UV divergences* (poles in ϵ, with the universal constant $\gamma_E - \ln 4\pi$ subtracted in $\overline{\text{MS}}$). No finite-part tuning is imposed to make the physical pole residue equal to one. Thus the renormalized propagator generally has

$$i\Delta_{F,R}^{\overline{\text{MS}}}(p^2) \xrightarrow{p^2 \to m^2_{\text{phys}}} \frac{iZ_{\text{pole}}^{\overline{\text{MS}}}(\mu)}{p^2 - m^2_{\text{phys}} + i\epsilon} + (\text{regular}), \qquad Z_{\text{pole}}^{\overline{\text{MS}}}(\mu) \neq 1 \quad (8.230)$$

in general. Consequence: the LSZ residue factors must be kept explicitly as finite external-leg factors multiplying the amputated connected Green function. Equivalently, the finite derivative of the self-energy at the pole enters through the LSZ residue; one should not add separate amputated diagrams with self-energy insertions on external on-shell legs in addition to this factor, or the effect is double counted. The advantage is that the scheme is mass-independent and well suited to studying the renormalisation-group running of couplings with μ – hence its ubiquity in perturbative QCD.

Same physics, different bookkeeping. The physical S-matrix is scheme-independent. The choice OS vs. $\overline{\text{MS}}$ only reshuffles where the finite parts live: in the OS scheme they are buried inside the field counterterm so that the renormalized propagator has unit pole residue; in $\overline{\text{MS}}$ the finite pole residue remains visible as an LSZ factor. Numerical predictions agree once self-energies and external-leg factors are accounted for consistently.

Scheme	Z	Effect on LSZ
On-shell (OS)	pole residue of φ_R fixed to 1	no separate $\sqrt{Z_{\text{pole}}}$ in amplitudes
$\overline{\text{MS}}$, MOM, …	pole residue generally finite and μ-dep.	LSZ residue remains as a finite external-leg factor

The LSZ formula at work: tree-level ϕ^4 scattering

To see how the reduction formula operates in practice, consider the simplest non-trivial example: $2 \to 2$ scattering in the theory $\mathcal{L}_{\text{int}} = -(\lambda/4!)\,\varphi^4$, at lowest order in λ.

The connected four-point Green's function at tree level consists of a single vertex with four propagators attached to it:

$$\widetilde{G}_{\text{conn.}}(p_1,p_2,p_3,p_4) = (2\pi)^4\delta^{(4)}\left(\sum_{i=1}^{4} p_i\right)\prod_{i=1}^{4}\frac{i}{p_i^2-m^2+i\epsilon}\cdot(-i\lambda)\,, \tag{8.231}$$

where all four momenta are taken as incoming, subject to $\sum_i p_i = 0$. At tree level there are no self-energy corrections, so $Z = 1 + O(\lambda)$ and the full and free propagators coincide.

Applying the LSZ prescription (8.226), and suppressing the overall momentum-conserving delta function and external normalisation factors, we multiply by $i(m^2 - p_i^2)$ for each external leg and take $p_i^2 \to m^2$:

$$\begin{aligned} S_{fi}^{\text{conn.}} &= \prod_{i=1}^{4}\left[i(m^2-p_i^2)\frac{i}{p_i^2-m^2+i\epsilon}\right](-i\lambda)\Bigg|_{p_i^2=m^2} \\ &= \prod_{i=1}^{4}(+1)\cdot(-i\lambda) = -i\lambda\,. \end{aligned} \tag{8.232}$$

Each factor $(m^2 - p_i^2)$ cancels the corresponding propagator pole, leaving the finite stripped result $i\mathcal{M} = -i\lambda$, hence $\mathcal{M} = -\lambda$. This is precisely the Feynman rule for the vertex read off directly from the Lagrangian: the LSZ machinery, though seemingly elaborate, reproduces the intuitive result.

The power of the formalism becomes apparent at higher orders: when self-energy insertions dress the external two-point function, the propagator poles acquire a finite residue. The LSZ factors then extract precisely that residue from the full Green function. Whether this finite residue is visible as an explicit external-leg factor or absorbed into the field counterterm depends on the renormalisation scheme (see the box "Renormalisation scheme and the fate of $\sqrt{Z}$" above): in an on-shell scheme the renormalized field is normalized to unit pole residue, while in $\overline{\text{MS}}$ the finite pole residue remains and must be included through LSZ [1, 6].

8.8 The reduction formula for Dirac fields

We now carry out the full LSZ reduction for Dirac spinor fields, following the same logic as for the scalar case but keeping careful track of the spinor structure.

Asymptotic fields and the Yang–Feldman equation

The interacting Dirac field $\psi(x)$ satisfies the Dirac equation with the *physical* mass m:

$$(i\partial\!\!\!/ - m)\psi(x) = \tilde{j}(x), \qquad \tilde{j}(x) = j(x) - (m - m_0)\psi(x)\,, \tag{8.233}$$

where $j(x)$ is the interaction current (e.g. $j = e_0 \not{A}\psi$ in the bare QED equation, later rewritten in terms of renormalized parameters and counterterms) and the mass counterterm $(m - m_0)\psi$ ensures the physical mass is m. The formal solution is:

$$\psi(x) = \sqrt{Z_2}\,\psi_{\text{in}}(x) + \int d^4y\, S_{\text{ret}}(x-y;m)\,\tilde{j}(y)\,, \tag{8.234}$$

where $S_{\text{ret}}(x-y;m)$ is the retarded Dirac Green's function satisfying

$$(i\not{\partial}_x - m)S_{\text{ret}}(x-y;m) = \delta^{(4)}(x-y), \qquad S_{\text{ret}}(x-y) = 0 \text{ for } x^0 < y^0\,. \tag{8.235}$$

The field renormalisation constant Z_2 and the free in-field ψ_{in} are defined so that ψ_{in} satisfies the free Dirac equation $(i\not{\partial} - m)\psi_{\text{in}} = 0$ and creates/annihilates single-particle states with unit norm. The **weak asymptotic condition** for Dirac fields states that, inside matrix elements between physical states:

$$\psi(x) \xrightarrow{x^0 \to -\infty} \sqrt{Z_2}\,\psi_{\text{in}}(x)\,, \qquad \psi(x) \xrightarrow{x^0 \to +\infty} \sqrt{Z_2}\,\psi_{\text{out}}(x)\,. \tag{8.236}$$

Fourier expansion of the in-field

Since $\psi_{\text{in}}(x)$ satisfies the free Dirac equation, its Fourier expansion is:

$$\psi_{\text{in}}(x) = \int d^3p \sum_{s=\pm} \Big[b_{\text{in}}(p,s)\,U_{ps}(x) + d^{\dagger}_{\text{in}}(p,s)\,V_{ps}(x) \Big]\,, \tag{8.237}$$

with the positive- and negative-frequency solutions:

$$U_{ps}(x) = \frac{1}{(2\pi)^{3/2}}\sqrt{\frac{m}{E_p}}\,u(p,s)\,e^{-ip\cdot x}, \qquad V_{ps}(x) = \frac{1}{(2\pi)^{3/2}}\sqrt{\frac{m}{E_p}}\,v(p,s)\,e^{ip\cdot x}\,. \tag{8.238}$$

The inverse relations that extract the operators from the field are:

$$b_{\text{in}}(p,s) = \int d^3x\, U^{\dagger}_{ps}(x)\,\psi_{\text{in}}(x)\,, \quad d^{\dagger}_{\text{in}}(p,s) = \int d^3x\, V^{\dagger}_{ps}(x)\,\psi_{\text{in}}(x)\,. \tag{8.239}$$

Proof of (8.239). We verify the first relation. Using the orthogonality of the spinor solutions at equal time:

$$\int d^3x\, U^{\dagger}_{ps}(x)\,U_{p's'}(x) = \delta_{ss'}\delta^{(3)}(\vec{p}-\vec{p}\,')\,, \qquad \int d^3x\, U^{\dagger}_{ps}(x)\,V_{p's'}(x) = 0\,, \tag{8.240}$$

The first identity follows from the spinor normalisation $u^{\dagger}(p,s)u(p,s') = (E_p/m)\delta_{ss'}$ and the prefactor m/E_p in the modes. For the mixed term, the spatial integral gives $\delta^{(3)}(\vec{p}+\vec{p}\,')$ because $U^{\dagger} \propto e^{+ip\cdot x}$ while $V \propto e^{+ip'\cdot x}$; the spinor factor is then

$u^\dagger(p,s)v(-p,s') = 0$. Substituting (8.237) into $\int d^3x\, U^\dagger_{ps}\psi_{\rm in}$ and using (8.240), only the $b_{\rm in}$ term survives, giving $b_{\rm in}(p,s)$. The second relation follows analogously.

The conjugate relations for $\bar\psi_{\rm in} = \psi^\dagger_{\rm in}\gamma^0$ are:

$$\begin{aligned} b^\dagger_{\rm in}(p,s) &= \int d^3x\, \bar\psi_{\rm in}(x)\,\gamma^0\, U_{ps}(x)\,, \\ d_{\rm in}(p,s) &= \int d^3x\, \bar\psi_{\rm in}(x)\,\gamma^0\, V_{ps}(x)\,. \end{aligned} \tag{8.241}$$

General in-states with fermions and antifermions are constructed as:

$$\begin{aligned} &|(\bar p_k \bar s_k)\cdots(\bar p_1 \bar s_1);(p_j s_j)\cdots(p_1 s_1)\ \text{in}\rangle \\ &\qquad = d^\dagger_{\rm in}(\bar p_k,\bar s_k)\cdots b^\dagger_{\rm in}(p_1,s_1)|0\rangle\,. \end{aligned} \tag{8.242}$$

Removing an incoming electron from the in-state

Consider the S-matrix element with an electron of momentum p and spin s in the in-state:

$$S_{\beta,\alpha(ps)} = \langle \beta\ \text{out}|\alpha(ps)\ \text{in}\rangle\,, \tag{8.243}$$

where $|\alpha(ps)\ \text{in}\rangle = b^\dagger_{\rm in}(p,s)|\alpha\ \text{in}\rangle$. Proceeding as in the scalar case, we split:

$$\begin{aligned} \langle \beta\ \text{out}|\alpha(ps)\ \text{in}\rangle &= \langle \beta\ \text{out}|b^\dagger_{\rm in}(p,s)|\alpha\ \text{in}\rangle \\ &= \langle \beta\ \text{out}|b^\dagger_{\rm out}(p,s)|\alpha\ \text{in}\rangle \\ &\quad + \langle \beta\ \text{out}|\big[b^\dagger_{\rm in}(p,s) - b^\dagger_{\rm out}(p,s)\big]|\alpha\ \text{in}\rangle\,. \end{aligned} \tag{8.244}$$

The first term gives the disconnected contribution $\langle \beta - (ps)\ \text{out}|\alpha\ \text{in}\rangle$.

For the second term, we need the expression for $b^\dagger_{\rm in}(p,s)$ in terms of the field. From (8.241):

$$b^\dagger_{\rm in}(p,s) = \int d^3x\, \bar\psi_{\rm in}(x)\,\gamma^0\, U_{ps}(x)\,. \tag{8.245}$$

Note that unlike the scalar case, the inverse relation involves $\bar\psi\gamma^0$ rather than $\overleftrightarrow{\partial_0}$, because the Dirac equation is first order. For free fields the corresponding equal-time Dirac inner product is time-independent:

$$\frac{d}{dx^0}\int d^3x\, \bar\psi_{\rm in}(x)\,\gamma^0\, U_{ps}(x) = 0\,, \tag{8.246}$$

since both $\psi_{\rm in}$ and U_{ps} satisfy the same free Dirac equation.

Proof of (8.246). The statement (8.246) is the conservation of the Dirac inner product. We show that the current

$$j^\mu(x) \;\equiv\; \bar\psi_{\rm in}(x)\,\gamma^\mu\, U_{ps}(x) \tag{8.247}$$

is divergenceless, $\partial_\mu j^\mu = 0$; integrating j^0 over $\vec{x}$ then gives the conserved "charge" (8.246).

Both $\psi_{\rm in}$ and U_{ps} satisfy the free Dirac equation of the physical mass m:

$$(i\gamma^\mu\partial_\mu - m)\,\psi_{\rm in} = 0\,, \qquad (i\gamma^\mu\partial_\mu - m)\,U_{ps} = 0\,. \tag{8.248}$$

Taking the Dirac adjoint of the first equation (Hermitian conjugate, then multiplication by γ^0 from the right) gives

$$\bar\psi_{\rm in}\big(i\overleftarrow{\partial}_\mu\gamma^\mu + m\big) = 0 \quad\Longleftrightarrow\quad i\,(\partial_\mu\bar\psi_{\rm in})\,\gamma^\mu = -m\,\bar\psi_{\rm in}\,. \tag{8.249}$$

Now apply Leibniz to $\partial_\mu j^\mu$:

$$\begin{aligned}\partial_\mu(\bar\psi_{\rm in}\gamma^\mu U_{ps}) &= \underbrace{(\partial_\mu\bar\psi_{\rm in})\gamma^\mu}_{=im\,\bar\psi_{\rm in}}\,U_{ps} \;+\; \bar\psi_{\rm in}\,\underbrace{\gamma^\mu(\partial_\mu U_{ps})}_{=-im\,U_{ps}} \\ &= im\,\bar\psi_{\rm in}\,U_{ps} - im\,\bar\psi_{\rm in}\,U_{ps} = 0\,,\end{aligned} \tag{8.250}$$

where the first identity uses (8.249) and the second uses (8.248) rewritten as $\gamma^\mu\partial_\mu U_{ps} = -im\,U_{ps}$. The two mass terms cancel exactly: $\partial_\mu j^\mu = 0$.

Separating temporal and spatial parts and integrating over $\vec{x}$:

$$\begin{aligned}\frac{d}{dx^0}\int d^3x\,\bar\psi_{\rm in}\gamma^0 U_{ps} &= -\int d^3x\,\nabla\cdot(\bar\psi_{\rm in}\vec\gamma\,U_{ps}) \\ &= -\oint_{\partial V} d\mathbf{S}\cdot(\bar\psi_{\rm in}\vec\gamma\,U_{ps}) \;=\; 0\,,\end{aligned} \tag{8.251}$$

the spatial boundary term vanishing for normalisable wave packets (or, for the sharp plane-wave choice, with the standard adiabatic / $i\epsilon$ prescription). This proves (8.246).

Geometric interpretation. The bilinear $\langle\phi,\chi\rangle_D \equiv \int d^3x\,\bar\phi\gamma^0\chi = \int d^3x\,\phi^\dagger\chi$ is the ordinary L^2 inner product on Dirac spinors. The statement (8.246) says that this inner product is *conserved* by the free Dirac flow: the time evolution is unitary on $L^2(\mathbb{R}^3)\otimes\mathbb{C}^4$, and orthonormal mode functions remain orthonormal at every time. This is the spinorial counterpart of the Klein–Gordon identity used in the scalar reduction with $\overleftrightarrow{\partial_0}$.

Subtracting the in and out expressions:

$$\begin{aligned}&\langle\beta\ \text{out}|b^\dagger_{\rm in}(p,s) - b^\dagger_{\rm out}(p,s)|\alpha\ \text{in}\rangle \\ &\quad= \int d^3x\,\langle\beta\ \text{out}|\big[\bar\psi_{\rm in}(x) - \bar\psi_{\rm out}(x)\big]|\alpha\ \text{in}\rangle\,\gamma^0 U_{ps}(x)\,.\end{aligned} \tag{8.252}$$

Applying the weak asymptotic condition. Using (8.236), $\bar{\psi}_{\text{in}}(x) \to \bar{\psi}(x)/\sqrt{Z_2}$ as $x^0 \to -\infty$ and $\bar{\psi}_{\text{out}}(x) \to \bar{\psi}(x)/\sqrt{Z_2}$ as $x^0 \to +\infty$:

$$\begin{aligned}\langle \beta \text{ out}|\alpha(ps) \text{ in}\rangle = \langle \beta - (ps) \text{ out}|\alpha \text{ in}\rangle \\ + \frac{1}{\sqrt{Z_2}}\Big(\lim_{x^0\to-\infty} - \lim_{x^0\to+\infty}\Big) \\ \int d^3x \, \langle \beta \text{ out}|\bar{\psi}(x)|\alpha \text{ in}\rangle \, \gamma^0 U_{ps}(x)\,. \end{aligned} \tag{8.253}$$

Note the sign: (in) − (out) here, opposite to the scalar case, because we are computing $b^\dagger_{\text{in}} - b^\dagger_{\text{out}}$ rather than $a^\dagger_{\text{in}} - a^\dagger_{\text{out}}$.

Converting to a four-dimensional integral. Define $\bar{g}(x) \equiv \langle \beta \text{ out}|\bar{\psi}(x)|\alpha \text{ in}\rangle$ (a 1×4 row spinor). We prove the following identity, which is the Dirac analogue of (8.190):

$$\Big(\lim_{x^0\to-\infty} - \lim_{x^0\to+\infty}\Big) \int d^3x \, \bar{g}\,\gamma^0 U_{ps} = i \int d^4x \, \bar{g}\,(i\overleftarrow{\not\partial} + m)\, U_{ps}\,, \tag{8.254}$$

where $\bar{g}\, i\overleftarrow{\not\partial} \equiv i(\partial_\mu \bar{g})\gamma^\mu$ and U_{ps} satisfies $(i\not\partial - m)U_{ps} = 0$.

Proof. We start from the right-hand side. Multiplying out:

$$i\int d^4x \, \bar{g}\,(i\overleftarrow{\not\partial} + m)\, U_{ps} = -\int d^4x \, (\partial_\mu \bar{g})\, \gamma^\mu U_{ps} + im \int d^4x \, \bar{g}\, U_{ps}\,. \tag{8.255}$$

Splitting into temporal and spatial parts:

$$\begin{aligned} = &-\int d^4x \, (\partial_0 \bar{g})\, \gamma^0 U_{ps} \\ &-\int d^4x \, (\partial_k \bar{g})\, \gamma^k U_{ps} \\ &+ im \int d^4x \, \bar{g}\, U_{ps}\,. \end{aligned} \tag{8.256}$$

Step 1: Integrate the spatial term by parts (boundary terms vanish at spatial infinity):

$$-\int d^4x \, (\partial_k \bar{g})\, \gamma^k U_{ps} = +\int d^4x \, \bar{g}\, \gamma^k \, (\partial_k U_{ps})\,. \tag{8.257}$$

Step 2: Use the free Dirac equation. From $(i\gamma^\mu \partial_\mu - m)U_{ps} = 0$ we have $i\gamma^0 \partial_0 U_{ps} + i\gamma^k \partial_k U_{ps} = m U_{ps}$, hence:

$$\gamma^k \partial_k U_{ps} = -\gamma^0 \partial_0 U_{ps} - im U_{ps}\,. \tag{8.258}$$

Step 3: Substitute (8.258) into (8.257):

$$+\int d^4x\,\bar{g}\,\gamma^k\,(\partial_k U_{ps}) = -\int d^4x\,\bar{g}\,\gamma^0\,(\partial_0 U_{ps}) - im\int d^4x\,\bar{g}\,U_{ps}\,. \tag{8.259}$$

Step 4: Substitute (8.259) back into (8.256):

$$\begin{aligned}
&= -\int d^4x\,(\partial_0\bar{g})\,\gamma^0\,U_{ps} - \int d^4x\,\bar{g}\,\gamma^0\,(\partial_0 U_{ps}) \\
&\quad \underbrace{-im\int d^4x\,\bar{g}\,U_{ps} + im\int d^4x\,\bar{g}\,U_{ps}}_{=\,0} \\
&= -\int d^4x\,\left[(\partial_0\bar{g})\,\gamma^0\,U_{ps} + \bar{g}\,\gamma^0\,(\partial_0 U_{ps})\right] = -\int d^4x\,\partial_0\left[\bar{g}\,\gamma^0\,U_{ps}\right],
\end{aligned} \tag{8.260}$$

where the last step is just the product rule in reverse. Finally:

$$-\int_{-\infty}^{+\infty} dx^0\,\partial_0\int d^3x\,\bar{g}\,\gamma^0\,U_{ps} = \left(\lim_{x^0\to-\infty} - \lim_{x^0\to+\infty}\right)\int d^3x\,\bar{g}\,\gamma^0\,U_{ps}\,. \quad \square \tag{8.261}$$

Assembling the final result. Substituting (8.254) into (8.253):

$$\begin{aligned}
\langle\beta\ \text{out}|\alpha(ps)\ \text{in}\rangle &= \langle\beta - (ps)\ \text{out}|\alpha\ \text{in}\rangle \\
&\quad + \frac{i}{\sqrt{Z_2}}\int d^4x\,\langle\beta\ \text{out}|\bar{\psi}(x)|\alpha\ \text{in}\rangle\,(i\overleftarrow{\not\partial} + m)\,U_{ps}(x)\,.
\end{aligned} \tag{8.262}$$

Writing $(i\overleftarrow{\not\partial} + m) = -(-i\overleftarrow{\not\partial} - m)$:

Box 8.4

$$\begin{aligned}
\langle\beta\ \text{out}|\alpha(ps)\ \text{in}\rangle &= \langle\beta - (ps)\ \text{out}|\alpha\ \text{in}\rangle \\
&\quad - \frac{i}{\sqrt{Z_2}}\int d^4x\,\langle\beta\ \text{out}|\bar{\psi}(x)|\alpha\ \text{in}\rangle\,(-i\overleftarrow{\not\partial} - m)\,U_{ps}(x)\,.
\end{aligned} \tag{8.263}$$

This is the Dirac analogue of (8.198). The operator $(-i\overleftarrow{\not\partial} - m)$ plays the role of $(\Box + m^2)$ in the scalar case: acting on the Green function it supplies the inverse external fermion propagator, while after integration by parts it vanishes on the free external wave function by the Dirac equation.

Removing an incoming positron from the in-state

For an incoming positron with momentum $\bar{p}$ and spin $\bar{s}$, we have $|\alpha(\bar{p}\bar{s})\ \text{in}\rangle = d^\dagger_{\text{in}}(\bar{p},\bar{s})|\alpha\ \text{in}\rangle$. From (8.239):

$$d^\dagger_{\text{in}}(\bar{p},\bar{s}) = \int d^3x\, V^\dagger_{\bar{p}\bar{s}}(x)\,\psi_{\text{in}}(x)\,. \tag{8.264}$$

Proceeding as before — splitting into $d^\dagger_{\text{out}} + (d^\dagger_{\text{in}} - d^\dagger_{\text{out}})$, applying the weak asymptotic condition, and converting to a 4D integral — we obtain:

$$\begin{aligned}&\langle\beta\ \text{out}|d^\dagger_{\text{in}}(\bar{p},\bar{s}) - d^\dagger_{\text{out}}(\bar{p},\bar{s})|\alpha\ \text{in}\rangle \\ &\quad = \int d^3x\, V^\dagger_{\bar{p}\bar{s}}(x)\,\langle\beta\ \text{out}|\big[\psi_{\text{in}}(x) - \psi_{\text{out}}(x)\big]|\alpha\ \text{in}\rangle\,.\end{aligned} \tag{8.265}$$

Note that here ψ (not $\bar{\psi}$) appears, because $d^\dagger$ is extracted from ψ (the negative-frequency part). Using the asymptotic condition and converting to a 4D integral with the free Dirac equations

$$(i\partial\!\!\!/ - m)U_{ps} = 0, \qquad (i\partial\!\!\!/ - m)V_{ps} = 0\,, \tag{8.266}$$

where the second equation is equivalent to $(p\!\!\!/ + m)v(p,s) = 0$ because $V_{ps} \propto e^{+ip\cdot x}$, gives:

Box 8.5

$$\begin{aligned}\langle\beta\ \text{out}|\alpha(\bar{p}\bar{s})\ \text{in}\rangle &= \langle\beta - (\bar{p}\bar{s})\ \text{out}|\alpha\ \text{in}\rangle \\ &\quad + \frac{i}{\sqrt{Z_2}}\int d^4x\,\bar{V}_{\bar{p}\bar{s}}(x)\,(i\partial\!\!\!/ - m)\,\langle\beta\ \text{out}|\psi(x)|\alpha\ \text{in}\rangle\,.\end{aligned} \tag{8.267}$$

Here $(i\partial\!\!\!/ - m)$ acts to the right on the matrix element. The sign difference with respect to the electron case (8.263) arises from the column/row placement of ψ versus $\bar{\psi}$ and from the orientation of the asymptotic time limits.

Removing an outgoing electron from the out-state

To remove an outgoing electron with momentum p', spin s' from $\langle\gamma(p's')\ \text{out}|$, we use $\langle\gamma(p's')\ \text{out}| = \langle\gamma\ \text{out}|b_{\text{out}}(p',s')$. From (8.239):

$$b_{\text{out}}(p',s') = \int d^3x\, U^\dagger_{p's'}(x)\,\psi_{\text{out}}(x)\,. \tag{8.268}$$

We write:

$$\begin{aligned}&\langle\gamma(p's')\ \text{out}|\varphi(y)\cdots|\alpha\ \text{in}\rangle = \langle\gamma\ \text{out}|b_{\text{out}}(p',s')\,\varphi(y)\cdots|\alpha\ \text{in}\rangle\\&= \langle\gamma\ \text{out}|\varphi(y)\cdots b_{\text{in}}(p',s')|\alpha\ \text{in}\rangle\\&\quad+\langle\gamma\ \text{out}|\big[b_{\text{out}}(p',s')\,\varphi(y)\cdots-\varphi(y)\cdots b_{\text{in}}(p',s')\big]|\alpha\ \text{in}\rangle\,.\end{aligned}\tag{8.269}$$

The first term gives the disconnected piece. For the second term, the asymptotic condition gives:

- As $x^0\to+\infty$: $b_{\text{out}}(p',s')$ involves $\psi_{\text{out}}(x)\to\psi(x)/\sqrt{Z_2}$, and $\psi_{\text{out}}(x)\,\varphi(y)\cdots=T(\psi(x)\varphi(y)\cdots)$ since $x^0>y^0$.
- As $x^0\to-\infty$: $b_{\text{in}}(p',s')$ involves $\psi_{\text{in}}(x)\to\psi(x)/\sqrt{Z_2}$, and $\varphi(y)\cdots\psi_{\text{in}}(x)=T(\psi(x)\varphi(y)\cdots)$ since $x^0<y^0$.

If O contains other fermionic fields, the time-ordered product includes the usual minus signs from permuting fermion operators; throughout this section a fixed external-operator ordering is understood. In both limits, the result involves $T(\psi(x)\cdots)$, so:

$$\begin{aligned}&\langle\gamma\ \text{out}|\big[\psi_{\text{out}}(x)\,O-O\,\psi_{\text{in}}(x)\big]|\alpha\ \text{in}\rangle\\&\quad=\frac{1}{\sqrt{Z_2}}\Big(\lim_{x^0\to+\infty}-\lim_{x^0\to-\infty}\Big)\langle\gamma\ \text{out}|T(\psi(x)\,O)|\alpha\ \text{in}\rangle\,.\end{aligned}\tag{8.270}$$

Converting to a 4D integral and using $(i\partial\!\!\!/_x-m)U_{p's'}(x)=0$, after integration by parts on the Dirac operator we obtain:

Box 8.6

$$\begin{aligned}&\langle\gamma(p's')\ \text{out}|O|\alpha\ \text{in}\rangle = \langle\gamma\ \text{out}|O|\alpha-(p's')\ \text{in}\rangle\\&\quad-\frac{i}{\sqrt{Z_2}}\int d^4x\,\bar U_{p's'}(x)\,(i\partial\!\!\!/-m)\\&\qquad\times\langle\gamma\ \text{out}|T(\psi(x)\,O)|\alpha\ \text{in}\rangle\,.\end{aligned}\tag{8.271}$$

Here $(i\partial\!\!\!/-m)$ acts on the matrix element to its right.

Removing an outgoing positron from the out-state

For an outgoing positron, $\langle\gamma(\bar p'\bar s')\ \text{out}|=\langle\gamma\ \text{out}|d_{\text{out}}(\bar p',\bar s')$, and from (8.241):

$$d_{\text{out}}(\bar p',\bar s')=\int d^3x\,\bar\psi_{\text{out}}(x)\,\gamma^0\,V_{\bar p'\bar s'}(x)\,.\tag{8.272}$$

The same procedure (split, asymptotic condition, 4D integral, integration by parts) yields:

Box 8.7

$$\langle \gamma(\bar{p}'\bar{s}')\ \text{out}|O|\alpha\ \text{in}\rangle = \langle \gamma\ \text{out}|O|\alpha - (\bar{p}'\bar{s}')\ \text{in}\rangle + \frac{i}{\sqrt{Z_2}}\int d^4x\,\langle \gamma\ \text{out}|T(\bar{\psi}(x)\,O)|\alpha\ \text{in}\rangle \times(-i\overleftarrow{\not\partial} - m)\,V_{\bar{p}'\bar{s}'}(x)\,. \tag{8.273}$$

The operator $(-i\overleftarrow{\not\partial} - m)$ acts to the left on the matrix element.

Summary and comparison with the scalar case

Collecting the four cases, the pattern is:

Particle	Operator	Free-field solution
Incoming e^- $(b^\dagger_{\text{in}})$	$\langle\bar{\psi}|\,(-i\overleftarrow{\not\partial} - m)$	$U_{ps}(x)$
Incoming e^+ $(d^\dagger_{\text{in}})$	$\bar{V}\,(i\not\partial - m)\,\|\psi\rangle$	$V_{\bar{p}\bar{s}}(x)$
Outgoing e^- (b_{out})	$\bar{U}\,(i\not\partial - m)\,\|T\psi\rangle$	$U_{p's'}(x)$
Outgoing e^+ (d_{out})	$\langle T\bar{\psi}|\,(-i\overleftarrow{\not\partial} - m)$	$V_{\bar{p}'\bar{s}'}(x)$

The correspondence with the scalar case is:

$$\underbrace{(\Box + m^2)}_{\text{scalar}} \longleftrightarrow \underbrace{(i\not\partial - m)}_{\text{Dirac}}, \qquad \underbrace{f_p(x)}_{\text{scalar}} \longleftrightarrow \underbrace{U_{ps}(x),\ V_{\bar{p}\bar{s}}(x)}_{\text{Dirac}}\,. \tag{8.274}$$

The close correspondence between the electron formula (8.263) and the positron formula (8.273) reflects the field-theoretic statement: positron processes are calculated by propagating negative-energy electrons backward in time.

The general Dirac reduction formula

Repeatedly applying the reduction steps to remove all fermions from the in- and out-states, we arrive at the Dirac analogue of (8.211), written in the same bracketed style ([LEFT block] $\langle 0|T(\ldots)|0\rangle$ [RIGHT block]) used for the scalar and photon

master formulas:

$$
\begin{aligned}
&\langle (p_1' s_1') \cdots (p_n' s_n');\, (\bar{p}_1' \bar{s}_1') \cdots (\bar{p}_{\bar{n}}' \bar{s}_{\bar{n}}') \text{ out}| \\
&\qquad (p_1 s_1) \cdots (p_m s_m);\, (\bar{p}_1 \bar{s}_1) \cdots (\bar{p}_{\bar{m}} \bar{s}_{\bar{m}}) \text{ in}\rangle \\
&= \left(\frac{-i}{\sqrt{Z_2}}\right)^{m+n} \left(\frac{i}{\sqrt{Z_2}}\right)^{\bar{m}+\bar{n}} \\
&\quad \times \left[\prod_{j=1}^{n} \int d^4 y_j \; \bar{U}_{p_j' s_j'}(y_j) \overrightarrow{(i\partial\!\!\!/_{y_j} - m)} \prod_{k=1}^{\bar{m}} \int d^4 \bar{x}_k \; \bar{V}_{\bar{p}_k \bar{s}_k}(\bar{x}_k) \overrightarrow{(i\partial\!\!\!/_{\bar{x}_k} - m)} \right] \\
&\quad \times \langle 0 | T \big(\psi(y_1) \cdots \psi(y_n) \psi(\bar{x}_1) \cdots \psi(\bar{x}_{\bar{m}}) \\
&\qquad\qquad \bar{\psi}(x_1) \cdots \bar{\psi}(x_m) \bar{\psi}(\bar{y}_1) \cdots \bar{\psi}(\bar{y}_{\bar{n}}) \big) | 0 \rangle \\
&\quad \times \left[\prod_{i=1}^{m} \int d^4 x_i \; \overleftarrow{(-i\partial\!\!\!/_{x_i} - m)} U_{p_i s_i}(x_i) \prod_{l=1}^{\bar{n}} \int d^4 \bar{y}_l \; \overleftarrow{(-i\partial\!\!\!/_{\bar{y}_l} - m)} V_{\bar{p}_l' \bar{s}_l'}(\bar{y}_l) \right] .
\end{aligned}
\tag{8.275}
$$

The displayed ordering fixes the overall fermionic sign convention. If one chooses a different ordering of external fermion operators, the formula must be multiplied by the sign of the corresponding permutation.

In momentum space, after inserting the explicit forms of U_{ps} and $V_{\bar{p}\bar{s}}$ and integrating by parts (exactly as in the scalar case discussed above; cf. (8.218)), the Dirac operators become inverse external fermion propagators, with the sign fixed by the momentum flow. For example,

$$
(i\partial\!\!\!/ - m)\, e^{-ip\cdot x}\, u(p,s) = (p\!\!\!/ - m)\, u(p,s)\, e^{-ip\cdot x} = 0 \quad \text{on shell,} \tag{8.276}
$$

whereas for the negative-frequency wave function,

$$
(i\partial\!\!\!/ - m)\, e^{+ip\cdot x}\, v(p,s) = -(p\!\!\!/ + m)\, v(p,s)\, e^{+ip\cdot x} = 0 \quad \text{on shell.} \tag{8.277}
$$

Off shell, the corresponding inverse Dirac operator cancels the external propagator pole. For an electron line with momentum p,

$$
(p\!\!\!/ - m)\, S_F(p) = (p\!\!\!/ - m) \frac{i(p\!\!\!/ + m)}{p^2 - m^2 + i\epsilon} \xrightarrow[p^2 \to m^2]{} i\,, \tag{8.278}
$$

up to the standard $i\epsilon$ prescription. Thus the LSZ operator removes the external fermion propagator and leaves the appropriate external spinor, just as $(m^2 - p^2) \cdot i/(p^2 - m^2) = -i$ removes an external scalar propagator.

The reduction formula for the photon field A^μ

The construction for the electromagnetic field A^μ proceeds in exact parallel with the scalar and Dirac cases, with two minor adjustments specific to massless gauge bosons.

Asymptotic condition and free field. The interacting electromagnetic field has, by the same arguments as in the scalar case, a weak asymptotic limit

$$A^\mu(x) \xrightarrow{x^0\to-\infty} \sqrt{Z_3}\,A^\mu_{\text{in}}(x)\,, \qquad A^\mu(x) \xrightarrow{x^0\to+\infty} \sqrt{Z_3}\,A^\mu_{\text{out}}(x)\,, \tag{8.279}$$

where Z_3 is the photon wave-function renormalisation constant and the in/out fields are massless free vector fields satisfying $\Box A^\mu_{\text{in/out}}(x) = 0$ (in Feynman–like covariant gauges; gauge-fixing terms contribute only to unphysical polarisations and drop out of physical S-matrix elements via the Ward identity). The mode expansion is

$$A^\mu_{\text{in}}(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \sum_\lambda \Big[a_{\text{in}}(k,\lambda)\,\varepsilon^\mu(k,\lambda)\,e^{-ik\cdot x} + a^\dagger_{\text{in}}(k,\lambda)\,\varepsilon^{\mu*}(k,\lambda)\,e^{+ik\cdot x}\Big], \tag{8.280}$$

with $\omega_k = |\vec{k}|$ and the polarisation 4-vectors $\varepsilon^\mu(k,\lambda)$ from Chapter 1.

Inverting the mode expansion. For a positive-frequency photon plane wave $A^\mu_p(x) = \varepsilon^\mu(p,\lambda)\,e^{-ip\cdot x}/\sqrt{(2\pi)^3 2\omega_p}$, the same Klein–Gordon-style overlap as in the scalar case (massless, applied component by component) gives

$$a_{\text{in}}(p,\lambda) = i\int d^3x\, A^{\mu*}_p(x) \overleftrightarrow{\partial_0} A^{\text{in}}_\mu(x)\,, \qquad a^\dagger_{\text{in}}(p,\lambda) = -i\int d^3x\, A^\mu_p(x) \overleftrightarrow{\partial_0} A^{\text{in}}_\mu(x)\,. \tag{8.281}$$

Single-photon reduction. Removing an outgoing photon with momentum p and polarisation λ from the state, and following step-by-step the scalar derivation (8.201)–(8.210) with $(\Box + m^2) \to \Box$ and $f_p \to \varepsilon^\mu(p,\lambda)\,e^{-ip\cdot x}/\sqrt{(2\pi)^3 2\omega_p}$, one obtains

$$\begin{aligned}\langle \gamma\, p\lambda \text{ out}|\alpha \text{ in}\rangle &= \langle \gamma \text{ out}|\alpha - p\lambda \text{ in}\rangle \\ &\quad + \frac{i}{\sqrt{Z_3}}\int d^4y\, \frac{\varepsilon^{\mu*}(p,\lambda)\,e^{+ip\cdot y}}{\sqrt{(2\pi)^3 2\omega_p}}\, \Box_y \langle \gamma \text{ out}|T(A_\mu(y)\cdots)|\alpha \text{ in}\rangle\,.\end{aligned} \tag{8.282}$$

General photon LSZ formula. Iterating the reduction on all photons in the in- and out-states, for a process with m incoming photons (momenta q_i, polarisations λ_i) and n outgoing photons (momenta p_j, polarisations λ'_j) one finds

$$\begin{aligned}&\langle p_1\lambda'_1 \cdots p_n\lambda'_n \text{ out}|q_1\lambda_1\cdots q_m\lambda_m \text{ in}\rangle \\ &= \left(\frac{i}{\sqrt{Z_3}}\right)^{m+n}\left[\prod_{i=1}^m \int d^4x_i\, \frac{\varepsilon^{\mu_i}(q_i,\lambda_i)\,e^{-iq_i\cdot x_i}}{\sqrt{(2\pi)^3 2\omega_{q_i}}}\overrightarrow{\Box_{x_i}}\right] \\ &\quad \times \langle 0|T\big(A_{\mu_1}(x_1)\cdots A_{\mu_m}(x_m)\,A_{\nu_1}(y_1)\cdots A_{\nu_n}(y_n)\big)|0\rangle \\ &\quad \times \left[\prod_{j=1}^n \int d^4y_j \overleftarrow{\Box_{y_j}}\, \frac{\varepsilon^{\nu_j*}(p_j,\lambda'_j)\,e^{+ip_j\cdot y_j}}{\sqrt{(2\pi)^3 2\omega_{p_j}}}\right],\end{aligned} \tag{8.283}$$

valid for all $p_j \neq q_i$. In momentum space (Fourier transforming the Green's function), each $\Box$ becomes $-p^2$ and the amputation factor for a photon leg is simply $-p^2$, evaluated in the on-shell limit $p^2 \to 0$. Inserting the explicit plane waves,

$$S_{fi}^{\text{conn.}} = \left(\frac{i}{\sqrt{Z_3}}\right)^{m+n} \prod_{i=1}^{m} \frac{\varepsilon^{\mu_i}(q_i,\lambda_i)\,(-q_i^2)}{\sqrt{(2\pi)^3 2\omega_{q_i}}} \prod_{j=1}^{n} \frac{\varepsilon^{\nu_j *}(p_j,\lambda'_j)\,(-p_j^2)}{\sqrt{(2\pi)^3 2\omega_{p_j}}} \Bigg|_{q_i^2,p_j^2\to 0}$$
$$\times \widetilde{G}_{\mu_1\cdots\mu_m\nu_1\cdots\nu_n}(q_1,\ldots,q_m;\,p_1,\ldots,p_n)\,, \tag{8.284}$$

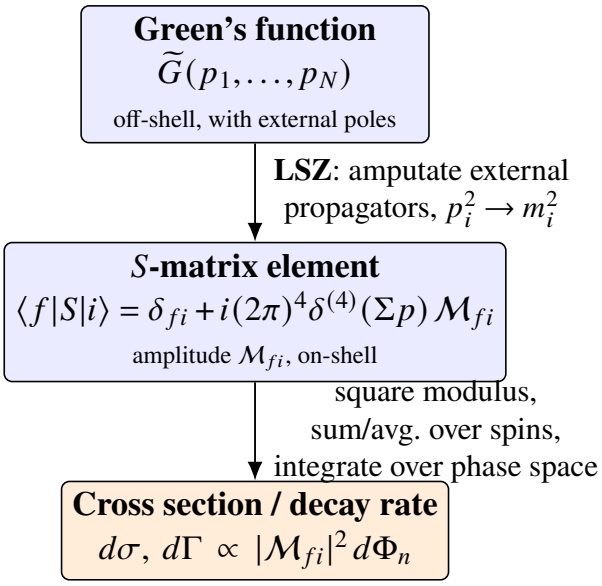
where $\widetilde{G}$ is the Fourier-transformed connected Green function of $m+n$ photon fields, with external indices left uncontracted until the polarisation vectors are attached.

Practical content. The structure is identical to the scalar formula (8.218): each external photon leg contributes a factor of $i/\sqrt{Z_3}$, an amputation factor $-p^2$ that kills the external pole, an on-shell polarisation ε^μ (or $\varepsilon^{\mu *}$ for outgoing), and a normalisation $1/\sqrt{(2\pi)^3 2\omega_p}$. As in the scalar case, the procedure is best understood as a residue prescription: continue the external photon momenta off-shell, multiply by $(-p^2)$, and only at the end take $p^2 \to 0$ with the appropriate transverse polarisation. The Ward identity guarantees that unphysical (longitudinal/scalar) polarisations decouple from the physical S-matrix, so the sum over λ in practical applications runs only over the two transverse helicities.

8.9 Green's functions and the S-matrix

In the previous sections we derived the LSZ reduction formula, which expresses S-matrix elements in terms of vacuum expectation values of time-ordered products of field operators. These vacuum expectation values are called **Green's functions** (or correlation functions). In this section we explain precisely what Green's functions are, how to compute them using perturbation theory, and how the LSZ formula connects them to physical scattering amplitudes.

The overall logical flow from correlation functions to observables can be summarised as follows:



The first arrow is the content of the LSZ formula derived in this chapter; the second arrow is the standard transition from amplitudes to observable rates, treated separately when computing cross sections.

Why are they called "Green's functions"? Connection with PDE theory

Before defining the QFT object precisely, it is worth pausing on the name. The term *Green's function* is borrowed from the classical theory of linear partial differential equations, where given a linear operator L one defines $G(x,y)$ by

$$L_x\, G(x,y) = \delta^{(d)}(x-y)\,, \tag{8.285}$$

i.e. the response of the system $L\phi = 0$ to a point-like source at y. The crucial fact is that the QFT object (8.295) is *not* merely homonymous with this classical notion: in the free case the two coincide *literally*, and in the interacting case the QFT version is a coherent generalisation through the Schwinger–Dyson equations and the generating functional.

Free case: literal identification. For the free scalar field, the two-point time-ordered VEV is precisely the Feynman Green's function of the Klein–Gordon operator:

$$i\Delta_F(x-y) \;\equiv\; \langle 0|T\big(\varphi(x)\varphi(y)\big)|0\rangle\,, \qquad (\Box_x + m^2)\, i\Delta_F(x-y) \;=\; -i\,\delta^{(4)}(x-y)\,. \tag{8.286}$$

The left-hand identity is a VEV of operators, the right-hand one is the classical defining equation $LG = \delta$ for $L = \Box + m^2$. The only non-trivial choice is which fundamental solution of $(\Box + m^2)$ one takes (retarded, advanced, Wightman, Feynman, …): different choices differ by homogeneous solutions, i.e. by how the

poles at $p^0 = \pm\omega_p$ are closed in the complex plane. The time-ordered product naturally selects the Feynman $i\epsilon$ prescription:

$$i\Delta_F(x-y) \;=\; \int \frac{d^4p}{(2\pi)^4}\,\frac{i\,e^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon}\,. \tag{8.287}$$

Same mathematical object as in the PDE theory; the QFT vacuum simply fixes the contour.

Interacting case: Schwinger–Dyson equations. For an interacting Lagrangian $\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2 - V(\varphi)$, the full two-point function

$$G^{(2)}(x-y) \;=\; \langle\Omega|T\big(\varphi(x)\varphi(y)\big)|\Omega\rangle \tag{8.288}$$

no longer satisfies the free $L\,G = \delta$ equation. Instead, it obeys the *Schwinger–Dyson* relation

$$(\Box_x + m^2)\,G^{(2)}(x-y) \;=\; -i\,\delta^{(4)}(x-y) \;-\; \langle\Omega|T\big(V'(\varphi(x))\,\varphi(y)\big)|\Omega\rangle\,, \tag{8.289}$$

which is the operator version of the classical equation of motion sourced by a δ, with the time-ordering producing the extra interaction term on the right. Iterating (8.289) couples $G^{(2)}$ to higher correlators. In perturbation theory the same content is organized by the *Dyson equation* for the exact scalar propagator,

$$i\Delta_F^{\text{full}}(p) = i\Delta_F^0(p) + i\Delta_F^0(p)\,\big[-i\Pi(p^2)\big]\,i\Delta_F^{\text{full}}(p), \tag{8.290}$$

where $i\Delta_F^0(p) = i/(p^2-m_0^2+i\epsilon)$ and $-i\Pi(p^2)$ is the scalar 1PI self-energy insertion. Equivalently,

$$i\Delta_F^{\text{full}}(p) = \frac{i}{p^2 - m_0^2 - \Pi(p^2) + i\epsilon}\,,$$

so the dressed inverse kinetic operator is shifted by the self-energy. The pole moves from the bare to the physical mass, and the pole residue is the wave-function renormalisation factor already encountered in the LSZ formula.

Synthesis: generating functional and path integral. The link is made fully systematic by the generating functional

$$\mathcal{Z}[J] = \int \mathcal{D}\varphi\; e^{iS[\varphi] + i\int d^4x\,J(x)\varphi(x)}\,, \qquad W[J] = -i\ln\mathcal{Z}[J]\,, \tag{8.291}$$

where $\mathcal{Z}[J]$ generates the full Green functions, while $W[J]$ generates the connected ones:

$$\begin{aligned} G^{(n)}(x_1,\dots,x_n) &= \frac{1}{i^n\,\mathcal{Z}[0]}\,\frac{\delta^n\mathcal{Z}[J]}{\delta J(x_1)\cdots\delta J(x_n)}\bigg|_{J=0}\,, \\ G_c^{(n)}(x_1,\dots,x_n) &= \frac{1}{i^{n-1}}\,\frac{\delta^n W[J]}{\delta J(x_1)\cdots\delta J(x_n)}\bigg|_{J=0}\,. \end{aligned} \tag{8.292}$$

For the free theory the path integral is Gaussian and yields explicitly

$$\mathcal{Z}_{\text{free}}[J] \;\propto\; \exp\left[-\frac{1}{2}\int d^4x\, d^4y\; J(x)\, i\Delta_F(x-y)\, J(y)\right], \tag{8.293}$$

whose kernel is exactly the classical Klein–Gordon Feynman Green's function, with our convention $i\Delta_F = -i(\Box + m^2)^{-1}_{\text{Feynman}}$. Wick's theorem then says that every free n-point function is a sum of products of $i\Delta_F$'s — diagrammatically, the free Green's functions are built entirely out of the classical PDE Green's function. In the interacting theory one writes

$$\mathcal{Z}[J] = \exp\left[i\, S_{\text{int}}\left(-i\frac{\delta}{\delta J}\right)\right]\mathcal{Z}_{\text{free}}[J]\,, \tag{8.294}$$

so the perturbative expansion is a series of interaction operators applied to the free Gaussian, producing the familiar Feynman diagrams: vertices joined by lines, where each line is the Feynman Green's function $i\Delta_F$.

Physical reading. The classical Green's function $G(x,y)$ represents the response of a linear system at x to a point-like source placed at y. The QFT Green's function $G^{(2)}(x,y)$ represents the quantum amplitude for a field quantum to propagate from y to x. In the free theory these two statements are mathematically the same; in the interacting theory the QFT Green's function generalises the classical one to include all virtual processes (self-interactions, loops, …) that dress the propagation. The terminology is therefore not metaphorical: it is a genuine extension of the PDE notion to the operator-valued, multi-point setting of QFT.

Summary table.

Aspect	**Classical PDE**	**QFT**
Defining equation	$L\,G(x,y) = \delta(x-y)$	$(\Box + m^2)i\Delta_F = -i\delta$ (free); Schwinger–Dyson (8.289) (interacting)
Meaning	response to point source	quantum propagation amplitude
Free 2-point function	literal identification	literal identification
Interacting 2-pt.	—	Green's function of the self-energy-dressed inverse kinetic operator
$n > 2$-point functions	—	generated by $\mathcal{Z}[J]$; perturbatively built from free $i\Delta_F$ connected by vertices

What is a Green's function?

A Green's function is the vacuum expectation value of a time-ordered product of *interacting* (Heisenberg-picture) field operators. Physically, it describes the amplitude for creating particles at certain spacetime points and detecting them at others, with all possible interactions included.

For a scalar field theory, the **n-point Green's function** is:

$$G(x_1,\dots,x_n) = \langle\Omega|T\big(\varphi(x_1)\cdots\varphi(x_n)\big)|\Omega\rangle\,, \tag{8.295}$$

where $|\Omega\rangle$ is the true vacuum of the interacting theory (the ground state of the full Hamiltonian H) and $\varphi(x)$ is the Heisenberg-picture field. The time-ordering T places operators at later times to the left.

For QED, the Green's function contains one field operator for each external particle:

$$G^{\mu\cdots}(x,y,z,\dots) = \langle\Omega|T\{A^\mu(x)\cdots\psi(y)\cdots\bar\psi(z)\cdots\}|\Omega\rangle\,, \tag{8.296}$$

with A^μ for each photon, ψ for each electron created (or positron destroyed), and $\bar\psi$ for each electron destroyed (or positron created).

Why are Green's functions useful?

- They are the natural objects produced by Feynman diagrams.
- They can be computed order-by-order in perturbation theory using Wick's theorem.
- The LSZ formula tells us exactly how to extract the physical S-matrix from them.
- They encode all information about the interacting theory: masses, coupling constants, bound states, etc.

Computing Green's functions: the interaction-picture formula

The Green's function (8.295) is defined in the Heisenberg picture, where states are time-independent and operators evolve with the full Hamiltonian. To compute it using perturbation theory, we need to express it in the *interaction picture*, where operators evolve with the free Hamiltonian and the interaction appears explicitly in the time-evolution operator.

The result (derived in detail in Peskin–Schroeder Ch. 4, or Mandl–Shaw Ch. 6) is:

$$\boxed{G(x_1,\dots,x_n) = \frac{\langle 0|T\big\{S\,\varphi_I(x_1)\cdots\varphi_I(x_n)\big\}|0\rangle}{\langle 0|S|0\rangle}}\,, \tag{8.297}$$

where:

- $|0\rangle$ is the *free* vacuum (ground state of H_0),
- $\varphi_I(x)$ is the field in the interaction picture (evolves with H_0, so it's a free field),
- $S = T\exp\left(-i\int d^4x\,\mathscr{H}_I(x)\right)$ is the interaction-picture Dyson operator, i.e. the perturbative representative of the S-matrix.

Where does this formula come from? The key steps are:

1. The Heisenberg field is related to the interaction-picture field by $\varphi_H(x) = U^\dagger(t,t_0)\,\varphi_I(x)\,U(t,t_0)$, where $U(t,t_0) = T\exp\left(-i\int_{t_0}^{t} dt'\,H_I(t')\right)$ is the time-evolution operator in the interaction picture.
2. The true vacuum $|\Omega\rangle$ is related to the free vacuum $|0\rangle$ by the Gell-Mann–Low formula: $|\Omega\rangle \propto U(0,-\infty)|0\rangle$, up to the adiabatic $i\epsilon$ prescription, normalisation and phase discussed in Appendix I.
3. After inserting these relations and simplifying, all the U operators combine into the S-matrix, and the phase factors produce the denominator $\langle 0|S|0\rangle$.

Box 8.8 Role of the denominator $\langle 0|S|0\rangle$

The denominator has a simple diagrammatic interpretation. When we expand the numerator of (8.297) using Wick's theorem, we get Feynman diagrams of two types:

1. Diagrams *connected* to the external field insertions $\varphi_I(x_1),\ldots,\varphi_I(x_n)$.
2. *Vacuum bubbles*: closed diagrams with no external legs, multiplying the connected diagrams.

The expansion of $\langle 0|S|0\rangle$ in the denominator generates *exactly the same* vacuum-bubble factor. Dividing cancels all vacuum bubbles. What remains are diagrams with no vacuum-bubble component: every connected component contains at least one external insertion. The full Green function may still factorise into disconnected pieces involving disjoint groups of external points; the fully connected part is selected separately by passing to G_c. This is the **linked-cluster theorem**.

Perturbative evaluation: Feynman diagrams

Expanding the S-matrix to order n:

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d^4z_1 \cdots d^4z_n\, T\{\mathscr{H}_I(z_1)\cdots\mathscr{H}_I(z_n)\}, \tag{8.298}$$

and substituting into (8.297), we apply Wick's theorem to evaluate the time-ordered product. Each contraction between two fields produces a Feynman propagator, and

each uncontracted interaction vertex produces a coupling constant. The systematic rules that emerge are the **Feynman rules** for the theory.

The Green's function $G(x_1,\dots,x_n)$ is therefore the sum of all Feynman diagrams with n external points at $x_1,\dots,x_n$, with all vacuum bubbles removed. In momentum space (after Fourier transforming each external coordinate) it is useful to separate the overall conservation delta function:

$$\tilde{G}(p_1,\dots,p_n) = (2\pi)^4\delta^{(4)}\left(\sum_i p_i\right) G_{\text{strip}}(p_1,\dots,p_n), \tag{8.299}$$

where the delta function enforces overall energy-momentum conservation and G_{strip} denotes the remaining momentum-dependent distribution.

Structure of the n-point function

The momentum-space Green's function has the following pole structure near the external one-particle shells. For a scalar field, with $n \geq 3$ external points and all external momenta conventionally taken as incoming, its most singular part is the product of one factor $iZ/(p_i^2 - m^2 + i\epsilon)$ for each external leg, multiplied by the amputated Green function $G_{\text{amp.}}(p_1,\dots,p_n)$; the remainder $R(p_1,\dots,p_n)$ is less singular in at least one external invariant $p_i^2 - m^2$. The two-point function is the exceptional case: it is itself the full propagator and contains only one one-particle pole. For $n \geq 3$, in perturbative language one says that the connected Green function is an amputated kernel multiplied by one full propagator for each external leg. Here $iD_F^{\text{full}}(p)$ is the *full* (exact, interacting) Feynman propagator, i.e. the Fourier transform of the two-point connected Green function (8.288) of the field $\varphi(x)$, including all self-energy corrections. Concretely, the Dyson resummation gives $iD_F^{\text{full}}(p) = i/[p^2 - m_0^2 - \Pi(p^2) + i\epsilon]$, with m_0 the bare mass and $\Pi(p^2)$ the 1PI self-energy (the systematic derivation appears in (8.301) below); near the physical mass shell, iD_F^{full} exhibits the pole $iZ/(p^2 - m_{\text{phys}}^2)$ familiar from the Källén–Lehmann representation. The factor $G_{\text{amp.}}$ is the truncated, or amputated, connected Green function obtained by stripping one full propagator from each external line. For two- and three-point proper functions this object is closely related to the usual one-particle-irreducible (1PI) vertex $\Gamma^{(n)}$. For a general scattering process, however, the amputated connected Green function may still contain internal propagators and several 1PI subgraphs; "amputated" only means that the propagators attached to the external field insertions have been removed.

This gives the useful dictionary between the Mandl–Shaw and Peskin–Schroeder presentations. Mandl–Shaw build the Green functions from the interaction-picture expansion and then identify the external legs diagrammatically. Peskin–Schroeder emphasize that the same external legs are the isolated poles of the exact Green function. LSZ is the operation that takes the residue of those poles.

The S-matrix from Green's functions

The LSZ reduction formula shows that the S-matrix element is obtained by:

1. Taking the n-point Green's function.
2. Isolating the external one-particle poles.
3. Amputating the external propagators: multiply by $(m^2 - p_i^2)$, with the sign fixed by the Fourier convention used above, for each scalar leg.
4. Putting the external momenta on-shell: $p_i^2 \to m^2$.
5. Multiplying by wave function factors: $u(p,s)$ and $\bar{u}(p,s)$ for external electrons, $v(p,s)$ and $\bar{v}(p,s)$ for external positrons, with the precise placement fixed by the incoming/outgoing assignment.
6. Including the wave function renormalization factors, e.g. $(i/\sqrt{Z})^n$ for n scalar external legs before accounting for convention-dependent signs.

This procedure isolates the physical scattering amplitude from the full Green's function. The factors $(\Box + m^2)$ in the LSZ formula become $(m^2 - p^2)$ in momentum space. They are applied before the on-shell limit: their role is to cancel the poles $1/(p^2 - m^2)$ from the external propagators, leaving a finite residue as $p^2 \to m^2$.

Box 8.9 Physical Interpretation

The Green's function describes the propagation and interaction of virtual particles at all values of p^2. The S-matrix describes the scattering of *real, on-shell* particles with $p^2 = m^2$.

The amputation procedure "removes" the external propagators because real particles don't need to propagate from infinity—they are already there as asymptotic states. The on-shell condition selects only those contributions where the external particles satisfy the relativistic dispersion relation.

The factor $\sqrt{Z}$ for each external leg accounts for the overlap between the local interacting field $\varphi(x)$ and the isolated one-particle state. With canonical normalization this overlap satisfies $0 < Z \leq 1$, and it is strictly smaller than one only when continuum states carry some of the spectral weight.

The key insight of LSZ is that we don't need to distinguish initial from final states when computing the Green's function—the time-ordered product handles everything automatically. The projection onto specific initial and final states is achieved by the $(\Box + m^2)$ factors and the exponentials $e^{\pm ip\cdot x}$.

Connected and disconnected diagrams

A Feynman diagram is **connected** if every vertex can be reached from every other vertex by following internal lines. The Green's function contains both connected

and disconnected contributions:

$$G(x_1,\dots,x_n) = G_c(x_1,\dots,x_n) + \text{disconnected pieces}. \tag{8.300}$$

For S-matrix elements, the disconnected pieces correspond to processes where some particles don't interact at all (they pass through unchanged). The interesting physics—the actual scattering—is contained in the connected part.

Cluster decomposition says that the full Green function factorizes when subsets of points are separated by large spacelike distances. Equivalently, the connected part tends to zero in that limit. This property is essential for the physical interpretation of scattering amplitudes.

The Dyson equation and one-particle-irreducible diagrams

The perturbative evaluation of Green's functions is organized systematically by introducing the concept of **one-particle-irreducible** (1PI) diagrams—those connected diagrams that remain connected after cutting any single internal line. Every connected diagram is built from 1PI blocks joined by full propagators; this observation leads to powerful resummation techniques [1, 6].

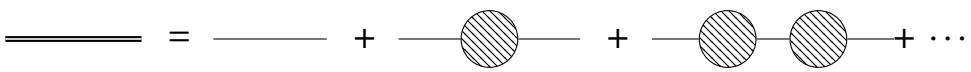
For the two-point function the construction is particularly transparent. The full (connected) propagator can be decomposed as a geometric series of 1PI self-energy insertions:

$$iD_F^{\text{full}}(p) = iD_F(p) + iD_F(p)\big[-i\Pi(p^2)\big]iD_F(p) + \cdots = \frac{i}{p^2 - m_0^2 - \Pi(p^2) + i\epsilon}, \tag{8.301}$$

where $D_F(p) = 1/(p^2 - m_0^2 + i\epsilon)$ is the free propagator, m_0 is the bare mass, and $-i\Pi(p^2)$ is the sum of all 1PI two-point diagrams (the self-energy). This is the **Dyson equation** for the scalar propagator, already encountered in (8.155). Its strength is that the full propagator—an object involving diagrams of arbitrarily high order—is determined once the 1PI self-energy is known.

Diagrammatically, the Dyson equation takes the recursive form (full = free + free × 1PI × full):

where the double line represents the full propagator, the single line the free propagator, and the shaded blob the 1PI self-energy $-i\Pi(p^2)$. Iterating the recursion generates the geometric series:



which sums to the closed-form expression (8.301).

The same structure holds for the Dirac field. The Dyson equation for the full fermion propagator is

$$iS_F^{\text{full}}(p) = \frac{i}{\not{p} - m_0 - \Sigma(p) + i\epsilon}, \tag{8.302}$$

where $-i\Sigma(p)$ is the sum of all 1PI insertions in the fermion line, in agreement with (8.181).

From full to connected to 1PI: hierarchy of Green's functions

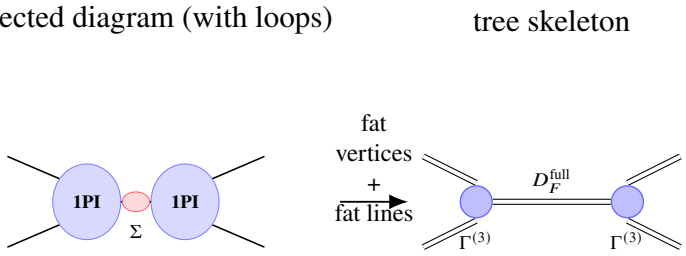
The classification of diagrams as full, connected, or 1PI defines a hierarchy of Green's functions that plays a central role in both the Mandl–Shaw and Peskin–Schroeder treatments. For an n-point function in a scalar theory:

1. The **full Green's function** $G(x_1, \ldots, x_n) = \langle \Omega | T(\varphi(x_1) \cdots \varphi(x_n)) | \Omega \rangle$ includes all diagrams, connected and disconnected, with vacuum bubbles removed by the denominator in (8.297) when it is evaluated perturbatively.
2. The **connected Green's function** G_c is the subset of diagrams in which all external points are linked. It is obtained from the full Green's function by subtracting all factorisations into disjoint groups.
3. The **1PI vertex function** $\Gamma^{(n)}$ (also called the proper vertex) is obtained from the connected Green's function by amputating all external propagators and restricting to diagrams that cannot be split into two pieces by cutting a single internal line.

Idea: replace loops with "fat" blocks. The decomposition that connects the three levels above is best understood pictorially. Take any connected Feynman diagram for G_c. We can systematically identify:

- *1PI sub-diagrams*: maximal sub-graphs that *cannot* be disconnected by cutting a single internal line. These contain all the loops and resum into the proper vertices $\Gamma^{(n)}$.
- *Self-energy sub-graphs*: 1PI two-point insertions. Resumming them along any line turns a free propagator D_F into a *full* propagator D_F^{full}.

If we represent each 1PI sub-diagram by a single "fat vertex" $\bullet$ and each chain of self-energy insertions by a single "fat line" $\Longleftrightarrow$, what is left is a graph with no loops — a **tree**, in the standard graph-theoretic sense:



The **tree skeleton** on the right is the same diagram in which every loop and every self-energy chain has been absorbed into a fat vertex ($\Gamma^{(n)}$) or a fat propagator (D_F^{full}). What remains is, by construction, loop-free: that is what "tree" means here. The skeleton has the same topology as a tree graph, even though each fat element internally encodes infinitely many ordinary diagrams.

Decomposition formula. For $n \geq 3$, the general statement is:

$$G_c(p_1,\ldots,p_n) = \sum_{\substack{\text{tree skeletons}\\ \text{with fat 1PI vertices}}} \prod_{\text{ext. legs } i} iD_F^{\text{full}}(p_i) \times \prod_{\text{int. lines } j} iD_F^{\text{full}}(k_j) \prod_{\text{vertices } v} i\Gamma^{(n_v)}(k_{v,1},\ldots,k_{v,n_v}), \quad (8.303)$$

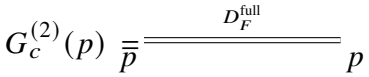
with momentum conservation at each fat vertex. "Tree skeleton" means a graph topology in which:

- each node is a 1PI vertex $\Gamma^{(n_v)}$ (any number of legs $n_v \geq 3$),
- each edge is a full propagator D_F^{full},
- cutting any single edge disconnects the graph (no closed loops at the skeleton level).

All the loop content lives inside the fat vertices and the full propagators; the sum in (8.303) only runs over the few *tree topologies* that connect the n external points.

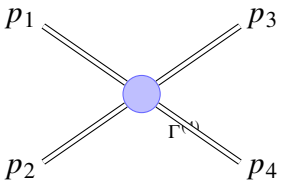
Concrete examples.

$n = 2$. The only possible tree topology with two external points has no fat vertices and a single full propagator joining them:



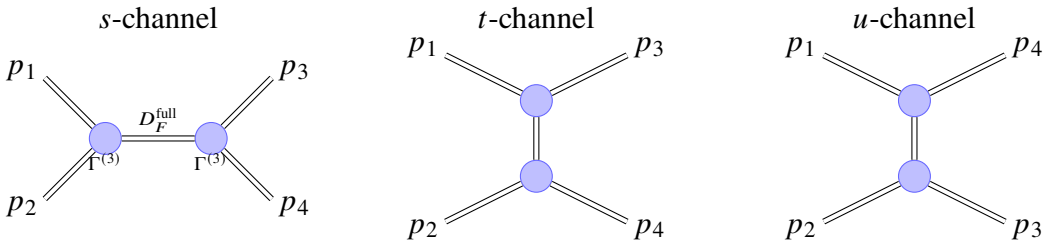
This is just the Dyson-resummed propagator: $G_c^{(2)}(p) = iD_F^{\text{full}}(p)$. This is the special case excluded from the product formula (8.303): there is no 1PI vertex with external propagators attached; the whole connected two-point function is the single full line whose resummation is given by the Dyson equation (8.301).

$n = 4$ *in* ϕ^4 *theory.* The only 1PI vertex with the right symmetry is $\Gamma^{(4)}$ (at tree level $i\Gamma^{(4)} = -i\lambda$). The tree skeleton has a single fat vertex with four full external propagators:

$G_c^{(4)} =$ 

Notice that the usual s-, t-, u-channel diagrams of the perturbative expansion of $G_c^{(4)}$ are *not* separate skeletons here: they all live *inside* the loop expansion of the single 1PI vertex $\Gamma^{(4)}$.

$n = 4$ in a theory with a cubic interaction (e.g. ϕ^3 or $\phi^4 + \phi^3$). Two distinct tree topologies contribute. The first is the contact topology, exactly as above. The second is the *exchange topology*: two 1PI three-vertices joined by one internal full propagator. There are three such graphs (s, t, u channels), corresponding to the three ways of partitioning $\{p_1, p_2, p_3, p_4\}$ into two pairs:



Each of these is a single skeleton with two fat vertices, four fat external legs, and one fat internal line. Cutting the internal line disconnects the graph (that is precisely the "tree" property).

What is and is not a tree skeleton. The key word is "skeleton": all loops have already been absorbed inside the fat blocks. Therefore the following ordinary diagrams *do not* appear in the skeleton sum:

- self-energy bubbles on an internal or external line — they are already absorbed into the corresponding D_F^{full};
- triangle, box, or higher loops with $n \le n_v$ external lines — they are already absorbed into the corresponding $\Gamma^{(n_v)}$.

The only freedom left is the choice of how the fat blocks are linked together at the level of the skeleton, and that information is fully captured by the tree topology.

Why this rearrangement is useful. (8.303) makes the structure of an interacting QFT modular: all genuine loop physics is encapsulated in a finite set of 1PI vertices and one full propagator; the skeletons are then trees, of which there are only finitely many for each n. This is the starting point of the *effective action* formalism [6]: the generating functional $\Gamma[\varphi_c]$ of 1PI vertices is the Legendre transform of the generating functional $W[J]$ of connected Green functions, and tree-level diagrams of Γ reproduce the full connected G_c. This formalism, though beyond our scope

here, provides the most systematic framework for renormalisation, spontaneous symmetry breaking and the analysis of vacuum stability.

Box 8.10 Why 1PI diagrams?

The decomposition into 1PI building blocks separates two conceptually distinct aspects of the dynamics. The 1PI vertices encode the "hard" interaction physics—the irreducible scattering amplitudes—while the full propagators encode propagation, including the effects of mass and wave function renormalization. In the renormalization programme (Chapters 6–7), this separation is essential: the divergences of the theory can be absorbed into a finite number of 1PI functions (the primitively divergent vertices), and the Dyson equation then propagates these renormalized quantities to all orders. The counting of primitively divergent diagrams is the content of the power-counting theorem; for a renormalizable theory, only a finite number of $\Gamma^{(n)}$ are superficially divergent.

8.10 Summary and physical interpretation

The key results of this chapter can be summarized as follows:

1. **Asymptotic states.** At $t \to \pm\infty$, the interacting fields approach free fields $\varphi \to \sqrt{Z}\,\varphi_{\text{in/out}}$ in the weak sense of matrix elements.
2. **Spectral density.** The exact two-point function contains an isolated one-particle pole plus a multiparticle continuum. The residue of the pole is the spectral weight Z.
3. **The S-matrix.** The transition amplitude $S_{\beta\alpha} = \langle \beta \text{ out} | \alpha \text{ in} \rangle$ connects asymptotic free-particle states.
4. **The LSZ reduction formula.** S-matrix elements are residues of time-ordered Green functions at their external one-particle poles, equivalently Green functions with external legs amputated and momenta put on-shell.
5. **Wave function renormalization.** For a canonically normalized scalar field with an isolated one-particle pole, the spectral weight satisfies $0 < Z \leq 1$. It is the probability weight for the local field to create the single physical particle from the vacuum; any remaining weight belongs to continuum states.

The LSZ formalism provides a rigorous foundation for the Feynman diagram technique. It shows that:

- The perturbation expansion of $\langle 0|T(S\,\text{fields})|0\rangle / \langle 0|S|0\rangle$ generates Feynman diagrams without vacuum bubbles.

- External leg factors arise from taking the residues of the exact propagator poles.
- The connection between Green's functions and S-matrix elements is exact under the LSZ assumptions; weak-coupling perturbation theory is only one way of evaluating the Green functions.

The physical picture is clear: scattering experiments involve preparing well-separated wave packets at $t \to -\infty$, letting them interact in some finite spacetime region, and observing well-separated final states at $t \to +\infty$. The LSZ formalism provides the mathematical framework for computing the transition amplitudes between such asymptotic states.

Part IV

Supersymmetry