10 Introduction to Supersymmetry

In this chapter we develop the foundations of supersymmetry using two-component Weyl spinors. We build on the spinor formalism established in Chapter 9, where the transformation properties of $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ spinors were derived. The key results we will use are the transformation matrices M and $(M^{-1})^\dagger$, and the sigma matrix identities. After establishing the motivations and the Weyl spinor apparatus, we construct the Wess-Zumino model as the simplest supersymmetric field theory, derive the SUSY algebra, and develop the theory of interactions through the superpotential.

The classic reference for supersymmetry is Wess and Bagger [32]. For a pedagogical introduction, Martin's primer [33] is highly recommended. A comprehensive treatment of spinor conventions and their connection to supersymmetry can be found in Dreiner, Haber, and Martin [34].

10.1 Why Supersymmetry Still Matters

Before developing the technical machinery of supersymmetry, it is worth asking: *why should we care?* After two decades of LHC data without direct evidence for superpartners [35, 36, 37], what remains of the original motivations?

The answer is nuanced. The Large Hadron Collider discovered the Higgs boson in 2012, confirming the Standard Model as a remarkably successful description of physics up to the TeV scale. Yet the same discovery deepened the *hierarchy problem*: a fundamental scalar with mass $m_h \simeq 125$ GeV is quadratically sensitive to any heavy physics, and without some protective mechanism, quantum corrections should push m_h to the Planck scale. The absence of new particles at the LHC has pushed the simplest supersymmetric scenarios to higher masses, but it has not eliminated the underlying motivations.

Supersymmetry remains compelling for several interconnected reasons:

1. **Naturalness**: SUSY protects scalar masses from quadratic divergences, addressing the hierarchy problem—though the degree of fine-tuning has become a subject of ongoing debate.
2. **Unification**: The gauge couplings unify with remarkable precision in the MSSM, a quantitative success that survives independently of naturalness concerns.

3. **Dark matter**: SUSY with conserved R-parity provides stable dark matter candidates with the right properties for thermal freeze-out.
4. **Theoretical structure**: SUSY extends spacetime symmetry itself, leading to powerful non-renormalization theorems and deep connections to string theory and quantum gravity.

In the following subsections, we examine each motivation in detail, emphasizing what has been tested, what has changed, and what remains robust.

Naturalness and the Higgs mass

Historically, a central motivation for TeV-scale SUSY was the stabilization of the electroweak scale against large radiative corrections. In the SM treated as an effective field theory with cutoff Λ, the Higgs mass parameter receives quadratic sensitivity to Λ,

$$\delta m_H^2 \sim \frac{\Lambda^2}{16\pi^2}\left(6\lambda + \frac{9}{4}g^2 + \frac{3}{4}g'^2 - 6y_t^2 + \cdots\right), \tag{10.1}$$

where λ is the Higgs quartic coupling ($V \supset \lambda(\Phi^\dagger\Phi)^2$), g and g' are the $SU(2)_L$ and $U(1)_Y$ gauge couplings, and y_t is the top Yukawa coupling, which provides the numerically dominant contribution. Supersymmetry replaces the quadratically sensitive structure with a cancellation between fermionic and bosonic loops when superpartners are included. In softly broken SUSY, quadratic divergences cancel, and the remaining corrections are controlled by the soft-breaking masses (and logarithms of UV scales),

$$\delta m_{H_u}^2 \sim -\frac{3y_t^2}{8\pi^2}\left(m_{\tilde{Q}_3}^2 + m_{\tilde{u}_3}^2 + |A_t|^2\right)\ln\frac{\Lambda_{\rm UV}}{m_{\rm soft}}, \tag{10.2}$$

schematically in MSSM-like theories. Thus, electroweak symmetry breaking in SUSY ties the Higgs potential parameters to the superpartner spectrum. A standard qualitative expectation was therefore that stops and higgsinos should not be far above the TeV scale if one demands little fine-tuning.

What LHC null results imply. After extensive searches [35, 36, 37], broad classes of "vanilla" weak-scale SUSY are constrained: colored superpartners are pushed well above the TeV scale in many simplified scenarios with standard missing-energy decay chains, and electroweak states are constrained in the $O(100\ \text{GeV} - 1\ \text{TeV})$ range depending on decay topologies and mass splittings. The direct consequence is that the simplest naturalness narrative—light stops and gluinos with large missing energy—is under significant pressure.

However, this does not eliminate the naturalness motivation; it reshapes it. There are three complementary perspectives in 2026:

1. **Naturalness as a heuristic rather than a theorem.** The SM itself contains parameters (e.g. the cosmological constant) that are far more “unnatural” by naive estimates. One may still value models that *reduce* tuning even if they do not eliminate it.
2. **Natural SUSY is not identical to “easy SUSY”.** Even if higgsinos and third-generation squarks are relatively light, experimental sensitivity can be reduced by compressed spectra, complex cascades, or unconventional final states. The absence of signals constrains *some* corners but leaves others open.
3. **Moderate tuning may be acceptable.** A tuning at the level of 10^{-2}–10^{-3} is often viewed differently than 10^{-30}. If the price of maintaining a structured UV framework is moderate fine-tuning, SUSY can still be compelling.

The Higgs mass at 125 GeV: a challenge for minimal SUSY. The discovery of the Higgs boson at 125 GeV created an interesting tension with the simplest supersymmetric models. In the MSSM, the quartic Higgs coupling is not a free parameter but is determined by gauge couplings. This leads to a striking tree-level prediction:

$$m_h^2 \leq m_Z^2 \cos^2 2\beta \leq m_Z^2 \approx (91\ \text{GeV})^2 \tag{10.3}$$

This bound is in clear conflict with the observed value of 125 GeV—at tree level, the MSSM cannot accommodate the data.

The resolution comes from radiative corrections, particularly from the stop sector. The dominant one-loop contribution scales as

$$\Delta m_h^2 \sim \frac{3 y_t^4 v^2}{8\pi^2} \left[\ln \frac{m_{\tilde{t}}^2}{m_t^2} + \frac{X_t^2}{m_{\tilde{t}}^2} \left(1 - \frac{X_t^2}{12 m_{\tilde{t}}^2} \right) \right], \tag{10.4}$$

where $X_t = A_t - \mu \cot\beta$ is the stop mixing parameter and $m_{\tilde{t}}$ denotes a characteristic stop mass scale. To reach 125 GeV requires either heavy stops ($m_{\tilde{t}} \gtrsim$ few TeV, depending on the spectrum) and/or substantial mixing ($|X_t| \sim \sqrt{6}\, m_{\tilde{t}}$, the “maximal mixing” scenario). Both options increase fine-tuning: heavy stops contribute to $\delta m_{H_u}^2$, while large A_t requires specific UV boundary conditions.

In 2026, it is common to view the MSSM as a benchmark rather than a definitive target. Extensions that add new contributions to the Higgs quartic—such as the NMSSM with a gauge singlet—can raise the tree-level bound and reduce tuning. Alternatively, one can accept moderate tuning as the price of maintaining a structured UV framework.

Gauge coupling unification

Perhaps the most compelling *quantitative* motivation for supersymmetry is the unification of gauge couplings. The one-loop running of the three gauge couplings

α_1, α_2, α_3 is governed by

$$\alpha_i^{-1}(Q) = \alpha_i^{-1}(m_Z) + \frac{b_i}{2\pi} \ln \frac{Q}{m_Z}, \tag{10.5}$$

where the coefficients b_i depend on the particle content of the theory. A useful one-loop diagnostic for whether the slopes can make the three couplings meet is

$$x \equiv \frac{b_2 - b_3}{5(b_1 - b_2)}. \tag{10.6}$$

The same combination extracted from the measured low-energy couplings is $x_{\text{exp}} \simeq 0.144$.

In the **Standard Model** with one Higgs doublet ($N_h = 1$), one finds $b_3 = -7$, $b_2 = -19/6$, $b_1 = 41/10$, yielding

$$x_{\text{SM}} = \frac{23}{218} \simeq 0.106. \tag{10.7}$$

This is significantly below the experimental value—*gauge coupling unification fails* in the Standard Model. Furthermore, the SU(3) and U(1) couplings would meet at $\sim 7 \times 10^{14}$ GeV, while SU(2) and U(1) meet at $\sim 10^{13}$ GeV: there is no single unification scale.

In the **MSSM** with two Higgs doublets ($N_h = 2$), the superpartner contributions modify the beta function coefficients to $b_3 = -3$, $b_2 = 1$, $b_1 = 33/5$, giving

$$x_{\text{MSSM}} = \frac{1}{7} \simeq 0.143. \tag{10.8}$$

This is remarkably close to the experimental value. The three couplings converge at

$$M_{\text{GUT}} \simeq 2 \times 10^{16} \text{ GeV}, \qquad \alpha_U^{-1} \simeq 24 \tag{10.9}$$

which is comfortably below the Planck scale (avoiding quantum gravity complications) and comfortably in the perturbative regime. For dimension-six GUT gauge-boson exchange, the rough scaling $\tau_p \sim M_{\text{GUT}}^4/(\alpha_U^2 m_p^5)$ gives very long proton lifetimes, of order 10^{35} years. In supersymmetric GUTs this statement must be supplemented by model-dependent dimension-five operators, which can impose stronger constraints.

The key point in 2026 is that this success is *less sensitive* to the precise superpartner masses than naturalness arguments. Whether colored superpartners are at 2 TeV or 10 TeV shifts the details via threshold corrections, but the overall picture—that SUSY dramatically improves unification—remains robust. Even if naturalness is partly deprioritized, unification keeps SUSY attractive as a UV-organizing principle.

Dark matter

Supersymmetry naturally provides stable dark matter candidates when R-parity (or another suitable symmetry) is imposed: the lightest supersymmetric particle (LSP) becomes stable, and weakly interacting neutral states can yield the observed relic abundance via thermal freeze-out. The neutralino (a mixture of bino, wino, higgsinos) is the canonical example.

Where the classic WIMP story stands in 2026. Direct detection experiments and collider searches have put significant pressure on simple regions of neutralino parameter space, but the SUSY dark matter narrative is not exhausted:

- **Higgsino- or wino-like LSPs.** These can be viable thermally at masses near the TeV scale (higgsino) or multi-TeV (wino), with distinctive phenomenology such as compressed spectra and soft decay products at colliders.
- **Well-tempered neutralinos.** A pure bino LSP typically annihilates too weakly (overproducing dark matter), while a pure higgsino annihilates too efficiently unless it is near the TeV scale. A carefully tuned *mixture* of bino, wino, and higgsino components can yield the observed relic abundance $\Omega h^2 \simeq 0.12$; such mixed states are called "well-tempered" neutralinos. The same mixing usually increases the couplings to the Higgs and Z bosons, so direct-detection bounds put much of this parameter space under strong pressure. Viable regions typically rely on heavier masses, blind spots, resonant annihilation, coannihilation, or non-standard cosmology.
- **Funnel scenarios.** When the neutralino mass sits near a resonance, $m_\chi \approx m_X/2$ with $X = h, Z$, or the heavy MSSM pseudoscalar A, s-channel annihilation proceeds through a near-on-shell propagator and the thermally averaged cross section is greatly enhanced. This drives the relic density down to the observed value even for neutralino compositions (e.g. nearly pure bino) that would otherwise overproduce dark matter. The name "funnel" refers to the narrow, funnel-shaped allowed regions in the $(m_0, m_{1/2})$ parameter plane of constrained MSSM scenarios. Both well-tempered and funnel mechanisms can produce the correct relic abundance with suppressed detection rates, depending on the precise couplings and mixing angles.
- **Non-thermal histories.** If the early Universe differs from the simplest radiation-dominated thermal picture (late entropy injection, moduli decays), a broader range of LSP masses/couplings can match the relic density.

Thus, dark matter remains a meaningful motivation, but one should be careful: SUSY does not *predict* the dark matter properties uniquely; it provides a consistent and testable arena in which dark matter emerges naturally and is tightly connected to electroweak and Higgs physics.

SUSY and UV structure

Even if one sets aside the expectation of TeV-scale discovery, supersymmetry remains uniquely valuable as a tool for understanding the UV behavior of quantum field theories:

1. **Control over radiative corrections.** Non-renormalization theorems and holomorphy constraints in SUSY theories strongly restrict how couplings run and how effective operators are generated, providing calculational control in strongly coupled regimes.
2. **Connections to dualities and non-perturbative dynamics.** SUSY enables sharp statements about confinement, chiral symmetry breaking, and moduli spaces, and provides laboratories where dual descriptions can be proven rather than conjectured.
3. **Compatibility with quantum gravity expectations.** Many developments in string theory and quantum gravity suggest that consistent UV completions impose constraints that resemble or favor supersymmetric structures, at least in some regimes. Even if low-energy SUSY is broken at high scales, SUSY remains a crucial organizing principle in UV model building.

In a lecture-notes context, it is worth stressing that SUSY is not merely "another BSM model" but a symmetry that upgrades the structure of QFT itself: it enlarges the Poincaré algebra, relates particle spins, and yields powerful constraints on dynamics.

The new SUSY paradigm

A key lesson from two decades of LHC is that if SUSY exists near the TeV scale, it may not appear in the simplest textbook signatures. Several generic mechanisms can render SUSY difficult to see:

- **Compressed spectra.** Small mass splittings reduce missing energy and visible momenta, making triggers and selections inefficient.
- **Long-lived particles.** If decays are slow (small couplings, small splittings), displaced vertices or detector-stable charged/colored tracks become relevant; these require dedicated analyses.
- ***R*-parity violation (RPV).** If the LSP decays hadronically, missing energy is reduced and signatures blend into QCD backgrounds, though multi-jet resonances or displaced decays may arise.
- **Electroweak production dominance.** If colored states are heavy, the production of charginos/neutralinos or sleptons can dominate, but cross sections are smaller and final states more subtle.

The implication is not that SUSY is unfalsifiable, but that its discovery (or exclusion) requires a richer experimental program: precision measurements, dedicated LLP searches, improved sensitivity to soft objects, and global combinations of collider and astroparticle constraints.

Taken together, a balanced 2026 view is that naturalness is treated less dogmatically, unification and UV structure remain robust motivations, and dark matter is still compelling but not unique. At the same time, discovery strategies have shifted toward targeted signatures (compressed spectra, LLPs, RPV, electroweak production) rather than textbook missing-energy searches.

From a pedagogical standpoint, this motivates a modern SUSY course structure: one should still teach SUSY as a symmetry of nature that solves (or ameliorates) the hierarchy problem and enables unification and dark matter, but one should also train students to interpret null results correctly and to appreciate the broader theoretical significance of supersymmetry as an organizing principle for quantum field theory.

Looking ahead. The central message is not that SUSY is "ruled out" or "guaranteed," but that it remains a coherent and fertile framework. The two decades of LHC data have raised the evidentiary bar and refined our expectations: if SUSY exists, it may be more subtle than the earliest minimal benchmarks, and its discovery may come from a combination of collider innovation, precision observables, and astroparticle signals rather than from a single spectacular channel.

10.2 Two-component spinor conventions (recap)

We use the two-component conventions established in Chapter 9 and only collect the identities needed in the SUSY construction. In the chiral basis,

$$\Psi = \begin{pmatrix} \chi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}, \qquad \sigma^\mu = (\mathbb{1}_2, \vec{\sigma}), \qquad \bar{\sigma}^\mu = (\mathbb{1}_2, -\vec{\sigma}), \tag{10.10}$$

and the spinor metric raises/lowers indices via $\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta$ with $\epsilon^{12} = -\epsilon_{12} = +1$.

Lorentz-invariant products and vector bilinears are

$$\psi\chi \equiv \psi^\alpha \chi_\alpha, \qquad \bar{\psi}\bar{\chi} \equiv \bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}}, \qquad \chi\sigma^\mu\bar{\psi} \equiv \chi^\alpha (\sigma^\mu)_{\alpha\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}, \tag{10.11}$$

with $\chi\sigma^\mu\bar{\psi} = -\bar{\psi}\bar{\sigma}^\mu\chi$ for anticommuting spinors.

Remark. [Constructing Lorentz invariants from a single chirality] A crucial observation for supersymmetry: the combination $i\sigma_2\chi^*$ transforms as a right-handed spinor. This follows from $\sigma_2\vec{\sigma}^*\sigma_2 = -\vec{\sigma}$, which implies that under a Lorentz transformation $\chi \to M\chi$, we have $i\sigma_2\chi^* \to (M^{-1})^\dagger(i\sigma_2\chi^*)$. This allows us to construct Lorentz invariants using only left-handed spinors.

10.3 The free Wess-Zumino model

The simplest supersymmetric model in four dimensions consists of a complex scalar field ϕ coupled to a left-handed Weyl spinor χ. The free Lagrangian takes the form

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi + \bar{\chi} i \bar{\sigma}^\mu \partial_\mu \chi \,, \tag{10.12}$$

where the fermionic kinetic term arises from the Dirac Lagrangian restricted to left-handed spinors. The mass dimensions are $[\phi] = 1$ and $[\chi] = 3/2$, as required by the canonical structure of the kinetic terms. Note that with a left-handed spinor χ_α, the appropriate sigma matrix is $\bar{\sigma}^\mu$ with index structure $\bar{\sigma}^{\mu\,\dot{\alpha}\alpha}\chi_\alpha$.

We seek field transformations that leave the action $S = \int d^4x\, \mathcal{L}$ invariant while mixing bosons with fermions, schematically $\delta\phi \sim \chi$ and $\delta\chi \sim \phi$. Since the variation of a scalar must itself be a scalar, and since χ carries spin-$\frac{1}{2}$, the generator of supersymmetry must be a spinorial object.

For the scalar field, dimensional analysis and Lorentz invariance lead us to try the ansatz $\delta\phi = \epsilon\chi$, where ϵ is a Grassmann-valued left-handed spinor parameter with dimension $[\epsilon] = -1/2$. The product $\epsilon\chi = \epsilon^\alpha \chi_\alpha$ transforms as a Lorentz scalar, as required. The conjugate transformation is $\delta\phi^\dagger = \bar{\epsilon}\bar{\chi}$.

The transformation of the spinor field requires more care. Since $\delta\chi$ must have dimension $3/2$ and carry spinor indices, dimensional analysis forces us to include a derivative of ϕ. However, ∂_μ carries a Lorentz index that must be contracted with a sigma matrix. To match the index structure of a left-handed spinor χ_α, we need the combination $\sigma^\mu_{\alpha\dot{\alpha}}\bar{\epsilon}^{\dot{\alpha}}\partial_\mu\phi$, which requires introducing the conjugate parameter $\bar{\epsilon}^{\dot{\alpha}}$. We therefore write

$$\delta\chi_\alpha = A(\sigma^\mu\bar{\epsilon})_\alpha\, \partial_\mu\phi, \qquad \delta\bar{\chi}_{\dot{\alpha}} = A^*\, \epsilon^\beta \sigma^\mu_{\beta\dot{\alpha}}\, \partial_\mu\phi^\dagger \,, \tag{10.13}$$

where the constant A is to be determined by requiring invariance of the action (it will turn out to be $A = -i$). The conjugate law is written with a lower dotted index because this is the index position that appears in the kinetic term $\bar{\chi}_{\dot{\alpha}} i\bar{\sigma}^{\mu\,\dot{\alpha}\alpha}\partial_\mu\chi_\alpha$.

Derivation of $\delta\bar{\chi}$ from $\delta\chi$ by complex conjugation. Take the complex conjugate of $\delta\chi_\alpha = A\,\sigma^\mu_{\alpha\dot{\beta}}\,\bar{\epsilon}^{\dot{\beta}}\,\partial_\mu\phi$, using $\bar{\chi}_{\dot{\alpha}} \equiv (\chi_\alpha)^*$:

$$\delta\bar{\chi}_{\dot{\alpha}} = A^*\,(\sigma^\mu_{\alpha\dot{\beta}})^*\,(\bar{\epsilon}^{\dot{\beta}})^*\,\partial_\mu\phi^\dagger \,.$$

Two ingredients convert this into the second relation of (10.13):

1. *Hermiticity of σ^μ* ((9.68)): $(\sigma^\mu_{\alpha\dot{\beta}})^* = \sigma^\mu_{\beta\dot{\alpha}}$.

2. *Conjugation of* $\bar\epsilon$: since $\bar\epsilon^{\dot\beta} = \epsilon^{\dot\beta\dot\gamma}(\epsilon_\gamma)^*$, one has $(\bar\epsilon^{\dot\beta})^* = \epsilon^{\beta\gamma}\epsilon_\gamma = \epsilon^\beta$.

Substituting them gives

$$\delta\bar\chi_{\dot\alpha} = A^*\,\epsilon^\beta\,\sigma^\mu_{\beta\dot\alpha}\,\partial_\mu\phi^\dagger\,.$$

For the value fixed below, $A = -i$, hence $A^* = +i$. If one raises the dotted index, the sigma–epsilon relation gives

$$\epsilon^{\dot\alpha\dot\beta}\epsilon^\gamma\sigma^\mu_{\gamma\dot\beta} = -(\bar\sigma^\mu\epsilon)^{\dot\alpha},$$

so the raised-index form used later is

$$\delta\bar\chi^{\dot\alpha} = -i\,\partial_\mu\phi^\dagger(\bar\sigma^\mu\epsilon)^{\dot\alpha}.$$

□

Varying the Lagrangian under these transformations, the scalar kinetic term contributes

$$\delta\mathcal{L}_s = \partial_\mu(\bar\epsilon\bar\chi)\partial^\mu\phi + \partial_\mu\phi^\dagger\cdot\epsilon\partial^\mu\chi\,, \tag{10.14}$$

while the fermionic kinetic term gives

$$\delta\mathcal{L}_F = iA^*\,\partial_\nu\phi^\dagger\,\epsilon\sigma^\nu\bar\sigma^\mu\partial_\mu\chi + iA\,\bar\chi\bar\sigma^\mu\sigma^\nu\bar\epsilon\,\partial_\mu\partial_\nu\phi\,. \tag{10.15}$$

Step-by-step variation (for a first SUSY course). Starting from

$$\mathcal{L} = \partial_\mu\phi^\dagger\,\partial^\mu\phi + \bar\chi_{\dot\alpha}\,i\bar\sigma^{\mu\,\dot\alpha\beta}\partial_\mu\chi_\beta, \qquad \delta\phi = \epsilon\chi, \quad \delta\phi^\dagger = \bar\epsilon\bar\chi, \quad \delta\chi_\alpha = A(\sigma^\nu\bar\epsilon)_\alpha\,\partial_\nu\phi,$$

we compute variations term by term.

Scalar kinetic term. Using the product rule,

$$\begin{aligned}\delta\mathcal{L}_s &= \partial_\mu(\delta\phi^\dagger)\,\partial^\mu\phi + \partial_\mu\phi^\dagger\,\partial^\mu(\delta\phi)\\ &= \partial_\mu(\bar\epsilon\bar\chi)\,\partial^\mu\phi + \partial_\mu\phi^\dagger\,\partial^\mu(\epsilon\chi)\\ &= (\partial_\mu\bar\chi)\bar\epsilon\,\partial^\mu\phi + \partial_\mu\phi^\dagger\,\epsilon\,\partial^\mu\chi\,,\end{aligned} \tag{10.16}$$

where $\partial_\mu\epsilon = \partial_\mu\bar\epsilon = 0$ for global supersymmetry.

Fermion kinetic term. Varying $\bar\chi i\bar\sigma^\mu\partial_\mu\chi$ gives

$$\begin{aligned}\delta\mathcal{L}_F &= (\delta\bar\chi_{\dot\alpha})\,i\bar\sigma^{\mu\,\dot\alpha\beta}\partial_\mu\chi_\beta + \bar\chi\,i\bar\sigma^\mu\partial_\mu(\delta\chi)\\ &= \Big(A^*\,\epsilon\sigma^\nu\,\partial_\nu\phi^\dagger\Big)\,i\bar\sigma^\mu\partial_\mu\chi + \bar\chi\,i\bar\sigma^\mu\partial_\mu\Big(A\,\sigma^\nu\bar\epsilon\,\partial_\nu\phi\Big)\\ &= iA^*\,\partial_\nu\phi^\dagger\,\epsilon\sigma^\nu\bar\sigma^\mu\partial_\mu\chi + iA\,\bar\chi\bar\sigma^\mu\sigma^\nu\bar\epsilon\,\partial_\mu\partial_\nu\phi\,,\end{aligned} \tag{10.17}$$

which matches the compact expression written above.

To show invariance, first integrate by parts the second term in $\delta\mathcal{L}_F$:

$$\int d^4x\, iA\,\bar{\chi}\bar{\sigma}^\mu\sigma^\nu\bar{\epsilon}\,\partial_\mu(\partial_\nu\phi)$$
$$= -\int d^4x\, iA\,(\partial_\mu\bar{\chi})\bar{\sigma}^\mu\sigma^\nu\bar{\epsilon}\,\partial_\nu\phi \;+\; \text{boundary term}\,. \tag{10.18}$$

Thus, inside the action and modulo a boundary term,

$$\delta\mathcal{L}_F \doteq iA^*\,\partial_\nu\phi^\dagger\,\epsilon\sigma^\nu\bar{\sigma}^\mu\partial_\mu\chi - iA\,(\partial_\mu\bar{\chi})\bar{\sigma}^\mu\sigma^\nu\bar{\epsilon}\,\partial_\nu\phi\,. \tag{10.19}$$

Here $\doteq$ denotes equality in the action up to a total derivative.

Now use the fundamental σ–$\bar{\sigma}$ identity

$$\bar{\sigma}^\mu\sigma^\nu + \bar{\sigma}^\nu\sigma^\mu = 2g^{\mu\nu}\mathbb{1}, \qquad \sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu = 2g^{\mu\nu}\mathbb{1}\,. \tag{10.20}$$

The antisymmetric parts of the sigma-matrix products in (10.19) are total derivatives in the action. For example,

$$\int d^4x\,\partial_\nu\phi^\dagger\,\epsilon(\sigma^\nu\bar{\sigma}^\mu - \sigma^\mu\bar{\sigma}^\nu)\partial_\mu\chi = -\int d^4x\,\partial_\mu\partial_\nu\phi^\dagger\,\epsilon(\sigma^\nu\bar{\sigma}^\mu - \sigma^\mu\bar{\sigma}^\nu)\chi = 0,$$

because $\partial_\mu\partial_\nu\phi^\dagger$ is symmetric in μ,ν while the matrix coefficient is antisymmetric. Hence only the symmetric pieces contribute:

$$\partial_\nu\phi^\dagger\,\epsilon\sigma^\nu\bar{\sigma}^\mu\partial_\mu\chi \doteq \partial_\mu\phi^\dagger\,\epsilon\,\partial^\mu\chi, \tag{10.21}$$
$$(\partial_\mu\bar{\chi})\bar{\sigma}^\mu\sigma^\nu\bar{\epsilon}\,\partial_\nu\phi \doteq (\partial_\mu\bar{\chi})\bar{\epsilon}\,\partial^\mu\phi. \tag{10.22}$$

Therefore

$$\delta\mathcal{L}_F \doteq iA^*\,\partial_\mu\phi^\dagger\,\epsilon\,\partial^\mu\chi - iA\,(\partial_\mu\bar{\chi})\bar{\epsilon}\,\partial^\mu\phi.$$

Comparing with $\delta\mathcal{L}_s$ from (10.16), the non-total-derivative pieces cancel term by term if and only if

$$\boxed{A = -i}\,, \tag{10.23}$$

so that $iA^* = -1$ and $-iA = -1$. The surviving terms are total divergences and integrate to zero for the usual boundary conditions. This fixes the normalisation of the SUSY transformations.

Having established invariance of the action, we now investigate the algebraic structure of supersymmetry by computing the commutator of two transformations. Acting on the scalar field produces a spacetime translation, while acting on the spinor yields extra terms proportional to its equation of motion $i\bar{\sigma}^\mu\partial_\mu\chi = 0$. The algebra therefore closes only *on shell*; the translation part of the superalgebra is derived in Section 10.4.

The off-shell problem and the auxiliary field F

The on-shell closure signals a deeper issue: the bosonic and fermionic *off-shell* degrees of freedom do not match.

Field	Off-shell (real d.o.f.)	On-shell (real d.o.f.)
ϕ (complex scalar)	2	2
χ_α (Weyl spinor)	4	2

On shell, the equation of motion $i\bar{\sigma}^\mu \partial_\mu \chi = 0$ removes 2 of the 4 real components of χ, restoring the boson–fermion balance $2 = 2$. But off-shell (before imposing the equations of motion), bosons have 2 components while fermions have 4. SUSY, which must map bosons into fermions and vice versa, cannot close algebraically unless the off-shell count also matches. The mismatch is $4 - 2 = 2$ real (= 1 complex) bosonic degrees of freedom.

The fix: an auxiliary field F. The remedy is to add a new complex scalar field $F(x)$ with no kinetic term. Its Lagrangian is purely algebraic:

$$\mathcal{L}_F = F^\dagger F \,. \tag{10.24}$$

Because F has no derivatives in $\mathcal{L}_F$, it is *non-propagating*: its Euler–Lagrange equation is

$$\frac{\partial \mathcal{L}_F}{\partial F^\dagger} = F = 0 \,,$$

so F vanishes on shell and contributes nothing to the physical (on-shell) spectrum. Off shell, however, F supplies the missing 2 real bosonic components, restoring the balance:

Field	Off-shell (real d.o.f.)	On-shell (real d.o.f.)
ϕ	2	2
F	2	0
χ_α	4	2
Total bosonic	4	2
Total fermionic	4	2

Why this particular Lagrangian? The choice $\mathcal{L}_F = F^\dagger F$ (rather than, say, $m^2 F^\dagger F$ or $F^\dagger \Box F$) is forced by three requirements:

- *Dimension*: since $[\mathcal{L}] = 4$ and $[F] = 2$ (mass dimension, in $d = 4$), the only renormalisable term without derivatives is $F^\dagger F$ (with a dimensionless coefficient, normalised to 1).