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Correlation Functions and Matrix Permanents

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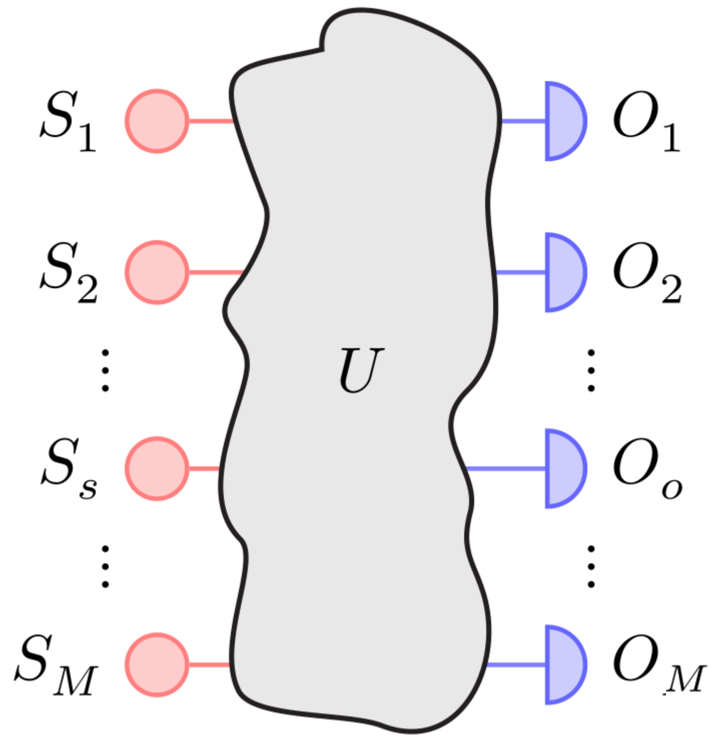
Multi-mode quantum interference in an M-order Boson Interferometer

Single- photon Sources



Quantum interference of an exponential number of indistinguishable ways for M single photons to trigger M detectors: exponentially hard to simulate!

Can multi-photon quantum interference lead to Exponential speed-up in computations?

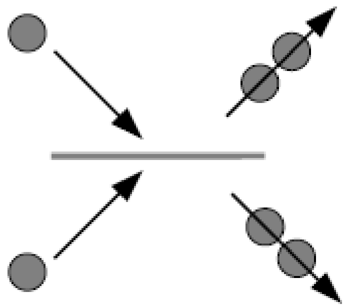
Multi-photon Interference and Permanents

Caianello 1953: Correspondence between the amplitudes of n-boson processes and the permanents of n x n matrixes

$$\mathrm{Per}\, U = \sum_{\sigma \in S} \prod_{i=1}^{n} U_{i,\sigma_i}$$

E. R. Caianiello. On quantum field theory, 1: explicit solution of Dyson's equation in electrodynamics without use of Feynman graphs. *Nuovo Cimento*, 10:1634–1652, 1953.

SA-HOM effect



$$\frac{|2,0\rangle - |0,2\rangle}{\sqrt{2}}$$

50:50 BS

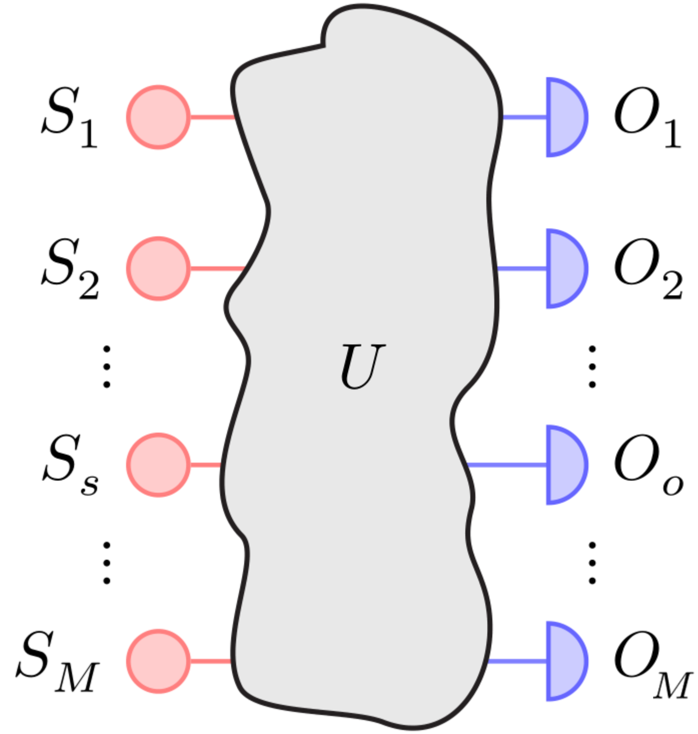
$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Amplitude for the two input photons to exit in separate spatial modes

$$\mathrm{Per} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = 0$$

Destructive Quantum Interference!

Multi-mode quantum interference in a generic linear optics network



n_{tot} input photons are distributed in the output modes of the interferometer

$$\sum_{s=1}^{M} n_s = n_{tot} = \sum_{o=1}^{M} m_o$$

Single photons in each of the input and output modes: $\mathrm{Per}\, U = {}^{\otimes M}\langle 1|\hat{U}|1\rangle^{\otimes M}$

Amplitude for a generic distribution in the input and output modes:

$$\langle m_1, m_2, \ldots, m_M|\hat{U}|n_1, n_2, \ldots, n_M\rangle = \left[\prod_s n_s!\right]^{-1/2}\left[\prod_o m_o!\right]^{-1/2} \mathrm{Per}\, U[\Omega'|\Omega]$$

$$\Omega = (1^{n_1}, 2^{n_2}, \ldots, M^{n_M})$$
$$\Omega' = (1^{m_1}, 2^{m_2}, \ldots, M^{m_M})$$

S. Scheel. Permanents in linear optical networks. quant-ph/0406127, 2004.

Multi-boson Interference and Permanents

Valiant, 1979: Computational complexity of permanents for large values of n:
Problem believed to be harder than factoring large integers!

L. G. Valiant. The complexity of computing the permanent. *Theoretical Comput. Sci.*, 8(2):189–201, 1979.

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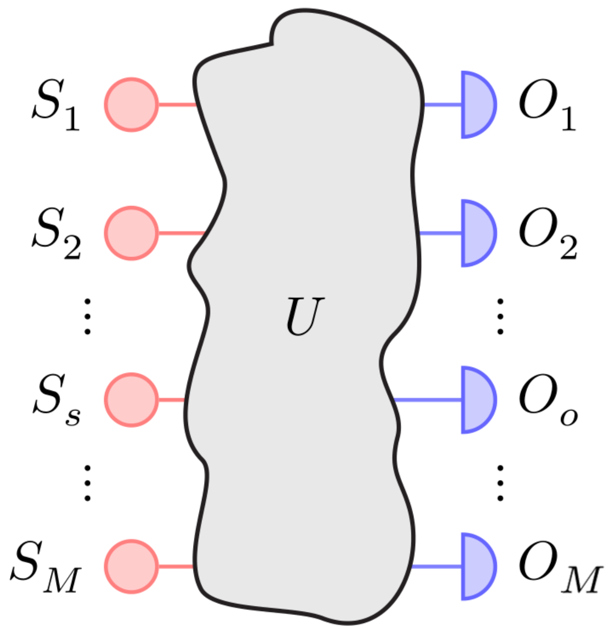


Boson Sampling Quantum Machine:

Probability for a generic distribution of all the input photons in the output modes of an M-mode linear interferometer believed to be exponentially hard to compute at the increasing of M.

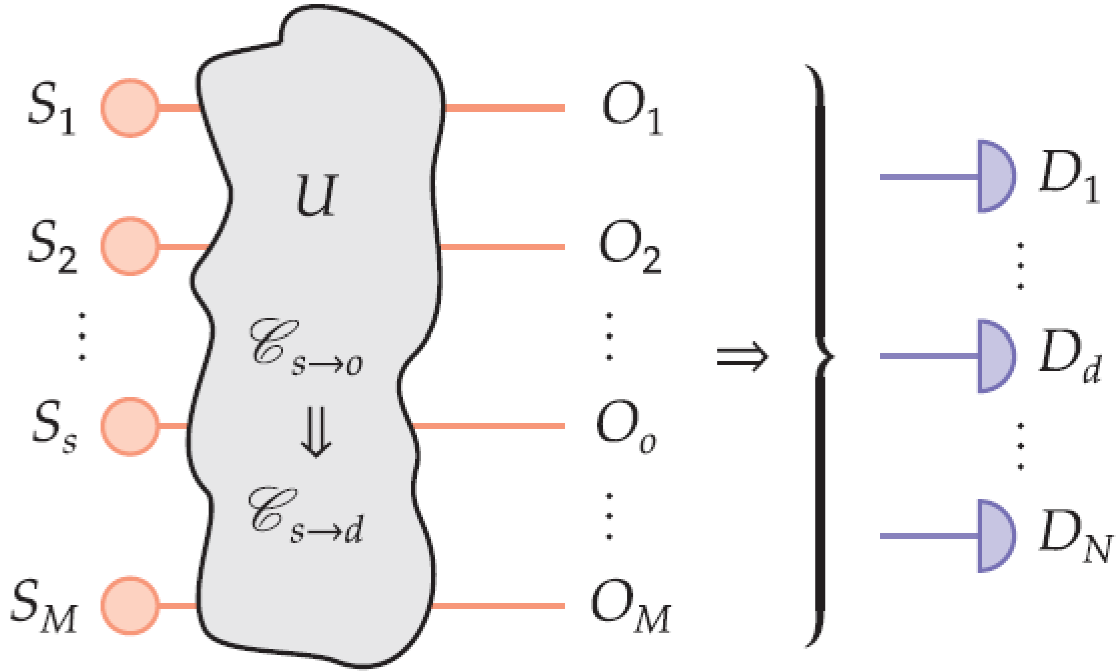
S. Scheel. Permanents in linear optical networks. quant-ph/0406127, 2004.

L. Troyansky and N. Tishby. Permanent uncertainty: On the quantum evaluation of the determinant and the permanent of a matrix. In *Proceedings of PhysComp*, 1996.

S. Aaronson and A. Arkhipov, Proceedings of ACM Symposium on the Theory of Computing, STOC, pp. 333-342 (Association for Computing Machinery, New York, 2011).

N^{th}-order correlation functions and permanents in a multi-mode Photon-Sampling Interferometer

$$\hat{\rho} = \bigotimes_{s=1}^{M} \hat{\rho}_s$$



N single-photon detectors

$$G^{(N)}(t_1, t_2, \ldots, t_N, t_N, \ldots, t_1) = \mathrm{tr}\left\{\hat{\rho}\left[\prod_{d=1}^{N} \hat{\mathcal{E}}_d^{(-)}(t_d)\right]\left[\prod_{d=1}^{N} \hat{\mathcal{E}}_d^{(+)}(t_d)\right]\right\}$$

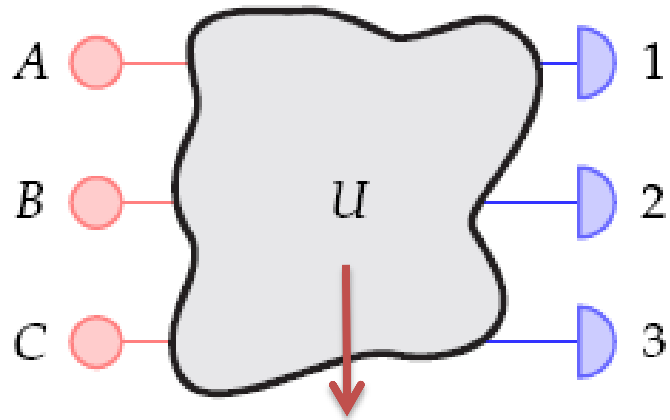
$$\hat{\mathcal{E}}_d^{(+)}(t_d) = \sum_{s=1}^{M} \mathscr{C}_{s\to d} \hat{E}_s^{(+)}(t_d) \qquad \hat{E}_s^{(+)}(t_d) = \mathrm{i}E_0 \int \mathrm{d}\omega e^{-\mathrm{i}\omega(t_d - t_{s\to d})} \hat{a}_s(\omega)$$

Is there a connection between correlation functions and permanents?

3rd-order correlation function

Optical sources:

1. Fock States
2. Independent Thermal Sources



N = 3 single-photon detectors

Unitary evolution of the fields modes from the sources s = A,B,C to the detectors d=1,2,3 in presence of generic linear optics devices

$$\begin{pmatrix} \mathscr{C}_{A\to 1} & \mathscr{C}_{B\to 1} & \mathscr{C}_{C\to 1} \\ \mathscr{C}_{A\to 2} & \mathscr{C}_{B\to 2} & \mathscr{C}_{C\to 2} \\ \mathscr{C}_{A\to 3} & \mathscr{C}_{B\to 3} & \mathscr{C}_{C\to 3} \end{pmatrix}$$

$$\hat{\mathcal{E}}_1^{(+)}(t_1) = \mathscr{C}_{A\to 1}\hat{E}_A^{(+)}(t_1) + \mathscr{C}_{B\to 1}\hat{E}_B^{(+)}(t_1) + \mathscr{C}_{C\to 1}\hat{E}_C^{(+)}(t_1)$$

$$\hat{\mathcal{E}}_2^{(+)}(t_2) = \mathscr{C}_{A\to 2}\hat{E}_A^{(+)}(t_2) + \mathscr{C}_{B\to 2}\hat{E}_B^{(+)}(t_2) + \mathscr{C}_{C\to 2}\hat{E}_C^{(+)}(t_2)$$

$$\hat{\mathcal{E}}_3^{(+)}(t_3) = \mathscr{C}_{A\to 3}\hat{E}_A^{(+)}(t_3) + \mathscr{C}_{B\to 3}\hat{E}_B^{(+)}(t_2) + \mathscr{C}_{C\to 3}\hat{E}_C^{(+)}(t_3)$$

$$\underline{G}^{(3)}(t_1, t_2, t_3, t_3, t_2, t_1)$$

$$= \left\langle \hat{\mathcal{E}}_1^{(-)}(t_1)\hat{\mathcal{E}}_1^{(-)}(t_2)\hat{\mathcal{E}}_1^{(-)}(t_3)\hat{\mathcal{E}}_1^{(+)}(t_3)\hat{\mathcal{E}}_1^{(+)}(t_2)\hat{\mathcal{E}}_1^{(+)}(t_1) \right\rangle_\rho$$

3^{rd}-order correlation functions: Fock states sources

$$t_{d-s\to d} = t_d - t_{s\to d}$$

$$\Upsilon_N(n_s) = \begin{cases} \frac{n_s!}{(n_s-N)!} & \text{for } n_s \geq N \\ 0 & \text{for } n_s < N \end{cases}$$

Multi-Photon Interference

$$G^{(3)}_{\mathrm{Fock}}\left(\left|t_{d-s\to d} - t_{0s}\right| \ll \tau\right)$$

$$= \tilde{E}_0^6 \Big\{ n_A\, n_B\, n_C \left|\mathrm{per}\, U_{ABC}\right|^2$$

Only term for $n_A = n_B = n_C = 1$

$$+ \Upsilon_2(n_A)\, n_B \frac{1}{(2!)^2} \left|\mathrm{per}\, U_{AAB}\right|^2 + \Upsilon_2(n_A)\, n_C \frac{1}{(2!)^2} \left|\mathrm{per}\, U_{AAC}\right|^2$$

$$+ \Upsilon_2(n_B)\, n_A \frac{1}{(2!)^2} \left|\mathrm{per}\, U_{BBA}\right|^2 + \Upsilon_2(n_B)\, n_C \frac{1}{(2!)^2} \left|\mathrm{per}\, U_{BBC}\right|^2$$

$$+ \Upsilon_2(n_C)\, n_A \frac{1}{(2!)^2} \left|\mathrm{per}\, U_{CCA}\right|^2 + \Upsilon_2(n_C)\, n_B \frac{1}{(2!)^2} \left|\mathrm{per}\, U_{CCB}\right|^2$$

$U_{sss'}$ terms vanish for $n_S < 2$

$s \neq s' \in \{A, B, C\}$

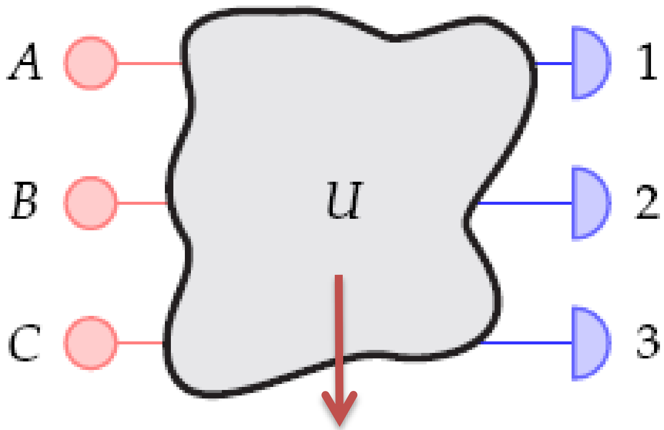
$$+ \Upsilon_3(n_A) \frac{1}{(3!)^2} \left|\mathrm{per}\, U_{AAA}\right|^2 + \Upsilon_3(n_B) \frac{1}{(3!)^2} \left|\mathrm{per}\, U_{BBB}\right|^2$$

$$+ \Upsilon_3(n_C) \frac{1}{(3!)^2} \left|\mathrm{per}\, U_{CCC}\right|^2 \Big\}$$

U_{sss} terms vanish for $n_S < 3$

3rd-order correlation functions: Fock states sources

Multi-photon Interference → $|\,per\, U_{s,s',s''}\,|^2$



$$\begin{pmatrix} \mathscr{C}_{A\to1} & \mathscr{C}_{B\to1} & \mathscr{C}_{C\to1} \\ \mathscr{C}_{A\to2} & \mathscr{C}_{B\to2} & \mathscr{C}_{C\to2} \\ \mathscr{C}_{A\to3} & \mathscr{C}_{B\to3} & \mathscr{C}_{C\to3} \end{pmatrix}$$

$$U_{ss's''} = \begin{pmatrix} \mathscr{D}_{s\to1} & \mathscr{D}_{s'\to1} & \mathscr{D}_{s''\to1} \\ \mathscr{D}_{s\to2} & \mathscr{D}_{s'\to2} & \mathscr{D}_{s''\to2} \\ \mathscr{D}_{s\to3} & \mathscr{D}_{s'\to3} & \mathscr{D}_{s''\to3} \end{pmatrix}$$

$$s, s', s'' \in \{A, B, C\}$$

$$d \in \{1, 2, 3\}$$

$$\mathscr{D}_{s\to d} = \mathscr{C}_{s\to d}\, e^{i\omega_0 t_{s\to d}}$$

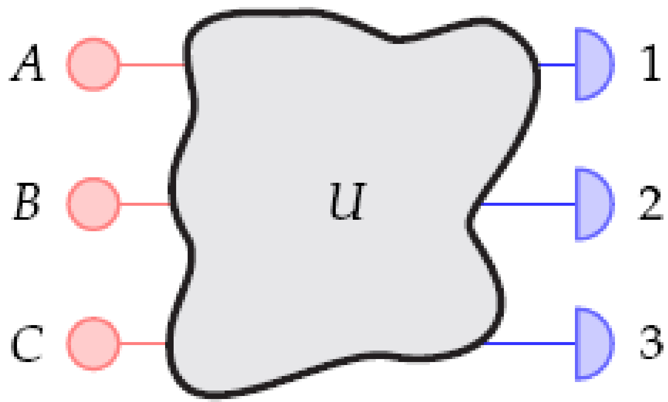
Equal interferometric paths $c\, t_{s\text{->}d}$ from the sources to the detectors

3rd-order correlation functions: independent thermal sources

3 independent thermal sources $s \in \{A, B, C\}$ with average photon number $\overline{n}_s$

$$\underline{G}^{(3)}_{\text{Ther}}\left(\left|t_{d-s\to d} - t_{d'-s\to d'}\right| \ll \tau\right) = E_0^6 \Big\{ \overline{n}_A\,\overline{n}_B\,\overline{n}_C \left|\operatorname{per} U_{ABC}\right|^2$$
$$+ \frac{\overline{n}_A^2}{2!}\,\overline{n}_B \left|\operatorname{per} U_{AAB}\right|^2 + \frac{\overline{n}_A^2}{2!}\,\overline{n}_C \left|\operatorname{per} U_{AAC}\right|^2$$
$$+ \frac{\overline{n}_B^2}{2!}\,\overline{n}_A \left|\operatorname{per} U_{BBA}\right|^2 + \frac{\overline{n}_B^2}{2!}\,\overline{n}_C \left|\operatorname{per} U_{BBC}\right|^2$$
$$+ \frac{\overline{n}_C^2}{2!}\,\overline{n}_A \left|\operatorname{per} U_{CCA}\right|^2 + \frac{\overline{n}_C^2}{2!}\,\overline{n}_B \left|\operatorname{per} U_{CCB}\right|^2$$
$$+ \frac{\overline{n}_A^3}{3!} \left|\operatorname{per} U_{AAA}\right|^2 + \frac{\overline{n}_B^3}{3!} \left|\operatorname{per} U_{BBB}\right|^2$$
$$+ \frac{\overline{n}_C^3}{3!} \left|\operatorname{per} U_{CCC}\right|^2 \Big\}$$

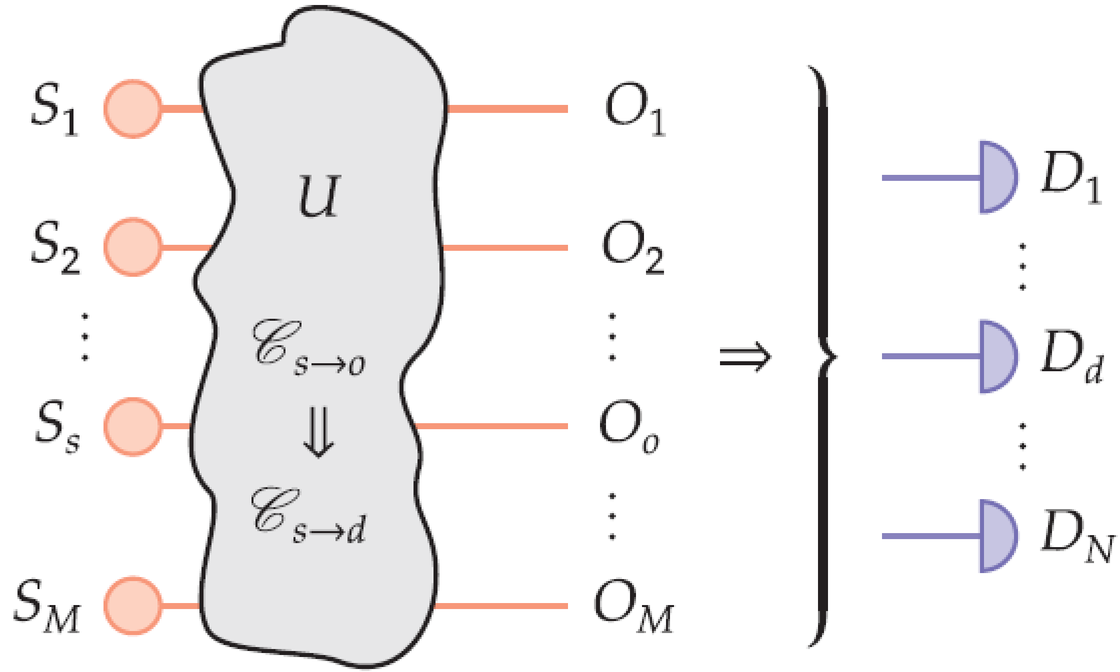
$$t_{d-s\to d} = t_d - t_{s\to d}$$



$\mathbf{N^{th}}$-order correlation function and permanents

$$\hat{\rho} = \bigotimes_{s=1}^{M} \hat{\rho}_s$$

$$N_1 + N_2 + \ldots + N_M = N$$



$$G^{(N)}(t_1, t_2, \ldots, t_N, t_N, \ldots, t_1) = \sum_{\substack{\text{ord. part.} \\ (N_1, \ldots, N_M)}} G^{(N)}_{N_1, \ldots, N_M}(t_1, t_2, \ldots, t_N, t_N, \ldots, t_1)$$

N^{th} order correlation function given by the incoherent sum of all the terms

$$G^{(N)}_{N_1, \ldots, N_M}(t_1, t_2, \ldots, t_N, t_N, \ldots, t_1) = \operatorname{tr}\left\{\hat{\rho} \left\| \left[\prod_{s=1}^{M} \frac{1}{N_s!}\right] \operatorname{per} \hat{U}_{N_1, \ldots, N_M} \right\|^2 \right\}$$

defined by how many times N_s each source S_s can contribute to an N-fold detection

"N^th^-order correlation Matrix Operators"

For any given set of values N_s , with s=1,2,,…,M, defining how many times N_s each source S_s can contribute to an N-fold detection :

$$\hat{U}_{N_1,N_2,\ldots,N_M}(t_1,\ldots,t_N) \qquad N_1+N_2+\ldots+N_M=N$$

$$= \begin{pmatrix} \mathscr{C}_{1\to1}\hat{E}_1^{(+)}(t_1) & \mathscr{C}_{2\to1}\hat{E}_2^{(+)}(t_1) & \ldots & \mathscr{C}_{s\to1}\hat{E}_i^{(+)}(t_1) & \ldots & \mathscr{C}_{M\to1}\hat{E}_M^{(+)}(t_1) \\ \mathscr{C}_{1\to2}\hat{E}_1^{(+)}(t_2) & \mathscr{C}_{2\to2}\hat{E}_2^{(+)}(t_2) & \ldots & \mathscr{C}_{s\to2}\hat{E}_i^{(+)}(t_2) & \ldots & \mathscr{C}_{M\to2}\hat{E}_M^{(+)}(t_2) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathscr{C}_{1\to d}\hat{E}_1^{(+)}(t_d) & \mathscr{C}_{2\to d}\hat{E}_2^{(+)}(t_d) & \ldots & \mathscr{C}_{s\to d}\hat{E}_i^{(+)}(t_d) & \ldots & \mathscr{C}_{M\to d}\hat{E}_M^{(+)}(t_d) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \underbrace{\mathscr{C}_{1\to N}\hat{E}_1^{(+)}(t_N)}_{N_1\text{ times}} & \underbrace{\mathscr{C}_{2\to N}\hat{E}_2^{(+)}(t_N)}_{N_2\text{ times}} & \ldots & \underbrace{\mathscr{C}_{s\to N}\hat{E}_s^{(+)}(t_N)}_{N_s\text{ times}} & \ldots & \underbrace{\mathscr{C}_{M\to N}\hat{E}_M^{(+)}(t_N)}_{N_M\text{ times}} \end{pmatrix}$$

Multi-photon quantum interference

Complete indisitinguishability ⟷ Complete overlapping between multi-photon amplitudes

$$t_{d-s\to d} = t_d - t_{s\to d}$$

⇩

$$\Upsilon_N(n_s) = \begin{cases} \dfrac{n_s!}{(n_s - N)!} & \text{for} \quad n_s \geq N \\ 0 & \text{for} \quad n_s < N \end{cases}$$

- M Fock states with a generic number of photons

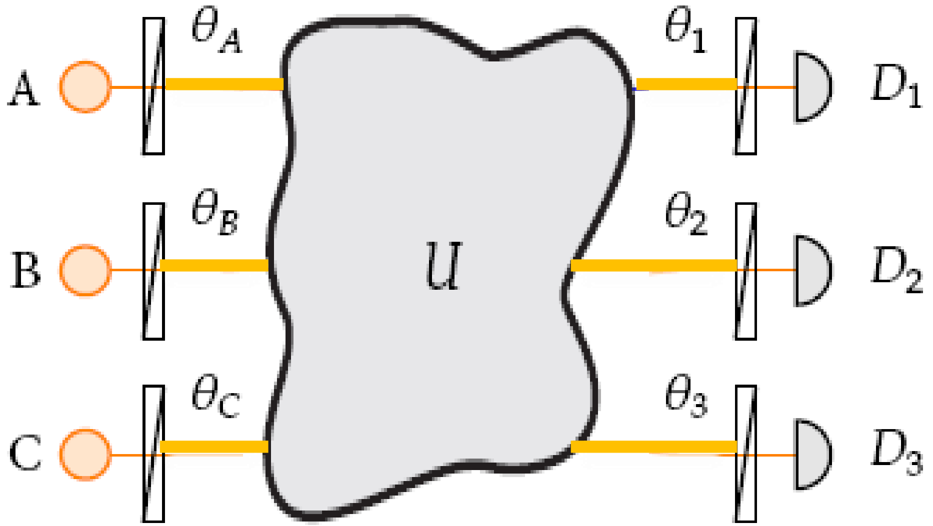
$$\underline{G}^{(N)}_{\text{Fock}}\left(\left|t_{d-s\to d} - t_{0s}\right| \ll \tau\right) = \tilde{E}_0^{2N} \sum_{\substack{\text{ord. part.} \\ (N_1,\ldots,N_M)}} \left[\prod_s \frac{1}{(N_s!)^2} \Upsilon_{N_s}(n_s)\right] \left|\text{per}\, U_{N_1,\ldots,N_M}\right|^2$$

- M independent thermal sources with a generic average number of photons

Multi-Photon Interference!

$$\underline{G}^{(N)}_{\text{Ther}}\left(\left|t_{d-s\to d} - t_{d'-s'\to d'}\right| \ll \tau\right) = E_0^{2N} \sum_{\substack{\text{ord. part.} \\ (N_1,\ldots,N_M)}} \left[\prod_s \frac{1}{N_s!} (\overline{n}_s)^{N_s}\right] \left|\text{per}\, U_{N_1,\ldots,N_M}\right|^2$$

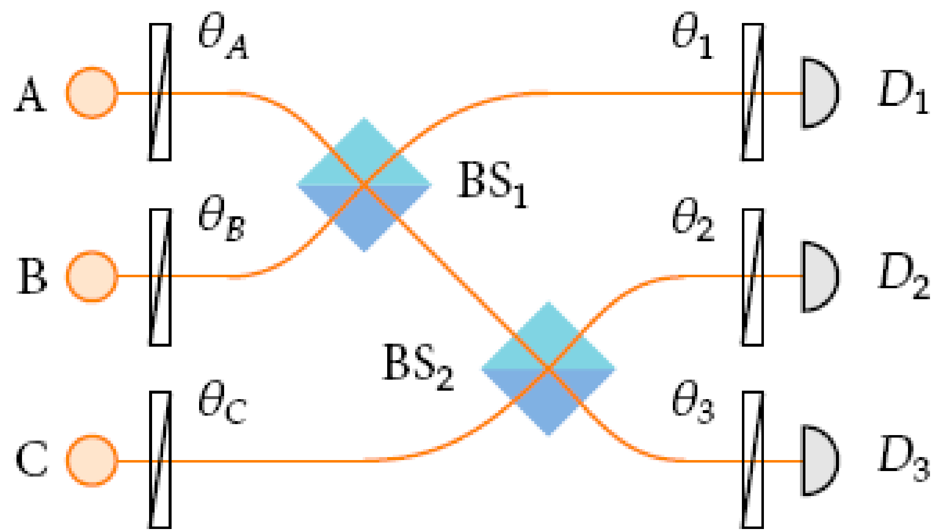
Polarization correlations with independent polarized sources



$$U_{ABC} = \begin{pmatrix} \mathcal{C}_{A\to1}\,\rho_{A\to1} & \mathcal{C}_{B\to1}\,\rho_{B\to1} & \mathcal{C}_{C\to1}\,\rho_{C\to1} \\ \mathcal{C}_{A\to2}\,\rho_{A\to2} & \mathcal{C}_{B\to2}\,\rho_{B\to2} & \mathcal{C}_{C\to2}\,\rho_{C\to2} \\ \mathcal{C}_{A\to3}\,\rho_{A\to3} & \mathcal{C}_{B\to3}\,\rho_{B\to3} & \mathcal{C}_{C\to3}\,\rho_{C\to3} \end{pmatrix}$$

$$\rho_{S\to D} = \cos(\theta_S - \theta_D)$$

Polarization correlations with independent polarized sources: example



$$BS_1 = \begin{pmatrix} \sqrt{\frac{1}{3}} & i\sqrt{\frac{2}{3}} \\ i\sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} \end{pmatrix}$$

$$BS_2 = \begin{pmatrix} \sqrt{\frac{1}{2}} & i\sqrt{\frac{1}{2}} \\ i\sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \end{pmatrix}$$

$\Rightarrow$

$$U_{ABC} = \begin{pmatrix} i\sqrt{\frac{2}{3}}\,\ell_{A\to 1} & \sqrt{\frac{1}{3}}\,\ell_{B\to 1} & 0 \\ i\sqrt{\frac{1}{6}}\,\ell_{A\to 2} & -\sqrt{\frac{1}{3}}\,\ell_{B\to 2} & \sqrt{\frac{1}{2}}\,\ell_{C\to 2} \\ \sqrt{\frac{1}{6}}\,\ell_{A\to 3} & i\sqrt{\frac{1}{3}}\,\ell_{B\to 3} & i\sqrt{\frac{1}{2}}\,\ell_{C\to 3} \end{pmatrix}$$

with $\ell_{S\to D} = \cos(\theta_S - \theta_D)$

Outlook

- Connection between N^{th}-order correlation functions for a generic multi-mode interferometer and Permanents of N X N matrixes:

1. Generic Fock states sources
2. Statistically independent thermal sources

- Independent polarized sources allow to implement generic polarization correlations measurements

- Open questions:

1. What permanents really tell us about the complexity of a physical system ?
3. Are there any “ad hoc” detection techniques able to select different “permanent” terms in the N-order correlation function for different input states ?

Merry Christmas!!!