Summary

Collecting all four transformation laws for a Lorentz transformation $\Lambda \leftrightarrow \pm M \in SL(2,\mathbb{C})$:

$$\boxed{\begin{array}{llr} \chi_\alpha \longrightarrow M_\alpha{}^\beta\, \chi_\beta & & \text{(L-type, lower undotted)} \\ \chi^\alpha \longrightarrow (M^{-1})^\alpha{}_\beta\, \chi^\beta & & \text{(L-type, upper undotted)} \\ \bar\chi^{\dot\alpha} \longrightarrow [(M^{-1})^\dagger]^{\dot\alpha}{}_{\dot\beta}\, \bar\chi^{\dot\beta} & & \text{(R-type, upper dotted)} \\ \bar\chi_{\dot\alpha} \longrightarrow (M^*)_{\dot\alpha}{}^{\dot\beta}\, \bar\chi_{\dot\beta} = \epsilon_{\dot\alpha\dot\gamma}\, [(M^{-1})^\dagger]^{\dot\gamma}{}_{\dot\delta}\, \epsilon^{\dot\delta\dot\beta}\, \bar\chi_{\dot\beta} & & \text{(R-type, lower dotted)} \end{array}} \tag{9.96}$$

The pattern is symmetric between the two chiralities. Within each sector, the spinor metric ϵ switches between a representation and its contragredient:

$$\underbrace{M}_{\text{acts on } \chi_\alpha} \xrightarrow{\epsilon} \underbrace{(M^{-1})}_{\text{acts on } \chi^\alpha}, \qquad \underbrace{(M^{-1})^\dagger}_{\text{acts on } \bar\chi^{\dot\alpha}} \xrightarrow{\epsilon} \underbrace{M^*}_{\text{acts on } \bar\chi_{\dot\alpha}} .$$

Complex conjugation maps between chiralities: $M_\alpha{}^\beta \leftrightarrow (M^*)_{\dot\alpha}{}^{\dot\beta}$ and $(M^{-1})^\alpha{}_\beta \leftrightarrow [(M^{-1})^\dagger]^{\dot\alpha}{}_{\dot\beta}$.

With these transformation laws in hand, we can now systematically construct all Lorentz-invariant and Lorentz-covariant spinor bilinears.

9.4 Bilinear covariants

Spinor products and contractions

Lorentz-invariant spinor products

Box 9.4 The descending/ascending staircase rule (NW–SE vs. SW–NE)

The shorthand notations $\chi\eta$ and $\bar\chi\bar\eta$ are not symmetric in the index height: each is defined by a *fixed* contraction pattern that depends on the spinor type. Stated graphically:

- **Undotted (left-handed, $(\frac{1}{2},0)$) spinors are contracted "descending" — from *NorthWest* (upper-left) to *SouthEast* (lower-right)**:

$$\chi\eta \equiv \chi^\alpha \eta_\alpha = \chi^\alpha{\searrow}\eta_\alpha .$$

The left spinor carries the index *upstairs*, the right one carries it *downstairs*; the contraction goes top-left $\longrightarrow$ bottom-right.

- **Dotted (right-handed, $(0, \frac{1}{2})$) spinors are contracted "ascending" — from *SouthWest* (lower-left) to *NorthEast* (upper-right)**:

$$\bar{\chi}\bar{\eta} \equiv \bar{\chi}_{\dot{\alpha}}\bar{\eta}^{\dot{\alpha}} = \bar{\chi}_{\dot{\alpha}} \nearrow \bar{\eta}^{\dot{\alpha}}.$$

The left spinor carries the index *downstairs*, the right one carries it *upstairs*; the contraction goes bottom-left $\rightarrow$ top-right.

The reason for the asymmetry is the relative sign $\epsilon^{12} = +1$, $\epsilon_{12} = -1$ between the metric used to raise undotted and lower dotted indices (Aitchison–Wess–Bagger convention; the same conventions are recorded in (†)).
Reversing the staircase changes the sign of the bilinear. Concretely:

$$\chi_\alpha \eta^\alpha = -\chi^\alpha \eta_\alpha = -\chi\eta.$$

The proof is one line. Raise the index on η using $\eta^\alpha = \epsilon^{\alpha\beta}\eta_\beta$:

$$\chi_\alpha \eta^\alpha = \chi_\alpha \epsilon^{\alpha\beta} \eta_\beta = \epsilon^{\alpha\beta} \chi_\alpha \eta_\beta.$$

Now relabel the dummy indices $\alpha \leftrightarrow \beta$ and use the antisymmetry $\epsilon^{\beta\alpha} = -\epsilon^{\alpha\beta}$:

$$\epsilon^{\alpha\beta} \chi_\alpha \eta_\beta = \epsilon^{\beta\alpha} \chi_\beta \eta_\alpha = -\epsilon^{\alpha\beta} \chi_\beta \eta_\alpha = -\chi^\alpha \eta_\alpha = -\chi\eta,$$

where in the last step we used the definition (9.97). The sign flip comes entirely from the antisymmetry of ϵ — not from any reordering of χ and η. (For Grassmann spinors there is a *separate* sign from the anticommutation, treated below in (9.101); the two effects must be tracked independently.) An analogous calculation for dotted indices gives $\bar{\chi}^{\dot{\alpha}}\bar{\eta}_{\dot{\alpha}} = -\bar{\chi}_{\dot{\alpha}}\bar{\eta}^{\dot{\alpha}} = -\bar{\chi}\bar{\eta}$.
This staircase rule is the single most frequent source of sign errors in two-component spinor calculations; once it is internalised, the unindexed shorthand $\chi\eta, \bar{\chi}\bar{\eta}$ is unambiguous.

Given two $(\frac{1}{2}, 0)$ fermion fields χ_α and η_α, the bilinear combination:

$$\boxed{\chi\eta \equiv \chi^\alpha \eta_\alpha = \epsilon^{\alpha\beta} \chi_\beta \eta_\alpha} \qquad (\text{NW} \rightarrow \text{SE}) \tag{9.97}$$

is invariant under Lorentz transformations (proved below). The ϵ tensor with $\epsilon^{12} = -\epsilon_{12} = +1$ is used to raise and lower spinor indices.

Similarly, for the R-type partners $\bar{\chi}_{\dot{\alpha}}$ and $\bar{\eta}_{\dot{\alpha}}$ defined via (9.43):

$$\boxed{\bar{\chi}\bar{\eta} \equiv \bar{\chi}_{\dot{\alpha}}\bar{\eta}^{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}} \bar{\chi}^{\dot{\beta}} \bar{\eta}^{\dot{\alpha}}} \qquad (\text{SW} \rightarrow \text{NE}). \tag{9.98}$$

Proof of Lorentz invariance of $\chi\eta$ and $\bar{\chi}\bar{\eta}$. Under $\chi_\alpha \to M_\alpha{}^\beta \chi_\beta$, $\eta_\alpha \to M_\alpha{}^\beta \eta_\beta$:

$$(\chi\eta)' = \epsilon^{\alpha\beta}\chi'_\beta\eta'_\alpha = \epsilon^{\alpha\beta}(M_\beta{}^\gamma\chi_\gamma)(M_\alpha{}^\delta\eta_\delta) = \underbrace{\epsilon^{\alpha\beta}M_\alpha{}^\delta M_\beta{}^\gamma}_{\epsilon^{\delta\gamma}\text{ by (9.92)}}\chi_\gamma\eta_\delta$$
$$= \epsilon^{\delta\gamma}\chi_\gamma\eta_\delta = \chi^\delta\eta_\delta = \chi\eta\,.$$

For the dotted product, an identical computation using $(M^*)_{\dot\alpha}{}^{\dot\beta}$ (which also has unit determinant, so the analogous identity (9.93) holds) gives $(\bar{\chi}\bar{\eta})' = \bar{\chi}\bar{\eta}$. Alternatively, invariance of $\bar{\chi}\bar{\eta}$ follows by complex conjugation of $(\chi\eta)' = \chi\eta$. □

Commuting and anticommuting spinors

Box 9.5 What do we mean by "commuting" vs "anticommuting" spinors?

The same symbol χ_α is used in this chapter for two *different* kinds of mathematical objects, which obey opposite statistics:

- **Anticommuting (Grassmann-valued) spinors.** The fundamental quantum fermion fields appearing in a Lagrangian, e.g. the operator $\hat{\chi}_\alpha(x)$ that creates/annihilates a left-handed Weyl particle, are Grassmann-odd: they satisfy

$$\{\hat{\chi}_\alpha(x), \hat{\eta}_\beta(y)\} = 0, \qquad \{\hat{\chi}_\alpha(x), \hat{\chi}^\dagger_\beta(y)\}\big|_{\text{eq. time}} \sim \delta^3(\vec{x}-\vec{y}),$$

  and *anticommute past each other*: $\chi_\alpha\eta_\beta = -\eta_\beta\chi_\alpha$. This is the Pauli exclusion principle written algebraically; it applies to fermionic quantum fields, to Grassmann SUSY parameters θ^α (Chapter 10), and to the spinorial integration variables in path integrals.
- **Commuting (c-number-valued) spinors.** The plane-wave amplitudes $u_\alpha(p,s)$, $v_\alpha(p,s)$ that emerge when one evaluates a Feynman diagram (after the field operators have been contracted with external states) are ordinary $\mathbb{C}$-valued 2-component objects. They obey ordinary number multiplication: $u_\alpha v_\beta = v_\beta u_\alpha$. The same applies to coherent-state wavefunctions and to "test" spinors used in proving Fierz identities.

The two cases differ *only* in the sign that arises when one swaps adjacent spinors in a product. To handle both uniformly we introduce the symbol $(-1)^A$ defined in (9.99) below: $(-1)^A = -1$ for Grassmann (anticommuting) spinors and $(-1)^A = +1$ for $\mathbb{C}$-valued (commuting) spinors. Every identity in (9.100) then specialises to the correct sign in either context.
A surprising and important consequence: for *anticommuting* spinors, $\chi\eta = \eta\chi$ (not $-\eta\chi$!), proved in (9.101) below. The two minus signs — one from antisymmetry of ϵ, one from Grassmann swap — combine to a plus. This is what makes $\chi\eta$ a sensible mass term in a fermionic Lagrangian.

In manipulating expressions, we often deal with both anticommuting fermion quantum fields and commuting spinor wave functions (that arise when evaluating Feynman rules). It is convenient to introduce the notation:

$$(-1)^A \equiv \begin{cases} +1 & \text{commuting spinors} \\ -1 & \text{anticommuting spinors} \end{cases}. \qquad (9.99)$$

The following identities hold for generic spinors z_i:

$$\boxed{\begin{aligned} z_1 z_2 &= -(-1)^A z_2 z_1 \\ \bar{z}_1 \bar{z}_2 &= -(-1)^A \bar{z}_2 \bar{z}_1 \\ z_1 \sigma^\mu \bar{z}_2 &= (-1)^A \bar{z}_2 \bar{\sigma}^\mu z_1 \\ z_1 \sigma^\mu \bar{\sigma}^\nu z_2 &= -(-1)^A z_2 \sigma^\nu \bar{\sigma}^\mu z_1 \\ \bar{z}_1 \bar{\sigma}^\mu \sigma^\nu \bar{z}_2 &= -(-1)^A \bar{z}_2 \bar{\sigma}^\nu \sigma^\mu \bar{z}_1 \\ \bar{z}_1 \bar{\sigma}^\mu \sigma^\rho \bar{\sigma}^\nu z_2 &= (-1)^A z_2 \sigma^\nu \bar{\sigma}^\rho \sigma^\mu \bar{z}_1 \end{aligned}}\,. \tag{9.100}$$

For **anticommuting** two-component fermion fields (the case $(-1)^A = -1$):

$$\chi\eta = \eta\chi, \qquad \bar{\chi}\bar{\eta} = \bar{\eta}\bar{\chi}\,. \tag{9.101}$$

Proof. Using the antisymmetry of $\epsilon^{\alpha\beta}$:

$$\chi\eta = \epsilon^{\alpha\beta}\chi_\beta\eta_\alpha = -\epsilon^{\alpha\beta}\eta_\alpha\chi_\beta = \epsilon^{\beta\alpha}\eta_\alpha\chi_\beta = \eta\chi$$

where we used $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}$ and relabeled the dummy indices. The proof for $\bar{\chi}\bar{\eta}$ is analogous. □

Under hermitian conjugation (valid for both commuting and anticommuting spinors):

$$(\chi\eta)^\dagger = \bar{\eta}\bar{\chi}\,. \tag{9.102}$$

Combining with (9.101) for anticommuting spinors yields:

$$(\chi\eta)^\dagger = \bar{\chi}\bar{\eta}\,. \tag{9.103}$$

Bilinear covariants and Fierz identities

Four-vector bilinears

The spinor products:

$$\begin{aligned} &\chi\sigma^\mu\bar{\eta} \equiv \chi^\alpha\sigma^\mu_{\alpha\dot{\beta}}\bar{\eta}^{\dot{\beta}}, \\ &\bar{\eta}\bar{\sigma}^\mu\chi \equiv \bar{\eta}_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\chi_\alpha \end{aligned} \tag{9.104}$$

transform as Lorentz four-vectors.

Proof. Using the transformation laws $\chi^\alpha \to (M^{-1})^\alpha{}_\beta\chi^\beta$ and $\bar{\eta}^{\dot{\alpha}} \to ((M^{-1})^\dagger)^{\dot{\alpha}}{}_{\dot{\beta}}\bar{\eta}^{\dot{\beta}}$:

$$\begin{aligned} \chi'(x')\sigma^\mu\bar{\eta}'(x') &= \chi^\alpha(x)[M^{-1}\sigma^\mu(M^{-1})^\dagger]_{\alpha\dot{\alpha}}\bar{\eta}^{\dot{\alpha}}(x) \\ &= \Lambda^\mu{}_\nu\chi(x)\sigma^\nu\bar{\eta}(x) \end{aligned} \tag{9.105}$$

where we used (9.59). □

For anticommuting spinors:

$$\bar{\eta}\bar{\sigma}^\mu\chi = -\chi\sigma^\mu\bar{\eta}, \qquad (\chi\sigma^\mu\bar{\eta})^\dagger = -\bar{\chi}\bar{\sigma}^\mu\eta\,. \tag{9.106}$$

Complete list of bilinear covariants

The independent bilinear covariants constructed from two-component spinors are:

Bilinear	Components	Lorentz transformation
$\chi\eta$	1	Scalar
$\bar\chi\bar\eta$	1	Scalar
$\chi\sigma^\mu\bar\eta$	4	Four-vector
$\bar\chi\bar\sigma^\mu\eta$	4	Four-vector
$\chi\sigma^{\mu\nu}\eta$	3	Self-dual antisymmetric tensor
$\bar\chi\bar\sigma^{\mu\nu}\bar\eta$	3	Anti-self-dual antisymmetric tensor

The total is 16 independent components, matching the dimension of the product space of two independent 2×2 matrix spaces.

Fierz identities

Products of two bilinear covariants can be simplified using Fierz identities. The following hold for *both* commuting and anticommuting two-component spinors z_i (in each identity, the spinor rearrangements involve an even number of transpositions, so the net sign is statistics-independent):

$$(z_1z_2)(z_3z_4) = -(z_1z_3)(z_4z_2) - (z_1z_4)(z_2z_3) \tag{9.107}$$
$$(z_1\sigma^\mu\bar z_2)(\bar z_3\bar\sigma_\mu z_4) = -2(z_1z_4)(\bar z_2\bar z_3) \tag{9.108}$$
$$(\bar z_1\bar\sigma^\mu z_2)(\bar z_3\bar\sigma_\mu z_4) = 2(\bar z_1\bar z_3)(z_4z_2) \tag{9.109}$$
$$(z_1\sigma^\mu\bar z_2)(z_3\sigma_\mu\bar z_4) = 2(z_1z_3)(\bar z_4\bar z_2)\,. \tag{9.110}$$

Proof of (9.108). Using (9.70):

$$\begin{aligned}(z_1\sigma^\mu\bar z_2)(\bar z_3\bar\sigma_\mu z_4) &= z_1^\alpha\sigma^\mu_{\alpha\dot\beta}\bar z_2^{\dot\beta}\cdot\bar z_{3\dot\gamma}\bar\sigma_\mu^{\dot\gamma\delta}z_{4\delta}\\ &= z_1^\alpha\bar z_2^{\dot\beta}\bar z_{3\dot\gamma}z_{4\delta}\cdot\underbrace{\sigma^\mu_{\alpha\dot\beta}\bar\sigma_\mu^{\dot\gamma\delta}}_{2\delta_\alpha^\delta\delta_{\dot\beta}^{\dot\gamma}}\\ &= 2\,z_1^\alpha\bar z_2^{\dot\beta}\bar z_{3\dot\beta}z_{4\alpha}\,.\end{aligned} \tag{9.111}$$

Rearranging: move z_4 past the two barred spinors (two transpositions, net sign $(-1)^{2A}=1$ for either statistics):

$$= 2\,(z_1^\alpha z_{4\alpha})\,(\bar z_2^{\dot\beta}\bar z_{3\dot\beta}) = -2\,(z_1z_4)\,(\bar z_2\bar z_3)\,,$$

where the final minus sign comes from the dotted staircase reversal $\bar z_2^{\dot\beta}\bar z_{3\dot\beta} = -\bar z_{2\dot\beta}\bar z_3^{\dot\beta} = -\bar z_2\bar z_3$ (cf. box 9.4), not from the spinor rearrangement. □

9.5 Four-component spinors

In this section, we establish the connection between the two-component spinor formalism developed above and the more familiar four-component Dirac spinor formalism. The motivation for introducing four-component spinors is analogous to the motivation for introducing complex scalar fields: when two real, mass-degenerate scalar fields ϕ_1 and ϕ_2 are present, it is convenient to combine them into a single complex field $\Phi = (\phi_1 + i\phi_2)/\sqrt{2}$. Similarly, when two mass-degenerate two-component spinors χ and η of opposite U(1) charge are present, it is natural to combine them into a single four-component spinor.

The Dirac spinor and gamma matrices

Definition of the four-component Dirac spinor

A **four-component Dirac spinor field** $\Psi(x)$ is constructed from two mass-degenerate two-component spinor fields $\chi_\alpha(x)$ and $\eta_\alpha(x)$ of opposite U(1) charge:

$$\boxed{\Psi(x) \equiv \begin{pmatrix} \chi_\alpha(x) \\ \bar{\eta}^{\dot{\alpha}}(x) \end{pmatrix}}. \tag{9.112}$$

The upper two components transform under the $(\frac{1}{2},0)$ representation, while the lower two components transform under the $(0,\frac{1}{2})$ representation. Thus, Ψ transforms in the reducible $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation of the Lorentz group.

Dirac gamma matrices

The **Dirac gamma matrices** γ^μ are 4×4 matrices that can be expressed in 2×2 block form using the sigma matrices:

$$\boxed{\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu_{\alpha\dot{\beta}} \\ \bar{\sigma}^{\mu\,\dot{\alpha}\beta} & 0 \end{pmatrix}}. \tag{9.113}$$

This is known as the **chiral representation** (or Weyl representation) of the gamma matrices.

Proposition 9.5 (Clifford algebra). The gamma matrices satisfy the anticommutation relation:

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}_4}, \tag{9.114}$$

where $\mathbb{1}_4$ is the 4×4 identity matrix.

Proof. Computing the anticommutator in block form:

$$\begin{aligned}\{\gamma^\mu,\gamma^\nu\} &= \gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu \\ &= \begin{pmatrix}\sigma^\mu\bar\sigma^\nu+\sigma^\nu\bar\sigma^\mu & 0 \\ 0 & \bar\sigma^\mu\sigma^\nu+\bar\sigma^\nu\sigma^\mu\end{pmatrix}\end{aligned} \tag{9.115}$$

Using the identities (9.73) and (9.74):

$$[\sigma^\mu\bar\sigma^\nu+\sigma^\nu\bar\sigma^\mu]_\alpha{}^\beta = 2g^{\mu\nu}\delta_\alpha^\beta, \quad [\bar\sigma^\mu\sigma^\nu+\bar\sigma^\nu\sigma^\mu]^{\dot\alpha}{}_{\dot\beta} = 2g^{\mu\nu}\delta^{\dot\alpha}_{\dot\beta}$$

Therefore $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}_4$. □

The 4×4 identity matrix in the chiral representation has the block form:

$$\mathbb{1}_4 = \begin{pmatrix}\delta_\alpha{}^\beta & 0 \\ 0 & \delta^{\dot\alpha}{}_{\dot\beta}\end{pmatrix}. \tag{9.116}$$

The chirality matrix γ^5

The **chirality matrix** γ^5 is defined by:

$$\boxed{\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma}, \tag{9.117}$$

where $\epsilon^{0123} = +1$. In the chiral representation:

$$\gamma^5 = \begin{pmatrix}-\delta_\alpha{}^\beta & 0 \\ 0 & \delta^{\dot\alpha}{}_{\dot\beta}\end{pmatrix} = \begin{pmatrix}-\mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2\end{pmatrix}. \tag{9.118}$$

Proposition 9.6 (Properties of γ^5). Independently of the representation of the gamma matrices:

$$(\gamma^5)^2 = \mathbb{1}_4, \qquad \{\gamma^\mu,\gamma^5\} = 0. \tag{9.119}$$

Proof. Using the definition (9.117) and the Clifford algebra (9.114):

$$\begin{aligned}(\gamma^5)^2 &= (i)^2\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 \\ &= -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3\end{aligned} \tag{9.120}$$

Using $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu + 2g^{\mu\nu}$ repeatedly to move γ^0 to the right (picking up three sign changes from anticommuting with $\gamma^1,\gamma^2,\gamma^3$), then $(\gamma^0)^2 = \mathbb{1}_4$, and similarly for the other matrices, one obtains $(\gamma^5)^2 = \mathbb{1}_4$.

For the anticommutator, $\gamma^\mu\gamma^5 = i\gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3$. Moving γ^μ through the four gamma matrices produces $(-1)^3 = -1$ sign changes (one for each γ^ν with $\nu \neq \mu$), giving $\gamma^\mu\gamma^5 = -\gamma^5\gamma^\mu$. □

Chiral projection operators

The **chiral projection operators** are defined as:

$$\boxed{P_L \equiv \frac{1}{2}(\mathbb{1}_4 - \gamma^5), \qquad P_R \equiv \frac{1}{2}(\mathbb{1}_4 + \gamma^5)}. \tag{9.121}$$

In the chiral representation:

$$P_L = \begin{pmatrix} \delta_\alpha{}^\beta & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_R = \begin{pmatrix} 0 & 0 \\ 0 & \delta^{\dot\alpha}{}_{\dot\beta} \end{pmatrix}. \tag{9.122}$$

Proposition 9.7 (Properties of projection operators).

$$P_L^2 = P_L, \quad P_R^2 = P_R, \quad P_L P_R = P_R P_L = 0, \quad P_L + P_R = \mathbb{1}_4. \tag{9.123}$$

Proof. These follow directly from $(\gamma^5)^2 = \mathbb{1}_4$:

$$P_L^2 = \tfrac{1}{4}(\mathbb{1}_4 - \gamma^5)^2 = \tfrac{1}{4}(\mathbb{1}_4 - 2\gamma^5 + (\gamma^5)^2) = \tfrac{1}{2}(\mathbb{1}_4 - \gamma^5) = P_L \tag{9.124}$$

$$P_L P_R = \tfrac{1}{4}(\mathbb{1}_4 - \gamma^5)(\mathbb{1}_4 + \gamma^5) = \tfrac{1}{4}(\mathbb{1}_4 - (\gamma^5)^2) = 0 \tag{9.125}$$

The remaining identities follow similarly. □

The **left-handed** and **right-handed Weyl spinor fields** are defined as the eigenstates of γ^5:

$$\Psi_L(x) \equiv P_L\Psi(x) = \begin{pmatrix} \chi_\alpha(x) \\ 0 \end{pmatrix}, \quad \Psi_R(x) \equiv P_R\Psi(x) = \begin{pmatrix} 0 \\ \bar\eta^{\dot\alpha}(x) \end{pmatrix}, \tag{9.126}$$

satisfying $\gamma^5\Psi_L = -\Psi_L$ and $\gamma^5\Psi_R = +\Psi_R$.

Lorentz structure and transformations

Lorentz generators in four-component form

The generators of the Lorentz group in the $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation are:

$$\boxed{\frac{1}{2}\Sigma^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu,\gamma^\nu] = \begin{pmatrix} (\sigma^{\mu\nu})_\alpha{}^\beta & 0 \\ 0 & (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \end{pmatrix}}. \tag{9.127}$$

Proof. Computing the commutator in block form:

$$\begin{aligned}[][\gamma^\mu,\gamma^\nu] &= \gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu \\ &= \begin{pmatrix} \sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu & 0 \\ 0 & \bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu \end{pmatrix}\end{aligned} \tag{9.128}$$

From eqs. (9.83) and (9.84):

$$\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu = -2i(\sigma^{\mu\nu} - \sigma^{\nu\mu}) = -4i\sigma^{\mu\nu}$$

and similarly for the dotted indices. Therefore $\frac{i}{4}[\gamma^\mu,\gamma^\nu] = \mathrm{diag}(\sigma^{\mu\nu},\bar{\sigma}^{\mu\nu})$, or equivalently $\Sigma^{\mu\nu} = 2\,\mathrm{diag}(\sigma^{\mu\nu},\bar{\sigma}^{\mu\nu})$. □

The duality conditions (9.87) for $\sigma^{\mu\nu}$ and $\bar{\sigma}^{\mu\nu}$ imply:

$$\gamma^5\Sigma^{\mu\nu} = \frac{i}{2}\epsilon^{\mu\nu\rho\kappa}\Sigma_{\rho\kappa}\,. \tag{9.129}$$

The spin-$\frac{1}{2}$ angular momentum matrices are $\{\frac{1}{2}\Sigma^i\}$, $i = 1,2,3$, where:

$$\Sigma^i \equiv \frac{1}{2}\epsilon^{ijk}\Sigma_{jk} = \gamma^0\gamma^i\gamma^5 = \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}. \tag{9.130}$$

Gamma matrix identities

The following identities are useful for calculations:

$$\begin{aligned}\gamma^\mu\gamma^\nu\gamma^\rho &= g^{\mu\nu}\gamma^\rho - g^{\mu\rho}\gamma^\nu + g^{\nu\rho}\gamma^\mu \\ &\quad + i\epsilon^{\mu\nu\rho\kappa}\gamma_\kappa\gamma^5 \end{aligned} \tag{9.131}$$

$$\Sigma^{\mu\nu}\gamma^\rho = i\big(g^{\nu\rho}\gamma^\mu - g^{\mu\rho}\gamma^\nu\big) - \epsilon^{\mu\nu\rho\kappa}\gamma_\kappa\gamma^5 \tag{9.132}$$

$$\gamma^\rho\Sigma^{\mu\nu} = i\big(g^{\mu\rho}\gamma^\nu - g^{\nu\rho}\gamma^\mu\big) - \epsilon^{\mu\nu\rho\kappa}\gamma_\kappa\gamma^5\,. \tag{9.133}$$

Proof of (9.131). Using the Clifford algebra $\gamma^\mu\gamma^\nu = g^{\mu\nu} - i\Sigma^{\mu\nu}$:

$$\begin{aligned}\gamma^\mu\gamma^\nu\gamma^\rho &= (g^{\mu\nu} - i\Sigma^{\mu\nu})\gamma^\rho \\ &= g^{\mu\nu}\gamma^\rho - i\Sigma^{\mu\nu}\gamma^\rho\end{aligned} \tag{9.134}$$

Using eq. (9.132) and simplifying yields the result. □

Gamma matrix trace identities

Proposition 9.8 (Trace formulas). The following trace identities hold:

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu} \tag{9.135}$$

$$\begin{aligned}\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\kappa) = 4(&g^{\mu\nu}g^{\rho\kappa} - g^{\mu\rho}g^{\nu\kappa}\\ &+ g^{\mu\kappa}g^{\nu\rho})\end{aligned} \tag{9.136}$$

$$\mathrm{Tr}(\Sigma^{\mu\nu}\Sigma^{\rho\kappa}) = 4(g^{\mu\rho}g^{\nu\kappa} - g^{\nu\rho}g^{\mu\kappa}) \tag{9.137}$$

$$\mathrm{Tr}(\Sigma^{\mu\nu}) = 0 \tag{9.138}$$

$$\begin{aligned}\mathrm{Tr}(\gamma^5) = \mathrm{Tr}(\gamma^5\gamma^\mu) &= \mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu)\\ &= \mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho) = 0\end{aligned} \tag{9.139}$$

$$\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\kappa) = -4i\epsilon^{\mu\nu\rho\kappa}\,. \tag{9.140}$$

Traces of an odd number of gamma matrices vanish.

Proof of (9.135). Using the Clifford algebra:

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = \tfrac{1}{2}\mathrm{Tr}(\{\gamma^\mu,\gamma^\nu\}) = \tfrac{1}{2}\mathrm{Tr}(2g^{\mu\nu}\mathbb{1}_4) = 4g^{\mu\nu}$$

□

Proof that odd traces vanish. For any string of gamma matrices Γ:

$$\mathrm{Tr}(\Gamma) = \mathrm{Tr}((\gamma^5)^2\Gamma) = \mathrm{Tr}(\gamma^5\Gamma\gamma^5)$$

If Γ consists of an odd number of gamma matrices, then $\gamma^5\Gamma\gamma^5 = -\Gamma$ (using $\{\gamma^\mu,\gamma^5\} = 0$ an odd number of times). Therefore $\mathrm{Tr}(\Gamma) = -\mathrm{Tr}(\Gamma)$, which implies $\mathrm{Tr}(\Gamma) = 0$. □

The Dirac adjoint spinor

The **Dirac adjoint** (or **Dirac conjugate**) spinor is defined as:

$$\boxed{\bar\Psi(x) \equiv \Psi^\dagger(x)\gamma^0 = (\eta^\alpha(x),\, \bar\chi_{\dot\alpha}(x))}\,, \tag{9.141}$$

where we have used the explicit form of γ^0 in the chiral representation:

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^0 \\ \bar\sigma^0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}. \tag{9.142}$$

Remark. [Consistency with two-component formalism] Note that $\bar{\Psi}\Psi = \eta^\alpha \chi_\alpha + \bar{\chi}_{\dot{\alpha}}\bar{\eta}^{\dot{\alpha}} = \eta\chi + \bar{\chi}\bar{\eta}$, which matches the scalar bilinear in the two-component formalism.

Remark. [The bar notation: $\bar{\Psi}$ vs. $\bar{\chi}, \bar{\eta}, \bar{\psi}$] The overline symbol carries *two distinct meanings* in these notes, and the two should not be confused:

- **Dirac adjoint of a four-component spinor.** The symbol $\bar{\Psi}$ denotes the Dirac conjugate $\bar{\Psi} \equiv \Psi^\dagger\gamma^0$ of a four-component field Ψ. It is a *row* object built from $\Psi^\dagger$ by multiplication with γ^0. The bar is, in this case, shorthand for the composite operation $(\,\cdot\,)^\dagger\gamma^0$.
- **Dotted-index (R-type) two-component spinor.** The symbols $\bar{\chi}^{\dot{\alpha}}, \bar{\eta}^{\dot{\alpha}}, \bar{\psi}^{\dot{\alpha}}$ denote two-component Weyl spinors transforming in the $(0, \frac{1}{2})$ representation; they are the dotted-index partners of the undotted $(\frac{1}{2}, 0)$ spinors $\chi_\alpha, \eta_\alpha, \psi_\alpha$. In some textbooks (Aitchison's Lorentz chapter, early Weinberg) this object is instead written $\chi^{\dagger\dot{\alpha}}$, $\eta^{\dagger\dot{\alpha}}$, $\psi^{\dagger\dot{\alpha}}$ ("dagger notation"). We have unified on the bar notation throughout to match the modern conventions used in Chapter 10 (Wess–Bagger, Aitchison's SUSY chapter, Srednicki).

The two usages are disambiguated by the *type of the parent field*: if the symbol under the bar is a two-component $(\frac{1}{2}, 0)$ Weyl spinor (lowercase Greek $\chi_\alpha, \eta_\alpha, \psi_\alpha, \xi_\alpha, \ldots$), then the bar denotes its $(0, \frac{1}{2})$ partner $\bar{\chi}_{\dot{\alpha}} \equiv (\chi_\alpha)^*$ as in (9.43); if the symbol is a four-component Dirac field (capital Ψ), then the bar denotes the Dirac adjoint $\Psi^\dagger\gamma^0$. The typographical collision is genuine and worth flagging once, but the parent field always determines the meaning unambiguously.

Equivalently, one can read $\bar{\Psi}$ as the Dirac analogue of the Weyl bar: just as $\bar{\chi}$ pairs an undotted L-type with its dotted R-type, $\bar{\Psi}$ pairs a four-spinor with its "charge–parity rearranged" adjoint $\Psi^\dagger\gamma^0$. In the chiral basis, $\bar{\Psi}$ explicitly consists of the bar components of the underlying two-component fields: $\bar{\Psi} = (\eta^\alpha, \bar{\chi}_{\dot{\alpha}})$, see (9.141).

Lorentz transformation of four-component spinors

Under a Lorentz transformation $x'^\mu = \Lambda^\mu{}_\nu x^\nu$, the four-component spinor transforms as:

$$\Psi'(x') = \mathbb{M}\Psi(x), \qquad \bar{\Psi}'(x') = \bar{\Psi}(x)\mathbb{M}^{-1}, \tag{9.143}$$

where the 4×4 representation matrix is:

$$\mathbb{M} = \begin{pmatrix} M & 0 \\ 0 & (M^{-1})^\dagger \end{pmatrix} = \exp\left(-\frac{i}{4}\theta_{\mu\nu}\Sigma^{\mu\nu}\right), \tag{9.144}$$

with M and $(M^{-1})^\dagger$ given by eqs. (9.44) and (9.46).

Transformation law for $\bar{\Psi}$. From $\bar{\Psi} = \Psi^\dagger\gamma^0$ and using the identity $\gamma^0\mathbb{M}\gamma^0 = (\mathbb{M}^{-1})^\dagger$:

$$\begin{aligned}\bar{\Psi}'(x') &= \Psi'^\dagger(x')\gamma^0 = (\mathbb{M}\Psi)^\dagger\gamma^0 = \Psi^\dagger\mathbb{M}^\dagger\gamma^0 \\ &= \Psi^\dagger\gamma^0(\gamma^0\mathbb{M}^\dagger\gamma^0) = \bar{\Psi}\mathbb{M}^{-1}\end{aligned} \tag{9.145}$$

□

It follows immediately that $\bar{\Psi}\Psi$ is a Lorentz scalar. More generally, the gamma matrices transform as:

$$\mathbb{M}^{-1}\gamma^\mu\mathbb{M} = \Lambda^\mu{}_\nu\gamma^\nu, \tag{9.146}$$

which ensures that $\bar{\Psi}\gamma^\mu\Psi$ transforms as a four-vector, $\bar{\Psi}\Sigma^{\mu\nu}\Psi$ as an antisymmetric tensor, etc.

The Dirac equation

The field equation satisfied by $\Psi(x)$ is the **Dirac equation**:

$$\boxed{(i\gamma^\mu\partial_\mu - m)\Psi = 0}. \tag{9.147}$$

Derivation from two-component equations. The Dirac equation is equivalent to the following coupled two-component equations. Writing it out in block form:

$$i\begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}\partial_\mu\begin{pmatrix}\chi \\ \bar{\eta}\end{pmatrix} - m\begin{pmatrix}\chi \\ \bar{\eta}\end{pmatrix} = 0$$

This gives the coupled equations:

$$i\sigma^\mu\partial_\mu\bar{\eta} - m\chi = 0 \tag{9.148}$$

$$i\bar{\sigma}^\mu\partial_\mu\chi - m\bar{\eta} = 0 \tag{9.149}$$

which match the two-component field equations. □

Applying $(-i\gamma^\nu\partial_\nu - m)$ from the left:

$$(\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + m^2)\Psi = (\partial^2 + m^2)\Psi = 0 \tag{9.150}$$

using $\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu = g^{\mu\nu}\partial_\mu\partial_\nu = \partial^2$. Thus each component of Ψ satisfies the Klein-Gordon equation.

Charge conjugation and Majorana spinors

Charge conjugation

The **charge-conjugated spinor** is defined as:

$$\boxed{\Psi^C(x) \equiv C\bar{\Psi}^\top(x) = \begin{pmatrix} \eta_\alpha(x) \\ \bar{\chi}^{\dot{\alpha}}(x) \end{pmatrix}}, \tag{9.151}$$

where the **charge conjugation matrix** C in the chiral representation is:

$$C = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix} = i\gamma^0\gamma^2 . \tag{9.152}$$

The matrix C satisfies:

$$C^{-1}\gamma^\mu C = -(\gamma^\mu)^\top, \qquad C^\top = -C . \tag{9.153}$$

Proof. In the chiral representation, $(\gamma^\mu)^\top$ has sigma matrices transposed. Using $(\sigma^i)^\top = \sigma^{i*}$ for Pauli matrices, one can verify $C^{-1}\gamma^\mu C = -(\gamma^\mu)^\top$ directly. □

The condition $(\Psi^C)^C = \Psi$ is easily verified using $C^\top C = \mathbb{1}_4$.

Majorana spinors

A **Majorana spinor** is a four-component spinor that satisfies the **Majorana condition**:

$$\boxed{\Psi_M = \Psi_M^C}. \tag{9.154}$$

In the chiral representation, this implies:

$$\Psi_M(x) = \begin{pmatrix} \chi_\alpha(x) \\ \bar{\chi}^{\dot{\alpha}}(x) \end{pmatrix}, \tag{9.155}$$

where a single two-component field χ determines the entire four-component spinor. A Majorana fermion is its own antiparticle.

> **Remark.** [Degrees of freedom] A Dirac spinor has 4 complex components (8 real degrees of freedom, reduced to 4 on-shell). A Majorana spinor has only 2 complex components (4 real degrees of freedom, reduced to 2 on-shell), reflecting its self-conjugate nature.

Bilinears and Lagrangian

Bilinear covariants in four-component form

The **Dirac bilinear covariants** transform irreducibly as Lorentz tensors. Using two Dirac spinor fields Ψ_i and Ψ_j constructed from two-component fields $(\chi_i, \bar{\eta}_i)$ and $(\chi_j, \bar{\eta}_j)$, the bilinear covariants are:

> **Remark.** [Placement of the flavour indices i, j] The labels i, j in the table below are *flavour* (or family) indices distinguishing the two Dirac fields Ψ_i and Ψ_j; they are *not* spinor indices. Their up/down position follows the same Dreiner–Haber–Martin convention adopted for the two-component spinors throughout this chapter: **complex conjugation raises the flavour index**. Starting from
>
> $$\Psi_i = \begin{pmatrix} \chi_{i\,\alpha} \\ \bar{\eta}_i^{\dot{\alpha}} \end{pmatrix},$$
>
> with χ_i and $\bar{\eta}_i$ carrying i *down*, the Dirac adjoint is
>
> $$\bar{\Psi}^i = (\eta^{i\,\alpha},\ \bar{\chi}^i_{\dot{\alpha}}),$$
>
> where $\eta^i \equiv (\chi_i)^*$ and $\bar{\chi}^i \equiv (\bar{\eta}_i)^*$ now carry i *up*. This mirrors, at the level of flavour, the same staircase used for spinor indices, where conjugating an undotted lower spinor χ_α produces a dotted upper spinor $\bar{\chi}^{\dot{\alpha}}$. Hence each bilinear in the table appears in the form $\bar{\Psi}^i \Gamma \Psi_j$ with i up and j down, and the two-component reduction inherits exactly that placement (e.g. $\eta^i \chi_j$, $\bar{\chi}^i \bar{\eta}_j$). The convention is purely bookkeeping — it tracks which field has been conjugated — and involves no contraction with any flavour metric.

Four-component form	Two-component form	Lorentz representation
$\bar{\Psi}^i \Psi_j$	$\eta^i \chi_j + \bar{\chi}^i \bar{\eta}_j$	Scalar
$\bar{\Psi}^i \gamma^5 \Psi_j$	$-\eta^i \chi_j + \bar{\chi}^i \bar{\eta}_j$	Pseudoscalar
$\bar{\Psi}^i \gamma^\mu \Psi_j$	$\bar{\chi}^i \bar{\sigma}^\mu \chi_j + \eta^i \sigma^\mu \bar{\eta}_j$	Vector
$\bar{\Psi}^i \gamma^\mu \gamma^5 \Psi_j$	$-\bar{\chi}^i \bar{\sigma}^\mu \chi_j + \eta^i \sigma^\mu \bar{\eta}_j$	Axial vector
$\bar{\Psi}^i \Sigma^{\mu\nu} \Psi_j$	$2(\eta^i \sigma^{\mu\nu} \chi_j + \bar{\chi}^i \bar{\sigma}^{\mu\nu} \bar{\eta}_j)$	Antisymmetric tensor

Derivation of vector bilinear. Using the definition (9.141) and the block form of

γ^μ:

$$
\begin{aligned}
\bar{\Psi}^i \gamma^\mu \Psi_j &= (\eta^{i\alpha}, \bar{\chi}^i_{\dot{\alpha}}) \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \begin{pmatrix} \chi_{j\beta} \\ \bar{\eta}^{\dot{\beta}}_j \end{pmatrix} \\
&= \eta^{i\alpha} \sigma^\mu_{\alpha\dot{\beta}} \bar{\eta}^{\dot{\beta}}_j + \bar{\chi}^i_{\dot{\alpha}} \bar{\sigma}^{\mu\dot{\alpha}\beta} \chi_{j\beta} \\
&= \eta^i \sigma^\mu \bar{\eta}_j + \bar{\chi}^i \bar{\sigma}^\mu \chi_j
\end{aligned} \tag{9.156}
$$

The other bilinear covariants follow similarly. □

The chiral projections of the bilinear covariants are particularly useful:

$$\bar{\Psi}^i P_L \Psi_j = \eta^i \chi_j, \qquad \bar{\Psi}^i P_R \Psi_j = \bar{\chi}^i \bar{\eta}_j \tag{9.157}$$

$$\bar{\Psi}^i \gamma^\mu P_L \Psi_j = \bar{\chi}^i \bar{\sigma}^\mu \chi_j, \qquad \bar{\Psi}^i \gamma^\mu P_R \Psi_j = \eta^i \sigma^\mu \bar{\eta}_j \tag{9.158}$$

$$\bar{\Psi}^i \Sigma^{\mu\nu} P_L \Psi_j = 2\eta^i \sigma^{\mu\nu} \chi_j, \qquad \bar{\Psi}^i \Sigma^{\mu\nu} P_R \Psi_j = 2\bar{\chi}^i \bar{\sigma}^{\mu\nu} \bar{\eta}_j\,. \tag{9.159}$$

Four-component form of the Dirac Lagrangian

The **Dirac Lagrangian** for a free massive spin-$\frac{1}{2}$ fermion is:

$$\boxed{\mathscr{L}_{\text{Dirac}} = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi = i\bar{\Psi}\gamma^\mu \partial_\mu \Psi - m\bar{\Psi}\Psi}\,. \tag{9.160}$$

This compact four-component formula is the form most commonly used in QFT calculations. The next section constructs the corresponding two-component Majorana and Dirac Lagrangians directly; the short check below explains how the four-component expression reduces to that two-component form.

Equivalence with two-component form. Expanding using eqs. (9.141) and (9.113):

$$
\begin{aligned}
\bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi &= i\bar{\Psi}\gamma^\mu \partial_\mu \Psi - m\bar{\Psi}\Psi \\
&= i(\eta^\alpha \sigma^\mu_{\alpha\dot{\beta}} \partial_\mu \bar{\eta}^{\dot{\beta}} + \bar{\chi}_{\dot{\alpha}} \bar{\sigma}^{\mu\dot{\alpha}\beta} \partial_\mu \chi_\beta) \\
&\quad - m(\eta^\alpha \chi_\alpha + \bar{\chi}_{\dot{\alpha}} \bar{\eta}^{\dot{\alpha}})
\end{aligned} \tag{9.161}
$$

Using integration by parts on the first term,

$$i\eta\sigma^\mu \partial_\mu \bar{\eta} = -i(\partial_\mu \eta)\sigma^\mu \bar{\eta} + (\text{total derivative}) = i\bar{\eta}\bar{\sigma}^\mu \partial_\mu \eta + (\text{total derivative}),$$

where the last equality uses the anticommuting spinor identity $(\partial_\mu \eta)\sigma^\mu \bar{\eta} = -\bar{\eta}\bar{\sigma}^\mu \partial_\mu \eta$. Thus, up to a total derivative, this becomes:

$$\mathscr{L} = i\bar{\chi}\bar{\sigma}^\mu \partial_\mu \chi + i\bar{\eta}\bar{\sigma}^\mu \partial_\mu \eta - m(\chi\eta + \bar{\chi}\bar{\eta})$$

which matches the two-component Dirac Lagrangian (9.167). □

9.6 Lagrangians for free spin-1/2 fermions

We now construct the free fermion Lagrangians directly in two-component notation. This fills in the object that the previous subsection rewrote compactly as the four-component Dirac Lagrangian: first a single neutral Weyl field gives a Majorana fermion, and then two independent Weyl fields of opposite charge give a Dirac fermion.

The Majorana Lagrangian

Consider a theory of a single two-component fermion ξ. The quantity:

$$i\bar{\xi}_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\beta}\partial_\mu\xi_\beta \equiv i\bar{\xi}\bar{\sigma}^\mu\partial_\mu\xi \tag{9.162}$$

is a Lorentz scalar. The factor of i is inserted since the hermitian quantity:

$$\tfrac{1}{2}i\bar{\xi}\bar{\sigma}^\mu \overleftrightarrow{\partial}_\mu\xi \equiv \tfrac{1}{2}i\left[\bar{\xi}\bar{\sigma}^\mu(\partial_\mu\xi) - (\partial_\mu\bar{\xi})\bar{\sigma}^\mu\xi\right] \tag{9.163}$$

differs from $i\bar{\xi}\bar{\sigma}^\mu\partial_\mu\xi$ by a total divergence. Hence, eq. (9.162) is a candidate for a kinetic energy term in the Lagrangian.

A second hermitian Lorentz scalar:

$$m\xi\xi + m^*\bar{\xi}\bar{\xi} \tag{9.164}$$

is a candidate for a mass term. Without loss of generality, we can absorb the phase of the parameter m into the definition of the field ξ. Henceforth, we shall work in a convention in which m is real and nonnegative.

The Lagrangian for a **free Majorana fermion** is:

$$\boxed{\mathscr{L} = i\bar{\xi}\bar{\sigma}^\mu\partial_\mu\xi - \frac{1}{2}(m\xi\xi + m\bar{\xi}\bar{\xi})}\,. \tag{9.165}$$

The corresponding field equations (equations of motion) are:

$$\boxed{\begin{aligned} &i\bar{\sigma}^\mu\partial_\mu\xi - m\bar{\xi} = 0 \\ &i\sigma^\mu\partial_\mu\bar{\xi} - m\xi = 0 \end{aligned}}\,, \tag{9.166}$$

where the second equation is the hermitian conjugate of the first.

Remark. [Physical interpretation] This is the Lagrangian for a neutral (Majorana) spin-$\frac{1}{2}$ fermion. A Majorana fermion is its own antiparticle. Neutrinos may be Majorana fermions, although this has not yet been experimentally confirmed.

The Dirac Lagrangian in two-component form

For a **Dirac fermion**, which carries a conserved charge (such as electric charge), we need two independent two-component spinors χ and η. The Lagrangian is:

$$\boxed{\mathscr{L} = i\bar{\chi}\bar{\sigma}^{\mu}\partial_{\mu}\chi + i\bar{\eta}\bar{\sigma}^{\mu}\partial_{\mu}\eta - m\left(\chi\eta + \bar{\chi}\bar{\eta}\right)}\,. \tag{9.167}$$

The field equations are:

$$i\bar{\sigma}^{\mu}\partial_{\mu}\chi - m\bar{\eta} = 0 \tag{9.168}$$
$$i\bar{\sigma}^{\mu}\partial_{\mu}\eta - m\bar{\chi} = 0\,. \tag{9.169}$$

Remark. [Connection to four-component formalism] The Dirac spinor Ψ is constructed from two Weyl spinors:

$$\Psi = \begin{pmatrix} \chi_{\alpha} \\ \bar{\eta}^{\dot{\alpha}} \end{pmatrix}. \tag{9.170}$$

In this notation, the Dirac Lagrangian (9.167) takes the familiar four-component form $\mathscr{L} = \bar{\Psi}(i\gamma^{\mu}\partial_{\mu} - m)\Psi$.

Energy-momentum tensor

The canonical energy-momentum tensor and angular momentum density can be derived using the Noether procedure. For a theory with N two-component $(\frac{1}{2},0)$ spinor fields $\xi_{\alpha j}$ and their conjugates $\bar{\xi}^{\dot{\alpha} j}$ (where $j \in \{1,2,\ldots,N\}$):

$$\begin{aligned} T^{\mu}{}_{\nu} = & \frac{\partial\mathscr{L}}{\partial(\partial_{\mu}\xi_{\alpha j})}\partial_{\nu}\xi_{\alpha j} + \frac{\partial\mathscr{L}}{\partial(\partial_{\mu}\bar{\xi}^{\dot{\alpha} j})}\partial_{\nu}\bar{\xi}^{\dot{\alpha} j} \\ & - \delta^{\mu}_{\nu}\mathscr{L}\,. \end{aligned} \tag{9.171}$$

$$\begin{aligned} M^{\mu}{}_{\nu\lambda} = {} & x_{\nu}T^{\mu}{}_{\lambda} - x_{\lambda}T^{\mu}{}_{\nu} \\ & - i\frac{\partial\mathscr{L}}{\partial(\partial_{\mu}\xi_{\alpha j})}(\sigma_{\nu\lambda})_{\alpha}{}^{\beta}\xi_{\beta j} \\ & - i\frac{\partial\mathscr{L}}{\partial(\partial_{\mu}\bar{\xi}^{\dot{\alpha} j})}(\bar{\sigma}_{\nu\lambda})^{\dot{\alpha}}{}_{\dot{\beta}}\bar{\xi}^{\dot{\beta} j}\,. \end{aligned} \tag{9.172}$$

These satisfy the conservation laws $\partial_{\mu}T^{\mu}{}_{\nu} = 0$ and $\partial_{\mu}M^{\mu}{}_{\nu\lambda} = 0$ when the field equations are satisfied.

For the Majorana Lagrangian (9.165):

$$\frac{\partial \mathscr{L}}{\partial(\partial_\mu \xi_\alpha)} = i\bar{\xi}_{\dot{\beta}}\bar{\sigma}^{\mu\dot{\beta}\alpha}, \qquad \frac{\partial \mathscr{L}}{\partial(\partial_\mu \bar{\xi}^{\dot{\alpha}})} = 0. \tag{9.173}$$

The corresponding time-independent conserved quantities are the momentum four-vector and the total angular momentum tensor:

$$P_\nu = \int d^3x\, T^0{}_\nu, \qquad J_{\nu\lambda} = \int d^3x\, M^0{}_{\nu\lambda}. \tag{9.174}$$