- *No kinetic term*: if F had a kinetic term $\partial_\mu F^\dagger \partial^\mu F$, it would propagate and add physical particles to the spectrum; we need F to be purely auxiliary.
- *Positive scalar potential after elimination*: the sign of $F^\dagger F$ is fixed so that, once the auxiliary field is eliminated in the interacting theory, the on-shell scalar potential is a sum of absolute squares, $V_F = |W_i|^2 \geq 0$. Since F has no kinetic term, the off-shell term $F^\dagger F$ should not be interpreted as a propagating energy density by itself.

Dimension of F and its SUSY transformation. The dimension $[F] = 2$ is determined by the requirement that $\delta_\epsilon \chi$ be dimensionally consistent. Since $[\chi] = 3/2$, $[\epsilon] = -1/2$, and $[\partial_\mu \phi] = 2$, the original term $-i\sigma^\mu \bar{\epsilon}\, \partial_\mu \phi$ in $\delta_\epsilon \chi$ has dimension $2 + (-1/2) = 3/2$ — correct. The new term ϵF must have the same dimension: $[\epsilon F] = -1/2 + [F]$, so $[F] = 2$.

Lorentz invariance then dictates the form of $\delta_\epsilon F$. The transformation must be a Lorentz scalar of dimension $[F] = 2$, built from the SUSY parameter and the fields of the multiplet with at most one derivative (to keep the algebra first-order). The available building blocks with the right quantum numbers are $\bar{\epsilon}$ ($[\bar{\epsilon}] = -1/2$, right-handed) and $\partial_\mu \chi$ ($[\partial_\mu \chi] = 5/2$, left-handed with a Lorentz index). The combination $\bar{\epsilon}\bar{\sigma}^\mu \partial_\mu \chi$ contracts all indices into a scalar and has dimension $-1/2 + 5/2 = 2$. This is the unique such combination:

$$\delta_\epsilon F = -i\bar{\epsilon}\bar{\sigma}^\mu \partial_\mu \chi, \qquad \delta_\epsilon F^\dagger = i\partial_\mu \bar{\chi}\,\bar{\sigma}^\mu \epsilon\,. \tag{10.25}$$

Correspondingly, the spinor transformation acquires the additional term: $\delta_\epsilon \chi = -i\sigma^\mu \bar{\epsilon}\, \partial_\mu \phi + \epsilon F$. With these modifications, the supersymmetry algebra closes *off-shell* on all three fields of the chiral multiplet (ϕ, χ, F).

The complete transformation laws are

$$\boxed{\begin{aligned} \delta_\epsilon \phi &= \epsilon \chi \\ \delta_\epsilon \chi &= -i\sigma^\mu \bar{\epsilon}\, \partial_\mu \phi + \epsilon F \\ \delta_\epsilon F &= -i\bar{\epsilon}\bar{\sigma}^\mu \partial_\mu \chi \end{aligned}} \qquad \boxed{\begin{aligned} \delta_\epsilon \phi^\dagger &= \bar{\epsilon}\bar{\chi} \\ \delta_\epsilon \bar{\chi}^{\dot{\alpha}} &= -i\partial_\mu \phi^\dagger\, (\bar{\sigma}^\mu \epsilon)^{\dot{\alpha}} + \bar{\epsilon}^{\dot{\alpha}} F^\dagger \\ \delta_\epsilon F^\dagger &= +i\partial_\mu \bar{\chi}\,\bar{\sigma}^\mu \epsilon \end{aligned}} \tag{10.26}$$

and the algebra $(\delta_\epsilon \delta_\eta - \delta_\eta \delta_\epsilon)\Phi = (\epsilon\sigma^\mu \bar{\eta} - \eta\sigma^\mu \bar{\epsilon})(-i\partial_\mu \Phi)$ now holds for $\Phi = \phi, \chi, F$ without invoking the equations of motion.

10.4 The SUSY algebra and supermultiplets

A fundamental aspect of any symmetry (other than a U(1) symmetry) is the algebra associated with the symmetry generators. For example, the generators T_i of SU(2) satisfy the commutation relations $[T_i, T_j] = i\epsilon_{ijk} T_k$, which constitute the 'SU(2) algebra'. The whole theory of angular momentum in quantum mechanics—in particular the multiplet structure—can be developed from these commutation relations alone. In the same way, to proceed systematically with SUSY and to understand what kind of 'supermultiplets' occur, we must determine the SUSY algebra.

Deriving the algebra: the SU(2) example

Before tackling supersymmetry, it is instructive to see the method in a familiar context. Consider an SU(2) doublet of fields $q = (u, d)^\top$ with the Lagrangian invariant under infinitesimal transformations

$$q \to q' = (1 - i\boldsymbol{\epsilon} \cdot \boldsymbol{\tau}/2)q \equiv q + \delta_\epsilon q\,, \qquad \delta_\epsilon q = -i\boldsymbol{\epsilon} \cdot \boldsymbol{\tau}/2\, q\,, \tag{10.27}$$

where $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)$ are the Pauli matrices and $\boldsymbol{\epsilon}$ are three real infinitesimal parameters.

There are two equivalent descriptions of this transformation, and the distinction is crucial for what follows.

Matrix level. The doublet $q = (u, d)^\top$ is a column vector of fields (classical or quantum). The transformation (10.27) acts by matrix multiplication: $q' = U_{\text{fund}}\, q$, where $U_{\text{fund}} = \exp(-i\boldsymbol{\epsilon} \cdot \boldsymbol{\tau}/2)$ is a 2×2 unitary matrix (the expression $1 - i\boldsymbol{\epsilon} \cdot \boldsymbol{\tau}/2$ in (10.27) is its linearisation). The matrices $\tau_i/2$ act on the *internal* (doublet) indices of q and know nothing about Fock space.

Operator level. In the quantum theory, $q(x)$ is a field *operator* acting on Fock space. The same physical transformation is implemented by conjugation with a unitary Fock-space operator $U_{\text{op}} = \exp(i\boldsymbol{\epsilon} \cdot \mathbf{T})$, where $\mathbf{T} = (T_1, T_2, T_3)$ are the conserved Noether charges (operators on Fock space, not 2×2 matrices). Infinitesimally:

$$q' = U_{\text{op}}\, q\, U_{\text{op}}^{-1} = (1 + i\boldsymbol{\epsilon} \cdot \mathbf{T})\, q\, (1 - i\boldsymbol{\epsilon} \cdot \mathbf{T}) = q + i\boldsymbol{\epsilon} \cdot [\mathbf{T}, q] + O(\epsilon^2)\,. \tag{10.28}$$

The two descriptions must agree: the commutator $[\mathbf{T}, q]$ on Fock space must reproduce the matrix multiplication $(-\boldsymbol{\tau}/2)\, q$ on the doublet indices. Comparing with (10.27):

$$\delta_\epsilon q = i\boldsymbol{\epsilon} \cdot [\mathbf{T}, q] = -i\boldsymbol{\epsilon} \cdot \boldsymbol{\tau}/2\, q\,. \tag{10.29}$$

This equation bridges the two levels: the left-hand side is the Fock-space commutator; the right-hand side is the 2×2 matrix action on the doublet.

Deriving the algebra. Consider two transformations with parameters ϵ_1 and ϵ_2:

$$\delta_{\epsilon_1} q = i\epsilon_1 [T_1, q] = -i\epsilon_1 (\tau_1/2) q, \qquad \delta_{\epsilon_2} q = i\epsilon_2 [T_2, q] = -i\epsilon_2 (\tau_2/2) q \tag{10.30}$$

We calculate $(\delta_{\epsilon_1}\delta_{\epsilon_2} - \delta_{\epsilon_2}\delta_{\epsilon_1})q$ in two ways.

Method 1: Using the explicit form of δq. We have

$$\begin{aligned}\delta_{\epsilon_1}\delta_{\epsilon_2} q &= \delta_{\epsilon_1}\{-i\epsilon_2(\tau_2/2)q\} = -i\epsilon_2(\tau_2/2)\delta_{\epsilon_1} q \\ &= -i\epsilon_2(\tau_2/2)\cdot(-i\epsilon_1(\tau_1/2)q) = -\frac{1}{4}\epsilon_1\epsilon_2\tau_2\tau_1 q\end{aligned} \tag{10.31}$$

Similarly, $\delta_{\epsilon_2}\delta_{\epsilon_1} q = -\frac{1}{4}\epsilon_1\epsilon_2\tau_1\tau_2 q$. Hence

$$\begin{aligned}(\delta_{\epsilon_1}\delta_{\epsilon_2} - \delta_{\epsilon_2}\delta_{\epsilon_1})q &= -\frac{1}{4}\epsilon_1\epsilon_2(\tau_2\tau_1 - \tau_1\tau_2)q = \frac{1}{4}\epsilon_1\epsilon_2[\tau_1,\tau_2]q \\ &= \frac{1}{4}\epsilon_1\epsilon_2(2i\tau_3)q = i\epsilon_1\epsilon_2(\tau_3/2)q \\ &= -i\epsilon_1\epsilon_2[T_3, q]\end{aligned} \tag{10.32}$$

where we used $[\tau_1/2, \tau_2/2] = i\tau_3/2$.

Method 2: Using the generator form. We have

$$\delta_{\epsilon_1}\delta_{\epsilon_2} q = \delta_{\epsilon_1}\{i\epsilon_2[T_2, q]\} = i\epsilon_2\delta_{\epsilon_1}[T_2, q] = i\epsilon_1 i\epsilon_2[T_1, [T_2, q]] \tag{10.33}$$

Similarly, $\delta_{\epsilon_2}\delta_{\epsilon_1} q = -\epsilon_1\epsilon_2[T_2, [T_1, q]]$. Hence

$$(\delta_{\epsilon_1}\delta_{\epsilon_2} - \delta_{\epsilon_2}\delta_{\epsilon_1})q = -\epsilon_1\epsilon_2\{[T_1, [T_2, q]] - [T_2, [T_1, q]]\} \tag{10.34}$$

Using the Jacobi identity $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$, we write

$$[T_1, [T_2, q]] - [T_2, [T_1, q]] = [T_1, [T_2, q]] + [T_2, [q, T_1]] = [[T_1, T_2], q] \tag{10.35}$$

Therefore:

$$(\delta_{\epsilon_1}\delta_{\epsilon_2} - \delta_{\epsilon_2}\delta_{\epsilon_1})q = -\epsilon_1\epsilon_2[[T_1, T_2], q] \tag{10.36}$$

Comparing (10.32) and (10.36), we deduce $[T_1, T_2] = iT_3$, exactly as in the SU(2) algebra. This is the method we shall use to derive the SUSY algebra.

Supersymmetry generators and their algebra

For supersymmetry, we need the analogue of (10.29): a Fock-space operator G_ξ such that $\delta_\xi\phi = i[G_\xi, \phi]$.

Counting supercharges: what $\mathcal{N}$ means. The Standard Model symmetries (Poincaré plus internal gauge symmetries) are generated by *bosonic* charges — they all commute with the field operators in a way that preserves the boson/fermion nature of the state. Supersymmetry introduces an entirely new kind of generator: a *fermionic* charge that, when applied to a boson, gives a fermion, and vice versa. Such a charge must itself transform as a spinor under the Lorentz group.

The minimal choice is to introduce *one* spinor charge Q_α in the $(\frac{1}{2},0)$ representation, together with its right-handed partner $\bar{Q}_{\dot\alpha} \equiv (Q_\alpha)^\dagger$ (the dagger maps $(\frac{1}{2},0)$ to $(0,\frac{1}{2})$, just as complex conjugation does for spinor fields). This is called $\mathcal{N}=1$ supersymmetry: "$\mathcal{N}$" counts the number of *independent* pairs of spinor charges introduced. In general:

- $\mathcal{N}=1$: one pair $(Q_\alpha, \bar{Q}_{\dot\alpha})$. *This is the case treated in these notes.*
- $\mathcal{N}=2$: two independent pairs $(Q^I_\alpha, \bar{Q}^I_{\dot\alpha})$ with $I=1,2$.
- $\mathcal{N}=4$: four independent pairs (the maximum allowed for an interacting theory in 4D without higher-spin gravity-like complications: $\mathcal{N}=4$ super-Yang–Mills is the most symmetric four-dimensional field theory known).

Each additional pair of supercharges tightens the constraints on the spectrum: bigger $\mathcal{N}$ means smaller, more rigid multiplets. As we shall see at the end of this section, only $\mathcal{N}=1$ is compatible with the chiral structure of the Standard Model.

For the rest of this section we restrict to $\mathcal{N}=1$.

A note on F. In Section 10.3 we introduced the auxiliary field F to achieve off-shell closure. In the algebra derivation that follows, we work entirely with ϕ and its on-shell transformations $\delta_\xi\phi = \xi\chi$, $\delta_\xi\chi = -i\sigma^\mu\bar{\xi}\partial_\mu\phi$. This is sufficient because: (i) the algebra $\{Q_\alpha, \bar{Q}_{\dot\beta}\} = \sigma^\mu_{\alpha\dot\beta}P_\mu$ is extracted by computing $(\delta_\eta\delta_\xi - \delta_\xi\delta_\eta)\phi$ — a double variation on the *scalar* field, for which the algebra closes without F (the F-dependent terms in $\delta\chi$ would only matter when computing the commutator on χ itself); (ii) the resulting algebra is the same whether derived on-shell or off-shell, because the anticommutation relations between the charges Q_α and $\bar{Q}_{\dot\alpha}$ are operator identities valid in the full Hilbert space. The field F is essential for off-shell closure on χ, but does not affect the algebra itself.

Building G_ξ. Since the SUSY parameter ξ_α is a left-handed Grassmann spinor, G_ξ must be linear in ξ and $\bar{\xi}$ and a Lorentz scalar. The right-handed partner of Q_α is defined as

$$\bar{Q}_{\dot\alpha} \;\equiv\; (Q_\alpha)^\dagger\,.$$

This is *exactly* the same convention used for spinor *fields* in Chapter 9: the bar on a left-handed object χ_α produces its right-handed partner $\bar{\chi}_{\dot\alpha} \equiv (\chi_\alpha)^*$ (eq. (9.43)). For c-number fields the conjugation is complex conjugation ($*$); for Fock-space operators the conjugation is Hermitian conjugation ($\dagger$). Both map the $(\frac{1}{2},0)$ representation into the inequivalent $(0,\frac{1}{2})$ representation, which is why the index changes from undotted to dotted. The bar notation is the same in both cases: "bar = conjugate partner in the opposite chirality."

With this, the only Lorentz-invariant bilinears are

$$\xi\cdot Q \equiv \xi^\alpha Q_\alpha \qquad \text{and} \qquad \bar{\xi}\cdot\bar{Q} \equiv \bar{\xi}_{\dot\alpha}\bar{Q}^{\dot\alpha}\,,$$

contracted with the standard NW→SE and SW→NE staircases respectively (cf. box 9.4). *Both* terms are needed: $\xi \cdot Q$ alone would not be Hermitian (the generator of a unitary transformation must satisfy $G_\xi^\dagger = G_\xi$). To verify $(\xi \cdot Q)^\dagger = \bar{\xi} \cdot \bar{Q}$, we adopt the convention that Hermitian conjugation reverses the order of all factors *without* introducing extra Grassmann signs (i.e., Grassmann parameters are treated as "classical external objects" for the purpose of †): $(\xi^\alpha Q_\alpha)^\dagger = Q_\alpha^\dagger (\xi^\alpha)^* = \bar{Q}_{\dot\alpha} \bar{\xi}^{\dot\alpha} = \bar{\xi}_{\dot\alpha} \bar{Q}^{\dot\alpha} = \bar{\xi} \cdot \bar{Q}$, where in the last step we used the standard spinor-index gymnastics. The Hermitian combination is then

$$G_\xi \equiv \xi \cdot Q + \bar{\xi} \cdot \bar{Q}.$$

Equivalently, $\bar{Q}$ is required because $\delta_\xi \phi = \xi\chi$ involves only ξ (not $\bar{\xi}$), while $\delta_\xi \chi = -i\sigma^\mu \bar{\xi}\, \partial_\mu \phi$ involves only $\bar{\xi}$: a single spinor charge could not generate both transformations simultaneously.

With this notation, the operator-level form of the SUSY transformation of ϕ is:

$$\delta_\xi \phi = i[G_\xi, \phi] = i[\xi \cdot Q + \bar{\xi} \cdot \bar{Q}, \phi] = \xi \cdot \chi. \tag{10.37}$$

Step 1: Calculate $(\delta_\eta \delta_\xi - \delta_\xi \delta_\eta)\phi$ using the explicit transformations. Using $\delta_\xi \phi = \xi \cdot \chi$ and $\delta_\xi \chi = -i\sigma^\mu \bar{\xi}\, \partial_\mu \phi$:

$$\begin{aligned}\delta_\eta \delta_\xi \phi &= \delta_\eta(\xi \cdot \chi) = \xi \cdot (\delta_\eta \chi) = \xi^\alpha (-i\sigma^\mu \bar{\eta})_\alpha \partial_\mu \phi \\ &= -i\, \xi^\alpha (\sigma^\mu)_{\alpha\dot\alpha} \bar{\eta}^{\dot\alpha} \partial_\mu \phi \end{aligned} \tag{10.38}$$

There is *no* Grassmann sign here: δ_η is a field substitution ($\chi \to \chi + \delta_\eta \chi$), not a Grassmann-odd differential operator, and ξ^α is a constant external parameter that is simply carried through. (Although $G_\xi = \xi \cdot Q + \bar{\xi} \cdot \bar{Q}$ involves Grassmann parameters, the product of two Grassmann-odd objects makes G_ξ Grassmann-*even*, so $\delta_\xi \phi = i[G_\xi, \phi]$ is an ordinary, even transformation.) Similarly, $\delta_\xi \delta_\eta \phi = -i\, \eta^\alpha (\sigma^\mu)_{\alpha\dot\alpha} \bar{\xi}^{\dot\alpha} \partial_\mu \phi$. Taking the difference:

$$\begin{aligned}(\delta_\eta \delta_\xi - \delta_\xi \delta_\eta)\phi &= -i(\xi^\alpha (\sigma^\mu)_{\alpha\dot\alpha} \bar{\eta}^{\dot\alpha} - \eta^\alpha (\sigma^\mu)_{\alpha\dot\alpha} \bar{\xi}^{\dot\alpha}) \partial_\mu \phi \\ &= -i(\xi \sigma^\mu \bar{\eta} - \eta \sigma^\mu \bar{\xi})\, \partial_\mu \phi \end{aligned} \tag{10.39}$$

To rewrite in the $\bar{\sigma}$ basis, use the spinor-reversal identity for anticommuting spinors $z_1 \sigma^\mu \bar{z}_2 = -\bar{z}_2 \bar{\sigma}^\mu z_1$ (eq. (9.100) with $(-1)^A = -1$). Applying it to each term:

$$\xi \sigma^\mu \bar{\eta} = -\bar{\eta} \bar{\sigma}^\mu \xi, \qquad \eta \sigma^\mu \bar{\xi} = -\bar{\xi} \bar{\sigma}^\mu \eta,$$

so

$$\xi \sigma^\mu \bar{\eta} - \eta \sigma^\mu \bar{\xi} = -(\bar{\eta} \bar{\sigma}^\mu \xi - \bar{\xi} \bar{\sigma}^\mu \eta). \tag{10.40}$$

The overall minus from the reversal combines with the $-i$ from the variation to give $+i$:

$$\boxed{(\delta_\eta \delta_\xi - \delta_\xi \delta_\eta)\phi = +i(\bar{\eta} \bar{\sigma}^\mu \xi - \bar{\xi} \bar{\sigma}^\mu \eta)\, \partial_\mu \phi} \tag{10.41}$$

Step 2: Calculate $(\delta_\eta\delta_\xi - \delta_\xi\delta_\eta)\phi$ using the generators. Let $G_\xi \equiv \xi\cdot Q + \bar{\xi}\cdot\bar{Q}$. From (10.37), $\delta_\xi\phi = i[G_\xi, \phi]$. Using the Jacobi identity as in Aitchison:

$$[[G_\eta, G_\xi], \phi] = (\delta_\eta\delta_\xi - \delta_\xi\delta_\eta)\phi\,. \tag{10.42}$$

Step 3: Compare the two results. Introducing the 4-momentum operator P_μ with $[P_\mu, \phi] = -i\partial_\mu\phi$, equation (10.41) becomes

$$(\delta_\eta\delta_\xi - \delta_\xi\delta_\eta)\phi = -(\bar{\eta}\bar{\sigma}^\mu\xi - \bar{\xi}\bar{\sigma}^\mu\eta)[P_\mu, \phi] \tag{10.43}$$

Comparing with (10.42):

$$[G_\eta, G_\xi] = (\bar{\xi}\bar{\sigma}^\mu\eta - \bar{\eta}\bar{\sigma}^\mu\xi)P_\mu \tag{10.44}$$

Step 4: Extract the (anti)commutation relations. The right-hand side of (10.44) contains only terms of the form $\bar{\xi}\ldots\eta$ and $\bar{\eta}\ldots\xi$. There are no terms $\xi\ldots\eta$ or $\bar{\xi}\ldots\bar{\eta}$, which implies:

$$[\xi\cdot Q, \eta\cdot Q] = 0\,, \qquad [\bar{\xi}\cdot\bar{Q}, \bar{\eta}\cdot\bar{Q}] = 0\,. \tag{10.45}$$

Expanding the first commutator explicitly:

$$\begin{aligned} 0 &= (\xi^1Q_1 + \xi^2Q_2)(\eta^1Q_1 + \eta^2Q_2) - (\eta^1Q_1 + \eta^2Q_2)(\xi^1Q_1 + \xi^2Q_2) \\ &= -\xi^1\eta^1(2Q_1Q_1) - \xi^1\eta^2(Q_1Q_2 + Q_2Q_1) \\ &\quad -\xi^2\eta^1(Q_2Q_1 + Q_1Q_2) - \xi^2\eta^2(2Q_2Q_2) \end{aligned} \tag{10.46}$$

where we used the anticommutation of the Grassmann parameters. Since all parameter combinations are independent:

$$\boxed{\{Q_\alpha, Q_\beta\} = 0, \qquad \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0} \tag{10.47}$$

For the mixed term $[\eta\cdot Q, \bar{\xi}\cdot\bar{Q}]$, we write out components:

$$\bar{\xi}\cdot\bar{Q} = \bar{\xi}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}} = \xi_1^*Q_2^\dagger - \xi_2^*Q_1^\dagger \tag{10.48}$$

$$\eta\cdot Q = \eta^\alpha Q_\alpha = -\eta_1Q_2 + \eta_2Q_1 \tag{10.49}$$

Computing the commutator and using anticommutation of parameters:

$$\begin{aligned} [\eta\cdot Q, \bar{\xi}\cdot\bar{Q}] = \eta_1\xi_1^*(Q_2Q_2^\dagger + Q_2^\dagger Q_2) - \eta_1\xi_2^*(Q_2Q_1^\dagger + Q_1^\dagger Q_2) \\ -\eta_2\xi_1^*(Q_1Q_2^\dagger + Q_2^\dagger Q_1) + \eta_2\xi_2^*(Q_1Q_1^\dagger + Q_1^\dagger Q_1) \end{aligned} \tag{10.50}$$

Meanwhile, the right-hand side of (10.44) gives (using $c = -i\sigma_2$):

$$\begin{aligned} \bar{\xi}\bar{\sigma}^\mu\eta P_\mu &= \xi^\dagger(i\sigma_2)(\bar{\sigma}^\mu)\eta P_\mu \\ &= [\eta_2\xi_2^*(\sigma^\mu)_{11} - \eta_2\xi_1^*(\sigma^\mu)_{12} - \eta_1\xi_2^*(\sigma^\mu)_{21} + \eta_1\xi_1^*(\sigma^\mu)_{22}]P_\mu \end{aligned} \tag{10.51}$$

(where we used $c\bar{\sigma}^\mu c^{-1} = -\sigma^{\mu T}$ and transposed). Comparing coefficients of each parameter combination in (10.50) and the above:

$$\boxed{\{Q_\alpha, \bar{Q}_{\dot\beta}\} = (\sigma^\mu)_{\alpha\dot\beta} P_\mu} \tag{10.52}$$

This is Aitchison's normalization of the supercharges. The common alternative convention is obtained by the rescaling $Q \to \sqrt{2}Q$, which puts a factor 2 on the right-hand side.

This is the fundamental SUSY algebra. It must be supplemented by the commutator of the supercharges with the momentum operator.

Why $[Q_\alpha, P_\mu] = 0$. The supercharges are spatial integrals of a conserved current: $Q_\alpha = \int d^3x\, J^0_\alpha(x)$ (eq. (10.63)). Since translations are generated by P_μ via $[P_\mu, O(x)] = -i\partial_\mu O(x)$ for any local operator O:

$$[P_\mu, Q_\alpha] = \int d^3x\, [P_\mu, J^0_\alpha(x)] = -i \int d^3x\, \partial_\mu J^0_\alpha(x)\,.$$

For $\mu = i$ (spatial): for each fixed i, $\partial_i J^0_\alpha$ is a total derivative in the x^i direction. By the fundamental theorem of calculus:

$$\int d^3x\, \partial_i J^0_\alpha(x) = \int dx^j\, dx^k \left[J^0_\alpha\right]_{x^i=-\infty}^{x^i=+\infty} = 0\,,$$

because $J^0_\alpha \to 0$ as $x^i \to \pm\infty$ (the standard fall-off condition on the fields). For $\mu = 0$ (temporal):

$$-i\int d^3x\, \partial_0 J^0_\alpha = -i\frac{d}{dt}Q_\alpha = 0\,,$$

because Q_α is a conserved charge. To see this, use $\partial_\mu J^\mu_\alpha = 0$:

$$\dot{Q}_\alpha = \int d^3x\, \partial_0 J^0_\alpha = -\int d^3x\, \partial_i J^i_\alpha = 0\,,$$

where the last integral vanishes by the same total-derivative argument applied to each J^i_α. Therefore:

$$[Q_\alpha, P_\mu] = [\bar{Q}_{\dot\alpha}, P_\mu] = 0\,. \tag{10.53}$$

Physically: SUSY transformations and spacetime translations commute — performing a SUSY rotation and then translating gives the same result as translating first and then rotating. Combined with the anticommutators derived above, the supertranslation algebra is closed.

Box 10.1 The $\mathcal{N} = 1$ Supersymmetry Algebra

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = (\sigma^\mu)_{\alpha\dot\beta} P_\mu \quad \text{(Aitchison normalization)} \tag{10.54}$$

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0 \tag{10.55}$$

$$[Q_\alpha, P_\mu] = [\bar{Q}_{\dot\alpha}, P_\mu] = 0 \tag{10.56}$$

The SUSY generators are directly connected to the energy-momentum operator, justifying the view of SUSY as an extension of spacetime symmetry.

The Hamiltonian as a sum of squares

A striking consequence of the algebra is that the Hamiltonian can be expressed as a sum of squares of supercharges. From $\{Q_\alpha, \bar{Q}_{\dot\beta}\} = (\sigma^\mu)_{\alpha\dot\beta} P_\mu$ and recalling that $(\sigma^0)_{11} = (\sigma^0)_{22} = 1$, $(\sigma^3)_{11} = -(\sigma^3)_{22} = 1$, we have

$$\{Q_1, \bar{Q}_{\dot 1}\} = P_0 + P_3\,, \qquad \{Q_2, \bar{Q}_{\dot 2}\} = P_0 - P_3\,. \tag{10.57}$$

Following Aitchison, P_3 in these component formulas is the covariant component appearing in $\sigma^\mu P_\mu$; with the $(+---)$ metric, $P_3 = -P^3$. Adding these relations, the P_3 contributions cancel:

$$\boxed{H = P^0 = \frac{1}{2}(Q_1\bar{Q}_{\dot 1} + \bar{Q}_{\dot 1}Q_1 + Q_2\bar{Q}_{\dot 2} + \bar{Q}_{\dot 2}Q_2)} \tag{10.58}$$

This has a profound consequence: in any state $|\psi\rangle$,

$$\langle\psi|H|\psi\rangle = \frac{1}{2}\left(\|Q_1|\psi\rangle\|^2 + \|\bar{Q}_{\dot 1}|\psi\rangle\|^2 + \|Q_2|\psi\rangle\|^2 + \|\bar{Q}_{\dot 2}|\psi\rangle\|^2\right) \geq 0\,. \tag{10.59}$$

The energy is non-negative, and $E = 0$ if and only if all supercharges annihilate the state: $Q_\alpha|0\rangle = \bar{Q}_{\dot\alpha}|0\rangle = 0$. This is the condition for an *unbroken supersymmetric vacuum*.

The supersymmetry current

Review: Noether's theorem for exact and quasi-symmetries. Recall the logic of Noether's theorem. If a Lagrangian $\mathcal{L}(\phi_r, \partial_\mu\phi_r)$ is *exactly* invariant under $\phi_r \to \phi_r + \delta\phi_r$, then $\delta\mathcal{L} = 0$. On the other hand, the chain rule gives

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi_r}\delta\phi_r + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_r)}\partial_\mu(\delta\phi_r) = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_r)}\delta\phi_r\right) + \left(\frac{\partial\mathcal{L}}{\partial\phi_r} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_r)}\right)\delta\phi_r\,.$$

The second bracket vanishes by the Euler–Lagrange equations (on shell). Combining with $\delta\mathcal{L} = 0$:

$$\partial_\mu j^\mu = 0\,, \qquad j^\mu \equiv \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_r)}\,\delta\phi_r\,. \tag{10.60}$$

If the symmetry is only a *quasi-symmetry* — the Lagrangian is not invariant but changes by a total derivative, $\delta\mathcal{L} = \partial_\mu K^\mu$ for some K^μ — then the same argument gives $\partial_\mu(j^\mu - K^\mu) = 0$, so the conserved current is $j^\mu - K^\mu$. This is precisely the situation for supersymmetry, where the action is invariant but $\mathcal{L}$ shifts by a total derivative (as shown in Section 10.3).

Application to the Wess–Zumino model. For the free Wess–Zumino Lagrangian $\mathcal{L} = \partial_\mu\phi^\dagger\partial^\mu\phi + i\bar\chi\bar\sigma^\mu\partial_\mu\chi$, the fields are ϕ, $\phi^\dagger$, and χ_α (with $\bar\chi$ determined by χ). Since the SUSY current carries a spinor index (it is a spinor-valued vector), we write $j^\mu = \xi\cdot J^\mu + \bar\xi\cdot\bar J^\mu$, where J^μ_α and $\bar J^{\mu\dot\alpha}$ are the supercurrent and its conjugate. The Noether construction gives:

$$\xi\cdot J^\mu + \bar\xi\cdot\bar J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\delta\phi + \delta\phi^\dagger\,\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^\dagger)} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\chi_\alpha)}\,\delta\chi_\alpha - K^\mu\,. \tag{10.61}$$

Important sign convention for fermions. For the fermionic field χ, the derivative $\partial\mathcal{L}/\partial(\partial_\mu\chi_\alpha)$ must be taken as a *left* derivative. This means: identify $\partial_\mu\chi_\alpha$ in each term, and read off the coefficient that multiplies it *from the left*. Concretely, the fermionic kinetic term is

$$\mathcal{L}_F = i\;\underbrace{\bar\chi_{\dot\beta}\bar\sigma^{\mu\dot\beta\alpha}}_{\text{left coefficient}}\;\partial_\mu\chi_\alpha\,.$$

The object $\partial_\mu\chi_\alpha$ already sits at the *rightmost* position of the product, so its left coefficient is read off directly without moving any Grassmann field past another:

$$\frac{\partial^L\mathcal{L}_F}{\partial(\partial_\mu\chi_\alpha)} = i\bar\chi_{\dot\beta}\bar\sigma^{\mu\dot\beta\alpha}\,.$$

No sign flip occurs precisely because $\partial_\mu\chi_\alpha$ does not need to be commuted past $\bar\chi$ — it is already in the correct position. Had the Lagrangian been written in the opposite order, $\mathcal{L}_F = i(\partial_\mu\chi_\alpha)\bar\chi_{\dot\beta}\bar\sigma^{\mu\dot\beta\alpha}$, then $\partial_\mu\chi_\alpha$ would be at the *left* and extracting it would again give no sign; but the remaining coefficient $\bar\chi\bar\sigma^\mu$ would now be a *right* coefficient, and the distinction between left and right derivatives matters for the overall sign of the Noether current. The convention used here (and in Aitchison) is: always write $\mathcal{L}_F$ with $\partial_\mu\chi$ to the right, and take the left derivative. (See Weinberg I, §7.3 for a systematic treatment of fermionic functional derivatives.)

Computing each piece. The partial derivatives of $\mathcal{L}$ are:

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \partial^\mu\phi^\dagger\,, \qquad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^\dagger)} = \partial^\mu\phi\,, \qquad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\chi_\alpha)} = i\bar{\chi}_{\dot\beta}\bar{\sigma}^{\mu\dot\beta\alpha}\,.$$

The SUSY variations (with $A = -i$):

$$\delta\phi = \xi\chi\,, \qquad \delta\phi^\dagger = \bar{\xi}\bar{\chi}\,, \qquad \delta\chi_\alpha = -i(\sigma^\nu\bar{\xi})_\alpha\partial_\nu\phi\,.$$

Substituting into (10.61):

$$\begin{aligned}\xi\cdot J^\mu + \bar{\xi}\cdot\bar{J}^\mu &= \partial^\mu\phi^\dagger\cdot\xi\chi + \bar{\xi}\bar{\chi}\cdot\partial^\mu\phi + i\bar{\chi}\bar{\sigma}^\mu\cdot(-i\sigma^\nu\bar{\xi}\partial_\nu\phi) - K^\mu \\ &= \partial^\mu\phi^\dagger\,\xi\chi + \bar{\xi}\bar{\chi}\,\partial^\mu\phi + \bar{\chi}\bar{\sigma}^\mu\sigma^\nu\bar{\xi}\,\partial_\nu\phi - K^\mu\,.\end{aligned} \tag{10.62}$$

The total-derivative piece K^μ was determined during the invariance proof (Section 10.3); its role is to absorb the antisymmetric $\sigma^{\mu\nu}$ terms that arose there. After subtracting K^μ and simplifying using the equations of motion, the supercurrent takes the form:

$$\boxed{J^\mu_\alpha = (\sigma^\nu\bar{\sigma}^\mu\chi)_\alpha\,\partial_\nu\phi^\dagger} \tag{10.63}$$

with conjugate $\bar{J}^{\mu\dot\alpha} = (\bar{\sigma}^\nu\sigma^\mu\bar{\chi})^{\dot\alpha}\partial_\nu\phi$. Conservation $\partial_\mu J^\mu_\alpha = 0$ holds on shell (using the free equations of motion $\Box\phi = 0$ and $i\bar{\sigma}^\mu\partial_\mu\chi = 0$).

Improvement terms. This result is not unique: the Noether current is defined only up to *improvement terms*. An improvement is any tensor $\Sigma^{\mu\nu} = -\Sigma^{\nu\mu}$ (antisymmetric in $\mu\nu$), since

$$J'^\mu \;\equiv\; J^\mu + \partial_\nu\Sigma^{\mu\nu}$$

is automatically conserved whenever J^μ is: $\partial_\mu J'^\mu = \partial_\mu J^\mu + \partial_\mu\partial_\nu\Sigma^{\mu\nu} = 0$ (the second term vanishes because $\partial_\mu\partial_\nu$ is symmetric while $\Sigma^{\mu\nu}$ is antisymmetric). The integrated charge is also unaffected: $Q' = \int d^3x\,J'^0 = \int d^3x\,J^0 + \int d^3x\,\partial_i\Sigma^{0i} = Q + (\text{boundary term at spatial infinity}) = Q$. Therefore J^μ and J'^μ generate the same symmetry and are physically equivalent. Different textbooks may present supercurrents that differ by such improvements; the algebra and the physics are identical.

The supercharges. Integrating the time component:

$$Q_\alpha = \int d^3x\,J^0_\alpha\,. \tag{10.64}$$

Using $\bar{\sigma}^0 = \mathbb{1}_2$, so $(\sigma^\nu\bar{\sigma}^0\chi)_\alpha = (\sigma^\nu\chi)_\alpha$:

$$Q_\alpha = \int d^3x\,(\sigma^\nu\chi)_\alpha\,\partial_\nu\phi^\dagger\,. \tag{10.65}$$

These charges generate the correct transformations via $\delta_\xi\phi = i[\xi\cdot Q + \bar{\xi}\cdot\bar{Q}, \phi] = \xi\cdot\chi$, as can be verified using the canonical (anti)commutation relations $[\phi(\vec{x}), \dot{\phi}^\dagger(\vec{y})] = i\delta^3(\vec{x}-\vec{y})$ and $\{\chi_\alpha(\vec{x}), \chi^\dagger_\beta(\vec{y})\} = \delta_{\alpha\beta}\delta^3(\vec{x}-\vec{y})$.

Supermultiplets

The SUSY algebra has profound consequences for the structure of particle multiplets.

Mass degeneracy. From $[Q_\alpha, P_\mu] = 0$, the operator $P^2 = P_\mu P^\mu$ commutes with all generators. States in a supermultiplet, connected by the action of Q and $\bar{Q}$, must all have the *same mass*.

Helicity content. Since Q_α transforms as a left-handed spinor under Lorentz transformations, its action on a one-particle state changes the angular momentum quantum number by $\pm\frac{1}{2}$. For the massless supermultiplets that follow, the relevant quantum number is the *helicity* $\lambda = \vec{J}\cdot\hat{p}$ (projection of total angular momentum along the direction of motion), not the spin s (the Casimir label of massive Poincaré representations). The reason is physical: a massless particle has no rest frame, so spin-as-"angular momentum at rest" is not defined; the only Lorentz-invariant label is the helicity. For our standard massless momentum $p^\mu = (E, 0, 0, E)$ along z, the helicity operator is simply J_3. We therefore need $[J_3, Q_\alpha]$.

To determine the precise sign for each component, we use the Lorentz algebra.

The supercharges Q_α belong to the $(\frac{1}{2}, 0)$ representation of the Lorentz group. We now derive the commutator $[J^i, Q_\alpha]$.

In quantum field theory, a Lorentz transformation Λ acts on an operator $\mathcal{O}_a(x)$ that transforms in a representation D as:

$$U(\Lambda)\,\mathcal{O}_a(x)\,U(\Lambda)^{-1} = D(\Lambda^{-1})_a{}^b\,\mathcal{O}_b(\Lambda x)\,.$$

For an infinitesimal rotation by $\vec{\theta}$, expanding both sides to first order gives:

$$[J^i, \mathcal{O}_a(x)] = \underbrace{-i(x^j\partial^k - x^k\partial^j)\mathcal{O}_a(x)}_{\text{orbital part}} + \underbrace{(S^i)_a{}^b\,\mathcal{O}_b(x)}_{\text{spin part}}\,,$$

where S^i are the spin generators in the representation D.

For the supercharge $Q_\alpha = \int d^3x\, J^0_\alpha(x)$, the spatial integration kills the orbital part: terms like $\int d^3x\, x^j\partial^k J^0_\alpha$ can be integrated by parts in x^k (fundamental theorem of calculus), giving boundary contributions that vanish because $J^0_\alpha \to 0$ at spatial infinity. Only the spin part survives:

$$[J^i, Q_\alpha] = (S^i)_\alpha{}^\beta Q_\beta\,.$$

Since Q_α is in the $(\frac{1}{2}, 0)$ representation, $S^i = \frac{1}{2}\sigma^i$ (eq. (9.38)):

$$[J^i, Q_\alpha] = \frac{1}{2}(\sigma^i)_\alpha{}^\beta Q_\beta\,. \tag{10.66}$$

For $i = 3$: $(\sigma^3)_1{}^1 = +1$, $(\sigma^3)_2{}^2 = -1$, off-diagonal $= 0$. Therefore:

$$[J_3, Q_1] = +\frac{1}{2}Q_1\,, \qquad [J_3, Q_2] = -\frac{1}{2}Q_2\,. \tag{10.67}$$

The physical meaning: if $|\lambda\rangle$ is a helicity eigenstate with $J_3|\lambda\rangle = \lambda|\lambda\rangle$, then $Q_1|\lambda\rangle$ has helicity $\lambda + \frac{1}{2}$ (since $J_3(Q_1|\lambda\rangle) = ([J_3, Q_1] + Q_1 J_3)|\lambda\rangle = (\frac{1}{2}Q_1 + \lambda Q_1)|\lambda\rangle = (\lambda + \frac{1}{2})Q_1|\lambda\rangle$), and $Q_2|\lambda\rangle$ has helicity $\lambda - \frac{1}{2}$.

For the conjugate charges $\bar{Q}_{\dot\alpha} = Q^\dagger_\alpha$, which belong to the $(0, \frac{1}{2})$ representation: $Q^\dagger_1$ lowers helicity by $\frac{1}{2}$, and $Q^\dagger_2$ raises it by $\frac{1}{2}$.

In summary: Q_1 and $Q^\dagger_2$ raise the helicity by $\frac{1}{2}$; Q_2 and $Q^\dagger_1$ lower it by $\frac{1}{2}$.

Constructing massless supermultiplets. The strategy is to start from a lowest-helicity state and act on it with all possible supercharges, keeping only the non-zero states. The result is a short multiplet with very few states — far fewer than for massive particles.

Step 1: choose the reference momentum. For a massless particle, pick a standard momentum along the z-axis: $p^\mu = (E, 0, 0, E)$, so that $p_\mu = (E, 0, 0, -E)$ in the $(+---)$ convention. On this state:

$$P^0 = E\,, \quad P^3 = E\,, \quad P_0 = E\,, \quad P_3 = -E\,.$$

Step 2: evaluate the anticommutators on this state. The algebra $\{Q_\alpha, \bar{Q}_{\dot\beta}\} = (\sigma^\mu)_{\alpha\dot\beta}P_\mu$ becomes, on a state with momentum p:

$$\{Q_1, Q^\dagger_1\} = (\sigma^\mu)_{11}P_\mu = P_0 + P_3 = E + (-E) = 0\,, \tag{10.68}$$

$$\{Q_2, Q^\dagger_2\} = (\sigma^\mu)_{22}P_\mu = P_0 - P_3 = E - (-E) = 2E\,. \tag{10.69}$$

(Here we used $(\sigma^0)_{11} = (\sigma^0)_{22} = 1$, $(\sigma^3)_{11} = +1$, $(\sigma^3)_{22} = -1$, and $(\sigma^{1,2})_{11} = (\sigma^{1,2})_{22} = 0$.)

Equation (10.68) is remarkable: it says $\{Q_1, Q^\dagger_1\} = 0$. Since $\{Q_1, Q^\dagger_1\} = Q_1 Q^\dagger_1 + Q^\dagger_1 Q_1$, taking the expectation value in *any* state $|\psi\rangle$:

$$\|Q_1|\psi\rangle\|^2 + \|Q^\dagger_1|\psi\rangle\|^2 = 0\,.$$

Both terms are non-negative (norms squared), so both must vanish: $Q_1|\psi\rangle = Q^\dagger_1|\psi\rangle = 0$ for *every* state with this momentum. Thus Q_1 and $Q^\dagger_1$ act as zero on the entire massless representation. The only non-trivial generators are Q_2 and $Q^\dagger_2$.

Step 3: build the multiplet from the lowest-helicity state. Let $|p, \lambda_{\min}\rangle$ be a state with the lowest helicity $\lambda_{\min}$ in the multiplet. Since Q_2 lowers helicity by $\frac{1}{2}$ (eq. (10.67)), acting with it would produce a state of helicity $\lambda_{\min} - \frac{1}{2}$, which cannot exist if $\lambda_{\min}$ is the minimum. Therefore:

$$Q_2|p, \lambda_{\min}\rangle = 0\,.$$

Similarly, $Q_1^\dagger$ lowers helicity by $\frac{1}{2}$, but we already showed $Q_1^\dagger = 0$ on all massless states.

The only non-trivial action is $Q_2^\dagger$, which *raises* helicity by $\frac{1}{2}$:

$$|p, \lambda_{\min} + \tfrac{1}{2}\rangle \;\propto\; Q_2^\dagger|p, \lambda_{\min}\rangle\,.$$

This state is non-zero: from (10.69) with $Q_2|p, \lambda_{\min}\rangle = 0$,

$$\langle p, \lambda_{\min}|Q_2 Q_2^\dagger|p, \lambda_{\min}\rangle = 2E > 0\,,$$

confirming $Q_2^\dagger|p, \lambda_{\min}\rangle \neq 0$.

Step 4: the chain terminates. Acting with $Q_2^\dagger$ a second time: $(Q_2^\dagger)^2 = 0$ because $\{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0$ implies $(Q_2^\dagger)^2 = \frac{1}{2}\{Q_2^\dagger, Q_2^\dagger\} = 0$. So no further states are generated.

Result. Each massless $\mathcal{N} = 1$ supermultiplet consists of exactly **two** helicity states:

$$\boxed{\text{Massless supermultiplet: } |p, \lambda\rangle \text{ and } |p, \lambda + \tfrac{1}{2}\rangle} \tag{10.70}$$

where λ is the lower helicity. CPT invariance requires the inclusion of the conjugate states with helicities $-\lambda$ and $-\lambda - \frac{1}{2}$, so a CPT-complete massless supermultiplet has four states in total (or two, if the multiplet is self-conjugate under CPT).

Physical examples.

Chiral (matter) supermultiplet: $\lambda = 0$. The two states have helicities 0 and $+\frac{1}{2}$. CPT adds 0 and $-\frac{1}{2}$. The fields describing these states are:

- a complex scalar ϕ (helicity 0, two real degrees of freedom on shell),
- a Weyl fermion χ_α (helicities $\pm\frac{1}{2}$, two real degrees of freedom on shell).

This is exactly the Wess–Zumino multiplet (ϕ, χ, F) studied in Section 10.3 (with F the off-shell auxiliary field that vanishes on shell). Every quark and lepton of the Standard Model sits in a chiral supermultiplet, paired with a scalar "superpartner" (squark, slepton).

Vector (gauge) supermultiplet: $\lambda = -1$. The two states have helicities -1 and $-\frac{1}{2}$. CPT adds $+1$ and $+\frac{1}{2}$. The fields are:

- a massless gauge boson A_μ (helicities ± 1, as for the photon or gluon),
- a Weyl fermion λ_α called the *gaugino* (helicities $\pm\frac{1}{2}$).

Every gauge boson of the SM (photon, $W^\pm$, Z, gluons) sits in a vector supermultiplet, paired with a fermionic gaugino (photino, wino, zino, gluino).

Graviton supermultiplet: $\lambda = -2$. The two states have helicities -2 and $-\frac{3}{2}$. CPT adds $+2$ and $+\frac{3}{2}$. The fields are:

- the graviton $g_{\mu\nu}$ (helicities ± 2, the quantum of gravity),
- the *gravitino* ψ_μ (helicities $\pm\frac{3}{2}$, a spin-$\frac{3}{2}$ Rarita–Schwinger field).

This multiplet requires local (gauged) supersymmetry, i.e. *supergravity*, which is beyond the scope of these notes.

The boson–fermion degeneracy. In every supermultiplet, the number of bosonic and fermionic on-shell degrees of freedom is equal: 2 bosonic = 2 fermionic. This is a general theorem: SUSY maps bosons to fermions and vice versa, and since $[Q, P^2] = 0$, all states in a supermultiplet have the same mass. If SUSY were unbroken, every particle would have a superpartner of equal mass but spin differing by $\frac{1}{2}$. The absence of such partners at current collider energies means SUSY must be broken, with the superpartner masses pushed above the experimental reach.

Remark. [What does $\mathcal{N}$ mean, and why only $\mathcal{N} = 1$?] The integer $\mathcal{N}$ counts the number of *independent* sets of supercharges. In $\mathcal{N} = 1$ SUSY (the case treated in these notes), there is a single pair $(Q_\alpha, \bar{Q}_{\dot\alpha})$ — one left-handed and one right-handed spinor charge. In $\mathcal{N} = 2$ SUSY, there are two such pairs $(Q^1_\alpha, \bar{Q}^1_{\dot\alpha})$ and $(Q^2_\alpha, \bar{Q}^2_{\dot\alpha})$; more generally, $\mathcal{N}$ copies. Each additional pair of supercharges further constrains the multiplet structure and enlarges the minimal representations.

For example, in $\mathcal{N} = 2$ the smallest matter representation (the *hypermultiplet*) contains states with $\lambda = +\frac{1}{2}, -\frac{1}{2}$, and two $\lambda = 0$ states. The two fermionic components are tied together by the extended supersymmetry and therefore *cannot* be assigned different gauge quantum numbers. But in the Standard Model, $SU(2)_L$ treats left- and right-handed fermions differently (only left-handed fermions form doublets). This chiral structure is incompatible with $\mathcal{N} \geq 2$ matter multiplets: in such multiplets, the left- and right-handed components are locked into the same representation and cannot be separated.

Realistic four-dimensional weak-scale model building therefore starts from $\mathcal{N} = 1$, where left and right chiral multiplets can be introduced independently with different gauge charges. Extended SUSY ($\mathcal{N} \geq 2$) can still appear in gauge sectors ($\mathcal{N} = 2$ gauge theories have remarkable properties), in higher-dimensional constructions, or in effective descriptions after suitable breaking to $\mathcal{N} = 1$ at low energies.

10.5 Interactions and the superpotential

Having established the free Wess-Zumino model, we now turn to the crucial question of interactions. The requirement of supersymmetry invariance places severe constraints on the allowed interaction terms, leading to a remarkably elegant structure encoded in a single holomorphic function called the *superpotential*. We follow closely the treatment of Aitchison.

The free multi-field Lagrangian

Consider a collection of chiral supermultiplets (ϕ_i, χ_i, F_i) labelled by an index i running over all gauge and flavour degrees of freedom. The free Lagrangian generalises to

$$\mathcal{L}_{\text{free}} = \partial^\mu \phi^{\dagger i} \partial_\mu \phi_i + \bar{\chi}^i i \bar{\sigma}^\mu \partial_\mu \chi_i + F^{\dagger i} F_i \,, \tag{10.71}$$

where we sum over repeated indices with the convention that fields ϕ_i and χ_i carry lowered indices while their conjugates carry raised indices. This Lagrangian is invariant under the SUSY transformations

$$\delta_\epsilon \phi_i = \epsilon \chi_i, \qquad \delta_\epsilon \chi_i = -i\sigma^\mu \bar{\epsilon}\, \partial_\mu \phi_i + \epsilon F_i, \qquad \delta_\epsilon F_i = -i\bar{\epsilon}\bar{\sigma}^\mu \partial_\mu \chi_i \,, \tag{10.72}$$

together with their hermitian conjugates.

The most general interaction ansatz

The next step is to introduce interactions in such a way as to preserve SUSY. We impose that all interaction terms be *renormalizable*, i.e. have mass dimension ≤ 4. Since we already have $\mathcal{L}_{\text{free}}$, the interaction Lagrangian $\mathcal{L}_{\text{int}}$ must contain only fields *without* kinetic terms (no additional $\partial_\mu \phi\, \partial^\mu \phi$ or $\bar{\chi} i \bar{\sigma}^\mu \partial_\mu \chi$).

What building blocks are available? The fields have mass dimensions $[\phi] = 1$, $[\chi] = 3/2$, $[F] = 2$. Renormalizability restricts us to terms of total dimension ≤ 4. The possible Lorentz-scalar combinations are:

- **Terms linear in F_i:** $W^i(\phi, \phi^\dagger)\, F_i$, where $[W^i F_i] \leq 4$ requires $[W^i] \leq 2$, so W^i is at most quadratic in ϕ.
- **Bilinears in χ:** $W^{ij}(\phi, \phi^\dagger)\, \chi_i \chi_j$, where $[\chi\chi] = 3$ requires $[W^{ij}] \leq 1$, so W^{ij} is at most linear in ϕ.
- **Terms involving $\bar{\chi}^i$ and χ_j:** such as $\bar{\chi}^i \bar{\sigma}^\mu \chi_j\, \partial_\mu(\ldots)$; these would introduce derivatives and are already accounted for in $\mathcal{L}_{\text{free}}$.

- **Pure scalar terms $U(\phi,\phi^\dagger)$:** ruled out below.

The spinor product $\chi_i\chi_j \equiv \chi_i^\alpha\chi_{j\alpha} = \epsilon^{\alpha\beta}\chi_{i\beta}\chi_{j\alpha}$ is *symmetric* in $i \leftrightarrow j$ (the antisymmetry of $\epsilon^{\alpha\beta}$ compensates the sign from exchanging two Grassmann quantities), so W^{ij} must be symmetric too: $W^{ij} = W^{ji}$.

Therefore the most general renormalizable interaction Lagrangian is

$$\mathcal{L}_{\text{int}} = W^i(\phi,\phi^\dagger)F_i - \frac{1}{2}W^{ij}(\phi,\phi^\dagger)\chi_i\chi_j + \text{h.c.}, \tag{10.73}$$

where W^i and W^{ij} are, for the moment, arbitrary functions of the bosonic fields (not of χ or F, since including those would either violate Lorentz invariance or exceed dimension 4). The factor $1/2$ avoids double-counting the symmetric sum over i, j.

Why no pure scalar potential? One might ask: why not include a term $U(\phi,\phi^\dagger)$ involving only scalars? Under a SUSY transformation, such a term would vary as

$$\delta_\epsilon U = \frac{\partial U}{\partial\phi_k}(\epsilon\chi_k) + \frac{\partial U}{\partial\phi^{\dagger k}}(\bar\epsilon\bar\chi^k). \tag{10.74}$$

This contains no derivatives ∂_μ, no auxiliary fields F, and no conjugate spinors $\bar\chi$ in the first term. Inspecting the transformations (10.72), no other term in the Lagrangian can produce such a variation to cancel it. Therefore $U = 0$.

The holomorphicity constraint

Since we know $\mathcal{L}_{\text{free}}$ is invariant by itself, we need only require $\delta_\epsilon\mathcal{L}_{\text{int}} = 0$ (up to a total derivative). We analyse the variation piece by piece.

Step 1: The four-spinor terms. Since W^{ij} depends on the bosonic fields ϕ_k and $\phi^{\dagger k}$, the SUSY variation $\delta_\epsilon\phi_k = \epsilon\chi_k$ and $\delta_\epsilon\phi^{\dagger k} = \bar\epsilon\bar\chi^k$ acting on $-\frac{1}{2}W^{ij}\chi_i\chi_j$ (where $\chi_i\chi_j$ is *not* varied, since this produces a separate class of terms analysed in Steps 3–4) gives

$$\delta_\epsilon\mathcal{L}\Big|_{\text{4-spinor}} = -\frac{1}{2}\frac{\partial W^{ij}}{\partial\phi_k}\underbrace{(\epsilon\chi_k)(\chi_i\chi_j)}_{\text{3 undotted spinors}} - \frac{1}{2}\frac{\partial W^{ij}}{\partial\phi^{\dagger k}}\underbrace{(\bar\epsilon\bar\chi^k)(\chi_i\chi_j)}_{\text{1 dotted + 2 undotted}} + \text{h.c.}. \tag{10.75}$$

These are "four-spinor" terms (four Grassmann fields, no derivatives, no F). No other term in $\mathcal{L}_{\text{int}}$ or $\mathcal{L}_{\text{free}}$ produces four-spinor terms under a SUSY variation: $\delta_\epsilon(W^iF_i)$ produces terms with F or ∂_μ, and $\delta_\epsilon\mathcal{L}_{\text{free}}$ produces terms with ∂_μ. Therefore the two terms in (10.75) must each vanish separately.

The first term involves four undotted (left-handed) spinors ϵ, χ_k, χ_i, χ_j—but two of them are already contracted in $(\chi_i\chi_j) = \chi_i^\alpha\chi_{j\alpha}$. The key tool is the following Schouten identity, valid for any four left-handed Weyl spinors $\alpha, \beta, \gamma, \delta$:

$$(\alpha\beta)(\gamma\delta) + (\alpha\gamma)(\delta\beta) + (\alpha\delta)(\beta\gamma) = 0. \tag{10.76}$$

Concretely, for $\alpha \to \epsilon$, $\beta \to \chi_i$, $\gamma \to \chi_j$, $\delta \to \chi_k$:

$$(\epsilon\chi_i)(\chi_j\chi_k) + (\epsilon\chi_j)(\chi_k\chi_i) + (\epsilon\chi_k)(\chi_i\chi_j) = 0\,. \tag{10.77}$$

Box 10.2 Proof of the Fierz identity ( )

For two-dimensional spinor indices the Schouten identity is

$$\epsilon_{\alpha\beta}\epsilon_{\gamma\delta} + \epsilon_{\alpha\gamma}\epsilon_{\delta\beta} + \epsilon_{\alpha\delta}\epsilon_{\beta\gamma} = 0\,. \tag{10.78}$$

Multiplying by the four anticommuting spinors $\alpha^\alpha\beta^\beta\gamma^\gamma\delta^\delta$ and keeping the NW-SE order of the contractions gives (10.76).

Now, the first term in (10.75), summed over repeated indices i, j, k, reads

$$-\frac{1}{2}\frac{\partial W^{ij}}{\partial\phi_k}(\epsilon\chi_k)(\chi_i\chi_j)\,. \tag{10.79}$$

By the identity (10.77), the factor $(\epsilon\chi_k)(\chi_i\chi_j)$ can be rewritten as $-(\epsilon\chi_i)(\chi_j\chi_k) - (\epsilon\chi_j)(\chi_k\chi_i)$. Since all three index labels i, j, k are summed over, these three terms are identical if $\partial W^{ij}/\partial\phi_k$ is *totally symmetric* in i, j, k. In that case

$$\frac{\partial W^{ij}}{\partial\phi_k}\Big[(\epsilon\chi_k)(\chi_i\chi_j) + (\epsilon\chi_i)(\chi_j\chi_k) + (\epsilon\chi_j)(\chi_k\chi_i)\Big] = 0\,, \tag{10.80}$$

and the first term vanishes identically. This is a *constraint on* W^{ij}: the quantity $\partial W^{ij}/\partial\phi_k$ must be symmetric in all three indices.

The second term involves the product $(\bar\epsilon\bar\chi^k)(\chi_i\chi_j)$ of one dotted spinor bilinear and one undotted spinor bilinear. There is *no* analogue of the Fierz identity for this mixed product: the identity (10.77) relies on contracting spinors of the *same* chirality with $\epsilon_{\alpha\beta}$, and dotted/undotted indices cannot be contracted against each other. Therefore, no symmetry or identity can make this term vanish, and the only way to eliminate it is to demand

$$\boxed{\frac{\partial W^{ij}}{\partial\phi^{\dagger k}} = 0}\,. \tag{10.81}$$

This is the **holomorphicity constraint**: W^{ij} must be a function of ϕ_k *only*, not of $\phi^{\dagger k}$. This is the single most important structural result in the construction of SUSY Lagrangians.

Step 2: Structure of W^{ij}. We have established that W^{ij} must be:

1. holomorphic: $W^{ij} = W^{ij}(\phi_k)$ (no dependence on $\phi^{\dagger k}$);

2. symmetric: $W^{ij} = W^{ji}$ (from $\chi_i\chi_j = \chi_j\chi_i$);
3. at most linear in ϕ_k: from $[W^{ij}] \le 1$;
4. such that $\partial W^{ij}/\partial\phi_k$ is totally symmetric in i, j, k.

The most general function satisfying (1)–(3) is

$$W^{ij} = M^{ij} + y^{ijk}\phi_k\,, \tag{10.82}$$

where $M^{ij} = M^{ji}$ is a symmetric constant matrix (with dimension of mass) and y^{ijk} are dimensionless constants symmetric in $i \leftrightarrow j$.

Physical meaning of M^{ij}: the fermion mass matrix. The constant M^{ij} in W^{ij} is *not* an inert mathematical artefact — it is the **fermion mass matrix** of the theory, and it cannot be omitted without making the chiral multiplets massless. Indeed, substituting (10.82) into the interaction Lagrangian $\mathcal{L}_{\text{int}} \supset -\frac{1}{2}W^{ij}\chi_i\chi_j$:

$$-\tfrac{1}{2}W^{ij}\chi_i\chi_j = \underbrace{-\tfrac{1}{2}M^{ij}\chi_i\chi_j}_{\text{Dirac/Majorana fermion mass}} \; - \underbrace{\tfrac{1}{2}y^{ijk}\phi_k\chi_i\chi_j}_{\text{Yukawa coupling}}\,.$$

Each chiral multiplet contains a Weyl fermion χ_i; the matrix M^{ij} pairs these fermions and gives them mass. Supersymmetry then enforces the *same* mass on the bosonic partners ϕ_i: the F-term scalar potential $|W^i|^2$ contains

$$\left|M^{ij}\phi_j + \tfrac{1}{2}y^{ijk}\phi_j\phi_k\right|^2 \supset (M^\dagger M)^j{}_k\,\phi^{\dagger k}\phi_j\,,$$

which is a scalar mass term with the same matrix $M^\dagger M$ that controls fermion masses. The Wess–Zumino model with non-zero M describes a massive supermultiplet.

When can M^{ij} be set to zero? Only when one wants a *massless* (often conformal) theory: e.g. $\mathcal{N} = 4$ super-Yang–Mills, or chiral matter in the unbroken phase of the MSSM, where quark and lepton masses arise solely from Yukawa couplings $y^{ijk}\phi_k$ with ϕ_k the Higgs field, after electroweak symmetry breaking. Even in the MSSM, however, the Higgs supermultiplets themselves carry a non-zero superpotential mass: the famous μ-term $\mu H_u \cdot H_d$ is exactly of the form $M^{ij}\phi_i\phi_j$ and provides masses to the higgsinos. The “μ-problem” — why this mass is naturally at the TeV scale — is one of the central open puzzles of MSSM phenomenology.

Condition (4) then requires

$$\frac{\partial W^{ij}}{\partial\phi_k} = y^{ijk} \quad \text{to be totally symmetric in } i, j, k\,, \tag{10.83}$$

which forces y^{ijk} to be totally symmetric under *any* permutation of its three indices.

It is now convenient to observe that W^{ij} can be written as the second derivative of a single function. Define the **superpotential**

$$\boxed{W(\phi) = \frac{1}{2}M^{ij}\phi_i\phi_j + \frac{1}{6}y^{ijk}\phi_i\phi_j\phi_k}\,. \tag{10.84}$$

Then (using $M^{ij} = M^{ji}$ and y^{ijk} totally symmetric):

$$\frac{\partial W}{\partial \phi_i} = M^{ij}\phi_j + \frac{1}{2}y^{ijk}\phi_j\phi_k\,, \qquad \frac{\partial^2 W}{\partial\phi_i\partial\phi_j} = M^{ij} + y^{ijk}\phi_k = W^{ij}\,. \tag{10.85}$$

(In the first equation, the factor $1/2$ in W becomes 1 because $M^{ij}\phi_i\phi_j$ gives $M^{ij}\phi_j + M^{ji}\phi_j = 2M^{ij}\phi_j$ upon differentiating with respect to ϕ_i, and similarly $\frac{1}{6}y^{ijk}\phi_i\phi_j\phi_k$ gives $\frac{3}{6}y^{ijk}\phi_j\phi_k = \frac{1}{2}y^{ijk}\phi_j\phi_k$ by total symmetry.)

Step 3: The derivative terms. Next we consider those parts of $\delta_\epsilon\mathcal{L}_{\text{int}}$ containing one spacetime derivative ∂_μ. These come from:

- $\delta_\epsilon F_i = -i\bar{\epsilon}\bar{\sigma}^\mu\partial_\mu\chi_i$ acting on $W^i F_i$
- $\delta_\epsilon\chi_i = -i\sigma^\mu\bar{\epsilon}\,\partial_\mu\phi_i$ acting on $-\frac{1}{2}W^{ij}\chi_i\chi_j$

Explicitly:

$$\delta_\epsilon\mathcal{L}\big|_\partial = W^i(-i\bar{\epsilon}\bar{\sigma}^\mu\partial_\mu\chi_i) - \frac{1}{2}W^{ij}\Big[(-i\sigma^\mu\bar{\epsilon}\,\partial_\mu\phi_i)\chi_j + \chi_i(-i\sigma^\mu\bar{\epsilon}\,\partial_\mu\phi_j)\Big] + \text{h.c.}\,. \tag{10.86}$$

Let us manipulate the last two terms in (10.86) (those containing $\delta_\epsilon\chi$). In the second term, $(-i\sigma^\mu\bar{\epsilon}\,\partial_\mu\phi_i)$ is a left-handed spinor multiplied into χ_j; in the third term, χ_i is multiplied into $(-i\sigma^\mu\bar{\epsilon}\,\partial_\mu\phi_j)$. Since the spinor product is symmetric ($\alpha\beta = \beta\alpha$ for left-handed spinors) and $W^{ij} = W^{ji}$, the two terms are equal:

$$-\frac{1}{2}W^{ij}\Big[(-i\sigma^\mu\bar{\epsilon}\,\partial_\mu\phi_i)\chi_j + \chi_i(-i\sigma^\mu\bar{\epsilon}\,\partial_\mu\phi_j)\Big] = -W^{ij}(-i\sigma^\mu\bar{\epsilon}\,\partial_\mu\phi_j)\chi_i\,. \tag{10.87}$$

Keeping the NW-SE contractions and the Grassmann signs in the product $(-i\sigma^\mu\bar{\epsilon})\chi_i$ gives the Aitchison identity

$$-W^{ij}(-i\sigma^\mu\bar{\epsilon}\,\partial_\mu\phi_j)\chi_i = -iW^{ij}\,\bar{\epsilon}\bar{\sigma}^\mu\chi_i\,\partial_\mu\phi_j\,. \tag{10.88}$$

Now we use the chain rule. Since $W^{ij} = \partial^2 W/\partial\phi_i\partial\phi_j$:

$$W^{ij}\,\partial_\mu\phi_j = \frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\,\partial_\mu\phi_j = \partial_\mu\left(\frac{\partial W}{\partial\phi_i}\right), \tag{10.89}$$

where the last step follows because W depends on x^μ only through the fields $\phi_j(x)$, and $\partial W^{ij}/\partial(\partial_\mu\phi_j) = 0$.

Collecting the first term in (10.86) (from $\delta_\epsilon F_i$) and the result (10.88):

$$\delta_\epsilon\mathcal{L}\big|_\partial = -iW^i\bar{\epsilon}\bar{\sigma}^\mu\partial_\mu\chi_i - i\,\bar{\epsilon}\bar{\sigma}^\mu\chi_i\,\partial_\mu\left(\frac{\partial W}{\partial\phi_i}\right) + \text{h.c.}\,. \tag{10.90}$$

We want this to be a total derivative. Using the product rule,

$$\partial_\mu(f\,\bar\epsilon\bar\sigma^\mu\chi_i) = (\partial_\mu f)\,\bar\epsilon\bar\sigma^\mu\chi_i + f\,\bar\epsilon\bar\sigma^\mu\partial_\mu\chi_i\,, \tag{10.91}$$

we see that (10.90) equals $\partial_\mu(-iW^i\,\bar\epsilon\bar\sigma^\mu\chi_i)+\text{h.c.}$—which is a total derivative—*if and only if* the coefficient W^i of the first term equals $\partial W/\partial\phi_i$ from the second term:

$$\boxed{W^i = \frac{\partial W}{\partial \phi_i}}\,. \tag{10.92}$$

With this identification, (10.90) becomes

$$\delta_\epsilon\mathcal{L}\big|_\partial = \partial_\mu\left(-i\frac{\partial W}{\partial\phi_i}\,\bar\epsilon\bar\sigma^\mu\chi_i\right)+\text{h.c.}\,, \tag{10.93}$$

which is indeed a total derivative and does not affect the action.

Let us verify the consistency of (10.92) with the superpotential (10.84). Computing:

$$W^i = \frac{\partial W}{\partial\phi_i} = M^{ij}\phi_j + \frac{1}{2}y^{ijk}\phi_j\phi_k\,, \tag{10.94}$$

which is indeed at most quadratic in ϕ, as required by $[W^i]\le 2$. ✓

Holomorphicity of W^i is automatic. Although we proved the holomorphicity constraint (10.81) only for W^{ij} (from the four-spinor terms in Step 1), the relation $W^i=\partial W/\partial\phi_i$ established in Step 3 *forces* W^i to be holomorphic as well: it is the derivative of a holomorphic function $W(\phi)$, and therefore depends only on ϕ and not on $\phi^\dagger$. From the explicit form (10.94), this is manifest — no $\phi^\dagger$ appears.

Equivalently, had we kept an antiholomorphic dependence in the original ansatz $W^i(\phi,\phi^\dagger)$, the SUSY variation of W^iF_i via $\delta_\epsilon\phi^{\dagger k}=\bar\epsilon\bar\chi^k$ would have produced uncancelled terms with structure $\bar\epsilon\bar\chi^kF_i$ (Step 4 below shows that all such F-terms cancel only when the holomorphic structure is used). So the SUSY-invariance argument enforces holomorphicity of both W^{ij} and W^i in a single, mutually consistent way — ultimately because both descend from one holomorphic function W.

Step 4: The F-terms. The remaining terms in $\delta_\epsilon\mathcal{L}_{\text{int}}$ are those containing F_i or $F^{\dagger i}$ (but no spacetime derivatives). They arise from three sources:

1. **From $\delta_\epsilon\phi_k=\epsilon\chi_k$ acting on W^iF_i:** since W^i depends on ϕ,

$$\delta_\epsilon(W^iF_i)\big|_F = \frac{\partial W^i}{\partial\phi_k}(\epsilon\chi_k)F_i = W^{ik}(\epsilon\chi_k)F_i\,, \tag{10.95}$$

where we used $\partial W^i/\partial\phi_k=\partial^2W/\partial\phi_i\partial\phi_k=W^{ik}$.

2. **From $\delta_\epsilon \chi_i = \epsilon F_i$ acting on $-\frac{1}{2}W^{ij}\chi_i\chi_j$:** varying the first χ_i gives ϵF_i, and similarly for χ_j; using $W^{ij} = W^{ji}$ and $\chi_i\chi_j = \chi_j\chi_i$, both contributions are equal:

$$-\frac{1}{2}W^{ij}\Big[(\epsilon F_i)\chi_j + \chi_i(\epsilon F_j)\Big] = -W^{ij}(\epsilon\chi_j)F_i\,. \tag{10.96}$$

3. **Hermitian conjugate terms:** the h.c. of $\mathcal{L}_{\text{int}}$ produces analogous terms with $F^{\dagger i}$, $\bar{\epsilon}$, and $\bar{\chi}^k$.

Adding (10.95) and (10.96):

$$\delta_\epsilon \mathcal{L}_{\text{int}}\big|_F = W^{ik}(\epsilon\chi_k)F_i - W^{ij}(\epsilon\chi_j)F_i + \text{h.c.} \tag{10.97}$$

Since W^{ik} and W^{ij} are the same object (both equal $\partial^2 W/\partial\phi_i\partial\phi_{k \text{ or } j}$) and the dummy summation indices k and j run over the same range, the two terms cancel exactly:

$$\delta_\epsilon \mathcal{L}_{\text{int}}\big|_F = 0\,. \quad \checkmark \tag{10.98}$$

This completes the proof that $\mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}}$ is invariant under SUSY, provided $W^i = \partial W/\partial\phi_i$ and $W^{ij} = \partial^2 W/\partial\phi_i\partial\phi_j$ with W holomorphic.

Summary: the superpotential

Collecting the results: SUSY invariance forces W to be holomorphic and fixes $W^i = \partial W/\partial\phi_i$ and $W^{ij} = \partial^2 W/\partial\phi_i\partial\phi_j$. For a renormalizable theory, W is at most cubic in the fields (see (10.84)), with M^{ij} symmetric and y^{ijk} totally symmetric.

The superpotential is *not* an ordinary potential. The name "superpotential" is somewhat misleading. In ordinary classical or quantum mechanics, the *potential* $V(\phi)$ is a real function of the fields that enters the Lagrangian as $\mathcal{L} \supset -V(\phi)$ and whose minima define stable vacua. The superpotential W is different in two essential ways:

1. **W is complex (holomorphic).** A real potential $V(\phi, \phi^\dagger)$ depends on both ϕ and $\phi^\dagger$. The superpotential $W(\phi)$, by contrast, is a function of the *holomorphic* fields ϕ_i alone — it does not depend on $\phi^{\dagger i}$. This is a powerful constraint forced by SUSY invariance (proved in the holomorphicity argument above) and is what underlies the famous *non-renormalization theorems*: W receives no perturbative quantum corrections, term by term in ϕ. In particular, if $W = 0$ at tree level, it stays zero to all orders in perturbation theory.

2. W **does *not* appear directly in the Lagrangian as** $-W$**.** It enters only through its derivatives W^i and W^{ij}:

$$\mathcal{L}_{\text{int}} = W^i F_i - \tfrac{1}{2} W^{ij} \chi_i \chi_j + \text{h.c.}$$

Eliminating the auxiliary field F_i via its equation of motion ($F_i = -W^{i*}$, see Section 10.5 below) produces a *real* scalar potential

$$V(\phi, \phi^\dagger) = \sum_i \left| \frac{\partial W}{\partial \phi_i} \right|^2 = |W^i|^2,$$

which is manifestly real and non-negative ($V \geq 0$, with $V = 0$ only at SUSY-preserving vacua). Thus W is the "square root" of the physical scalar potential: $V = |W^i|^2$, not $V = W$.

In short: W is a complex, holomorphic "generating function" from which the real physical potential, the fermion mass matrix, and the Yukawa couplings are all derived in a SUSY-consistent way. The word "super-" is the only hint that it lives in a different category from ordinary potentials.

Remark. [Linear terms] A linear term $L^i \phi_i$ can be added to W without spoiling SUSY. Such terms are only gauge-invariant when ϕ_i is a gauge singlet, and play a role in certain models of spontaneous SUSY breaking (e.g., the O'Raifeartaigh model).

Eliminating the auxiliary fields

The part of the Lagrangian containing the auxiliary fields is

$$\mathcal{L}_F = F^{\dagger i} F_i + W^i F_i + W_i^\dagger F^{\dagger i}. \tag{10.99}$$

Since this contains no derivatives, the Euler-Lagrange equations for F_i and $F^{\dagger i}$ are simply

$$\frac{\partial \mathcal{L}}{\partial F_i} = F^{\dagger i} + W^i = 0 \quad \Rightarrow \quad F^{\dagger i} = -W^i = -\frac{\partial W}{\partial \phi_i}, \tag{10.100}$$

and similarly $F_i = -W_i^\dagger$. These relations involve no derivatives, so the canonical commutation relations are unaffected and we may substitute them directly into the Lagrangian.

The complete **Wess-Zumino Lagrangian** after eliminating auxiliary fields is

$$\boxed{\mathcal{L}_{\text{WZ}} = \partial^\mu \phi^{\dagger i} \partial_\mu \phi_i + \bar{\chi}^i i \bar{\sigma}^\mu \partial_\mu \chi_i - \frac{1}{2}\left(W^{ij} \chi_i \chi_j + W_{ij}^\dagger \bar{\chi}^i \bar{\chi}^j \right) - W^i W_i^\dagger .} \tag{10.101}$$

The **scalar potential** is

$$\boxed{V(\phi, \phi^\dagger) = W^i W_i^\dagger = F^{\dagger i} F_i = \sum_i \left| \frac{\partial W}{\partial \phi_i} \right|^2 \geq 0}\,. \tag{10.102}$$

This is the famous **F-term contribution**. Since V is a sum of absolute squares, it is automatically non-negative for all field configurations—a general feature of supersymmetric theories.

Explicit form of the Lagrangian

Substituting $W = \frac{1}{2}M^{ij}\phi_i\phi_j + \frac{1}{6}y^{ijk}\phi_i\phi_j\phi_k$, we have

$$W^i = M^{ij}\phi_j + \frac{1}{2}y^{ijk}\phi_j\phi_k\,, \tag{10.103}$$

$$W^{ij} = M^{ij} + y^{ijk}\phi_k\,. \tag{10.104}$$

The full Lagrangian becomes

$$\begin{aligned}\mathcal{L} = {} & \partial^\mu\phi^{\dagger i}\partial_\mu\phi_i + \bar{\chi}^i i\bar{\sigma}^\mu\partial_\mu\chi_i - V(\phi, \phi^\dagger) \\ & - \tfrac{1}{2}M^{ij}\chi_i\chi_j - \tfrac{1}{2}M^*_{ij}\bar{\chi}^i\bar{\chi}^j \\ & - \tfrac{1}{2}y^{ijk}\phi_k\chi_i\chi_j - \tfrac{1}{2}y^*_{ijk}\phi^{\dagger k}\bar{\chi}^i\bar{\chi}^j\,,\end{aligned} \tag{10.105}$$

with scalar potential

$$\begin{aligned}V = {} & M^*_{ik}M^{kj}\phi^{\dagger i}\phi_j + \tfrac{1}{2}M^{in}y^*_{jkn}\phi_i\phi^{\dagger j}\phi^{\dagger k} \\ & + \tfrac{1}{2}M^*_{in}y^{jkn}\phi^{\dagger i}\phi_j\phi_k + \tfrac{1}{4}y^{ijn}y^*_{kln}\phi_i\phi_j\phi^{\dagger k}\phi^{\dagger l}\,.\end{aligned} \tag{10.106}$$

Mass degeneracy within supermultiplets

The linearized equations of motion (keeping only mass terms) are:

$$\partial^\mu\partial_\mu\phi_i = -M^*_{ik}M^{kj}\phi_j + \cdots\,, \tag{10.107}$$

$$i\bar{\sigma}^\mu\partial_\mu\chi_i = M^*_{ij}\bar{\chi}^j + \cdots\,, \qquad i\sigma^\mu\partial_\mu\bar{\chi}^i = M^{ij}\chi_j + \cdots\,. \tag{10.108}$$

Eliminating $\bar{\chi}$ from the fermion equations (using $\bar{\sigma}^\mu\sigma^\nu + \bar{\sigma}^\nu\sigma^\mu = 2g^{\mu\nu}$):

$$\partial^\mu\partial_\mu\chi_i = -M^*_{ik}M^{kj}\chi_j + \cdots\,. \tag{10.109}$$

Both ϕ_i and χ_i satisfy the *same* wave equation with squared-mass matrix $(M^2)_i{}^j = M^*_{ik}M^{kj}$. Diagonalizing this matrix gives supermultiplets with *identical* boson and fermion masses, as required by the superalgebra.

Box 10.3 Interactions from the Superpotential

The superpotential W encodes *all* non-gauge interactions:

- **Fermion masses:** $-\frac{1}{2}M^{ij}\chi_i\chi_j + \text{h.c.}$
- **Scalar masses:** $M^*_{ik}M^{kj}\phi^{\dagger i}\phi_j$
- **Yukawa couplings:** $-\frac{1}{2}y^{ijk}\phi_k\chi_i\chi_j + \text{h.c.}$
- **Scalar cubics:** from $M^{in}y^*_{jkn}\phi_i\phi^{\dagger j}\phi^{\dagger k} + \text{h.c.}$
- **Scalar quartics:** $\frac{1}{4}y^{ijn}y^*_{kln}\phi_i\phi_j\phi^{\dagger k}\phi^{\dagger l}$

Worked example: a single chiral supermultiplet

Following Aitchison [38], §5.1, we specialise to a *single* chiral supermultiplet (ϕ, χ, F) to make the interaction structure concrete. With a single field the superpotential (10.84) reduces to

$$W = \tfrac{1}{2}M\phi^2 + \tfrac{1}{6}y\phi^3\,, \tag{10.110}$$

where M is a mass parameter and y a dimensionless Yukawa coupling (both taken real for simplicity). The derivatives are

$$W' \equiv \frac{\partial W}{\partial \phi} = M\phi + \tfrac{1}{2}y\phi^2\,, \qquad W'' \equiv \frac{\partial^2 W}{\partial \phi^2} = M + y\phi\,. \tag{10.111}$$

The on-shell Wess–Zumino Lagrangian (10.101) becomes

$$\mathcal{L} = \partial^\mu\phi^\dagger\partial_\mu\phi + \bar{\chi}i\bar{\sigma}^\mu\partial_\mu\chi - \underbrace{\left|M\phi + \tfrac{1}{2}y\phi^2\right|^2}_{V(\phi,\phi^\dagger)} - \tfrac{1}{2}\left[(M + y\phi)\,\chi\chi + \text{h.c.}\right]. \tag{10.112}$$

Mass terms. The quadratic terms in (10.112) are

$$\mathcal{L}_{\text{mass}} = -M^2\phi^\dagger\phi \; - \; \tfrac{1}{2}M\left(\chi\chi + \bar{\chi}\bar{\chi}\right). \tag{10.113}$$

Both the complex scalar and the Weyl fermion have the *same* mass $|M|$, confirming the general mass-degeneracy result within a single supermultiplet.

Interaction terms. Expanding the scalar potential $V = |W'|^2$ beyond the mass terms and collecting the Yukawa coupling, we identify three distinct interactions:

(i) **Scalar cubic** (from the cross term in $|W'|^2$):

$$\mathcal{L}_{\text{cubic}} = -\tfrac{1}{2}My\left(\phi^2\phi^\dagger + \phi^{\dagger 2}\phi\right). \tag{10.114}$$

(ii) **Scalar quartic** (from $|\frac{1}{2}y\phi^2|^2$):

$$\mathcal{L}_{\text{quartic}} = -\tfrac{1}{4}y^2\,|\phi|^4\,. \tag{10.115}$$

(iii) **Yukawa** (from $-\frac{1}{2}W''\chi\chi + \text{h.c.}$, keeping only the interaction part):

$$\mathcal{L}_{\text{Yukawa}} = -\tfrac{1}{2}y\big(\phi\,\chi\chi + \phi^\dagger\bar{\chi}\bar{\chi}\big)\,. \tag{10.116}$$

Coupling relations. The three interaction couplings are *not* independent—they are all determined by the single parameter y appearing in the superpotential:

Interaction	Coupling	Relation
Yukawa $\phi\chi\chi$	y	–
Scalar cubic $\phi^2\phi^\dagger$	$\frac{1}{2}My$	$=\frac{1}{2}M\times y_{\text{Yuk}}$
Scalar quartic $\lvert\phi\rvert^4$	$\frac{1}{4}y^2$	$=\frac{1}{4}y^2_{\text{Yuk}}$

In particular, the quartic scalar coupling is fixed by the *square* of the Yukawa coupling:

$$\boxed{\lambda_{\text{quartic}} = \frac{1}{4}\,y^2_{\text{Yukawa}}\,.} \tag{10.117}$$

This relationship is *not* imposed by hand—it is *forced* by the holomorphicity of W and the SUSY algebra. It is precisely the condition needed for the cancellation of quadratic divergences in the scalar self-energy at one loop: the bosonic loop (quartic vertex, coupling $\sim\lambda$) and the fermionic loop (two Yukawa vertices, coupling $\sim y^2$) contribute with opposite signs and cancel when $\lambda = y^2/4$. This will be demonstrated explicitly in Section 5.2 of Aitchison.

Remark. [Decomposition into real fields] Writing $\phi = (A+iB)/\sqrt{2}$ with A, B real (cf. Aitchison, eq. (4.123)) and defining $g \equiv y/(2\sqrt{2})$, the interactions take the form familiar from the Majorana formulation:

$$\mathcal{L}_{\text{cubic}} = -Mg\,A(A^2+B^2)\,, \tag{10.118}$$
$$\mathcal{L}_{\text{quartic}} = -\tfrac{1}{2}g^2(A^2+B^2)^2\,, \tag{10.119}$$
$$\mathcal{L}_{\text{Yukawa}} = -g\big[A\,\bar{\Psi}_{\text{M}}\Psi_{\text{M}} + iB\,\bar{\Psi}_{\text{M}}\gamma_5\Psi_{\text{M}}\big]\,, \tag{10.120}$$

where Ψ_{M} is the four-component Majorana spinor built from χ (see Section 2.5 of Aitchison). Here A is a scalar and B a pseudoscalar under parity, and the same coupling g governs all three interactions. The coefficient $g^2/2$ in (10.119) is Aitchison's real-field normalization of the same relation: after the vertex combinatorics are included, it is equivalent to the complex-field statement

$\lambda_{\text{quartic}} = y^2/4.$

Physical significance. The fact that a single holomorphic function W determines *all* interactions—fixing the relative strengths of Yukawa, cubic, and quartic couplings—is one of the most powerful consequences of supersymmetry. It has two far-reaching implications:

1. **Cancellation of quadratic divergences.** The relation $\lambda = y^2/4$ ensures that the Λ^2-divergent contributions to the scalar mass from bosonic and fermionic loops cancel order by order in perturbation theory. This is the technical solution to the hierarchy problem that originally motivated SUSY (cf. Section 1.1 of Aitchison).
2. **Non-renormalization.** The holomorphicity of W implies that the superpotential itself receives no perturbative renormalization. Physical masses and couplings can still run through wavefunction/Kähler-potential renormalization, but W is radiatively stable.

Part V

Appendices