Course: Theory of Fundamental Interactions

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Introduction to Fundamental Interactions

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Chapter 1

Introduction to Quantum Field Theory

1.1 From classical to quantum field theory

Quantum field theory arises from the need to combine quantum mechanics and relativity. In ordinary quantum mechanics, to deal with a system of N particles one solves the Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi \tag{1.1}$$

where, if one is working in the coordinate representation

$$\Psi = \Psi(q_1, ...q_{3N}, t) \tag{1.2}$$

and $\hat{H} = \hat{H}(q, p, t)$. This approach cannot be followed in relativistic quantum theory. The reason lies not only in the absence of relativistic covariance for eq. (1.1). In principle this defect could be corrected by adopting a covariant generalization, such as the Klein-Gordon equation. The reason is more profound and has its roots in the energy uncertainty principle. As is well known this principle restricts the validity of the energy conservation for small time intervals. More precisely suppose we measure twice the energy of a particle with a time delay Δt. Then the uncertainties ΔE and $\Delta E'$ satisfy

$$\Delta E - \Delta E' \sim \frac{\hbar}{\Delta t} \; . \tag{1.3}$$

For small Δt the energy fluctuations can be large enough as to pass the threshold for the creation of new particles $\sim mc^2$. Clearly the formalism of non relativistic quantum mechanics embodied in (1.2) cannot account for this behavior because the number of particles is fixed.

A further consequence of the uncertainty principle is a new limitation on the precision of measurements of momentum, as originally shown by Landau and Peierls. Let us follow their argument. Since $\Delta E = v\Delta P$ then from (1.3)

$$(v - v')\Delta P \sim \frac{\hbar}{\Delta t} \tag{1.4}$$

Since at most $v - v' = 2c$ then

$$\Delta P \sim \frac{\hbar}{c\Delta t} \tag{1.5}$$

When $\Delta t \to 0$, ΔP grows and and the particle momentum ceases to be an observable.

This behavior is not peculiar of momentum; also position cannot be measured with arbitrary precision. This is a consequence of the existence of negative-energy solutions of relativistic wave equations (Klein Gordon, Dirac equations). If one tries to localize the particle, then these unwanted solutions cannot be avoided. On the basis of dimensional arguments we can estimate the uncertainty on position Δq in the particle rest frame as

$$\Delta q \sim \frac{\hbar}{mc} \ . \tag{1.6}$$

From $\Delta q\Delta P \sim \hbar$ follows that $\Delta P \sim mc$, and this uncertainty on momentum is large enough as to reach the energy threshold for creation of new particles. If the particle moves with energy E, we have, instead of (1.6):

$$\Delta q \sim \frac{\hbar c}{E} \tag{1.7}$$

and for massless particles, such as the photon,

$$\Delta q \sim \frac{\hbar}{p} \ . \tag{1.8}$$

In other words for the photon the uncertainty in the localization is of the order of the de Broglie wavelength λ_{DB}. This means that only in the limit $\lambda_{DB} \to 0$, i.e. in the limit of geometrical optics, we can treat e.m. waves as composed of localized particles. For the other cases the notion of space coordinates for the photon has no meaning. Therefore in quantum field theory we expect a formalism where position, analogously to time, is not an observable, but has only the meaning of a label. It follows that position will not be described by an operator in Hilbert spaces. The operators will be constructed by new entities, the fields, i.e. functions $F(x) = F(\vec{r}, t)$. Therefore the natural formalism that takes into account the peculiarities of the relativistic quantum theory is quantum field theory. This might be a major development, but actually it is not. In fact a well known example of a classical field with a quantum behavior is the electromagnetic field in vacuum, whose quantum properties (photons) were discovered at the very beginning of quantum theory.

Electrodynamics is especially suited for relativistic quantum theory since the Maxwell equations are covariant. Therefore, we shall start from this theory. After two introductory subsections, as a first example of the transition from the classical to the quantum field theory we will consider below, in subsection 1.1.3, the quantization of the electromagnetic field in absence of matter.

There is another formalism for quantization, the path-integral approach of Dirac and Feynman. This method is naturally extended to field theory and we will give an outline of it in section 1.2 and 1.3 where we derive some properties of the quantum theory of scalar field with self-interactions. In section 1.4 we will briefly address the spin 1 and 2 fields. A few appendices conclude the chapter.

1.1.1 Summary of Classical Electromagnetism

Electromagnetic tensor:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \ . \tag{1.9}$$

Maxwell equations:

$$\epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0 \,, \qquad \partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu \ ; \tag{1.10}$$

they are invariant under gauge transformations:

$$A^\mu \to A' = A^\mu + \partial^\mu f \ . \tag{1.11}$$

Relation between 4−potential, scalar and vector potential:

$$A^\mu = (\phi, \vec{A}) \ . \tag{1.12}$$

Lorentz gauge:

$$\partial_\mu A^\mu = 0 \ . \tag{1.13}$$

Radiation gauge

$$\phi = 0 \, , \qquad \vec{\nabla} \cdot \vec{A} = 0 \ . \tag{1.14}$$

Fields $\vec{E}$, $\vec{B}$ in the radiation gauge

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} \ , \qquad \vec{B} = \vec{\nabla} \wedge \vec{A} \ . \tag{1.15}$$

Vacuum Maxwell equations in the Lorentz gauge coincide with the d'Alembert equation:

$$\Box A^\mu = 0 \ , \tag{1.16}$$

where the d'Alembert operator is: $\Box \equiv \partial_\mu \partial^\mu \equiv \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$.
Lagrangian for free e.m. field:

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \ . \tag{1.17}$$

1.1.2 Electromagnetic fields as ensembles of oscillators

Let us consider the e.m. field in the vacuum in the radiation gauge. To start with, we suppose that the radiation field is confined in a cube of volume $V = L^3$. The vector $\vec{A}$ can be decomposed in a Fourier series:

$$\vec{A} = \frac{1}{V} \sum_{\vec{k}} \left(\vec{a}_{\vec{k}} \, e^{i\vec{k}\cdot\vec{r}} \ + \ \vec{a}^*_{\vec{k}} \, e^{-i\vec{k}\cdot\vec{r}} \right) \tag{1.18}$$

where

$$k_x = \frac{2\pi n_x}{L} \ , \quad k_y = \frac{2\pi n_y}{L} \ , \quad k_z = \frac{2\pi n_z}{L} \ , \tag{1.19}$$

with n_x, n_y, n_z arbitrary integers. When $L \to \infty$ one replace the Fourier series by a Fourier integral and $\vec{k}$ becomes a continuous variable:

$$\vec{A} = \int \frac{d\vec{k}}{(2\pi)^3} \left(\vec{a}_{\vec{k}} \, e^{i\vec{k}\cdot\vec{r}} \ + \ \vec{a}^*_{\vec{k}} \, e^{-i\vec{k}\cdot\vec{r}} \right) . \tag{1.20}$$

In fact, for any function f, with $\Delta k_x = \Delta k_y = \Delta k_z = \frac{2\pi}{L} \to 0$ (for $L \to \infty$), one has:

$$\int \frac{d\vec{k}}{(2\pi)^3} f(\vec{k}) \ \approx \ \sum_{n_x, n_y, n_z} \frac{\Delta k_x \Delta k_y \Delta k_z}{(2\pi)^3} f(\Delta k_x n_x, \Delta k_y n_y, \Delta k_z n_z) =$$

$$= \frac{1}{V} \sum_{n_x, n_y, n_z} f(\vec{k}) \ . \tag{1.21}$$

Notice that $\vec{A}$ in (1.18) is real since $\vec{a}^*_{\vec{k}}$ is the conjugate of $\vec{a}_{\vec{k}}$. Since $\vec{\nabla} \cdot \vec{A} = 0$, then $\vec{a}_{\vec{k}}$ is proportional to $\vec{k}$:

$$\vec{a}_{\vec{k}} \cdot \vec{k} = 0 \ . \tag{1.22}$$

As to its time dependence, it must be harmonic:

$$\vec{a}_{\vec{k}} \propto e^{-i\omega t} \, , \qquad \omega = |\vec{k}| \ ,$$

so that each Fourier component satisfies the vacuum Maxwell equations exactly as $\vec{A}$ does, see (1.16). Notice that we work with units $c = 1$.

Let us define new variables:

$$\vec{q}_{\vec{k}} = \frac{1}{\sqrt{4\pi V}} \left(\vec{a}_{\vec{k}} + \vec{a}^*_{\vec{k}} \right) \ , \ \vec{p}_{\vec{k}} = -i\omega \frac{1}{\sqrt{4\pi V}} \left(\vec{a}_{\vec{k}} - \vec{a}^*_{\vec{k}} \right) = \frac{d\vec{q}_{\vec{k}}}{dt} \ , \tag{1.23}$$

so that

$$\vec{a}_{\vec{k}} = \sqrt{\pi V} \left(\vec{q}_{\vec{k}} + \frac{i}{\omega} \vec{p}_{\vec{k}} \right) \ , \qquad \vec{a}^*_{\vec{k}} = \sqrt{\pi V} \left(\vec{q}_{\vec{k}} - \frac{i}{\omega} \vec{p}_{\vec{k}} \right) \ . \tag{1.24}$$

By these variables we can identify the e.m. field as an ensemble of harmonic oscillators, which is very useful to perform quantization, see subsequent section 1.1.3, because the linear oscillator problem is exactly solvable in quantum mechanics. In fact, let us compute $\vec{E}$ and $\vec{B}$:

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} = -\sqrt{\frac{4\pi}{V}} \sum_{\vec{k}} \left(\vec{p}_{\vec{k}} \cos \vec{k}\vec{r} + \omega \vec{q}_{\vec{k}} \sin \vec{k}\vec{r} \right) \ , \tag{1.25}$$

$$\vec{B} = \vec{\nabla} \wedge \vec{A} = -\sqrt{\frac{4\pi}{V}} \sum_{\vec{k}} \left(\vec{k} \wedge \vec{q}_{\vec{k}} \sin \vec{k}\vec{r} + \frac{\vec{k}}{\omega} \wedge \vec{p}_{\vec{k}} \cos \vec{k}\vec{r} \right) \tag{1.26}$$

and the energy of the e.m. field:

$$H = \frac{1}{8\pi} \int d\vec{r} \left(\vec{E}^2 + \vec{B}^2 \right) \ . \tag{1.27}$$

The only contributions that, after integration do not vanish are those proportional to $\cos^2 \vec{k}\vec{r}$ or $\sin^2 \vec{k}\vec{r}$. The result is

$$H = \frac{1}{2} \sum_{\vec{k}} \left(\vec{p}_{\vec{k}}^{\,2} + \omega^2 \vec{q}_{\vec{k}}^{\,2} \right) \ . \tag{1.28}$$

We observe that $\vec{q}_{\vec{k}}$, $\vec{p}_{\vec{k}}$ satisfy $\vec{q}_{\vec{k}} \cdot \vec{k} = \vec{p}_{\vec{k}} \cdot \vec{k} = 0$ because of their definition and eq. (1.22). Therefore they have only two components that we call $q_{\vec{k}\alpha}$ and $p_{\vec{k}\alpha}$ ($\alpha = 1, 2$). These two directions coincide with two directions of $\vec{E}$, i.e. with two different polarizations of the e.m. wave with wave vector $\vec{k}$. We have therefore

$$H = \frac{1}{2} \sum_{\vec{k}\,\alpha} \left(p^2_{\vec{k}\,\alpha} + \omega^2 q^2_{\vec{k}\,\alpha} \right) , \tag{1.29}$$

which shows that the hamiltoniana of classical radiation field is written as sum of hamiltonians of an infinite number of independent linear oscillators with canonical variables $q_{\vec{k}\,\alpha}$ e $p_{\vec{k}\,\alpha}$.

Problem. Using the Hamilton equations for the system (1.29) prove that the oscillators are independent from each other.

Each term in (1.29) corresponds to a monochromatic plane wave with definite polarization, wave vector $\vec{k}$ and frequency ω coincident with the frequency of the linear oscillators.

If we consider the field momentum,

$$\vec{P} = \frac{1}{4\pi} \int d\vec{r}\, \vec{E} \wedge \vec{B} , \tag{1.30}$$

proceeding analogously to our treatment of energy one proves that

$$\vec{P} = \frac{1}{2} \sum_{\vec{k}\,\alpha} \left(p^2_{\vec{k}\,\alpha} + \omega^2 q^2_{\vec{k},\alpha} \right) \vec{n} , \tag{1.31}$$

with $\vec{n} = \vec{k}/\omega$. We notice that $H = c|\vec{P}|$.

1.1.3 Quantization of the electromagnetic field

In quantum mechanics observables are described by hermitian operators in Hilbert spaces. Since the components of the e.m. are observables, $\vec{E}$ and $\vec{B}$ will be observables as well. The quantum theory of light is an example of quantum field theory (QFT). QFT is significantly more complicated than quantum mechanics of a system of particles since $\vec{E}$ and $\vec{B}$ depend not only on t but also on $\vec{r}$, which means that we are dealing with infinitely many operators. However elementary particle physics can only be dealt with using this language, therefore in this chapter we will present some basic results and fundamental ideas of QFT. In this subsection we shall describe the main ideas needed to quantize the free [1] e.m. field, basically following Vol. IV of the Landau and Lifchitz textbook.

To start with we consider only the time-dependent free field, i.e. the fields that classically describe e.m. waves. In fact, as we discuss below, the static fields are always classic. It is useful to consider as variables $\vec{A}$, from which, in radiation gauge, both $\vec{E}$ and $\vec{B}$ can be derived using eq. (1.15). Our starting point are eqns. (1.29) and (1.31). We shall use the following units:

$$\hbar = c = 1 \ ,$$

Besides $\vec{E}$, $\vec{B}$ and $\vec{A}$ also the variables $q_{\vec{k}\alpha}$ and $p_{\vec{k}\alpha}$ are operators. To develop the quantum theory one can assume for H and $\vec{P}$ the classical formulae (1.29) and (1.31), but one has to postulate commutation relations between the operators $q_{\vec{k}\alpha}$ and $p_{\vec{k}\alpha}$. The form of (1.29) and (1.31) suggests the ansatz

$$[q_{\vec{k}\alpha},\ p_{\vec{k}'\alpha'}] \ = \ i\delta_{\vec{k}\vec{k}'}\delta_{\alpha\alpha'} \ . \tag{1.32}$$

As a consequence the energy of the e.m. field is quantized, assuming the values:

$$\mathcal{E} \ = \ \sum_{\vec{k},\alpha} \left(N_{\vec{k},\,\alpha} + \frac{1}{2} \right) \omega \ , \tag{1.33}$$

where $N_{\vec{k},\alpha}$ are arbitrary integers

$$N_{\vec{k},\,\alpha} \ = \ 0,\ 1,\ \dots,n,\ \dots \tag{1.34}$$

Instead of the operators $a_{\vec{k}\,\alpha}$ and $a^{\dagger}_{\vec{k}\,\alpha}$ let us introduce the operators

$$c_{\vec{k}\,\alpha} = \sqrt{\omega/2\pi V} a_{\vec{k}\,\alpha} = \frac{1}{\sqrt{2\omega}} \left(\omega q_{\vec{k}\,\alpha} + i p_{\vec{k}\,\alpha} \right) \tag{1.35}$$

$$c^{\dagger}_{\vec{k}\,\alpha} = \sqrt{\omega/2\pi V} a^{\dagger}_{\vec{k}\,\alpha} = \frac{1}{\sqrt{2\omega}} \left(\omega q_{\vec{k}\,\alpha} - i p_{\vec{k}\,\alpha} \right) \tag{1.36}$$

that are analogous to the annihilation and creation operators of the quantum linear oscillator problem as they satisfy

$$[c_{\vec{k}\,\alpha},\ c_{\vec{k}'\,\alpha'}] \ = \ 0 \tag{1.37}$$

$$[c^{\dagger}_{\vec{k}\,\alpha},\ c^{\dagger}_{\vec{k}'\,\alpha'}] \ = \ 0 \tag{1.38}$$

$$[c_{\vec{k}\,\alpha},\ c^{\dagger}_{\vec{k}'\,\alpha'}] \ = \ \delta_{\vec{k}\,\vec{k}'}\delta_{\alpha\,\alpha'} \ . \tag{1.39}$$

[1] This means with $J^{\mu} = 0$ in eq. (1.10).

One can easily prove that

$$H = \frac{1}{2}\sum_{\vec{k}\,\alpha}\left(p^2_{\vec{k}\,\alpha} + \omega^2 q^2_{\vec{k}\,\alpha}\right) = \frac{1}{2}\sum_{\vec{k}\,\alpha}\left(c_{\vec{k}\,\alpha}c^\dagger_{\vec{k}\,\alpha} + c^\dagger_{\vec{k}\,\alpha}c_{\vec{k}\,\alpha}\right)\omega =$$

$$= \sum_{\vec{k}\,\alpha}\left(\hat{N}_{\vec{k}\,\alpha} + \frac{1}{2}\right)\omega \ , \qquad (1.40)$$

where we have introduced the operator (*number* operator)

$$\hat{N}_{\vec{k}\,\alpha} = c^\dagger_{\vec{k}\,\alpha}c_{\vec{k}\,\alpha}$$

whose eigenvalues are in eq. (1.34). Similarly one can show that

$$\vec{P} = \sum_{\vec{k}\,\alpha}\left(\hat{N}_{\vec{k}\,\alpha} + \frac{1}{2}\right)\vec{k} \ . \qquad (1.41)$$

If $|N_{\vec{k}\,\alpha} >$ are eigenkets of $\hat{N}_{\vec{k}\,\alpha}$, one has

$$c_{\vec{k}\,\alpha}\,|\,N_{\vec{k}\,\alpha} > \;=\; \frac{1}{\sqrt{N_{\vec{k}\,\alpha}}}\,|\,N_{\vec{k}\,\alpha} - 1 > \qquad (1.42)$$

$$c^\dagger_{\vec{k}\,\alpha}\,|\,N_{\vec{k}\,\alpha} - 1 > \;=\; \frac{1}{\sqrt{N_{\vec{k}\,\alpha}}}\,|\,N_{\vec{k}\,\alpha} > \ . \qquad (1.43)$$

Using previous formulae it is easy to prove that

$$\vec{A}(\vec{r}) = \sum_{\vec{k}\,\alpha}\left(c_{\vec{k}\,\alpha}\vec{A}_{\vec{k}\,\alpha}(\vec{r}) + c^\dagger_{\vec{k}\,\alpha}\vec{A}^*_{\vec{k}\,\alpha}(\vec{r})\right) \qquad (1.44)$$

with

$$\vec{A}_{\vec{k}\,\alpha}(\vec{r}) = \sqrt{\frac{4\pi}{V}}\frac{\vec{e}^{\,\{\alpha\}}}{\sqrt{2\omega}}\,e^{i\vec{k}\vec{r}} \ , \qquad (1.45)$$

where $\vec{e}^{\,\{\alpha\}}$ (α (= 1, 2)) defines the polarization of the wave. In (1.44) c, $c^\dagger$ are operators and $\vec{A}_{\vec{k}\,\alpha}(\vec{r})$ numerical coefficients.

Eqns. (1.40) and (1.41) show that is more convenient to use the representation where the operators $N_{\vec{k}\,\alpha}$ are diagonal. We note that varying $\vec{k}$ and α, their common eigenkets:

$$|\,\{N_{\vec{k}\,\alpha}\} > = \;|\,N_{\vec{k}_1\,1},\ N_{\vec{k}_1\,2},\ N_{\vec{k}_2\,1},\ N_{\vec{k}_2\,2},\ N_{\vec{k}_3\,1},\ N_{\vec{k}_3\,2}... > \qquad (1.46)$$

form a complete orthonormal system. This representation is called second-quantization representation or occupation-number. The wave-function in this representation is

$$\Psi(\{N_{\vec{k}\,\alpha}\}) = < \{N_{\vec{k}\,\alpha}\}\;|\;\Psi > \ . \qquad (1.47)$$

In the Schrödinger scheme for temporal evolution this wavefunction satisfies the Schrödinger equation, but we prefer to work in the Heisenberg scheme that presents the advantage of dealing in the same way space and time coordinates. In the Heisenberg scheme operators depend on time: $\vec{A}(\vec{r},\ t)$. Using the Heisenberg equations one can prove that

$$\vec{A}(\vec{r},\ t)\ =\ \sum_{\vec{k}\,\alpha}\Big(c_{\vec{k}\,\alpha}\,\vec{A}_{\vec{k}\,\alpha}(\vec{r},\ t)\ +\ c^{\dagger}_{\vec{k}\,\alpha}\,\vec{A}^{*}_{\vec{k}\,\alpha}(\vec{r},\ t)\Big) \tag{1.48}$$

with

$$\vec{A}_{\vec{k}\,\alpha}(\vec{r},\ t)\ =\ \sqrt{\frac{4\pi}{V}}\,\frac{\vec{e}^{\,\{\alpha\}}}{\sqrt{2\omega}}\,e^{-i(\omega t-\vec{k}\vec{r})}\ . \tag{1.49}$$

In the second-quantization representation H and $\vec{P}$ are real numbers:

$$H\ =\ \sum_{\vec{k}\,\alpha}\left(N_{\vec{k}\,\alpha}\ +\ \frac{1}{2}\right)\ \omega\,,$$

$$\vec{P}\ =\ \sum_{\vec{k}\,\alpha}\left(N_{\vec{k}\,\alpha}\ +\ \frac{1}{2}\right)\ \vec{k}\ .$$

These equations can be interpreted as follows: In the state $\mid\{N_{\vec{k}\,\alpha}\}>$ energy and momentum of the e.m. wave are carried by particles having (I reintroduce for a while the factors $\hbar$, c) energy: $E=\hbar\omega$ and momentum $\hbar\vec{k}$. These particles are called photons. Notice that the energy of the states with one or more photons are computed from the vacuum energy

$$E_{vacuum}=\sum_{\vec{k}\,\alpha}\ \frac{1}{2}\ \omega\ . \tag{1.50}$$

E_{vacuum} is actually infinite, but it is constant and can be therefore removed safely.

A few comments are in order. To begin with we repeat that $N_{\vec{k}\,\alpha}$ are arbitrary. This means that more photons can possess the same quantum numbers ($\vec{k}$ and α). Therefore they are bosons. As well known bosons tend to occupy the same quantum states and this property is reflected in a property of the e.m. field. In fact we know that classical electromagnetism describes rather accurately an enormous amount of phenomena. This means that in all these cases quantum behavior should not differ much from classical description. For this to happen the operators should be c- numbers. Let us consider annihilation/creation operators $c^\dagger_{\vec{k}\,\alpha}$ and $c_{\vec{k}\,\alpha}$:

$$c_{\vec{k}\,\alpha} c^\dagger_{\vec{k}\,\alpha} \;=\; c^\dagger_{\vec{k}\,\alpha} c_{\vec{k}\,\alpha} \;+\; 1 \;=\; \hat{N}_{\vec{k}\,\alpha} \;+\; 1 \;. \tag{1.51}$$

In the second quantization representation $N_{\vec{k}\,\alpha}$ is a number. Only if this number is much larger than 1 we can write

$$c_{\vec{k}\,\alpha} c^\dagger_{\vec{k}\,\alpha} \;\simeq\; c^\dagger_{\vec{k}\,\alpha} c_{\vec{k}\,\alpha} \;. \tag{1.52}$$

and $c, c^\dagger$ are numbers as they commute with all the other operators. As a conclusion classical electromagnetism is a successful theory because of the bosonic nature of the photon. Let us consider the factor

$$\vec{A}_{\vec{k}\,\alpha}(\vec{r},\; t) \;=\; \sqrt{\frac{4\pi}{V}}\,\frac{\vec{e}^{\,\{\alpha\}}}{\sqrt{2\omega}}\, e^{-i(\omega t - \vec{k}\vec{r})}$$

appearing in (1.49). It is a plane wave describing a particle of definite momentum $\vec{p} = \hbar\vec{k}$ and energy $E = \hbar\omega$. We can interpret $\vec{A}_{\vec{k}\,\alpha}(\vec{r},\; t)$ as the *photon wavefunction*. The relation between the particle wavefunction and the field operator $\vec{A}(\vec{r},\; ,t)$ is as follows:

$$\vec{A}_{\vec{k}\,\alpha}(\vec{r},\; t) \;= < 0 \mid \vec{A}(\vec{r},\; ,t)\,|...,\; 1_{\vec{k}\,\alpha},\; ... > \;, \tag{1.53}$$

where $\mid 0 >$ is the vacuum and $|...,\; 1_{\vec{k}\,\alpha},\; ... >$, is the state containing one photon of quantum numbers $\vec{k}\,\alpha$. The photon wavefunction, differently from other particles, does not represent the probability amplitude to find, by a measurement, the photon in a given position $\vec{r}$; this is so because a measurement of the photon position means its destruction, e.g. by its absorption in the detector. In any event the photon wavefunction satisfies its own Schrödinger equation, i.e. the d'Alembert equation:

$$\nabla\vec{A} \;-\; \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} \;= 0 \;,$$

see (1.49).

Let us discuss mass, electric charge and spin. The photon mass vanishes because of the momentum-energy relation: $E = cp$. As for electric charge, photons do not interact directly or, in other words, a photon does not interact directly with the e.m. field. Therefore its charge vanishes. As to spin, the presence of the vector $\vec{e}^{\,\{\alpha\}}$ in the photon wavefunction shows that under rotations the photon wavefunction transforms as a vector. Such a behavior is peculiar of spin 1 particles with the same number of components of a vector $(3 = 2s+1)$. It should be noted that when we speak of photon spin we refer to the total photon angular momentum. because the identification of the spin with the angular momentum in the particle rest frame is meaningless for the light. However, differently from other spin 1 particles the photon spin components are two and not three. More precisely the components of the spin projection on the momentum direction are ± 1 (the $\vec{S}\cdot\vec{n} = 0$ component is absent). This is a consequence of the transverse nature of the e.m. waves, which implies that only two polarizations, both transverse, are possible.

As a concluding remark let us show that static fields are always classic. Let us suppose that the electric field $\vec{E}$ is almost time independent in the time interval Δt. Then its Fourier transform contains frequencies in the range $(0,\, 1/\Delta t)$. The total number of oscillators in $V = 1$ with frequencies lower than $1/\Delta t$ can be estimated:

$$\frac{1}{V}\int dn = \int \frac{d^3k}{(2\pi)^3} = \frac{1}{2\pi^2 c^3}\int_0^{1/\Delta t} \omega^2 d\omega \sim \frac{1}{(c\Delta t)^3} \ .$$

On the other hand the energy in the unitary volume is $\sim E^2$. Therefore the energy per oscillator is

$$E^2 c^3 \Delta t^3 \ .$$

Since each photon has energy $\hbar\omega$ the total number of photons is

$$N \ \sim \ \frac{E^2 c^3 \Delta t^4}{\hbar} \ .$$

The field is classic if $N \gg 1$, i.e. if

$$E^2 \ \gg \ \frac{\hbar}{c^2 \Delta t^4} \ .$$

For static fields one can take $\Delta t = \infty$ and this condition is always verified. On the other hand, rapidly varying fields and/or weak fields cannot be well described by classical electromagnetism.

1.2 Feynman path integral

1.2.1 Quantum mechanics

The Feynman path integral approach provides a way to compute the propagator in the coordinate representation:

$$U(q_F, t+T; q_I, t) \ = \ < q_F|U(t+T,\ t)|q_I >=< q_F|U(T)|q_I >$$

$$= \quad < q_F|e^{-iHT/\hbar}|q_I> , \tag{1.54}$$

where we suppose H independent of time and consider, for the time being, one dimensional motions.

With $T = N\delta t$, and putting $\hbar = 1$:

$$\begin{aligned} &< q_F|e^{-iHT}|q_I>=< q_F|\left(e^{-iH\delta t}\right)^N|q_I> \\ = \quad &< q_F|\int\left(\prod_{j=1}^{N-1} dq_j\right) < q_F|e^{-iH\delta t}|q_{N-1}>< q_{N-1}|...\times \\ \times \quad &...|q_1>< q_1|e^{-iH\delta t}|q_I> \quad . \end{aligned} \tag{1.55}$$

To begin with we compute

$$\sigma_j =< q_{j+1}|e^{-iH\delta t}|q_i> \tag{1.56}$$

for the free particle:

$$\begin{aligned} \sigma_j \quad &= \quad \int dp \; < q_{j+1}|e^{-i\delta t p^2/2m}|p>< p|q_i>= \\ &= \quad \int \frac{dp}{2\pi}\, e^{-i\delta t p^2/2m+ip(q_{j+1}-q_j)} = \sqrt{\frac{-im}{2\pi\delta t}} e^{\frac{im\delta t}{2}\left(\frac{q_{j+1}-q_j}{\delta t}\right)^2} \end{aligned} \tag{1.57}$$

where we have used (1.199). Therefore

$$\begin{aligned} < q_F|e^{-iHT}|q_I> \quad &= \quad \left(\frac{-im}{2\pi\delta t}\right)^{N/2}\left(\prod_{j=1}^{N-1}\int dq_j\right) \times \\ &\times \quad \exp\left\{\frac{im\delta t}{2}\sum_{j=1}^{N-1}\left(\frac{q_{j+1}-q_j}{\delta t}\right)^2\right\} \; . \end{aligned} \tag{1.58}$$

We define the measure

$$\int [Dq] = \lim_{N\to\infty}\left(\frac{-im}{2\pi\delta t}\right)^{N/2}\left(\prod_{j=1}^{N-1}\int dq_j\right) \quad . \tag{1.59}$$

In the limit $N = T/\delta t \to \infty$ one has

$$\frac{im\delta t}{2}\sum_{j=0}^{N-1}\left(\frac{q_{j+1}-q_j}{\delta t}\right)^2 \to i\int_0^T dt \frac{m\dot{q}^2}{2} \tag{1.60}$$

and therefore

$$
\begin{aligned}
< q_F|e^{-iHT}|q_I > &= \int [Dq] e^{iS[q]} \ , \qquad (1.61) \\
S[q] &= \int_0^T dt\, L(q,\dot q) \ . \qquad (1.62)
\end{aligned}
$$

In the present case the lagrangian is simply given by $L = E_{kin}$, however this formula holds also in presence of a potential energy. In fact in this case

$$
\begin{aligned}
\sigma_j &= < q_{j+1}|e^{-iE_{kin}\delta t} e^{-iV(q)\delta t}|q_i > \\
&= e^{-iV(q_j)\delta t} < q_{j+1}|e^{-iE_{kin}\delta t} e^{-iV(q)\delta t}|q_i > \qquad (1.63)
\end{aligned}
$$

where we have used the property

$$
e^{-i\delta t[E_{kin}+V(q)]} = e^{-i\delta t E_{kin}}\, e^{-i\delta t\, V(q)} \qquad (1.64)
$$

valid up to terms $\mathcal{O}(\delta t)^2$. This implies that (1.62) holds also for $L = E_{kin} - V$. This result can be generalized and gives the amplitude for the transition from an initial state $|I >$ at time 0 to the final state $|F >$ at time T:

$$
Z =< F|e^{-iHT}|I >= \int dq_F dq_I \int [Dq]\, \psi_F^*(q_F)\psi_I(q_I)\, e^{iS[q]} \ . \qquad (1.65)
$$

Integrals such as (1.61) or (1.65) are called path integrals (or Feynman integrals or functional integrals). We shall refer to Z also as the generating functional, for reasons that will be clear below. The name path integral follows from an interpretation of the measure. A particle can go from the initial position X' at time $t = 0$ to the final position X at time $t = T$ by any of the polygonal trajectories P_j depicted in fig. 1.1. All the paths begin and end in the same points. Since any path can be approximated by polygonal trajectories, the path-integral measure means that the transition amplitude is the sum over all the trajectories with fixed end point, weighted by the factor $\exp iS$.

It is not clear if the measure we have introduced in (1.59) makes generally sense, in other words if the limit exists. It exists however if one performs an analytic continuation and considers imaginary times $T \to -iT$. In fact

$$
\exp\left\{ i\int_0^T L\, dt \right\} \quad \to \quad \exp\left\{ i\int_0^{-iT} dt\, \left(\frac{m\dot q^2}{2} - V(q) \right) \right\}
$$

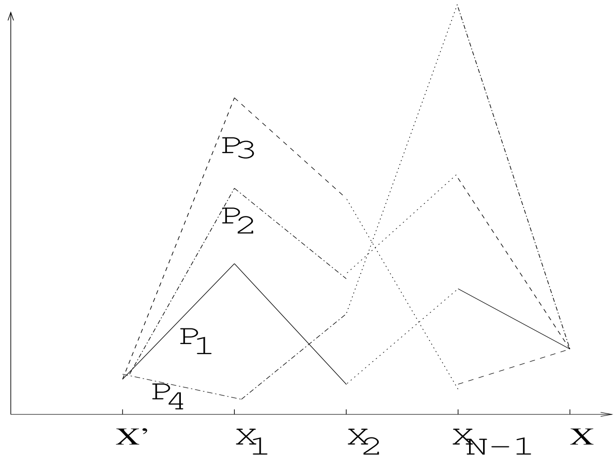


Figure 1.1: Path integral.

$$= \quad \exp\left\{-\int_0^T d\tau \left(\frac{m\dot{q}^2}{2} + V(q)\right)\right\} \quad (1.66)$$

($\tau = -it$), so that the integral

$$Z = \int dq_F dq_I \int [Dq]\, \psi_F^*(q_F)\psi_I(q_I)e^{-S_E[q]} \quad (1.67)$$

is well defined due to the quadratic terms in the exponential. Here the euclidean action is defined as

$$S_E[q] = \int_0^T d\tau \left(\frac{m\dot{q}^2}{2} + V(q)\right) \ . \quad (1.68)$$

Let us now derive the semiclassical ($\hbar \to 0$) limit. We can compute $< q_F|e^{-iHT/\hbar}|q_I >$ by the steepest descent (or stationary phase) method, see subsection 1.6; we get

$$< q|e^{-iHT/\hbar}|q_I > \propto e^{iS_c/\hbar} \quad (1.69)$$

where S_c is the action computed at the classical trajectory $q_c(t)$ i.e. the function $q(t)$ that starts at time t in q_I and ends at $t+T$ in q and satisfies $\delta S = 0$ or, equivalently, the Lagrange equations

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0 \ . \quad (1.70)$$

The form (1.69) is the well known semiclassical limit for the wavefunction at time $t{+}T$: $\psi(q,\, t{+}T)$, for a particle that at time t was localized in q_I. We interpret this result by saying that the path integral is dominated in the classical limit by the classical trajectory. In the following table we present the main result of this subsection.

$$\int [Dq] = \lim_{N\to\infty} \left(\frac{-im}{2\pi\delta t}\right)^{N/2} \left(\prod_{j=1}^{N-1} \int dq_j\right)$$

1.2.2 Fields

In subsection 1.1.3 we have discussed the quantization of the free e.m. field and we discussed the way to associate particles (photons) to it. Photons have spin 1, a consequence of the vector nature of the potential A_μ. We generalize this result stating that the association of particles and fields is a general feature; in particular we can imagine that spin 0 particles will be associated to a scalar field i.e. a function $\phi(x)$ ($x^\mu = (t, \vec{r})$) that is scalar under Lorentz transformations Λ:

$$\phi'(x') = \phi(x) \tag{1.71}$$

where $x' = \Lambda x$.

We will discuss quantization of the scalar field using the path integral approach. Before doing that, let us introduce the action for the scalar field by an analogy with the e.m. field, where we wrote the hamiltonian as a sum of hamiltonians for infinitely many oscillators. To this end we introduce a system of particles with mass m and potential energy V so that the lagrangian is

$$L = E_{kin} - V\ . \tag{1.72}$$

Let us consider L near an equilibrium point, where the first derivatives of V vanish. If q_a denotes the position of particle a with respect the equilibrium, then

$$L \approx \frac{1}{2}\sum_a m\dot{q}_a^2 - \frac{1}{2}\sum_{a\,b} K_{ab} q_a q_b - \sum_{a\,b\,c} F_{abc} q_a q_b q_c + \ldots \tag{1.73}$$

Neglecting the third order terms the equations of motion are

$$m\frac{d^2 q_a}{dt^2} = -\sum_b K_{a\,b} q_b + \mathcal{O}(q^2) \tag{1.74}$$

K can be diagonalized by an orthogonal matrix R so that, putting $Q = Rq$ we get

$$\frac{d^2 Q_a}{dt^2} = -\sum_a \omega_b^2 Q_a + \mathcal{O}(Q^2) \ , \tag{1.75}$$

where $m\omega_a^2$ are the eigenvalues of K. This shows that in harmonic approximation the system, exactly as for the e.m. field, is equivalent to a set of non interacting oscillators. Non harmonic terms correspond to interactions.

The generating functional for this system is a simple generalization of (1.65):

$$Z = \int \prod_a [Dq_a] e^{i\int_0^T dt\, L(q,\dot{q})} \tag{1.76}$$

where we have embodied the product of the wave functions in the definition of the measure and

$$L(q,\dot{q}) = \sum_a \frac{m\dot{q}_a^2}{2} - V(q) \ . \tag{1.77}$$

Let us define a field

$$\phi(t,\vec{r}) \tag{1.78}$$

whose values, on a discrete lattice, coincide with $q_a(t)$, e.g, if $a = (i,j,k)$, then $\phi(t,\ell i,\ell j,\ell k) = q_a(t)$ (ℓ = lattice spacing); when (i,j,k) vary in Z^3 we cover the entire 3-D space.

Suppose that $\ell \ll L$, where L denotes the smallest length scale we are interested in; then one can make the substitution $q_a(t) \to \phi(t,\vec{r})$. We have, if $\sigma = m/\ell^3$ is a mass density:

$$\sum_a m\dot{q}_a^2 = \sum_{ijk} \sigma\ell^3(\dot{\phi}(t,\ell i,\ell j,\ell k))^2 \to \int d\vec{r}\ \sigma\,[\dot{\phi}(t,\vec{r})]^2 \ . \tag{1.79}$$

Let us next consider the term containing $K_{ab}q_a q_b$ and let us suppose that K vanishes unless the triplets $a = (i,j,k)$ and $b = (i',j',k')$ differ by one unit along the same axis[2], e.g. $b = (i+1,j,k)$. For example with one space dimension we write

$$-\frac{1}{2}\sum_{ab} K_{ab}q_a q_b = \frac{1}{4}\sum_{ab} K_{ab}\left((q_a - q_b)^2 - q_a^2 - q_b^2\right) \tag{1.80}$$

[2] In other terms $K_{ijk;i'j'k'} = K(\delta_{i',i\pm1}\delta_{jj'}\delta_{kk'} + \delta_{i',i}\delta_{j,j'\pm1}\delta_{kk'} + ...)$

Now

$$(q_a - q_b)^2 = \ell^2 \left(\frac{\phi(a\ell) - \phi(a\ell - \ell)}{\ell} \right)^2 \rightarrow \ell^2 \left(\frac{\partial \phi}{\partial x} \right)^2 \tag{1.81}$$

Therefore we get

$$S[q] \rightarrow S[\phi] = \int_0^T dt \int d^D x \mathcal{L}(\phi, \partial_\mu \phi) \tag{1.82}$$

and the lagrangian density (or simply the lagrangian) is given by

$$\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2} \left[\sigma \left(\frac{\partial \phi}{\partial t} \right)^2 - \rho |\vec{\nabla} \phi|^2 - A\phi^2 - B\phi^3 - C\phi^4 ... \right] , \tag{1.83}$$

where the terms ϕ^3, ϕ^4, ... arise from anharmonic terms in (1.73).

With the definition

$$\phi' = \sqrt{\rho}\phi, \qquad c^2 = \frac{\rho}{\sigma}, \qquad \mu^2 = \frac{A}{\rho}, \ ... \tag{1.84}$$

we get (calling again ϕ the field ϕ'):

$$\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2} (\partial \phi)^2 - V(\phi) \tag{1.85}$$

where

$$(\partial \phi)^2 = \frac{1}{c^2} \left(\frac{\partial \phi}{\partial t} \right)^2 - |\vec{\nabla} \phi|^2 \tag{1.86}$$

and

$$V(\phi) = \frac{m^2}{2} \phi^2 + \frac{g}{3!} \phi^3 + \frac{\lambda}{4!} \phi^4 + ... \tag{1.87}$$

If we want to describe a Lorentz invariant physical system and ϕ is a Lorentz scalar, the lagrangian must be a scalar as well, $D = 3$ and c must be interpreted as the velocity of light in vacuum. Therefore

$$(\partial \phi)^2 = (\partial_\mu \phi)(\partial^\mu \phi) \ . \tag{1.88}$$

The generating functional is obtained from (1.76)

$$Z = \int [D\phi] e^{i/\hbar S[\phi]} \tag{1.89}$$

with

$$S[\phi] = \int d^d x \left(\frac{1}{2} (\partial \phi)^2 - V(\phi) \right) \tag{1.90}$$

and

$$[D\phi] = \prod_{\vec{x}} [D\phi(t,\vec{x})] \ . \tag{1.91}$$

Here $d = D+1$ (in three dimensions $D = 3$) and

$$\int d^d x = \int_0^T dt \int d^D x \ . \tag{1.92}$$

Ordinary quantum mechanics is obtained for $D = 0$.

Let us derive, by the stationary phase method the classical field equations. In the classical limit ($\hbar \to 0$) the dominant contribution to the path integral is given by the field ϕ satisfying $\delta S = 0$. Therefore

$$\begin{aligned} 0 &= \delta S = \delta \int dt \int d^d x \mathcal{L}(\phi, \partial\phi) = \\ &= \int d^D x \left(\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu\phi) \right) = \\ &= \int d^D x \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right)\delta\phi + \int d^D x \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi \right) \ . \end{aligned} \tag{1.93}$$

The last term is transformed by the Gauss theorem in a surface term

$$\int d^D x \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi \right) = \int_\Sigma d\sigma_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi \right) \tag{1.94}$$

where Σ is an hypersurface extended at infinity. The integral (1.94) vanishes because we assume that at the space-time infinity the fields vanish. Since $\delta\phi$ is arbitrary the Euler-Lagrange equations follow:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0 \ . \tag{1.95}$$

In the following table we summarize the main result of this section.

$$Z = \int [D\phi] e^{i/\hbar S[\phi]}$$

$$S[\phi] = \int d^d x \left(\tfrac{1}{2}(\partial\phi)^2 - V(\phi) \right)$$

$$[D\phi] = \prod_{\vec{x}} [D\phi(t,\vec{x})]$$

1.3 Quantization of the scalar field theory

1.3.1 Free scalar field theory

Let us first assume

$$V(\phi) = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \ . \tag{1.96}$$

Then from (1.95) one gets

$$(\Box + m^2)\phi = -\frac{\lambda}{6}\phi^3 \tag{1.97}$$

For $\lambda = 0$ the scalar field is called *free*; its lagrangian is

$$\mathcal{L} = \frac{1}{2}\left[(\partial\phi)^2 - m^2\phi^2\right] \tag{1.98}$$

and the corresponding field equation is known as the Klein-Gordon equation:

$$(\Box + m^2)\phi = 0 \ . \tag{1.99}$$

Before the advent of the Quantum Field Theory this equation was interpreted as a covariant generalization of the Schrödinger equation; in the present formalism is the equation satisfied by the scalar field ϕ and is analogous to the Maxwell equation (1.16). One might quantize it along the lines followed in subsection 1.1.3. The field ϕ would be written as

$$\phi = \frac{1}{V}\sum_{\vec{k}}\left(a_{\vec{k}}e^{i(\omega_k t - \vec{k}\cdot\vec{r})} + a^*_{\vec{k}}e^{-i(\omega_k t - \vec{k}\cdot\vec{r})}\right) \tag{1.100}$$

where, to satisfy (1.99) one has

$$\omega_k = \sqrt{k^2 + m^2} \tag{1.101}$$

Similarly to the photon case the particles appear here as quanta of mass m because (1.101) is the energy-momentum relativistic relation typical of massive particles.

Since we have decided to follow the path integral approach, we shall not introduce creation/annihilation operators and we will see particles to emerge in a different way.

Let us go back to the functional generator in (1.89) which we write as follows:

$$Z \ = \ <0|e^{-iHT}|0>=$$

$$= \int [D\phi] \exp \left\{ i \int d^d x \left[-\frac{1}{2}\phi(\Box + m^2 - i\epsilon)\phi + J\phi \right] \right\} \quad (1.102)$$

Note that we have integrated by parts (1.89) neglecting a surface integral, so that

$$(\partial\phi)^2 \to -\phi\Box\phi \ ;$$

moreover we have changed

$$m^2 \to m^2 - i\epsilon \quad (1.103)$$

which guarantees the convergence in the Minkowsky space due to the Gaussian nature of the functional integral. Finally we have added an extra term

$$\sim J\phi \quad (1.104)$$

to the lagrangian, which represents the interaction of the field with an external source J. Only including this term we can get interesting results; in fact, without (1.104) the functional integral would be trivial because $|0>$ is the ground state of H for $J = 0$.

The integral (1.102) can be performed exactly. As a matter of fact let us discretize space variables $\vec{r}$ so that $\vec{r} = a\ell = (n_x\ell, n_y\ell, n_z\ell)$ and let us take $t = j\delta t = jT/N$. Then $\phi(t,\vec{r}) = \phi(j\delta t, a\ell) \equiv \phi_K$ with $K = (a,j) = (n_x, n_y, n_z, j)$. Then, writing $\prod_a \prod_{j=1}^{N-1} = \prod_K$ we have

$$Z \propto \int \left(\prod_K d\phi_K \right) \exp \left\{ i \left[\frac{1}{2}\phi_I A_{IK} \phi_K + J_K \phi_K \right] \right\} \quad (1.105)$$

where sum over repeated indexes is understood and the matrix A is the operator $-(\Box + m^2 - i\epsilon)$ on the lattice. Using (1.203) to evaluate (1.105) we get

$$Z \propto e^{\frac{-i}{2} J_I A^{-1}_{IK} J_K} \ , \quad (1.106)$$

and going back to the continuum

$$Z[J] \ = \ Z[0]\, e^{iW[J]} \, , \quad (1.107)$$

with

$$W[J] \ = \ -\frac{1}{2} \int d^d x_1 d^d x_2 \, J(x_1) D(x_1 - x_2) J(x_2) = -\frac{1}{2} < J_1 D_{12} J_2 > \ . \quad (1.108)$$

Here we have introduced the notation $<>_{ij}$ to denote integration over space time variables x_i, x_j.

Problem. Prove that

$$\frac{\delta^2 W}{\delta J(x)\delta J(y)} = -D(x-y) \tag{1.109}$$

Therefore higher order functional derivatives vanish.

We have written $D(x-y)$ and not $D(x,y)$ since D satisfies

$$-(\partial^2 + m^2 - i\epsilon)D = \mathbf{1}\ , \tag{1.110}$$

i.e.

$$-(\partial^2 + m^2 - i\epsilon)D(x-y) = \delta(x-y) \tag{1.111}$$

$D(x-y)$ is called the Feynman propagator for the scalar field and is given by

$$D(x-y) = \int \frac{d^4k}{(2\pi)^d} \frac{e^{ik(x-y)}}{k^2 - m^2 + i\epsilon}\ . \tag{1.112}$$

Problem. Prove that

$$D(x) = -i \int \frac{d^3k}{(2\omega_k\, 2\pi)^3} \left\{ e^{-i(\omega_k x_0 - \vec{k}\cdot\vec{x})}\, \theta(x_0) + e^{+i(\omega_k x_0 - \vec{k}\cdot\vec{x})}\, \theta(x_0) \right\}\ . \tag{1.113}$$

Hint: Use the residue theorem to perform the k_0 integration. If $x_0 > 0$ close the contour with a semi-circle in the upper half plane, where k_0 has a positive imaginary part; otherwise use the lower half-plane.

Besides (1.109) another useful formula for the free field propagator is as follows:

$$i\,D(x-y) = \frac{1}{Z[0]} \int [D\phi]\phi(x)\phi(y) \exp\left\{ i \int d^dx \left[-\frac{1}{2}\phi(\Box + m^2 - i\epsilon)\phi \right] \right\} \tag{1.114}$$

Problem. Prove (1.114). Comment on the relation between (1.114) and the result

$$i\,D(x-y) = < 0|T(\phi(x)\phi(y))|0 > \tag{1.115}$$

that is obtained for the propagator in the operatorial approach, see e.g. the Bjorken& Drell textbook, Appendix C.

Hint: To prove (1.114) use (1.107) and (1.109). To prove (1.115) use the definition of path integral that takes into account time ordering automatically.

Let us go back to (1.102) and let us assume that the external source is the form

$$J(x) = J(0,\vec{x}) = J_1(\vec{x}) + J_2(\vec{x}) = J\left[\delta(\vec{x}-\vec{x}_1) + \delta(\vec{x}-\vec{x}_2)\right]\ . \tag{1.116}$$

In other terms we have two static sources concentrated in $\vec{x}_1$ and $\vec{x}_2$. How the vacuum energy is disturbed? We expect a change in the hamiltonian $H \to H + E$; the new term E will be a potential energy and

$$e^{iW[J]} = \frac{Z[J]}{Z[0]} = \frac{< 0|e^{-iHT}|0 >}{< 0|0 >} = e^{-iET} \ . \qquad (1.117)$$

Evaluating $W[J]$ we consider only the two terms containing the product $J_1 J_2$ since we wish to compute the effect of having two sources at the same time in two different places. We have therefore

$$\begin{aligned}
-ET &= W_{12}[J] = -\frac{J^2}{2}\int d^4x d^4y \frac{d^4k}{(2\pi)^4}\frac{e^{(ik(x-y)}}{k^2 - m^2 + i\epsilon}\times \\
&\times \ [\delta(\vec{x}-\vec{x}_1)\delta(\vec{y}-\vec{x}_2) + \delta(\vec{x}-\vec{x}_2)\delta(\vec{y}-\vec{x}_1)] = \\
&= -J^2 \int dx_0 \int \frac{dk_0\, e^{ik_0x_0}}{2\pi}\int dy_0\, e^{-ik_0y_0}\int \frac{d\vec{k}}{(2\pi)^3}\frac{e^{i\vec{k}(\vec{x}_1-\vec{x}_2)}}{k^2 - m^2 + i\epsilon} \\
&= +J^2 \int dx_0 \int \frac{d\vec{k}}{(2\pi)^3}\frac{e^{i\vec{k}(\vec{x}_1-\vec{x}_2)}}{\vec{k}^{\,2} + m^2} \qquad (1.118)
\end{aligned}$$

Since $\int dx_0 = T$ we finally get

$$E = -J^2 \int \frac{d\vec{k}}{(2\pi)^3}\frac{e^{i\vec{k}(\vec{x}_1-\vec{x}_2)}}{\vec{k}^{\,2} + m^2} \qquad (1.119)$$

and, performing the last integral (see section 1.9),

$$E = -\frac{J^2}{4\pi r}e^{-mr} \qquad (1.120)$$

where $r = |\vec{r}_1 - \vec{r}_2|$. With m =pion mass ($\sim$ 140 MeV) this is the famous Yukawa formula giving the potential between two nucleons in a nucleus. It is attractive and its typical range is 1 Fermi.

Two static sources coupled to the scalar field in two different points produce an energy potential and therefore a force acting on each source. This static potential corresponds to an instantaneous force. To produce results compatible with special relativity we need to consider time-dependent sources; let us therefore consider again

$$J(x) = J_1(x) + J_2(x) \qquad (1.121)$$

and the effect of the product $J_1 J_2$ on W:

$$W_{12}[J] = -\frac{1}{2}\int d^4x d^4y\, J_1(y)D(x-y)J_2(x) =$$

$$
\begin{aligned}
&= -\frac{1}{2}\int d^4x d^4y \frac{d^4\ell\, e^{i\ell\cdot y} J_1(\ell)}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}\frac{e^{ik(x-y)}}{k^2-m^2+i\epsilon}\frac{d^4q\, e^{iq\cdot x} J_2(q)}{(2\pi)^4} \\
&= -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\frac{J_1(k)J_2(-k)}{k^2-m^2+i\epsilon}\,. \qquad (1.122)
\end{aligned}
$$

We have here introduced $J(q)$, the Fourier transform of the source, related to $J(x)$ by

$$
J(x) = \int \frac{d^4q}{(2\pi)^4}\, e^{iq\cdot x} J(q)\,. \qquad (1.123)
$$

Note that since $J(x)$ is real, $J^*(q) = J(-q)$. Eq. (1.122) shows that the sources can perturb significantly the vacuum only if their Fourier transform are numerically large at $\pm k$ and k^μ is not far away from the *mass shell* condition

$$
k^2 = k_0^2 - \vec{k}^2 = m^2\ . \qquad (1.124)
$$

This is the condition satisfied by a particle of mass m and we can interpret this equation saying that (1.122) represents the creation of a particle of mass m and momentum k at the position 2 by the source J_2 that absorbs a momentum $-k$; then the particle travels to 1 where it is absorbed by the source J_1 to which it releases its momentum, see Fig.1.2

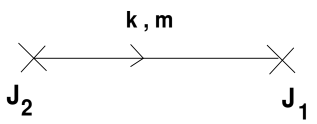


Figure 1.2: The virtual particle of momentum k and mass m going from source J_2 to source J_1.

Note that at the integral (1.122) contribute also off-shell momenta, for this reason we call in general the particle travelling between sources J_2 and J_1 a virtual particle, leaving the adjective real to particles satisfying (1.124). The existence of virtual particles, such that $E \neq \sqrt{\vec{k}^{\,2}+m^2}$ is possible only because the energy uncertainty principle allows violations of the energy conservation, as discussed in Section 1.1.

We can now reconcile eq. (1.120) with relativity. The instantaneous potential is only an approximation obtained when all velocities are much smaller than c. The relativistic picture is as follows: The source interacts with the field ϕ at point 2, then a particle is created that travels until is absorbed by the other source. Therefore the interaction is always mediated by particles that act as messengers.

Can we extract further information from the generating functional in the free field case? To give an answer let us perform an expansion in J of Z. We get, using (1.107):

$$Z[J] = Z[0] \sum_{n=0}^{\infty} \frac{(iW[J])^n}{n!} \ . \tag{1.125}$$

W can be interpreted as the process of creation of a virtual particle by a source, its propagation and its eventual absorption by another source. Each term of the series contains products of W's without nothing connecting them; for example, for $n = 2$,

$$\frac{1}{2}\left(\frac{-i}{2}\right)^2 < J_1 D_{12} J_2 >< J_3 D_{34} J_4 > \tag{1.126}$$

each term is represented pictorially by a line with ends in two sources and there is nothing connecting the pieces, see Fig. 1.3. Diagrams of this sort are called disconnected. Notice that in the free field theory all diagrams are disconnected.

Fig. 1.3 might represent the creation of two particles by two sources 1,2 and their eventual absorption by some detector in 3 and 4. During their lives these particles do not interact because we are assuming free particles. Therefore, to describe collisions we must include interacting terms in the lagrangian.

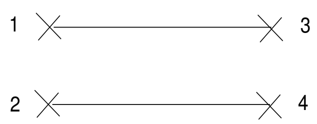


Figure 1.3: Disconnected diagrams.

In the following tables we summarize the main results of this section.

Notations

$$< AB >= \int d^4x\, A(x)B(x)$$

$$< A_i O_{ij} B_j >_{ij} = \int d^4x_i d^4x_j\, A(x_i)O(x_i, x_j)B(x_j)$$

Free scalar fields

Generating functional

$$Z[J] =< 0|e^{-iHT}|0 >= \int [D\phi]\, e^{i< -\frac{1}{2}\phi(\Box+m^2-i\epsilon)\phi+J\phi>}$$

$$Z[J] \;=\; Z[0]e^{iW[J]}\,; \qquad W[J] \;=\; -\tfrac{1}{2} < J_1 D_{12} J_2 >$$

Free scalar field propagator:

$$D(x-y) = \int \frac{d^4k}{(2\pi)^d} \frac{e^{ik(x-y)}}{k^2 - m^2 + i\epsilon}$$

1.3.2 Perturbative expansion for $\lambda\phi^4$: Green's functions

Including the self coupling we have to evaluate the generating functional

$$Z[J] = Z[0]e^{iW[J]} = \int [D\phi]\, e^{i< -\frac{1}{2}\phi(\Box+m^2-i\epsilon)\phi - \frac{\lambda}{4!}\phi^4 + J\phi>} \;. \qquad (1.127)$$

It is not an easy task to extract the relevant information from (1.127). The best we can do is to treat it perturbatively by expanding in J and λ. Let us first expand in J.

Since $Z[J]$ has a series expansion of the type (1.125), it clearly contains both disconnected and connected amplitudes; however now $W[J]$ is not simply given by the free field expression (1.122) and can be computed only as series expansion in λ. Before doing that, let us observe that the Taylor expansion of (1.127) gives

$$Z[J] = Z[0,0] \sum_{n=0}^{\infty} \frac{i^n}{n!} < G(1,2,...n)\, J_1 J_2 ... J_n >_{12...n} \ , \qquad (1.128)$$

where the function $G(1,2,...n) = G(x_1, x_2, ...x_n)$ is the $n-$point Green's function

$$G(x_1, x_2, ...x_n) = \frac{1}{i^n\, Z[0,0]} \frac{\delta^n Z}{\delta J(x_1) \delta J(x_2) ... \delta J(x_n)}\Big|_{J=0} \qquad (1.129)$$

and is given by

$$G(x_1, x_2, ...x_n) = \frac{1}{Z[0,0]} \int [D\phi] \phi(x_1)\phi(x_2)...\phi(x_n)\, e^{i<[\frac{1}{2}\phi D^{-1}\phi - \frac{\lambda}{4!}\phi^4>} \qquad (1.130)$$

with

$$D^{-1} = -(\partial^2 + m^2 - i\epsilon) \ . \qquad (1.131)$$

In these formulae we have extracted a factor $Z[0,0] = Z[J=0, \lambda=0]$. Examples. For $n=2$:

$$G(x_1, x_2) = \frac{1}{Z[0,0]} \int [D\phi] \phi(x_1)\phi(x_2)\, e^{i<[\frac{1}{2}\phi D^{-1}\phi - \frac{\lambda}{4!}\phi^4>} \qquad (1.132)$$

For $\lambda = 0$ the integration is easily performed using (1.205) with two variables. Clearly we have:

$$G^{(0)}(x_1, x_2) = iD(x_1 - x_2) \ , \qquad (1.133)$$

where the superscript (0) is a reminder that this result refers to the free case. Therefore, G_{12} and $G^{(0)}_{12}$ describe the propagation of a scalar particle between 1 and 2 respectively in presence and in absence of interaction.

For $n=4$:

$$\begin{aligned} G(x_1, x_2, x_3, x_4) &= \frac{1}{Z[0,0]} \int [D\phi] \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \times \\ &\times \ e^{i<\frac{1}{2}\phi D^{-1}\phi - \frac{\lambda}{4!}\phi^4>} \ . \end{aligned} \qquad (1.134)$$

Similarly to (1.133) also (1.134) depends only on the differences $x_k - x_j$ because of translation invariance. Let us compute the $4-$point Green's function in the free field case. Using again the Wick's theorem (1.205) with four variables

$$G^{(0)}(x_1, x_2, x_3, x_4) = G^{(0)}(x_1, x_2) G^{(0)}(x_3, x_4)$$

$$+G^{(0)}(x_1,x_3)G^{(0)}(x_2,x_4)+G^{(0)}(x_1,x_4)G^{(0)}(x_2,x_3)\ . \quad (1.135)$$

Each of these terms correspond to a diagram like that in fig.1.3 (without the factors corresponding to the sources. $G^{(0)}(1234)$ only contains disconnected pieces; considering $G(1234)$, that includes interactions, also connected contributions appear. Let us see how, by considering the first correction in the coupling constant λ:

$$G(1234)=\frac{1}{Z[0,0]}\int[D\phi]\phi_1\phi_2\phi_3\phi_4\, e^{i<\frac{1}{2}\phi D^{-1}\phi>}\left(1-\frac{i\lambda}{4!}<\phi_x^4>_x\right)$$
$$=G^{(0)}(1234)-\frac{i\lambda}{4!}\int\frac{[D\phi]}{Z[0,0]}\,e^{i<\frac{1}{2}\phi D^{-1}\phi>}<\phi_1\phi_2\phi_3\phi_4\phi_x^4>_x \quad (1.136)$$

Neglecting $G^{(0)}(1234)$ that we know already, let us apply the Wick's theorem to the last term:

$$-\frac{i\lambda}{4!}\int\frac{[D\phi]}{Z[0,0]}\,e^{i<\frac{1}{2}\phi D^{-1}\phi>}<\phi_1\phi_2\phi_3\phi_4\phi_x^4>_x=$$
$$=-\frac{i\lambda}{4!}\Big\{4!<G^{(0)}(1x)G^{(0)}(2x)G^{(0)}(3x)G^{(0)}(4x)>_x$$
$$+\frac{4!}{2}\Big(<G^{(0)}(1x)G^{(0)}(2x)>_x G^{(0)}(34)+$$
$$+<G^{(0)}(1x)G^{(0)}(3x)>_x G^{(0)}(24)+$$
$$+<G^{(0)}(1x)G^{(0)}(4x)>_x G^{(0)}(23)+$$
$$+<G^{(0)}(2x)G^{(0)}(3x)>_x G^{(0)}(14)+$$
$$+<G^{(0)}(2x)G^{(0)}(4x)>_x G^{(0)}(13)+$$
$$+<G^{(0)}(3x)G^{(0)}(4x)>_x G^{(0)}(12)\Big)<G^{(0)}(yy)>_y+$$
$$+3\Big(G^{(0)}(12)G^{(0)}(34)+G^{(0)}(13)G^{(0)}(24)+G^{(0)}(14)G^{(0)}(23)+\Big)\times$$
$$\times<G^{(0)}(xx)>_x<G^{(0)}(yy)>_y\ . \quad (1.137)$$

We see that the various terms can be organized as follows. There are terms that connect the four points where the sources are to a point x, see fig 1.4 (a); then we have terms where one particle propagates and the other also propagate but interact with itself, see fig 1.4 (b); finally there is the possibility that the particles propagate freely but particles are emitted and reabsorbed without interacting with the sources, see fig 1.4 (c). We see that the picture graphically represent a scattering process $1+2\to 3+4$ that, differently from the one depicted in fig.1.3 is not trivial. Clearly one can go ahead with the perturbative expansion; a typical $\mathcal{O}(\lambda^2)$ contribution is depicted in fig. 1.5.

It arises from

$$\frac{1}{2}\left(-\frac{i\lambda}{4!}\right)^2\int\frac{[D\phi]}{Z[0,0]}\,e^{i<\frac{1}{2}\phi D^{-1}\phi>}<\phi_1\phi_2\phi_3\phi_4\phi_x^4\phi_y^4>_{xy} \quad (1.138)$$

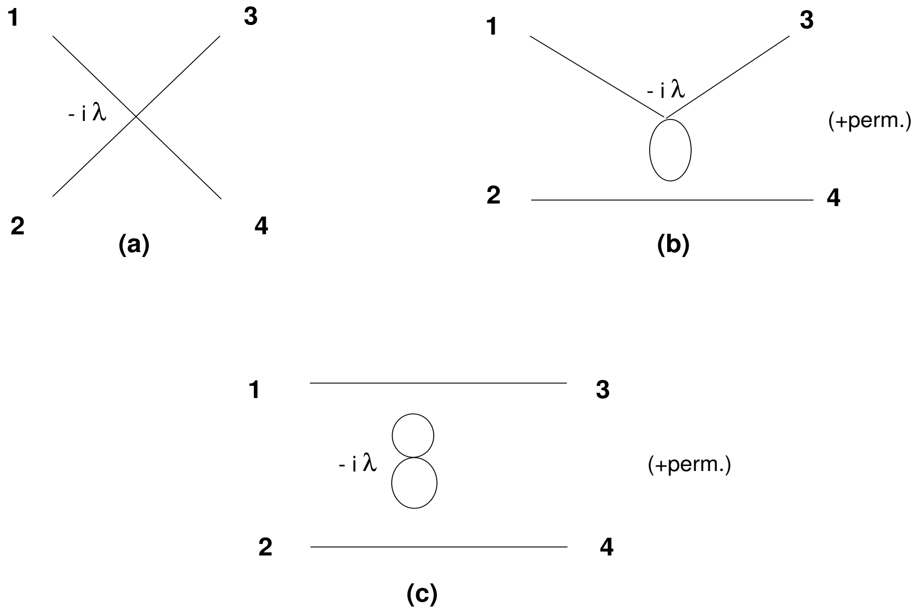


Figure 1.4: $\mathcal{O}$ (λ) perturbative expansion of $G(1234)$.

and is given by

$$
\begin{aligned}
&\frac{1}{2}\left(-\frac{i\lambda}{4!}\right)^2 2\cdot 2(4\cdot 3)^2\Big\{ \qquad\qquad (1.139)\\
&< G^{(0)}(1x)G^{(0)}(2x)G^{(0)}(3y)G^{(0)}(4y)G^{(0)}(xy)G^{(0)}(xy) >_{xy} +\\
&< G^{(0)}(1x)G^{(0)}(3x)G^{(0)}(2y)G^{(0)}(4y)G^{(0)}(xy)G^{(0)}(xy) >_{xy} +\\
&< G^{(0)}(1x)G^{(0)}(4x)G^{(0)}(3y)G^{(0)}(2y)G^{(0)}(xy)G^{(0)}(xy) >_{xy} \Big\}.
\end{aligned}
$$

Let us concentrate on the first of the three terms on the r.h.s. of this equation; this contribution is depicted in fig. 1.5, while the other contributions come from the so called crossed diagrams that are obtained from the previous one connecting the external legs in different ways. More precisely, the second term is obtained by connecting x_1 and x_3 in x and x_2 and x_4 in y; the third term is obtained connecting x_1 and x_4 in x and x_2 and x_3 in y. The numerical factor $(4 \times 3)^2 \times 2 \times 2$ is obtained as follows. The field ϕ_1 can be linked to ϕ_x in four ways, ϕ_2 to ϕ_x in three ways. This gives a factor 4×3 and a similar factor comes from linking ϕ_3 and ϕ_4 to ϕ_y. One of the remaining two ϕ_x can be linked to the two remaining ϕ_y in two ways, which gives another factor

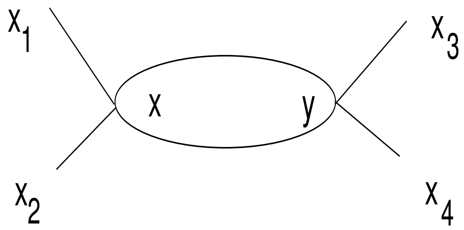


Figure 1.5: One of the contributions to the $\mathcal{O}\,(\lambda^2)$ connected part of $G(1234)$; it corresponds to the first of the three terms on the r.h.s. of eq. (1.139).

of 2, while the remaining ϕ_x can be only linked to the remaining ϕ_y. There is an overall factor of 2 obtained exchanging x and y.

Besides this connected contribution there are disconnected terms as well. It is clear however that the connected terms are more interesting, because the full amplitude can be written in terms of them only.

The generating functional $Z[J,\lambda]$ generates both connected and disconnected contributions. However if we write, analogously to (1.107) and (1.125),

$$Z[J.\lambda] = Z[0,\lambda]e^{iW[J]} \;=\; Z[0,\lambda]\sum_{n=0}^{\infty}\frac{(iW[J])^n}{n!} \tag{1.140}$$

we can identify the generating functional of the connected parts. In fact $Z[0,\lambda]$ generates diagrams without external lines ($J=0$), for example diagrams like Fig. 1.6.

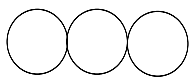


Figure 1.6: A diagram without external legs.

A term like $(W[J])^2$ is the product of two numbers and contains therefore two disconnected parts; $(W[J])^n$ contains n disconnected contributions and so on. It is therefore clear that the generating functional of connected graphs is $W[J]$.

Let us now evaluate in momentum space the amplitude depicted in Fig.1.5 and written down explicitly in (1.139). Using Fourier transform of the free propagators one gets:

$$
\begin{aligned}
& G(k_1, k_2, k_3, k_4) \\
= \quad & (2\pi)^4 \delta(k_1 + k_2 - k_3 - k_4) \prod_{j=1}^{4} \left(\frac{i}{k_j^2 - m^2 + i\epsilon} \right) \times \\
& \times \frac{(-i\lambda)^2}{2} \int \frac{d^4 q}{(2\pi)^4} \frac{i}{q^2 - m^2 + i\epsilon} \frac{i}{(k_1 + k_2 - q)^2 - m^2 + i\epsilon}
\end{aligned}
$$

The proof is left as an exercise. This contribution is depicted in fig. 1.7.

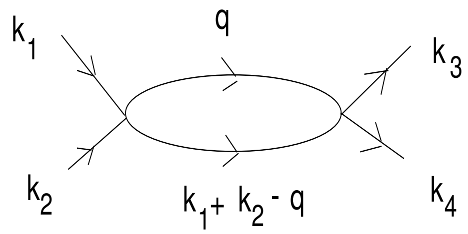


Figure 1.7: The $\mathcal{O}\ (\lambda^2)$ scattering amplitude in momentum space.

The crossed diagrams produce two other contributions; one is obtained by the exchange $k_2 \leftrightarrow -k_3$, the other one by the exchange $k_2 \leftrightarrow -k_4$. This is an example of the crossing symmetry discussed in more detail in section 1.7. It is useful in this context to introduce the Mandelstam variables

$$
\begin{aligned}
s &= (k_1 + k_2)^2 = (k_3 + k_4)^2 \ , \\
t &= (k_1 - k_3)^2 = (k_2 - k_4)^2 \ , \\
u &= (k_1 - k_4)^2 = (k_2 - k_3)^2 \ . \qquad (1.141)
\end{aligned}
$$

They satisfy, in general the relation

$$
s + t + u = \sum_{ext.particles} m_j^2 \qquad (1.142)
$$

and, in our case, $s + t + u = 4m^2$. Crossing symmetry in our example is equivalent to invariance under the exchanges $s \leftrightarrow t$, $s \leftrightarrow$, and $t \leftrightarrow u$.

It would be tedious and time consuming to derive each time expressions like these. Feynman helped the physics community inventing a set of practical rules; for the scalar theory they are written down in the subsequent section 1.3.3.

1.3.3 Feynman rules for $\lambda\phi^4$

Feynman rules allow to get Green' function easily. For $\lambda\phi^4$ and for connected Green's function $G(12...n)$ they are as follows:

1. Draw all the possible connected graphs with n external legs; to $\mathcal{O}(\lambda)^k$ include k vertices;
2. for each vertex a factor $(-i\lambda)(2\pi)^4\delta(\sum q_j - \sum q_n')$ if q, q' are all the momenta ingoing and outgoing;
3. a factor $\int d^4\ell/(2\pi)^4$ for each internal momentum ℓ;
4. for each line with momentum ℓ a Feynman propagator $i/(\ell^2 - m^2 + i\epsilon)$
5. a numerical symmetry factor that has to be evaluated directly from the definition.

It is common practice to omit the propagator on the external lines; in this case the external legs are *amputated.* As an example we consider the scattering amplitude of two pions into two pions: $\pi(k_1) + \pi(k_2) \rightarrow \pi(k_3) + \pi(k_4)$. The scattering amplitude is the $S-$matrix element once we have subtracted the unity matrix (corresponding to no scattering at all: $S - 1 = iT$. We denote the scattering amplitude as $i\mathcal{M}$. To the order $\mathcal{O}(\lambda)^2$ it is given by

$$i\mathcal{M} = -i\lambda + \frac{\lambda^2}{2}\left[A(s) + A(t) + A(u)\right] \tag{1.143}$$

where

$$A(s) = \int \frac{d^4k}{(2\pi)^4}\frac{1}{[k^2 - m^2 + i\epsilon][(k-K)^2 - m^2 + i\epsilon]} \tag{1.144}$$

with $K = k_1 + k_2$ and $s = K^2$. We note that this integral is logarithmically divergent for $k \rightarrow \infty$.

Formulae for cross sections can be found in section 1.8, where also the Feynman rules for QED and a few useful formulae are reported.

1.4 Spin 1 and spin 2

We derive here some results for spin 1 and spin 2 particles. The applications we have in mind are to photons and gravitons. The full-fledged quantization of these fields is complicated by requiring gauge invariance. Following Zee's treatment we will add a mass to the photon letting it go to zero at the end.

1.4.1 Spin 1 particles

Giving the photon a mass and adding a coupling to an external current transforms (1.17) into

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{m^2}{2}A_\mu A^\mu + J_\mu A^\mu \ . \tag{1.145}$$

The action takes the form (after an integration by parts)

$$S[A] = \int d^4x\mathcal{L} = \int d^4x \left(\frac{1}{2}\, A_\mu \left[(\Box + m^2)g^{\mu\nu} - \partial^\mu\partial^\nu\right] A_\nu + J_\mu A^\mu\right) \ , \tag{1.146}$$

and the generating functional

$$\begin{aligned} Z &= \ < 0|e^{-iHT}|0 >= \int [DA_\mu]e^{iS[A]} = \\ &= \int [DA_\mu]e^{i\int d^4x\left(\frac{1}{2}\, A_\mu\left[(\Box+m^2)g^{\mu\nu}-\partial^\mu\partial^\nu\right]A_\nu+J_\mu A^\mu\right)} \ . \end{aligned} \tag{1.147}$$

Notice the differences in comparison with the analogous relation for the spin 0 field, eq. (1.102), that we rewrite for the readers's convenience:

$$Z = \int [D\phi]e^{iS[\phi]} = \int [DA_\mu]e^{i\int d^dx\left[-\frac{1}{2}\phi(\Box+m^2-i\epsilon)\phi+J\phi\right]} \tag{1.148}$$

To get the massive spin-1 field propagator $D_{\sigma\lambda}(x)$ we simply have to compute the inverse of the operator

$$(\Box + m^2)g^{\mu\nu} - \partial^\mu\partial^\nu \tag{1.149}$$

i.e. solve the equation

$$\left[(\Box + m^2)g^{\mu\nu} - \partial^\mu\partial^\nu\right] D_{\nu\sigma}(x) = \delta^\mu_\sigma \delta(x) \tag{1.150}$$

Introducing the Fourier transform as in eq.(1.112)

$$D_{\nu\sigma}(x) = \int \frac{d^4k}{(2\pi)^d} D_{\nu\sigma}(k)e^{ikx} \tag{1.151}$$

we get

$$\left[(-k^2 + m^2)g^{\mu\nu} + k^\mu k^\nu\right] D_{\nu\sigma}(k) = \delta^\mu_\sigma \ . \tag{1.152}$$

Therefore the massive spin-1 field propagator in momentum space is

$$D_{\nu\sigma}(k) = \frac{-g_{\nu\sigma} + k_\nu k_\sigma/m^2}{k^2 - m^2} \ , \tag{1.153}$$

and for the generating functional of the connected Green's functions, instead of (1.122):

$$-\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} J(-k)\frac{1}{k^2-m^2+i\epsilon}J(k)$$

the result

$$W[J] = -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} J^\nu(-k)\frac{-g_{\nu\sigma}+k_\nu k_\sigma/m^2}{k^2-m^2}J^\sigma(k)\ . \qquad (1.154)$$

For the photon $\partial_\mu J^\mu(x) = 0$; therefore $k_\mu J^\mu(k) = 0$ and the term $\propto k_\nu k_\sigma$ vanishes; moreover $m \to 0$ so that one gets the result

$$W[J] = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} J_\mu(-k)\frac{1}{k^2}J^\mu(k)\ . \qquad (1.155)$$

We can now repeat the analysis with two static sources and get, instead of the Yukawa potential

$$E = -\frac{J^2}{4\pi r}e^{-mr}$$

the Coulomb long-range potential

$$E = +\frac{J^2}{4\pi r}\ . \qquad (1.156)$$

Notice the striking fact that equal electric charges repel each other[3], while equal nuclear charges attract. Therefore the character of the force between equal sources (attractive or repulsive) depends on the spin of the exchanged particle (spin 0 or 1). In the next subsection we shall consider gravitational force and we will discover that the attractive character of the gravitational force is related to the graviton spin ($s = 2$). For the time being let us discuss the numerator of (1.153) and introduce

$$G_{\nu\sigma}(k) = \sum_\lambda \epsilon_\nu^{\{\lambda\}*}\epsilon_\sigma^{\{\lambda\}}\ . \qquad (1.157)$$

Here $\epsilon_\mu^{\{\lambda\}}$ are, for $\lambda = 1,2,3$ three 4-vectors defined as follows. In the rest frame of the particle, when $k^\mu = (1,0,0,0)$, $\epsilon^{\{\lambda\}\,\mu} = (0, \vec{e}^{\{\lambda\}})$. The vectors $\vec{e}^{\{\lambda\}}$ are three orthonormal vectors, e.g. $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. They are the three independent spin eigenvectors and

[3]Clearly, if the electric charges are opposite in sign: $+J$ and $-J$, they produce an attractive forces.

appear, for a particle of polarization λ, as a multiplying factor of the wavefunction, exactly as the factor $\vec{e}^{\{\alpha\}}$ in the photon wavefunction. They satisfy

$$\begin{aligned} k^\mu \epsilon_\mu^{\{\lambda\}} &= 0 \\ \epsilon^{\{\lambda\}\mu} \epsilon_\mu^{\{\lambda\}} &= -1 \end{aligned} \tag{1.158}$$

in the rest frame and therefore in any other inertial reference frame.

We now prove that

$$G_{\nu\sigma}(k) = -g_{\nu\sigma} + \frac{k_\nu k_\sigma}{m^2} \ . \tag{1.159}$$

In fact we can generally write

$$G_{\nu\sigma}(k) = A g_{\nu\sigma} + B k_\nu k_\sigma \ . \tag{1.160}$$

Using (1.157) and (1.158), eq. (1.159) follows.

As a consequence we can write the massive spin 1 field propagator as follows:

$$D_{\nu\sigma}(k) = \frac{\sum_\lambda \epsilon_\nu^{\{\lambda\}*} \epsilon_\sigma^{\{\lambda\}}}{k^2 - m^2 + i\epsilon} \ . \tag{1.161}$$

We notice that it is impossible to construct three 4-vectors $\epsilon^{\{\lambda\}\,\mu}$ satisfying (1.158) for a massless particle; for example for $k = (1, 0, 0, 1)$ we can have only $\epsilon^\mu = (0, 1, 0, 0)$ or $(0, 1, 0, 0)$. This is why the photon has only two transverse (i.e. $\vec{k} \cdot \vec{e} = 0$) polarizations.

1.4.2 Spin 2 particles

Let us now compute the analogous of (1.157) for a particle of spin 2 (the graviton). Its polarization wavefunction must have $5 = 2s + 1$ components and therefore in general must be a rank 2 tensor. From a tensor $E_{\mu\nu}$ we can extract three tensors with definite symmetry properties, its antisymmetric part $E_{[\mu\nu]}$, its symmetric traceless part: $E_{\{\mu\nu\}}$ and its trace $g_{\mu\nu} E_\sigma^\sigma / 4$. They are apt to describe respectively a spin 1, a spin 2 and a spin 0 particle of momentum k, but the conditions $k^\mu E_{[\mu\nu]}$ and $k^\mu E_{\{\mu\nu\}}$ must be imposed. The former condition reduces the number of independent components from 6 to 3, the latter from $9 = 10 - 1$ to 5. The 5 graviton spin eigenfunctions will be denoted as $\epsilon_{\mu\nu}^{\{\lambda\}}$ $(\lambda = 1, ..., 5)$ with

$$\epsilon_{\mu\nu}^{\{\lambda\}} = \epsilon_{\nu\mu}^{\{\lambda\}} \, , \qquad \epsilon_\mu^{\{\lambda\}\mu} = 0 \, , \qquad k^\mu \epsilon_{\mu\nu}^{\{\lambda\}} = 0 \ . \tag{1.162}$$

The analogous of (1.157) for a particle of spin 2 is

$$\sum_\lambda \epsilon^{\{\lambda\}*}_{\mu\nu}\epsilon^{\{\lambda\}}_{\rho\sigma} = G_{\mu\rho}G_{\nu\sigma} + G_{\nu\rho}G_{\mu\sigma} - \frac{2}{3}G_{\mu\nu}G_{\rho\sigma} \ . \tag{1.163}$$

The proof is left as an exercise to the reader.

In analogy with (1.161), for the graviton propagator we write

$$D_{\mu\nu,\rho\sigma}(k) = \frac{\sum_\lambda \epsilon^{\{\lambda\}*}_{\mu\nu}\epsilon^{\{\lambda\}}_{\rho\sigma}}{k^2 - m^2 + i\epsilon} = \frac{G_{\mu\rho}G_{\nu\sigma} + G_{\nu\rho}G_{\mu\sigma} - \frac{2}{3}G_{\mu\nu}G_{\rho\sigma}}{k^2 - m^2 + i\epsilon} \tag{1.164}$$

Let us now derive a formula for the generating functional analogous to (1.154). The graviton field $\phi_{\mu\nu}$ (symmetric traceless) must be coupled in the action to a source $T^{\mu\nu}$. Since gravity is coupled to energy the natural candidate is the symmetric energy-momentum tensor $T^{\mu\nu}$, whose T^{00} component is the energy density. Therefore instead of eq. (1.165):

$$-\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} J^*(k)\frac{1}{k^2 - m^2 + i\epsilon}J(k)$$

one obtains the result

$$W[T] = -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} T^{\mu\nu *}(k)\frac{G_{\mu\rho}G_{\nu\sigma} + G_{\nu\rho}G_{\mu\sigma} - \frac{2}{3}G_{\mu\nu}G_{\rho\sigma}}{k^2 - m^2}T^{\sigma\rho}(k) \ . \tag{1.165}$$

Let us observe that because of the energy-momentum conservation, $k_\mu T^{\mu\nu} = 0$ and $G_{\mu\nu}$ reduces to $g_{\mu\nu}$ in (1.165). If we now want to derive the gravitational force between two static sources we must consider T^{00} and obtain

$$W[T] = -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} T^{00*}(k)\frac{1+1-\frac{2}{3}}{k^2 - m^2}T^{00}(k) \ , \tag{1.166}$$

i.e. an attractive force since $1 + 1 - \frac{2}{3} > 0$.

In a more rigorous treatment, that takes into account the massless nature of the graviton, the factor changes to 1, but the sign does not change and with it the attractive nature of the gravitational force.

1.5 Appendix I: Functional derivative

For a function of N variables $S(q) \equiv S(q_1, ...q_N)$, Taylor expansion around $q_c = (q_{c\,1}, q_{c\,2}, ..., q_{c\,N})$ is as follows

$$S(q) = S(q_c + \delta q) \approx S(q_c) + \sum_{k=1}^N \frac{\partial S(q_c)}{\partial q_k}\delta q_k + \frac{1}{2}\sum_{k,j=1}^N \frac{\partial^2 S(q_c)}{\partial q_k \partial q_j}\delta q_k \delta q_j + ... \tag{1.167}$$

An analogous formula for a functional $S[q]$ might be obtained by approximating the graph of the function $q = q(t)$, $t \in (0,T)$ by $N+1$ points q_0, q_1, q_2, $...q_N$, with $q_j = q(jT/N)$ for $j = 0, ...N$. This leads to the notion of functional derivative.

Suppose that the functional $S[q]$ is defined by the formula

$$S[q] = \int_0^T dt\, f(q(t)) \ , \tag{1.168}$$

then, for $q(t) = q_c(t) + \delta q(t)$ one has

$$\begin{aligned} S[q_c + \delta q] &= \int_0^T dt\, f(q_c(t) + \delta q(t)) \approx S[q_c] + \int_0^T dt\, \frac{\partial f}{\partial q}\Big|_{q=q_c} \delta q(t) \\ &+ \frac{1}{2}\int_0^T dt\, \frac{\partial^2 f}{\partial q^2}\Big|_{q=q_c} (\delta q(t))^2, +... \end{aligned} \tag{1.169}$$

$S[q]$ might have a a more complicated functional dependence, therefore we write in general:

$$\begin{aligned} S[q_c + \delta q] \approx S[q_c] + \int_0^T dt\, \frac{\delta S}{\delta q(t)}\Big|_{q=q_c} \delta q(t) + \\ + \frac{1}{2}\int_0^T dt\, \frac{\delta^2 S}{(\delta q(t))^2}\Big|_{q=q_c} (\delta q(t))^2 \ + ... \end{aligned} \tag{1.170}$$

$\dfrac{\delta S}{\delta q(t)}$, $\dfrac{\delta^2 S}{(\delta q(t))^2}$ are known as functional derivatives of the functional $S[q]$.

An equivalent definition of the functional derivative is as follows. for a function of n variables $S(q_j) = S(q_1, q_2, ..., q_n)$ we define

$$\frac{\partial S(q)}{\partial q_k} = \lim_{\epsilon \to 0} \frac{S(q_j + \epsilon \delta_{kj}) - S(q_j)}{\epsilon} \ . \tag{1.171}$$

We generalize to functionals as follows:

$$\frac{\delta S[q]}{\delta q(\tau)} = \lim_{\epsilon \to 0} \frac{S[q + \epsilon \delta_\tau] - S[q]}{\epsilon} \tag{1.172}$$

where

$$\forall t \in (0,T) : \delta_\tau(t) = \delta(t - \tau) \ . \tag{1.173}$$

It is easily seen that this property is equivalent to the definition given above.

Let us consider a few examples.

- $S[q]$ is a function, e.g $S[q] = q(t)$. It can be always written as follows

$$S[q] = \int d\tau \delta(t-\tau) q(\tau) = q(t) \tag{1.174}$$

In this case

$$\delta S = S[q+\delta q] - S[q] = \delta q(t) = \int dt \delta(t-\tau)\delta q(\tau) \tag{1.175}$$

and therefore

$$\frac{\delta q(t)}{\delta q(\tau)} = \delta(t-\tau) \tag{1.176}$$

- $S[q]$ is the action

$$S[q] = \int_0^T dt\, L[q(t), \dot{q}(t)] \; . \tag{1.177}$$

Then if $\delta q(T) = \delta q(T) = 0$, one has

$$\delta S[q] = S[q] - S[q_c] = \int_0^T dt\, \frac{\delta S}{\delta q(t)}\Big|_{q_c} \delta q(t) \; + \; \mathcal{O}(\delta q)^2 \; , \tag{1.178}$$

where

$$\frac{\delta S}{\delta q(t)} = \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \tag{1.179}$$

Clearly, if $\delta S = 0$, as stated by the action principle, then the Euler-Lagrange equations follow:

$$\frac{\delta S}{\delta q(t)} = 0 \; . \tag{1.180}$$

1.6 Appendix II: Steepest descent method

Let us consider the integral

$$I = \int_{-\infty}^{+\infty} dq\, g(q)\, e^{-\frac{1}{\hbar} f(q)} \, . \tag{1.181}$$

In the $\hbar \to 0$ limit I is dominated by minima of $f(q)$. Assuming one minimum at $q = a$, we write $f(q) \approx f(a) + \frac{1}{2} f^{(2)}(a)(q-a)^2$ with $f^{(2)}(a) > 0$. Therefore one has

$$I \approx g(a)\, e^{-\frac{1}{\hbar} f(a)} \int_{-\infty}^{+\infty} dq\; e^{-\frac{1}{2\hbar} f^{(2)}(a)(q-a)^2} = g(a)\, e^{-\frac{1}{\hbar} f(a)} \sqrt{\frac{2\pi\hbar}{f^{(2)}(a)}} \, . \tag{1.182}$$

For n variables using the result (1.202) and the development

$$f(q) \approx f(a) + \frac{1}{2} f^{(2)}_{i,j}(a)(q-a)_i(q-a)_j \tag{1.183}$$

with the matrix $f^{(2)}_{i,j}$ given by $f^{(2)}_{i,j}(a) = \partial^2 f/\partial q_i \partial q_j|_{q=a}$, one gets

$$I = \int_{-\infty}^{+\infty} d^n q\, g(q)\, e^{-\frac{1}{\hbar}f(q)} = g(a)\, e^{-\frac{1}{\hbar}f(a)} \sqrt{\frac{(2\pi\hbar)^n}{Det||f^{(2)}||}} \,. \tag{1.184}$$

Now we substitute the n-dimensional integral $\int dq$ with a functional integral $\int [Dq]$ and the function $f(q)$ with the functional $S[q]$:

$$I = \int [Dq] e^{-\frac{1}{\hbar}S[q]} \tag{1.185}$$

In the limit $\hbar \to 0$ the path integral is dominated by the path $q_c(t)$ corresponding to a minimum of $S[q]$. Let us expand I around the minimum q_c, which is such that:

$$\delta S \approx \int dt \frac{\delta S}{\delta q(t)} \delta q(t) = 0 \tag{1.186}$$

The variations $\delta q(t)$ vanish at the extrema of integration in t because all the paths in the functional integration start and end at the same points. Therefore, since $\delta q(t)$ is arbitrary, one recovers the Lagrange equations

$$\frac{\delta S}{\delta q(t)} = \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0 \,. \tag{1.187}$$

1.7 Appendix III: Crossing symmetry

Consider the scattering amplitude

$$\mathcal{M} = \mathcal{M}(A(p_a) + B(p_b) \to C(p_c) + D(p_d)...) \tag{1.188}$$

then $\mathcal{M}_{cross}$ is obtained by $\mathcal{M}$ by changing one (or more) particle into its antiparticle, changing the sign to its momentum and changing its place between initial and final state. For example

$$\mathcal{M}_{cross} = \mathcal{M}(A(p_a) + \bar{C}(-p_c) \to \bar{B}(-p_b) + D(p_d)...) \tag{1.189}$$

Crossing symmetry requires

$$\mathcal{M}_{cross} = \mathcal{M} \,. \tag{1.190}$$

Feynman rules automatically guarantee crossing symmetry.

1.8 Appendix IV: Feynman rules for QED and cross sections

QED Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(i\gamma_\mu(\partial^\mu + ieA^\mu) - m)\psi \tag{1.191}$$

Feynman rules:

$$\begin{array}{ll} \text{1. Vertex :} & -ie\gamma^\mu \\ \text{2. Dirac propagator :} & iS(p) = i\,\dfrac{\gamma\cdot p + m}{p^2 - m^2 + i\epsilon} \\ \text{3. Photon propagator} & iD_{\mu\nu}(q) = \dfrac{-ig_{\mu\nu}}{q^2 + i\epsilon} \\ \text{4. For any unconstrained internal momentum } k: & \displaystyle\int \frac{d^4k}{(2\pi)^4} \\ \text{5. Fermion loop :} & \text{factor } -1 \end{array} \tag{1.192}$$

6. A factor -1 between graphs differing by an exchange of external identical fermion lines
7. External initial fermions: a factor $u(p,s)$
8. External final fermions: a factor $\bar{u}(p,s))$
9. External final antifermions: a factor $v(p,s)$
10. External initial antifermions: a factor $\bar{v}(p,s)$.

Massive gauge boson propagator:

$$D_{\mu\nu}(q, m_W) = \frac{-g_{\mu\nu} + q_\mu q_\nu/m_W^2}{q^2 - m_W^2 + i\epsilon} \ . \tag{1.193}$$

Formulae for cross sections and widths

Cross section

For the process $A(q_a) + B(q_b) \to \sum_k C_k(p_k)$

$$d\sigma = \frac{1}{|\vec{v}_a - \vec{v}_b| 2E_a 2E_b}\left(\prod_k \frac{d^2p_k}{2E_k(2\pi)^3}\right)\overline{\sum}\left|\mathcal{M}\right|^2 \times (2\pi)^4\delta\left(q_a + q_b - \sum_k p_k\right) S \tag{1.194}$$

where $\overline{\sum}$ means sum over final spins and average over the initial spins and S is a statistical factor:

$$S = \prod_k \frac{1}{n_k!} \tag{1.195}$$

if there are n_k identical particles in the final state.
Decay width ($\Gamma = 1/\tau$)
For the decay process $A(q) \to \sum_k C_k(p_k)$

$$d\Gamma = \frac{1}{2m_A}\left(\prod_k \frac{d^2p_k}{(2\pi)^3}\right)\overline{\sum}\left|\mathcal{M}\right|^2 (2\pi)^4\delta\left(q-\sum_k p_k\right) S \ . \quad (1.196)$$

1.9 Problems

1. *Gaussian integrals.* If $\int dx f(x) \equiv \int_{-\infty}^{+\infty} dx f(x)$, prove the formulae

$$\int dx e^{-ax^2/2} = \sqrt{\frac{2\pi}{a}} \quad (1.197)$$

$$\int dx e^{-ax^2/2+Jx} = \sqrt{\frac{2\pi}{a}}\exp\frac{J^2}{2a} \quad (1.198)$$

$$\int dx e^{iax^2/2+iJx} = \sqrt{\frac{2\pi i}{a}}\exp\frac{-iJ^2}{2a} \quad (1.199)$$

Moreover,

$$< f(x) > \equiv \frac{\int dx f(x) e^{-ax^2/2}}{\int dx e^{-ax^2/2}} \ , \quad (1.200)$$

prove by induction that

$$< x^{2n} > \equiv \frac{\int dx x^{2n} e^{-ax^2/2}}{\int dx e^{-ax^2/2}} = \frac{(2n-1)!!}{a^n} \ . \quad (1.201)$$

If A is a symmetric matrix of rank n with $detA \neq 0$ and x, J are matrices $(n \times 1)$ prove that

$$\int d^n x e^{-x^T Ax/2+J^T x} = \frac{(2\pi)^{n/2}}{\sqrt{detA}} e^{J^T A^{-1} J/2} \quad (1.202)$$

$$\int d^n x e^{i(x^T Ax/2+J^T x)} = \sqrt{\frac{(2\pi i)^{n/2}}{detA}} e^{-iJ^T A^{-1} J/2} \quad (1.203)$$

2. If $x = (x_1, ...x_n)$ and $< f(x) >$ is defined as

$$< f(x) > \equiv \frac{\int d^n x f(x) e^{-x^T Ax/2}}{\int d^n x e^{-x^T Ax/2}} \ , \quad (1.204)$$

prove that

$$< x_i x_j ... x_k x_\ell >= \sum A^{-1}_{ab} ... A^{-1}_{cd} \qquad (1.205)$$

where the sum is over all the permutations $(a, b..., c, d)$ of $(i, j...k, \ell)$. This property is known as Wick's theorem in Quantum Field Theory. *Hint:* Derive repeatedly (1.202) with respect to J putting at the end $J = 0$.

3. Consider

$$\mathcal{L}(x,\, v,\, t) \;=\; \frac{mv^2}{2} \,+\, b(t)\, x\, v \,-\, \frac{1}{2}\, c(t)\, x^2 \,-\, e(t)\, x\ ,$$

with $v = \dfrac{dx}{dt}$. Prove that the classical equations of motion are

$$m\frac{d^2x}{d\tau^2} \,+\, [c(\tau) \,+\, \frac{db(\tau)}{d\tau}]\, x(\tau) \,+\, e(\tau) \;=\; 0\ .$$

Let $\bar{x}(\tau)$ be a solution of this equation with appropriate boundary conditions. Prove that

$$\begin{aligned} \delta S \;&=\; S[\bar{x} \,+\, \delta x] \,-\, S[\bar{x}] \;=\; \frac{1}{2}\int d\tau \frac{\delta^2 S}{\delta x(\tau)^2}\Big|_{x=\bar{x}} (\delta x(\tau))^2 = \\ &=\; \int_0^t d\tau \tilde{\mathcal{L}}(\delta x,\, \frac{d\delta x}{d\tau},\, \tau)\ , \end{aligned}$$

with

$$\tilde{\mathcal{L}}\left(y,\, \frac{dy}{d\tau},\, \tau\right) = \frac{m}{2}\left(\frac{dy}{d\tau}\right)^2 + b(\tau)\, y\, \frac{dy}{d\tau} - \frac{1}{2} c(\tau) y^2.$$

4. Prove that

$$\int \frac{d\vec{k}}{(2\pi)^3} \frac{e^{i\vec{k}(\vec{x}_1-\vec{x}_2)}}{\vec{k}^{\,2}+m^2} = +\frac{e^{-m|\vec{x}_1-\vec{x}_2|}}{4\pi|\vec{x}_1-\vec{x}_2|}\ . \qquad (1.206)$$

5. Prove that in QED in the massless limit, for $e^+e^- \to \mu^+\mu^-$

$$\frac{1}{4}\overline{\sum_{spin}}\Big|\mathcal{M}\Big|^2 = \frac{2e^4(t^2+u^2)}{s^2} \qquad (1.207)$$

and for $e^-\mu^- \to e^-\mu^-$

$$\frac{1}{4}\overline{\sum_{spin}}\Big|\mathcal{M}\Big|^2 = \frac{2e^4(s^2+u^2)}{t^2}\ . \qquad (1.208)$$

This gives an example of crossing symmetry.

6. Prove the formula

$$[P_\mu, \phi] = -i\partial_\mu \phi \tag{1.209}$$

that actually holds not only for scalar fields, but in general for any field or function of fields. *Hint:* As P^μ is the generator of space-time translations, $\phi(x + a) = e^{iP\cdot a}\phi(x)e^{-iP\cdot a}$. Use this property to prove (1.209).

References

For the role of energy uncertainty principle and section 1.1 see Vol. IV of the series of Theoretical Physics textbooks by L.Landau and E. Lifchitz. For canonical quantization one may consult any textbook on Quantum Field Theory, e.g. J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields*, McGraw-Hill, New York, 1965. For the rest of chapter see A. Zee, *Quantum Field Theory in a nutshell*, Princeton University Press, Princeton, 2003.

Chapter 2

Symmetries in Quantum Field Theory

2.1 Global and local symmetries

2.1.1 Nöther theorem

The relevance of symmetries in field theory is embodied in the Nöther theorem: *to each continuous symmetry of the lagrangian corresponds a conserved current.*
Proof. Consider a complex scalar field theory with lagrangian

$$\mathcal{L} = \mathcal{L}(\phi)\,, \qquad \phi = \{\phi\}_i \tag{2.1}$$

with $i = 1, ..., N$. Under

$$\phi \to \phi + \delta\phi \tag{2.2}$$

the lagrangian is invariant:

$$\mathcal{L}(\phi + \delta\phi) = \mathcal{L}(\phi) \tag{2.3}$$

Now

$$\begin{aligned} 0 &= \delta\mathcal{L} = \mathcal{L}(\phi+\delta\phi) - \mathcal{L}(\phi) = \frac{\partial\mathcal{L}}{\partial\phi_i}\delta\phi_i + \frac{\partial\mathcal{L}}{\partial_\mu\partial\phi_i}\delta\partial_\mu\phi_i = \\ &= \delta\phi_i\,\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)} + \frac{\partial\mathcal{L}}{\partial_\mu\partial\phi_i}\partial_\mu\delta\phi_i = \\ &= \partial_\mu J^\mu \end{aligned} \tag{2.4}$$

with

$$J^\mu = \delta\phi_i\frac{\partial\mathcal{L}}{\partial_\mu\partial\phi_i} = \delta\phi\frac{\partial\mathcal{L}}{\partial_\mu\partial\phi} \tag{2.5}$$

where in the last step we use a lighter notation. The theorem is proved. Even though we have considered the scalar field case, its validity is general.

Let us suppose that $\mathcal{L}$ is invariant[1] under

$$\phi \to \phi' = U\phi \tag{2.6}$$

with U is a matrix of rank n belonging to a unitary group, i.e. either $SU(n)$, a group of matrices $n \times n$ whose definition is

$$U \in SU(n) \iff UU^\dagger = 1 \quad \det U = 1 \ , \tag{2.7}$$

or $U(n)$ defined as

$$U \in U(n) \iff UU^\dagger = 1 \ . \tag{2.8}$$

In general we can write

$$U = e^{i\alpha_a T_a} \tag{2.9}$$

where, for $SU(n)$ (resp. $U(n)$), T_a are $n^2 - 1$ (resp. n^2) hermitean matrices of rank n. For $SU(n)$ T_a are traceless.

Problem. Prove this property.

If (2.6) is infinitesimal, then

$$\phi' \approx \phi + \delta\phi \ , \qquad \delta\phi = i\alpha_a T_a \phi \tag{2.10}$$

and there are as many conserved currents as there are generators in the group:

$$J_a^\mu = i\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} T_a \phi \ . \tag{2.11}$$

Since $\partial_\mu J_a^\mu = 0$, by integrating over all space we get

$$\frac{\partial}{\partial t} \int d^3x J_a^0 = - \int d\vec{\sigma} \cdot \vec{J}_a = 0 \tag{2.12}$$

where the surface integral is at the spatial infinity and vanishes since the fields and the currents vanish for $|\vec{x}| \to \infty$. Therefore, associated to the symmetry we have conserved charges

$$Q_a = \int d^3x J_a^0(\vec{x}, t) \tag{2.13}$$

$$\dot{Q}_a = 0 \qquad (a = 1, ..., n) \ . \tag{2.14}$$

[1]Here we consider only *internal* symmetries, leaving aside space time translations and rotations that give rise to 4-momentum and angular momentum conservation laws.

Let us go back to (2.9). If α_a are independent of space-time the symmetry is called a *global* gauge symmetry, otherwise it is a *local* gauge symmetry. We shall discuss examples of global symmetries in chapter 3. For the time being we shall discuss the role of local symmetries.

2.1.2 Local symmetries and the Yang-Mills construction

If the internal symmetry is a local one , then

$$\delta\phi = i\alpha_a(x)T_a\phi \tag{2.15}$$

then one can easily see that the lagrangian (2.1)is not invariant under local transformations. Since

$$\partial^\mu\phi \to \partial^\mu(\phi + \delta\phi) = \partial^\mu\phi + \delta\partial^\mu\phi \ , \tag{2.16}$$

then

$$\delta\partial^\mu\phi = \partial^\mu\delta\phi = i\alpha_a T_a\partial\phi + i(\partial^\mu\alpha_a)T_a\phi \ . \tag{2.17}$$

The breaking of the invariance arises from the derivative term (any polynomial term is invariant):

$$\begin{aligned}(\partial^\mu\phi)^\dagger(\partial_\mu\phi) &\to (\partial^\mu\phi)^\dagger(\partial_\mu\phi)+ \\ &+ i\partial_\mu\phi^\dagger(\partial^\mu\alpha_a)T_a\phi - i\phi^\dagger(\partial^\mu\alpha_a)T_a\partial_\mu\phi \ ,\end{aligned} \tag{2.18}$$

which shows that the origin of the symmetry breaking is in the derivative of the parameters $\alpha_a(x)$, i.e. in the second term on the r.h.s. of (2.17). To enforce local gauge symmetry one has to substitute the derivative ∂^μ with a *gauge covariant* derivative D^μ such that

$$\delta D^\mu\phi = i\alpha_a T_a D^\mu\phi = i\alpha D^\mu\phi \ , \tag{2.19}$$

i.e. without the offending term. We have put

$$\alpha = \alpha_a T_a \ . \tag{2.20}$$

In this way

$$(D^\mu\phi)^\dagger(D_\mu\phi) \tag{2.21}$$

is invariant[2]. Let us put

$$D^\mu = \partial^\mu + ig\mathbf{A}^\mu \tag{2.22}$$

[2]Note that $\delta(D^\mu\phi)^\dagger = -i\alpha D^\mu\phi^\dagger$.

We use here the notation

$$\mathbf{O} = O_a T_a \tag{2.23}$$

($O_a = A_a^\mu$ in the present case), with the normalization

$$\mathrm{Tr} T_a T_b = 2\delta_{ab} \ . \tag{2.24}$$

The fields A_a^μ are called gauge fields,

For eq. (2.18) to hold the field $\mathbf{A}^\mu$ must trasform in a peculiar way:

$$\mathbf{A}^\mu \to \mathbf{A}^\mu + \delta\mathbf{A}^\mu \tag{2.25}$$

so that

$$\begin{aligned} \delta D_\mu\phi &= D'_\mu\phi' - D_\mu\phi = [\partial_\mu + ig(\mathbf{A}^\mu + \delta\mathbf{A}^\mu)][\phi + i\alpha\phi] - \\ &- \partial_\mu\phi - ig\mathbf{A}^\mu\phi = \\ &= i\alpha\partial_\mu\phi + i(\partial_\mu\alpha)\phi - g\mathbf{A}^\mu\alpha\phi + ig\delta\mathbf{A}^\mu\phi \ . \end{aligned} \tag{2.26}$$

On the other hand

$$\delta D_\mu\phi = i\alpha\partial^\mu\phi - g\alpha\mathbf{A}^\mu\phi \ . \tag{2.27}$$

It follows:

$$ig\delta\mathbf{A}^\mu\phi = g\mathbf{A}^\mu\alpha\phi - g\alpha\mathbf{A}^\mu\phi - i(\partial_\mu\alpha)\phi \ . \tag{2.28}$$

Therefore

$$\delta\mathbf{A}^\mu = i[\alpha, \mathbf{A}^\mu] - \frac{1}{g}(\partial_\mu\alpha) \ , \tag{2.29}$$

which is the desired transformation law for the fields $\mathbf{A}^\mu$. The simplest case is the abelian case, with the $U(1)$ gauge group. Here U are complex numbers: $U = \exp(i\alpha)$, there is only one gauge field A_μ and the commutator in (2.29) is absent. This formalism describes ordinary electromagnetism, with A_μ the photon field. Note that A_μ is a dynamical field and therefore we have to add to the lagrangian the free photon field lagrangian of eq. (1.17) $-1/4F_{\mu\nu}F^{\mu\nu}$, with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

Let us consider the non abelian case (Yang-Mills case). Let us introduce the generalization of $F_{\mu\nu}$:

$$\begin{aligned} \mathbf{F}_{\mu\nu} &= \frac{1}{ig}[D_\mu, D_\nu] = [\partial_\mu + ig\mathbf{A}_\mu, \partial_\nu + ig\mathbf{A}_\nu] = \\ &= [\mathbf{A}_\mu, \partial_\nu] + [\partial_\mu, \mathbf{A}_\nu] \ + \ ig\,[\mathbf{A}_\mu, \mathbf{A}_\nu] = \\ &= \partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu \ + \ ig\,[\mathbf{A}_\mu, \mathbf{A}_\nu] \ . \end{aligned} \tag{2.30}$$

In terms of the components:

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu \ - \ g\, f_{abc} A^b_\mu A^c_\nu \ . \tag{2.31}$$

That $\mathbf{F}_{\mu\nu}$ is a suitable generalization of the abelian field tensor follows from the fact that out of it we can construct a suitable pure gauge lagrangian:

$$\mathcal{L}_{gauge} \;=\; -\frac{1}{2}\mathrm{Tr}\left(\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}\right) \tag{2.32}$$

which is gauge invariant because under the gauge transformation induced by $U = U(x)$:

$$\mathbf{F}_{\mu\nu} \rightarrow U\mathbf{F}_{\mu\nu}U^{\dagger} \; . \tag{2.33}$$

We prove (2.33) for infinitesimal transformation $U \approx 1 + i\alpha$. As a matter of fact one has

$$\delta\mathbf{F}_{\mu\nu} = i[\alpha, \mathbf{F}_{\mu\nu}] \tag{2.34}$$

and therefore

$$\mathbf{F}'_{\mu\nu} - \mathbf{F}_{\mu\nu} \;=\; i\,\alpha\mathbf{F}_{\mu\nu} \;-\; i\,\mathbf{F}_{\mu\nu}\alpha \tag{2.35}$$

which implies

$$\mathbf{F}'_{\mu\nu} \approx (1 + i\alpha)\mathbf{F}_{\mu\nu}(1 - i\alpha) \;\approx\; U\mathbf{F}_{\mu\nu}U^{\dagger} \tag{2.36}$$

Problem. Prove (2.34) using (2.30).

We conclude with two important remarks. First, for the gauge symmetry to hold, gauge fields must be strictly massless because terms like $m^2 A^a_\mu A^a_\mu$ is not invariant.

Problem. Prove that $D^\mu \rightarrow UD^\mu U^\dagger$, from which prove that the gauge fields can appear in the lagrangian only through gauge covariant derivatives.

The second remark is that in abelian gauge theories, due to the term proportional to the coupling constant g in (2.31) the pure gauge field lagrangian (2.32) contains besides the kinetic terms of the fields $F^{\mu\nu}_a$ also self-interactions of 3 and 4 gauge fields. They are absent only for abelian gauge theories.

2.2 Spontaneous symmetry breaking

We speak of spontaneous symmetry breaking (ssb) when

- the hamiltonian (or lagrangian) is invariant under transformations belonging to a given symmetry group (which means that also the Euler-Lagrange equations are symmetric).
- the physical states, in particular the ground state, are not symmetric.

Spontaneous symmetry breaking is a phenomenon different from explicit breaking that occurs when there is a symmetry breaking term in the lagrangian. Symmetry breaking is very common in nature. A well known example is given by ferromagnetism. The hamiltonian is rotationally invariant, but the ground state, that exhibits spontaneous magnetization, breaks space isotropy.

A simple example of ssb in classical mechanics is given by the one one-dimensional motion of a particle with potential energy $(\lambda > 0)$:

$$V(x) \;=\; \frac{\lambda}{4}(x^2 - a^2)^2 \;, \tag{2.37}$$

see fig. 2.1. Obviously the hamiltonian is parity invariant (the symmetry group is Z_2). Let us determine the ground state, i.e. the state of minimal energy. Clearly it would correspond to the particle at rest either in $x = a$ or in $x = -a$; in any case the ground state is not parity invariant. Notice that for ssb to occur, one needs degeneracy of the ground state. In our example there two different ground states i.e. states of motion with the same value of the energy (the minimum). If we consider both everything is symmetric, but since the particle can be only in one position, the physical state is not symmetric.

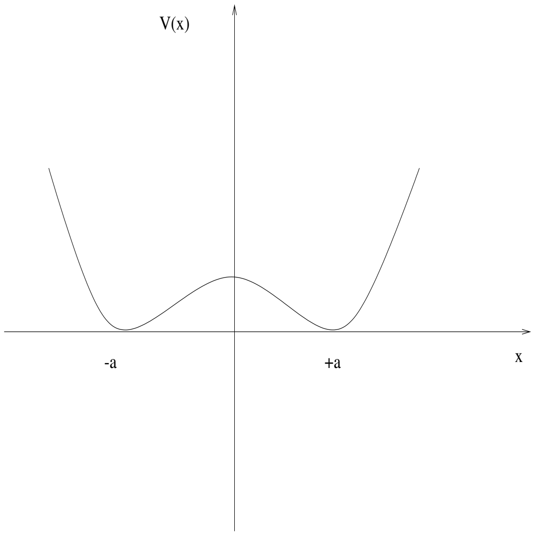


Figure 2.1: Double well potential.

A second example again taken from classical physics is the scalar field theory described by the lagrangian density

$$\mathcal{L}(\phi, \partial\phi) = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \tag{2.38}$$

where ϕ is a real scalar field and $V(\phi)$ has again the double-well shape of fig. 2.1:

$$V(\phi) \;=\; \frac{\lambda}{4}(\phi^2 - a^2)^2 \ , \tag{2.39}$$

The ground state has should have zero kinetic energy

$$E_{kin} = \frac{1}{2}\int dx\, \dot{\phi}^2 \tag{2.40}$$

and zero potential energy

$$E_{pot} = \int dx \left(\frac{1}{2}|\nabla\phi|^2 + V(\phi)\right) \tag{2.41}$$

It is therefore either the constant state

$$\phi = +a \tag{2.42}$$

or the constant state

$$\phi = -a \ . \tag{2.43}$$

In any case the symmetry is spontaneously broken.

Let us now consider the quantum case and to begin with we consider the quantum-mechanical version of the one-dimensional motion of a particle with potential energy (2.37) of fig. 2.1. From general properties of the Schrödinger equation for a potential such as (2.37), that allows only discrete spectrum, we know that all the energy eigenstates are not degenerate. This implies that there is not ssb. Parity invariance of the hamiltonian implies that there exist both even and odd eigenfunctions, but they correspond to different eigenvalues (in particular the ground state correspond to an even eigenfunction).

The reason why in the quantum mechanical problem the ground state is not degenerate is in the tunnelling phenomenon. It allows the particle to go from one minimum of fig. 2.1 to the other. Therefore, loosely speaking, the state of minimal energy is a superposition of two wave-packets, peaked at $x = +a$ and $x = -a$ and there is no spontaneous symmetry breaking. It is useful to stress that no tunnelling would take place were the two minima separated by an infinite energy barrier. This is why when we consider the quantum field theoretical version of the problem, thins are different. As a matter of fact, in quantum field theory with potential (2.39) the tunnelling barrier is not $V(0) - V(\pm a)$, but $\int d^3x[V(0) - V(\pm a)]$, which is infinite. Therefore also in the quantum case for the scalar field theory with potential (2.39) parity is spontaneously broken, exactly as in the classical field theory.

In quantum field theory the value a or $-a$ is the vacuum expectation value of the field. To prove this consider

$$< \phi >_0 = \frac{< 0|\phi|0 >}{< 0|0 >} = \frac{1}{Z} \int [d\phi]\Psi_0^*[\phi] \; \phi \; e^{-S_E[\phi]}\Psi_0[\phi] \tag{2.44}$$

where

$$Z = \int [d\phi] \; \Psi_0^*[\phi] \; e^{-S_E[\phi]} \; \Psi_0[\phi] = \int [D\phi] e^{-\int dx_E \{[\frac{1}{2}(\partial_E \phi)^2 + V(\phi)\}} \; . \tag{2.45}$$

Let us evaluate the functional integrals by the saddle point method. There are two saddle points, corresponding to the minima of $S_E[\phi]$, at $\phi = const. = \pm a$. The ground state functional $\Psi[\phi]$ is not parity invariant and will be peaked e.g. in $\phi = +a$. Then

$$< \phi >_0 = \frac{(+a) \; \int [d\phi] |\Psi_0[a]|^2 \; e^{-S_E[a]}}{\int [d\phi] \; |\Psi_0[a]|^2 e^{-S_E[a]}} = +a \; . \tag{2.46}$$

Therefore the minimum of the classical potential energy $\phi = +a$ corresponds (in the semiclassical limit $\hbar \to 0$) to the vacuum expectation value (vev) of the field in the quantum case. The quantum field will differ by the classical solution by small fluctuations $\mathcal{O}$ ($\hbar$):

$$\phi = a + \phi' \tag{2.47}$$

and ϕ' must be small for the perturbation theory to make sense. For this reason the other vacuum $-a$ cannot be reached by perturbation theory (it can give rise to non perturbative phenomena, however, but they lye outside the scope of this course).

Let us examine a few consequences of ssb in the field theory case. In evaluating the perturbation theory it is better to work with fields having zero vev. Therefore we consider the shifted field ϕ' defined in (2.47) so that the lagrangian becomes

$$\begin{aligned} \mathcal{L}(\phi, \partial\phi) \;\; &\to \;\; \mathcal{L}(\phi', \partial\phi') = \\ &= \;\; \frac{1}{2}(\partial\phi')^2 - m^2\phi'^2 - \lambda a \phi'^3 - \frac{\lambda}{4}\phi'^4 \end{aligned} \tag{2.48}$$

with $m^2 = \lambda a^2 > 0$. The field ϕ' is massive and has both cubic and quartic couplings. It is interesting to observe that the original ϕ had bo cubic coupling and the quadratic term had a negative coefficient $-\lambda a^2/2$ that could not be interpreted as a mass. last we observe that $\mathcal{L}(\phi', \partial\phi')$ is no longer symmetric, which is a consequence of the shift (2.47).

2.2.1 SSB of continuous symmetries

Let us consider a more general case, characterized by a continuous symmetry instead of the discrete symmetry (parity) of the previous subsection. We consider the lagrangian for the complex scalar field ϕ:

$$\mathcal{L}(\phi, \partial\phi) = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - V(\phi) , \qquad V(\phi) = \lambda \left(\phi^\dagger \phi - \frac{a^2}{2} \right)^2 . \quad (2.49)$$

$\mathcal{L}$ has $U(1)$ invariance, which, differently from the previous example, is continuous. Since ϕ has two real components:

$$\phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}} , \quad (2.50)$$

we can rewrite $\mathcal{L}$ as follows:

$$\mathcal{L}(\vec{\phi}, \partial\vec{\phi}) = \frac{1}{2} (\partial\vec{\phi})^2 \, - \, V(\vec{\phi}) \quad (2.51)$$

with $\vec{\phi} = (\phi_1, \phi_2)$ and

$$V(\vec{\phi}) = \frac{\lambda}{4} [(\vec{\phi})^2 - a^2]^2 . \quad (2.52)$$

This shows that $U(1)$ invariance is equivalent to $O(2)$ invariance. In the (ϕ_1, ϕ_2) plane the potential (2.52) has the form of a mexican hat as in fig. (2.2). There are infinite degenerate vacua given by

$$\phi_1^2 + \phi_2^2 = a^2 . \quad (2.53)$$

Suppose that the true vacuum has $\phi_1 = +a$, $\phi_2 = 0$. Then performing the shift

$$\phi_1 \to \phi_1 + a , \qquad \phi_2 \to \phi_2 \quad (2.54)$$

one gets

$$\mathcal{L}(\vec{\phi}, \partial\vec{\phi}) = \frac{1}{2} \left[(\partial\phi_1)^2 + \partial\phi_2)^2 - m^2 \phi_1^2 \right] - \frac{\lambda}{4} [(\phi_1^2 + \phi_2^2)^2 + 4a\phi_1^3] \quad (2.55)$$

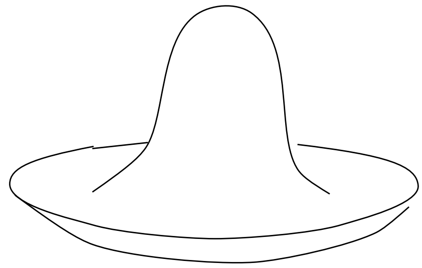


Figure 2.2: Mexican hat.

The interesting point is that out of the two fields, one, ϕ_1, has mass $m^2 = 2\lambda a^2 > 0$, and the other,ϕ_2, is massless.

The presence of a massless scalar field is an example of the Goldstone theorem, that states that in presence of a spontaneous breaking of a continuous symmetry there are as many massless bosons (called Nambu-Goldstone bosons, NGB) as are the broken generators. In particular if the group has n generators and is completely broken, then there are n NGB; if a subgroup with $k < n$ generators remains unbroken, then there are only $n - k$ NGB.

It is useful to derive this result again using the form (2.49) of the lagrangian. We write

$$\phi = \rho e^{i\theta} \tag{2.56}$$

The minimum of V is obtained for

$$|\phi| = \rho = \frac{a}{\sqrt{2}} \tag{2.57}$$

and therefore, performing the shift

$$\rho = \chi + v \tag{2.58}$$

one gets

$$\begin{aligned} \mathcal{L} &= \frac{a^2}{2}(\partial\theta)^2 + (\partial\chi)^2 - 2\lambda^2 a^2 \chi^2 \\ &- 2\sqrt{2}\lambda a \chi^3 \left(\sqrt{2}a\chi + \chi^2\right)(\partial\theta)^2 , \end{aligned} \tag{2.59}$$

which shows again the presence of a massless (θ) and a massive (χ) field, together with three different interaction terms.

2.2.2 Goldstone theorem

The Goldstone theorem is completely general. To begin with we prove that in presence of ssb the vacuum is degenerate. If $|0>$ denotes the vacuum ($H|0>= E_0|0>$) we can always add the constant $-E_0$ to the hamiltonian so that:

$$H|0>= 0 \ . \tag{2.60}$$

Let Q_a be the conserved charges associated to a symmetry group. Then

$$[H, Q_a] = 0 \, , \qquad a = 1, ...n \tag{2.61}$$

since the charges are conserved. If the vacuum is symmetric, then $Q_a|0>= 0$ because in this way $U|0>= \exp\{iT_a\alpha_a\}|0>= |0>$. Since it is not symmetric we have $Q_a|0>\neq 0$. Now

$$HQ_a|0>= [H, Q_a] = 0 = 0Q_a|0> \tag{2.62}$$

means that there is another eigenstate $Q_a|0>$ different from $|0|>$ corresponding to the same eigenvalue. Therefore the vacuum state is degenerate.

Let us consider a field theory, when, according to (2.13),

$$Q_a = \int d^3x J^0_a(\vec{x}, \, t) \tag{2.63}$$

Take the state

$$|\vec{k}, a>= \int d^3x e^{i\vec{k}\cdot\vec{x}} J^0_a(\vec{x}, \, t)|0> \tag{2.64}$$

We have

$$\begin{aligned}(P^i)^2|\vec{k}, a>= &\int d^3x e^{i\vec{k}\cdot\vec{x}}(P^i)^2 J^0_a(\vec{x}, \, t)|0>= \\ = \int d^3x e^{i\vec{k}\cdot\vec{x}}[(P^i)^2, J^0_a(\vec{x}, \, t)]|0>= &\int d^3x e^{i\vec{k}\cdot\vec{x}}(-\nabla^2) J^0_a(\vec{x}, \, t)|0>= \\ = \vec{k}^2 \int d^3x e^{i\vec{k}\cdot\vec{x}} J^0_a(\vec{x}, \, t)|0>& \end{aligned} \tag{2.65}$$

In the limit $\vec{k} \to 0$, this state vanishes. But in this limit also vanishes $P_0^2|\vec{k} \to 0>= H^2 Q_a|0>$. Therefore

$$P_\mu P^\mu|\vec{k}, a>= 0 \tag{2.66}$$

since it is a Lorentz scalar. This means that $|\vec{k}, a>$ describes a massless particle. Varying the index a we see that there are as many massless particles as broken generators. This ends the proof[3].

[3] $|\vec{k}, a>$ is a spin 0 object since under rotations it does not change.

We shall discuss below an example of SSB in particle physics iand we shall learn that pions can be understood as the Goldstone bosons of the spontaneously broken chiral symmetry of quarks.

References

For the content of this chapter any textbook on Quantum Field Theory with High Energy Physics orientation; for example H. Georgi *Weak Interactions and Modern Particle Theory*, Menlo Park, California, Benjamin/Cummings 1984, or A. Zee, cit.

Chapter 3

Quarks

3.1 Isotopic Spin

The proton mass ($m_p \simeq 938.3$ MeV/c^2) and the neutron mass $m_n \simeq$ 939.6 MeV/c^2 are quite similar. This difference can be produced either by a small term in the strong hamiltonian, which distinguishes between p and n, or by the different electromagnetic properties of the two nucleons. It is reasonable to neglect, at a first level of approximation, this tiny mass difference. In a similar vein one can assume not only that the masses are equal, but also that the strong interactions between nucleons are invariant under the exchange

$$p \leftrightarrow n \ . \tag{3.1}$$

A test is provided by the so called *mirror nuclei*, i.e. two nuclei that are obtained by the substitution (3.1). Let us consider the binding energy E_b of a nucleus (Z, A) formed by Z protons and $A - Z$ neutrons:

$$E_b \equiv (Zm_p + (A - Z)\, m_n - M)\, c^2 \ ,$$

where M is the nucleus mass. If nuclear interactions are invariant under (3.1), the binding energies E_b of the nuclei (Z, A) and $(A-Z, A)$ should be equal. The simplest mirror nuclei are H^3 (tritium: pnn) an He^3 (ppn); the binding energies are

$$E_b(H^3) = 8.482 \text{ MeV} \ ,$$
$$E_b(He^3) = 7.711 \text{ MeV} \ .$$

The difference might be attributed to the electrostatic repulsion between the two protons in He^3, which decreases E_b. It is useful to observe here that typical nuclear binding energies are of the order of a few MeV, about 10^6 larger than the atomic bonding energies.

Besides H^3 and He^3, other mirror nuclei have similar E_b and this leads to to assume that:

$$V_{pp} = V_{nn} \ . \tag{3.2}$$

On this basis one makes the following hypothesis. Strong nuclear forces[1] are invariant under the exchange (3.1).

The symmetry (3.1) is called *charge symmetry* of nuclear forces and can be described by a formalism similar to the spin formalism. Let us see it in detail. If strong interactions cannot distinguish between proton (p) and neutron (n), we can imagine p and n as two states of the same physical system, the *nucleon* N. Therefore we can write for the generic nucleon state

$$|N> = \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix} = \psi_p \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \psi_n \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ , \tag{3.3}$$

with $|\psi_p|^2 + |\psi_n|^2 = 1$, while

$$\begin{aligned} |p> \ &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ , \\ |n> \ &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ . \end{aligned} \tag{3.4}$$

Within this formalism the operator I_+ that transforms the n state into the p state, thus creating a unit of electric charge:

$$I_+ \, |n> = \, |p> \ , \tag{3.5}$$

is described by

$$I_+ \ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ , \tag{3.6}$$

while

$$I_- = I_+^\dagger \ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ , \tag{3.7}$$

transforms the p state into the n state and destroys a unit of electric charge:

$$I_- \, |p> = \, |n> \ .$$

[1]These forces, also called strong interactions, are responsible for the stability of the nucleus. They are distinct from weak nuclear forces, responsible for the β decays.

Let us introduce the Pauli matrices

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and three operators I_1, I_2, I_3 that satisfy

$$[I_i, I_j] = i\epsilon_{ijk} I_k \ , \tag{3.8}$$

while they commute with all the operators describing space properties ($\vec{r}, \vec{p}, \vec{L}$, etc.). In the orthonormal basis when $|1>=|p>$ e $|2>=|n>$ the matrices $\frac{\tau_j}{2}$ are a representation of the operators I_j while

$$I_\pm = \frac{1}{2}(\tau_1 \pm i\,\tau_2) \ .$$

The operators I_k act as generators of unitary transformations

$$U = e^{i\alpha_k I_k} \tag{3.9}$$

acting as follows

$$\psi \to U\psi \tag{3.10}$$

on a generic state ψ. If $I_k = \tau_k/2$, then U is a traceless, hermitean, 2×2 complex matrix; the U form a group called $SU(2)$ (for further examples of unitary groups see section 3.3). Given its analogy with spin the operator $\vec{I}$ is called I-spin or *isospin*.

Due to the invariance of the strong hamiltonian under the exchange $p \leftrightarrow n$ one has

$$[H, I_j] = 0 \ , \tag{3.11}$$

which implies a conservation rule

$$\frac{dI_j}{dt} = 0 \tag{3.12}$$

called *isospin conservation* in strong interactions. As a consequence the physical states can have definite energy and isospin at the same time. The states $|p>$ and $|n>$ have charge +1 and 0 (in units $|e|$). Therefore they are eigenkets of the electric charge Q that has the form:

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \ . \tag{3.13}$$

One has

$$Q = I_3 + \frac{1}{2}\mathbf{1} = \frac{\tau_3}{2} + \frac{1}{2}\mathbf{1} \ . \tag{3.14}$$

Physical states can have definite

$$I^2 = I_1^2 + I_2^2 + I_3^2 \tag{3.15}$$

and, e.g., I_3:

$$\begin{aligned} I^2\,|i,\,i_3> &= i(i+1)\,|i,\,i_3> \,, \\ I_3\,|i,\,i_3> &= i_3\,|i,\,i_3> \,. \end{aligned} \tag{3.16}$$

Moreover i is integer or half-integer and i_3 has $2i+1$ possible values, from $-i$ to $+i$. Nucleons have $i = 1/2$:

$$\begin{aligned} |p> &= \Big|\frac{1}{2}, +\frac{1}{2}> \,, \\ |n> &= \Big|\frac{1}{2}, -\frac{1}{2}> \,. \end{aligned} \tag{3.17}$$

For $i = 1$ one expects one or more multiplets with $3 = 2i + 1$ particles having the same mass. An example are the pions π^+, π^-, π^0 with masses in the range 135-140 MeV. Let us write

$$|\pi^+ >= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \,, \quad |\pi^0 >= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \,, \quad |\pi^- >= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \,. \tag{3.18}$$

For them

$$I_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \,, \quad I_- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \,; \tag{3.19}$$

let us consider

$$\begin{aligned} I_1 &= \frac{1}{\sqrt{2}}\,(I_+ + I_-) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \,, \\ I_2 &= \frac{-i}{\sqrt{2}}\,(I_+ - I_-) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \,, \\ I_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \,. \end{aligned} \tag{3.20}$$

These matrices satisfy (3.8) and constitute a representation D=3 of isospin. The charge matrix is

$$Q = I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \,; \tag{3.21}$$

it satisfy

$$Q = I_3 + \frac{1}{2}\mathbf{B}\,, \tag{3.22}$$

where B is an operator assuming the value 1 for nucleons (and other particles, called baryons) and 0 for pions (and other particles, called mesons). B is called bayonic number. This name has its origin in the fact that the first known hadrons with $B = 1$ were heavier than those with $B = 0$.

For the systems with several hadrons the total isospin is obtained by the same rules used to add angular momentum. For example, two nucleons can have total isospin 1 or 0; a system comprising a nucleon and a pion has isospin 3/2 or 1/2. On the other hand I_3 is the sum of the eigenvalues of I_3 for the constituents. The total bayonic number is additive as well. Therefore (3.22) holds also for the complex systems.

3.2 Strangeness and Hypercharge

If we consider mesons of increasing mass, we encounter, after the pions, the kaons K (mass $\simeq$ 500 MeV/c^2). They exist with three different charges: ±1 and 0. However two different neutral kaons there exist: the states K^0 and $\overline{K}^0$ (its antiparticle). They are truly different as it can be argued as follows. Let us consider the following reaction

$$\pi^- + p \rightarrow \Lambda^0 + K^0\,. \tag{3.23}$$

Here the hyperon Λ^0 is a baryon of spin 1/2 and mass 1116 MeV. The process (3.23) is mediated by strong nuclear interactions, as one can argue by the rather large value of the cross section. On the other hand the following process does not take place:

$$\pi^- + p \rightarrow \Lambda^0 + \overline{K}^0\,. \tag{3.24}$$

The explanation was found by Pais in the '50 of the last century. Pais argued that, differently from pions and nucleons, hyperons and kaons possess a new quantum number, called *strangeness*. It is an additive quantum number that is assumed to be conserved by strong interactions. Strangeness is really a new quantum number, different from the electric charge. In fact while the electric charge of an isolated system is always conserved strangeness is only conserved by strong interactions. One conventionally attributes strangeness $S = -1$ to the hyperon Λ^0 and $S = +1$ to K^0; moreover particle and antiparticle have opposite strangeness and therefore $\overline{K}^0$ has $S = -1$. By this mechanism while the process (3.23) can be mediated by strong interactions, the reaction (3.24) cannot.

In presence of strange particles eq. (3.22) is modified as follows:

$$Q = I_3 + \frac{Y}{2} , \tag{3.25}$$

where

$$Y = B + S \tag{3.26}$$

is called *hypercharge*. Clearly also hypercharge is conserved by strong interactions and one can use the conservation of hypercharge to characterize the behavior of strongly interacting strange particles.

Since the observable Y is conserved, its corresponding operator must commute with the strong interaction hamiltonian H. Clearly Y is an internal symmetry which is distinct from isospin: Therefore there are at least four generators of internal symmetries commuting with H. If we want to consider only unitary symmetry groups we are led to enlarge the isospin group $SU(2)_I$ and consider the next unitary group: $SU(3)$.

3.3 $SU(3)$ and $SU(n)$

For $SU(n)$

$$U \in SU(n) \iff UU^\dagger = 1 \qquad \det U = 1 . \tag{3.27}$$

in general we have

$$U \in SU(n) \iff U = \exp(i \sum_{a=1}^{n^2-1} \phi_a t_a) \tag{3.28}$$

The matrices t_a must be linearly independent, hermitean and traceless (for this reason their number is $n^2 - 1$). For $SU(3)$ we have

$$U \in SU(3) \iff U = \exp(i \sum_{a=1}^{8} \phi_a \lambda_a / 2). \tag{3.29}$$

The usual choice for the λ is by the Gell-Mann matrices:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{3.30}$$

They satisfy

$$\mathrm{Tr}\lambda_a\lambda_b = 2\delta_{ab} \ . \tag{3.31}$$

Moreover

$$\left[\frac{\lambda_a}{2}, \frac{\lambda_b}{2}\right] = if_{abc}\frac{\lambda_c}{2} \ . \tag{3.32}$$

The constants f_{abc} are the structure constants for the $SU(3)$ group. They are antisymmetric for exchange of any pair of indices.

Problem. Prove that

$$f_{abc} = \frac{1}{4i}\mathrm{Tr}\left([\lambda_a, \lambda_b]\lambda_c\right) \ . \tag{3.33}$$

It also useful to consider the anticommutator that is given by the formula

$$\{\lambda_a, \lambda_b\} = \frac{4}{3}\delta_{ab} \cdot \mathbf{1} + 2d_{abc}\lambda_c \ . \tag{3.34}$$

Problem. Prove (3.34).

Solution. The l.h.s. is a hermitean matrix of rank 3: therefore it is given by a linear combination of the 8 Gell-Mann and the matrix $\mathbf{1}$: $\{\lambda_a, \lambda_b\} = C_{ab} \cdot \mathbf{1} + 2d_{abc}\lambda_c$. Taking the trace one gets eq. (3.34).

Problem. Prove that

$$d_{abc} = \frac{1}{4}\mathrm{Tr}\left(\{\lambda_a, \lambda_b\}\lambda_c\right) \ , \tag{3.35}$$

which shows that the constants d_{abc} are totally symmetric.

The parameters f and d for $SU(3)$ are in the Table below.

ijk	f_{ijk} (antisym.)	ijk	d_{ijk} (symm.)
123	1	118	$1/\sqrt{3}$
147	1/2	146	1/2
156	-1/2	157	1/2
246	1/2	228	$1/\sqrt{3}$
257	1/2	247	-1/2
345	1/2	256	1/2
367	-1/2	338	$1/\sqrt{3}$
458	$\sqrt{3}/2$	344	1/2
678	$\sqrt{3}/2$	355	1/2
		366	-1/2
		377	-1/2
		448	$-1/2\sqrt{3}$
		558	$-1/2\sqrt{3}$
		668	$-1/2\sqrt{3}$
		778	$-1/2\sqrt{3}$
		888	$-1/\sqrt{3}$

We note that the analogous of the structure constants f_{ijk} for the group $SU(2)$ are the components of the tensor ϵ_{ijk}.

3.3.1 Representations

Let us consider $SU(n)$ and the mapping

$$D \ : U \in SU(n) \rightarrow \ D(U) \in C^{n\times n} \ . \tag{3.36}$$

$D(U)$ is a matrix (n, n) and (3.36) defines a representation iff

$$\begin{aligned} D(\mathbf{1}) &= \mathbf{1} && (3.37)\\ D(UV) &= D(U)D(V) \ . && (3.38)\end{aligned}$$

The simplest representations are

$$\begin{array}{rlll} \text{Trivial}: \ \mathbf{1} & U \ \rightarrow \mathbf{1}\,, & \text{a matrix}\,(1,1) & \\ \text{Fundamental}: \ \mathbf{3} & U \ \rightarrow U\,, & \text{a matrix}\,(3,3) & \\ \text{Conjugate}: \ \bar{\mathbf{3}} & U \ \rightarrow U^*\,, & \text{a matrix}\,(3,3) & (3.39) \end{array}$$

Suppose that $U \in SU(3)$ is infinitesimal, i.e.

$$U \sim \mathbf{1} + i\delta\phi_a \frac{\lambda_a}{2} \tag{3.40}$$

and consider a representation of dimension n. $SU(n)$ is a group whose element depend smoothly by a set of parameters ϕ_a. Groups with this property are called Lie groups and for them also the representations depend smoothly by the parameters. It follows that also $D(U)$ differs infinitesimally from $\mathbf{1}$:

$$D(U) \sim \mathbf{1} + i\delta\phi_a F_a \ . \tag{3.41}$$

The matrices F_a are called *generators of the representation* D and are matrices of rank n. It is easy to prove that

$$[F_a, F_b] = i f_{abc} F_c \ . \tag{3.42}$$

Problem. Prove eq. (3.42). Hint: Use (3.38) expanding $U \exp(i\phi_a t_a)$, $V \exp(i\varphi_a t_a)$ up to second order.

We note that in the fundamental representation $F_a \equiv \lambda_a/2$. It is useful to note that the set of generators commuting among themselves contain at most two elements: for example we can pick up F_3 and F_8:

$$[F_3, F_8] = 0 \ . \tag{3.43}$$

There is no other generator commuting with F_3, F_8.

Another important representation is the *adjunct.* Let us consider the matrix

$$C = C_a \frac{\lambda_a}{2} \tag{3.44}$$

We can write it as a ket $|C>$ in the C^8 vector space. More generally, for $SU(n)$, $|C>$ has n^2-1 components and belongs to the C^{n^2-1} vector space. The representative of U in adjunct representation is, by definition, the $D(U)$ matrix that, for each $|C>$ in the vector space C^{n^2-1}, works as follows:

$$D(U)|C> = |UCU^\dagger> \ . \tag{3.45}$$

Clearly this is a representation.

Problem. Prove it. Hint: Show that eqns. (3.37), (3.38) are satisfied.

Let us determine the generators of the adjunct representation. If $D(U)$ represents U in the adjunct, then

$$D(\mathbf{1} + i\delta\phi_a \frac{\lambda_a}{2}) = (\mathbf{1} + i\delta\phi_a F_a) \tag{3.46}$$

Moreover

$$D(\mathbf{1} + i\delta\phi_a \frac{\lambda_a}{2})|C> \ = \ |(\mathbf{1} + i\delta\phi_a\lambda_a/2)C(\mathbf{1} - i\delta\phi_a\lambda_a/2)> =$$

$$= |C + i\delta\phi_a[\frac{\lambda_a}{2}, C] > \quad \simeq \quad |C > +i\delta\phi_a| \, [\frac{\lambda_a}{2}, C] > \tag{3.47}$$

which gives

$$F_a|C >= |[\frac{\lambda_a}{2}, C] > ; \tag{3.48}$$

if $C = \lambda_c/2$, then

$$F_a|\lambda_c/2 >= if_{acb}|\lambda_b/2 > \ . \tag{3.49}$$

Written in terms of matrices this equation is

$$(F_a)_{mn} \, \delta_{nc} = if_{acb}\delta_{bm} \ , \tag{3.50}$$

i.e.

$$(F_a)_{mc} = if_{acm} \tag{3.51}$$

in other words the group structure constants provide the generators of the adjunct representation.

3.3.2 Irreducible representations and Young tableaux

Consider matrices $D(U)$ in a given representation of dimension n (the rank of the matrix). They are operators in a vector space $V = C^n$:

$$D : \xi^j \to \xi'^k = D^{kj}\xi^j \ . \tag{3.52}$$

We say that $D(U)$ is reducible if, for any U it can be written as follows:

$$D(U) = \begin{pmatrix} D_1(U) & \alpha(U) \\ 0 & D_2(U) \end{pmatrix} \tag{3.53}$$

Here $D_1(U)$, $D_2(U)$ are square matrices of rank n_1, n_2 (and $n_1 + n_2 = n$) and $\alpha(U)$ a matrix (n_1, n_2). In other words $D(U)$ is reducible if there are subspaces in V that are left invariant under the action of the group. If this does not happen we say that D is irreducible (IRR). In the example above the subspace which is left invariant is as follows:

$$W = \left\{ \psi \in V \,\middle|\, \psi = \begin{pmatrix} \chi \\ 0 \end{pmatrix} \right\} \ . \tag{3.54}$$

To give an example of reducible and irreducible representations, let us consider in $SU(3)$ the tensor product

$$\psi^i_j = \xi^i \eta_j \tag{3.55}$$

of two tensors, one (ξ^i) belonging to the fundamental representation **3** and the other (η_j) belonging to the conjugate $\bar{\mathbf{3}}$ (we distinguish **3** and $\bar{\mathbf{3}}$ by raing/lowering the index). Let us write

$$\psi^i_j = \hat{\psi}^i_j + \frac{\delta^i_j}{3}\mathrm{Tr}\psi \ , \tag{3.56}$$

where

$$\hat{\psi}^i_j = \psi^i_j - \frac{\delta^i_j}{3}\mathrm{Tr}\psi \, , \qquad \mathrm{Tr}\psi = \psi^i_i \ . \tag{3.57}$$

In other words $\hat{\psi}$ is traceless (it has therefore 8 components); moreover $\frac{\delta^i_j}{3}\mathrm{Tr}\psi$ is proportional to the unit matrix and is therefore a scalar. Since ψ has the transformation property $\psi^i_j \rightarrow \psi'^i_j = U_{i\ell}U^*_{jm}\psi^\ell_m$, or, in matrix form,

$$\psi \rightarrow \psi' = U\psi U^\dagger, \tag{3.58}$$

also $\hat{\psi}'$ is traceless; analogously the scalar maintains its form under the group transformations. Therefore ψ^i_j can be reduced in the sum of two are irreducible tensors; written as a column (with 9 rows) we have

$$\psi = \begin{pmatrix} \hat{\psi} \\ Tr\psi \end{pmatrix} \tag{3.59}$$

where the column $\hat{\psi}$ has 8 rows. Therefore the vector space to which ψ belongs has two invariant subspaces and the $D(U)$ acting ψ can generally be written as follows:

$$D(U) = \begin{pmatrix} D_1(U) & 0 \\ 0 & D_2(U) \end{pmatrix} \tag{3.60}$$

Here $D_1(U)$, $D_2(U)$ are square matrices of rank 8 and 1 respectively. (3.56) can be summarized as follows:

$$\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1} \ . \tag{3.61}$$

Problem. Prove that the **8** defined above coincides with adjunct representation. Hint: Use (3.44), (3.45) and (3.58).

Problem. Show that

$$\psi^{ij} = \xi^i \xi^j \tag{3.62}$$

transforms under a reducible representations; prove that the decomposition of (3.62) in IRR is obtained dividing ψ into its symmetric and antisymmetric part. Write down the analogous of (3.61).

The procedure we have outlined in the examples above is rather cumbersome. In general, higher dimension IRRs can be constructed starting from the fundamental representation and the conjugate representations $\mathbf{n}$ and $\bar{\mathbf{n}}$ of $SU(n)$ by the Young method.

Let us construct start by the n^k elements

$$\xi^{j_1}\xi^{j_2}...\xi^{j_k} \tag{3.63}$$

forming a tensor in the tensor space

$$\underbrace{V \times V \times ... \times V}_{\text{k terms}}$$

For simplicity we only consider now vector in fundamental representation (upper indices). Since we are taking a tensor product we denote the operation (3.63) as follows:

$$\underbrace{\mathbf{n} \times \mathbf{n} \times ... \times \mathbf{n}}_{\text{k terms}} \tag{3.64}$$

Under the group each vector ξ transforms according to eq. (3.52) and therefore the tensor (3.63) transforms as follows

$$\xi^{j_1}\xi^{j_2}...\xi^{j_k} \rightarrow \xi'^{j_1}\xi'^{j_2}...\xi'^{j_k} = D^{j_1 i_1}D^{j_2 i_2}...D^{j_k i_k}\xi^{i_1}\xi^{i_2}...\xi^{i_k}. \tag{3.65}$$

However $D^{j_1 i_1}D^{j_2 i_2}...D^{j_k i_k}$ is not IRR. To get IRR out of the product of the k IRRs in (3.64) we decompose the tensor (3.63) in IRR tensors having definite symmetry properties under index permutations. Indeed, as shown in the examples above, tensors having definite permutation symmetry properties or traceless tensors belong to IRR. The procedure of getting IRRs out of the product of other IRRs is based on a graphical method (Young tableaux) that we present without proof. We work in $SU(n)$. Let us take n_1 indices and symmetrize them. To each index j we associate a box. Let us denote this operation as in figure 3.1.

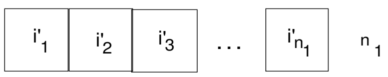


Figure 3.1: n_1 symmetrized indices.

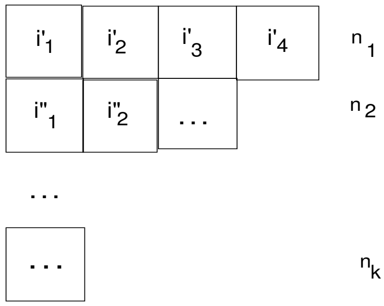


Figure 3.2: A Young tableau.

Let us repeat this operation with other $n_2 < n_1$ indices different from the previous ones and so forth. We obtain the figure 3.2 (in the example $n_1 = 4$).

Clearly

$$n_1 \geq n_2 \geq ... \geq n_k \,, \qquad \sum_{j=1}^{k} = n \,. \tag{3.66}$$

A table as in the previous picture is called a Young tableau. It corresponds to a possible partition of n into $n_1 \, n_2 \, ..., n_k$ integers. Indexes in any row must be symmetrized in the tensor corresponding to a given tableau. For example, if in fig 3.2 $n_1 = 3$, i.e. if the indexes to be symmetrized in $\xi^{i_1 i_2 ... i_n}$ are the first three, then we construct

$$\xi^{i_1 i_2 i_3 ... i_n} \to \frac{1}{6}\left(\xi^{i_1 i_2 i_3 ... i_n} + \xi^{i_1 i_3 i_2 ... i_n} + \xi^{i_2 i_1 i_3 ... i_n} + ...\right) \tag{3.67}$$

Once we have done this we antisymmetrize over all the columns, each column once, and independently from each other. Let us how it works in a simple case. Consider the tensor $\xi^{\alpha\beta\gamma\delta}$ that has no particular symmetry property under the permutation group and therefore it is not an IRR tensor. The Young tableau in fig. 3.3
allows to construct an IRR tensor, and therefore also an IRR representation, according to the formula

$$\begin{aligned} &\frac{1}{12}\Big(\xi^{\alpha\beta\gamma\delta} + \xi^{\alpha\gamma\beta\delta} + \xi^{\gamma\alpha\beta\delta} + \xi^{\gamma\beta\alpha\delta} + \xi^{\beta\gamma\alpha\delta} + \xi^{\beta\alpha\gamma\delta} \\ &-\xi^{\delta\beta\gamma\alpha} - \xi^{\delta\gamma\beta\alpha} - \xi^{\gamma\delta\beta\alpha} - \xi^{\gamma\beta\delta\alpha} - \xi^{\beta\gamma\delta\alpha} - \xi^{\beta\delta\gamma\alpha}\Big) \end{aligned} \tag{3.68}$$

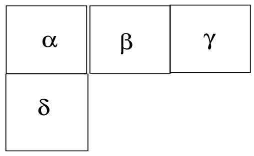


Figure 3.3: Another Young tableau.

The first six terms correspond to symmetrization of the first row in the tableau, the other six to antisymmetrization along the first column. The IRR tensor (3.68) has the following symmetry property: it is symmetric in β, γ and antisymmetric in α, δ. It is clear that we can use the Young tableau to denote the corresponding tensor. Note that in our example the two tensors depicted in 3.4 are identical.

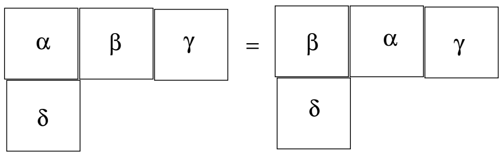


Figure 3.4: Two identical tensors.

The following properties hold:
Property 1: Tensors of any given symmetry class form an invariant subspace. In other words they constitute an IRR.
Property 2: Each IRR correspond to one and only one Young tableau.

Therefore there is a correspondence one-to-one between IRRs and Young tableaux. We do not prove these properties and prefer to consider in more detail the $SU(3)$ group. In this case, in a generic Young tableau, each column cannot have more than three boxes (the corresponding property in $SU(n)$ is that each column cannot have more than n boxes). Let us consider the Young tableaux in fig. 3.5. The corresponding tensor is

$$\xi^{ijk} = \epsilon^{ijk} \tag{3.69}$$

Under the group it transforms as follows

$$\epsilon^{ijk} \to U_{ii'} U_{jj'} U_{kk'} \epsilon^{i'j'k'} = \det U \epsilon^{ijk} = \epsilon^{ijk} \; . \tag{3.70}$$

This is a tensor of rank 0 (a scalar). The corresponding IRR is the trivial, of dimension 1. We write symbolically as in fig.3.6.

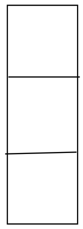


Figure 3.5: The tableau representing the singlet in $SU(3)$.

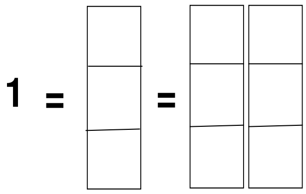


Figure 3.6: The singlet in $SU(3)$.

It is important to note that in any diagram a full column can be eliminated; an example of this procedure is in fig. 3.7.

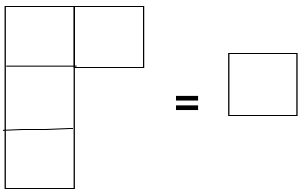


Figure 3.7: The procedure of eliminating an entire column.

The most general Young tableau in $SU(3)$ is depicted in fig. 3.8. The corresponding tensor can also be denoted as follows:

$$\left(\begin{array}{cccc|cccc} k_1 & k_2 & ... & k_m & i_1 & i_2 & ... & i_n \\ j_1 & j_2 & ... & j_m & & & & \end{array} \right) \tag{3.71}$$

The tensor (3.71) is completely fixed by a pair of integer numbers: $(m\,, n)$ such that there $n+m$ boxes in the first row and m boxes in

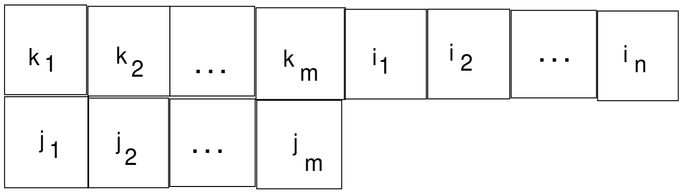


Figure 3.8: The most general Young tableau in $SU(3)$.

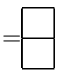
the second. Accordingly we shall call $D^{m,n}$ the corresponding IRR and by $D(m,n)$ its dimension. The IRR tensor (3.71) can be written in a more economic way:

$$\xi^{\ell_1 \ldots \ell_m}_{i_1 \ldots i_n} = \begin{pmatrix} k_1 & \ldots & k_m & | & i_1 & \ldots & i_n \\ j_1 & \ldots & j_m & | & & & \end{pmatrix} \epsilon^{\ell_1 k_1 j_1} \ldots \epsilon^{\ell_m k_m j_m} \tag{3.72}$$

The dimension of the representation (m,n) is

$$D(m,n) = \frac{(n+1)(m+1)(n+m+2)}{2} \tag{3.73}$$

(for the proof see section 3.8).

Problem. Prove that the representation $(m\,,n)$= $(1\,,0)$=  is the conjugate $\bar{\mathbf{3}}$.
Solution.

$$\begin{aligned} \eta_i &= \epsilon_{ijk}\psi^{jk} \rightarrow \epsilon_{ijk}U_{jm}U_{kn}\psi^{mn} = \epsilon_{\ell jk}U_{\ell a}U_{jm}U_{kn}U^{\dagger}_{ai}\psi^{mn} \\ &= U^*_{ia}\epsilon_{amn}\psi^{mn} = (U^*\eta)_i \end{aligned} \tag{3.74}$$

transforms according to the $\bar{\mathbf{3}}$.

It is useful to note that the representations (n,m) and (m,n) are conjugate:

$$(n,m) = \overline{(m,n)}\ .$$

Problem. In $SU(3)$ draw the Young tables and compute the dimensions of the IRRs defined by: $(m,n) = (0,1)$, $(0,2)$, $(2,0)$, $(1,1)$, $(1,2)$, $(2,1)$, $(3,0)$, $(0,3)$, $(0,4)$, $(4,0)$, $(1,3)$, $(3,1)$, $(2,2)$. Moreover write the tensors corresponding to these IRRs.
Hint: Use the property

$$D(m,n) = D(n,m)$$

that follows from (3.73)

3.3.3 Product of IRRs

We have already shown an example of decomposition of the product of two IRRs in the sum of other two IRRs in (3.61). Another example was given in the problem and eq. (3.62). It corresponds to the decomposition

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \bar{\mathbf{3}} \; . \tag{3.75}$$

The practical rule is as follows. Consider $(n, k) \otimes (\ell, m)$ and write the corresponding Young tableau inserting indexes in the boxes. Then arrange the boxes in all the possible ways such that the diagrams of the resulting IRRs maintain the previous antisymmetry among the indexes (if the indexes are still in the new diagrams and have not been eliminated by subtracting an entire column as in (3.7).

Problem. Prove by this procedure (3.61), (3.75). Prove that

$$\mathbf{8} \otimes \mathbf{8} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10} \oplus \overline{\mathbf{10}} \oplus \mathbf{27} \; . \tag{3.76}$$

Prove that this equation is written in terms of Young tables as in fig. 3.9.

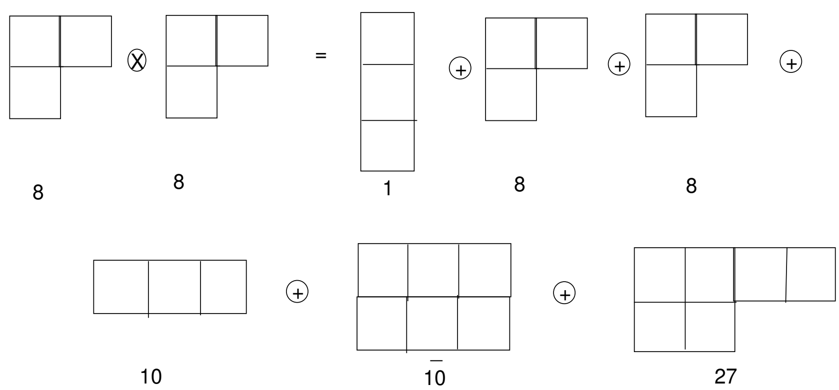


Figure 3.9: The decomposition (3.76).

3.4 $SU(3)_f$ symmetry and quarks

We now make the assumption that strong interactions are invariant under the group of transformations $SU(3)_f$ [2]. As it is shown in the previous paragraph, $SU(3)_f$ comprises a subgroup $SU(2)$ generated by

[2]We use the symbol $SU(3)_f$ for the global (flavor) symmetry group and $SU(3)_c$ for the local (color) symmetry group.

Table 3.1: Isospin and Hypercharge.

F_a	$I_i = F_i$ ($i = 1,2,3$: isospin)	$Y = \frac{2}{\sqrt{3}} F_8$ ($a = 8$: hypercharge)
3	$I_1 = \begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $I_2 = \begin{pmatrix} 0 & i/2 & 0 \\ i/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $I_3 = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$Y = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & -2/3 \end{pmatrix}$

the first three generators: F_a ($a = 1,2,3$). We identify this subgroup with the isospin group and denote it as $SU(2)_I$. We also identify F_8, up to a factor, with hypercharge. These results are summarized in table 3.1.

We shall see below that in the quark model $SU(3)_f$ symmetry is a consequence of the assumption that the three lighter quarks have the same mass. Since this is only approximately true, $SU(3)_f$ must be regarded as an approximate and not an exact symmetry.

Let us now discuss the consequences of this symmetry. Let us suppose that the hamiltonian of a system of hadrons can be approximated only by the part containing strong interactions H, i.e. we neglect gravitational, electromagnetic and weak interactions. Under a transformation of $SU(3)_f$, H remains unchanged

$$H \rightarrow H' = U^\dagger H U = H \,, \tag{3.77}$$

which implies in particular that

$$[H, F_a] = 0 \qquad (a = 1, ..., 8) \,. \tag{3.78}$$

As a consequence of these formulae for any representation of $SU(3)_f$ one has

$$D(U)H = HD(U) \qquad \forall U \in SU(3)_f \,, \tag{3.79}$$

and in particular this is true for IRR. This result is extremely important due to the following theorem due to Schur.
Theorem. Let

$$\begin{array}{lllr} U & \rightarrow & D_1(U) & \dim D_1 = n_1 \\ U & \rightarrow & D_2(U) & \dim D_2 = n_2 \end{array} \tag{3.80}$$

be two IRRs of a Lie group G. If there exists S such that

$$\forall U \in G : D_1(U)S = SD_2(U)\,, \tag{3.81}$$

then either $S = 0$ or

$$S \neq 0\,, \qquad det\, S \neq 0\,, \qquad n_1 = n_2\,.$$

Remark. We say that two representations $D_1(U)$, $D_2(U)$ are equivalent iff exists a matrix S such that $\forall U \in G$: $D_1(U) = SD_2(U)S^{-1}$. Therefore in the hypotheses of the Schur's theorem, the two representations are equivalent.
Corollary. Let $D(U)$ be an IRR of G of dimension n. If

$$\forall U \in G : D(U)S = SD(U)\,, \tag{3.82}$$

then S is proportional to the unity matrix:

$$S = \alpha \cdot \mathbf{1}\,,$$

with $\alpha \in C$. We prove only the corollary.
Proof. Let $|\alpha>$ be an eigenvector of S with eigenvalue α. Then, for all $U \in G$:

$$SD(U)|\alpha >= D(U)S|\alpha >= \alpha D(U)|\alpha > \ . \tag{3.83}$$

Varying $U \in G$ generates the whole vector space C^n, and therefore $S = \alpha \cdot \mathbf{1}$.

This important theorem (also called Schur's lemma) has an important application here. In fact within any assigned IRR of $SU(3)_f$ H is a number. If the states describe single particles at rest, this number is the particle mass. Therefore hadrons will appear grouped in multiplets of multiplicity equal to the dimension of the representation.

Because of (3.43) and (3.77) there exist states that are simultaneously eigenstates of H, $I_3 = F_3$, $Y = 2/\sqrt{3}F_8$; they describe single particles of mass m at rest

$$|m, i_3, y > \ . \tag{3.84}$$

The generators I_k and Y for the **3** are reported in Table 3.1. For fixed m, and variable i_3, y, the set $\{|m, i_3, y>\}$ is a basis of a vector space. We report these vectors on the $I_3 - Y$ plane and we get as a result a diagram, called the *weight diagram.*

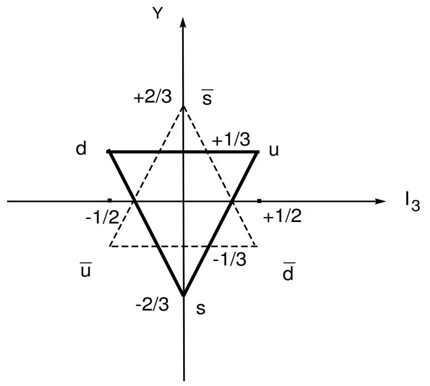


Fig. 3.4 The weight diagrams for the representations **3** (solid line) and $\bar{\mathbf{3}}$ (dashed line).

Let us consider for example the weight diagram for the fundamental representation **3**. It is depicted in fig. 3.4 in solid line. The three particles corresponding to the three base vectors are called *quarks*[3]. The state $|i_3, y>= |+1/2, +1/3>$ is called *up* (u); the state $|-1/2, +1/3>$ is the quark *down* (d); the state $|0, -2/3>$ is the quark *strange* (s), see fig. 3.4. To make contact with our previous notation we write the irreducible tensor of **3** as ξ^k, so that $< j|i_3, y>= \xi^j$ and

$$u^j = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} , \quad d^j = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} , \quad s^j = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} . \tag{3.85}$$

We attribute baryonic number $+1/3$ to the quarks. Therefore, using eqns. (3.25) and (3.26) one can see that the charges of the three quarks are as follows

$$e_u = +2/3 , \quad e_d = -1/3 , \quad e_s = -1/3 . \tag{3.86}$$

As to strangeness, the quarks u and d have zero strangeness and the strange quark has $S = -1$.

[3]The introduction of $SU(3)_f$ as an approximate symmetry of strong interactions is due to Gell-Mann and Zweig. The introduction of quarks is due to M. Gell-Mann.

In fig. 3.4 we have reported also the weight diagram for the $\bar{\mathbf{3}}$ representation. It can be derived by observing that

$$D^*(U) = U^* \approx 1 - i\delta\phi_a \frac{\lambda_a^*}{2}\,, \tag{3.87}$$

where we have used (3.41). Expressing $D^*(U)$ by its generators, according again the generic formula (3.41), one gets

$$F_a = -\frac{\lambda_a^T}{2}\,. \tag{3.88}$$

Therefore the hadrons in the $\bar{\mathbf{3}}$ multiplet, that are called *antiquarks* and are denoted by a bar: $\bar{q}$, have opposite I_3 and Y, see again fig. 3.4. They also have opposite baryonic number ($B = -1/3$) and charges with respect to the quarks. Finally the antiquark $\bar{s}$ has $S = +1$.

3.5 Constituent quark model

Differently from other hadrons, quarks have never been detected as free particles. There are cogent reasons to believe in their existence, but a well established fact is that baryons with fractional baryonic number B have never been observed. Within the quark model this result is assumed as an ansatz and mathematically formulated as follows: The only IRRs corresponding to hadrons in the physical spectrum are those whose Young tableaux have a number of boxes which is multiple of 3. Therefore, for example, 8, 10, $\overline{10}$ and 27 representations are admitted, while 3, $\bar{3}$, 6, 15 do not correspond to physical states. The empirical rule we have stated is also called *triality rule.*

Problem. Prove that the IRRs 3, $\bar{3}$, 6, 15 have fractional baryonic number, while the 8, 10, $\overline{10}$ and 27 have integer B.

Even though quarks are not observed as free particles, they are extremely useful as building blocks of the other hadrons. In other words quarks can be thought of as constituents of physical hadrons. There are several experimental reasons why hadrons cannot be considered as elementary particles; for example the nucleon magnetic dipole moments show that neither the proton nor the neutron are truly elementary. Other evidences will be discussed in more detail below. For the time being let us discuss how we can implement the idea of constituent quarks.

First of all let us observe that quark must have spin 1/2, since there are hadrons with half-integer spin (baryons) as well as hadrons with integer spin (mesons), but half-integer spins cannot be produced summing only integer spins and orbital angular momenta. Moreover, if a hadron h comprises k quarks and ℓ antiquarks, its $SU(3)_f$ wave-function (wf) will be of the form $\xi^{j_1}...\xi^{j_k}\eta_{i_1}...\eta_{i_\ell}$. Therefore it is clear that the wfs of all the hadrons can be obtained by forming products of irreducible tensors of **3** and $\bar{\mathbf{3}}$ representations, i.e. by products of quark and antiquark $SU(3)_f$ wfs (=tensors). Hadrons have however definite mass, therefore they are in IRRs of $SU(3)_f$. It follows that in forming the products of quark and antiquark tensors we must decompose them in IRRs. Let us consider a few examples.

a) $\mathbf{3} \bigotimes \mathbf{3}$

Since $\mathbf{3} \bigotimes \mathbf{3} = \mathbf{6} \bigoplus \overline{\mathbf{3}}$ and neither **6** nor $\overline{\mathbf{3}}$ satisfy the triality rule, this product does not produces hadron representations.

b) $\mathbf{3} \bigotimes \overline{\mathbf{3}}$

Hadrons made up by a quark and an antiquark are called *mesons*. Since $\mathbf{3} \bigotimes \overline{\mathbf{3}} = 8 \bigoplus 1$, we conclude that mesons are either in the singlet or in the octet of $SU(3)_f$. Mesons have baryonic number $B = 0$.

c) $\mathbf{3} \bigotimes \mathbf{3} \bigotimes \mathbf{3}$

Hadrons made up by three quarks are called *baryons*. Since $\mathbf{3} \bigotimes \mathbf{3} \bigotimes \mathbf{3} = 1 \bigoplus 8 \bigoplus 8 \bigoplus 10$, possible baryonic multiplets are the singlet, the octet and the decuplet. Baryons have baryonic number $B = 1$.

c) $\overline{\mathbf{3}} \bigotimes \overline{\mathbf{3}} \bigotimes \overline{\mathbf{3}}$

This combination produces the antibayons, i.e. the antiparticles of the baryons with $B = -1$. They are formed by three antiquarks and their possible multiplets are the singlet, the octet and the antidecuplet.

Let us discuss in more detail mesons and baryons.

3.5.1 Mesons

Mesons M are made up by a quark and an antiquark. Their spin is integer. If the orbital angular momentum of the pair is zero, the only possibilities for J^P is 0^- and 1^-. In fact remember that $P_M = (-1)^\ell P_q P_{\bar{q}}$; conventionally $P_q = +1$ therefore $P_q P_{\bar{q}} = -1$, because particles and antiparticles have opposite intrinsic parities. As to $SU(3)_f$, mesons are organized in octets and singlets. The weight diagram for the octet and singlet is depicted in fig. 3.10 together with the quark content of the different states.

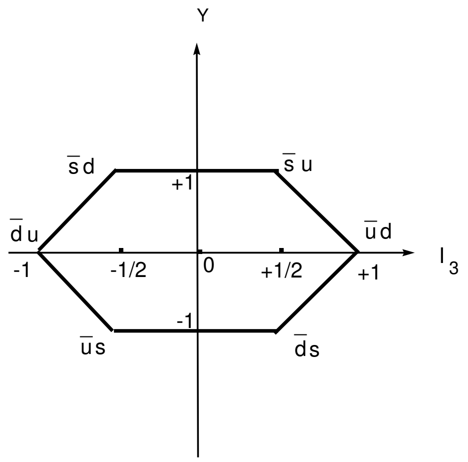


Figure 3.10: The weight diagrams for the mesonic octet and singlet .

Notice that in correspondence of the point $(I_3, Y) = (0,0)$ there are three different states:

$$|\mathbf{1}; Y=0, I=0, I_3=0> \; = \; \frac{\bar{u}u + \bar{d}d + \bar{s}s}{\sqrt{3}} \tag{3.89}$$

$$|\mathbf{8}; Y=0, I=0, I_3=0> \; = \; \frac{\bar{u}u + \bar{d}d - 2\bar{s}s}{\sqrt{6}} \tag{3.90}$$

$$|\mathbf{8}; Y=0, I=1, I_3=0> \; = \; \frac{\bar{u}u - \bar{d}d}{\sqrt{2}} \tag{3.91}$$

Low-lying mesons with $J^P = 0^-$ and $J^P = 1^-$ are reported in table 3.2.

On this basis we can write the quark content of the different meson states as follows.

0^- **mesons**:

$$\begin{aligned}
|\pi^+> &= |u\bar{d}>, \quad |\pi^- >= |d\bar{u}>, \quad |\pi^0>= \frac{1}{\sqrt{2}}|\left(\bar{u}u> -|\bar{d}d>\right), \\
|K^+> &= |u\bar{s}>, \qquad\quad |K^0>= |d\bar{s}>, \\
|K^-> &= |s\bar{u}>, \qquad\quad |\bar{K}^0>= |s\bar{d}>, \\
|\eta_8> &= \frac{1}{\sqrt{6}}\left(|\bar{u}u> + |\bar{d}d> -2|\bar{s}s>\right),
\end{aligned}$$

J^P	Particles (mass in MeV)
0^-	$\pi^0(135)$, $\pi^{\pm}(140)$, $\eta(547)$, $\eta'(958)$ $K^{\pm}(494)$, K^0, $\bar{K}^0(498)$
1^-	$\rho^{\pm,0}(771)$, $\omega(783)$ $K^*\pm$, K^{*0}, $\bar{K}^{*0}(892)$, $\Phi(1020)$
$\frac{1}{2}^+$	$p(938)$, $n(939)$, $\Lambda(1116)$, $\Sigma^{\pm,0}(1193)$, $\Xi^{0,-}(1318)$
$\frac{3}{2}^+$	Δ^{++}, Δ^+, Δ^0, $\Delta^-(1232)$ $\Sigma^{*\pm,0}(1385)$, $\Xi^{*\pm,0}(1530)$, $\Omega^-(1672)$

Table 3.2: Low-mass hadrons.

$$|\eta_0> \;=\; \frac{1}{\sqrt{3}}\left(|\bar{u}u> + |\bar{d}d> + |\bar{s}s>\right) \ . \tag{3.92}$$

In these equations we would like to identify the physical states η and η' with the $SU(3)_f$ states η_8 and η_0 since the η mass is closer to the masses of the other members of the octet. However out of the two states $|\eta_8>$ and $|\eta_0>$ we can construct, by the superposition principle, new states having the same quantum numbers Y and I. This procedure is called mixing and the actual content of the physical states η and η' is as follows

$$\begin{aligned} |\eta> &= \cos\theta|\eta_8> + \sin\theta|\eta_0> \\ |\eta'> &= -\sin\theta|\eta_8> + \cos\theta|\eta_0> \end{aligned} \tag{3.93}$$

The value of the mixing angle is

$$\theta \approx -20^o \tag{3.94}$$

and will be derived in a problem in section 3.8.

A useful representation of the pseudoscalar meson octet is by means of traceless 3×3 matrix. Using the fact that **8** is the adjunct representation, we write the octet as a matrix

$$||\mathcal{P}||$$

which is a linear combination of the 8 Gell-Mann matrices. Therefore a meson of quark content $q^i\bar{q}_j$ is represented by the matrix element $\mathcal{P}^i{}_j$. One gets

$$||\mathcal{P}|| = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta_8 \end{pmatrix} \ . \tag{3.95}$$

For the singlet one has

$$||\mathcal{P}_0|| = \frac{1}{\sqrt{3}} \begin{pmatrix} \eta_0 & 0 & 0 \\ 0 & \eta_0 & 0 \\ 0 & 0 & \eta_0 \end{pmatrix} . \tag{3.96}$$

1^- **mesons**:

$$\begin{aligned} |\rho^+> &= |u\bar{d}>, \quad |\rho^-> = |d\bar{u}>, \quad |\rho^0> = \frac{1}{\sqrt{2}}\left(|\bar{u}u> - |\bar{d}d>\right), \\ |K^{*+}> &= |u\bar{s}>, \qquad\qquad |K^{*0}> = |d\bar{s}>, \\ |K^{*-}> &= |s\bar{u}>, \qquad\qquad |\bar{K}^{*0}> = |s\bar{d}>, \\ |\rho_8> &= \frac{1}{\sqrt{6}}\left(|\bar{u}u> + |\bar{d}d> - 2|\bar{s}s>\right), \\ |\rho_0> &= \frac{1}{\sqrt{3}}\left(|\bar{u}u> + |\bar{d}d> + |\bar{s}s>\right) \end{aligned} \tag{3.97}$$

There are two physical states ω and ϕ having the same quantum numbers $Y = 0$ and $I = 0$ as the $SU(3)_f$ eigenstates ρ_8 and ρ_0. Also in this case mixing occurs and the actual content of the physical states ω and ϕ is as follows

$$|\omega> = \frac{1}{\sqrt{2}}\left(|\bar{u}u> + |\bar{d}d>\right), \qquad |\phi> = |\bar{s}s> . \tag{3.98}$$

This mixing (called ideal mixing) will be proved in section 3.6. The nonet (octet+singlet) of vector meson states $\mathcal{V}$ is represented by the matrix

$$||\mathcal{V}|| = \begin{pmatrix} \frac{\rho^0}{\sqrt{2}} + \frac{\omega}{\sqrt{2}} & \rho^+ & K^{*+} \\ \rho^- & -\frac{\rho^0}{\sqrt{2}} + \frac{\omega}{\sqrt{2}} & K^{*0} \\ K^{*-} & \bar{K}^{*0} & \phi \end{pmatrix} . \tag{3.99}$$

3.5.2 Baryons

Baryons are made up by three quarks. If the orbital angular momentum vanishes, the only possibilities for J^P is $1/2^+$ and $3/2^+$. As to $SU(3)_f$, baryons are organized in octets, singlets and decuplets.

The weight diagram for the octet is depicted in fig. 3.11. The low lying hadrons with $J^P = 1/2^+$ are in table 3.2. The quark content of the different states is as follows (we have grouped the states in isospin multiplets):

$$I = 1/2 : |p> = |u(ud)>, \qquad |n> = |d(du)>,$$

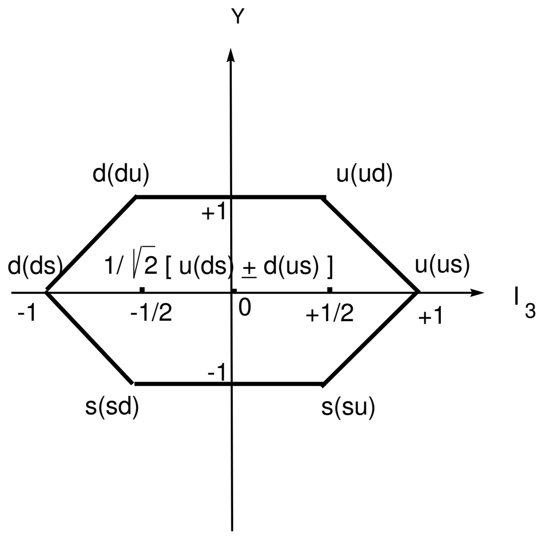


Figure 3.11: The weight diagrams for baryonic **8** and **1**.

$$
\begin{aligned}
I &= 1: \\
&\qquad |\Sigma^+ >= |u(us)>, \qquad |\Sigma^- >= |d(ds)>, \\
&\qquad |\Sigma^0 >= \frac{1}{\sqrt{2}} (|u(ds)> + |d(us)>), \\
I &= 0: \quad |\Lambda^0 >= \frac{1}{\sqrt{2}} (|u(ds)> - |d(us)>), \\
I &= 1/2: |\Xi^0 >= |s(su)>, \qquad |\Xi^- >= |s(sd)> . \qquad (3.100)
\end{aligned}
$$

Here, according to the rule (3.72), (qq') means

$$(qq') = q(1)q'(2) - q(2)q'(1) \qquad (3.101)$$

where the labels 1 and 2 identify the two different quarks. On this basis the state Λ^0 has not only $I_3 = 0$, but also $I = 0$ and differs from Σ^0 that has $I_3 = 0$, but $I = 1$. Using (3.72) we can represent the baryonic $\mathcal{B}$ octet by a traceless matrix as follows

$$\mathcal{B}^i_j = q^i (q^\ell q^k \epsilon_{j\,\ell\,k}) \qquad (3.102)$$

where i is the row index and j the column index. It follows that the matrix $||\mathcal{B}||$ is as follows:

$$||\mathcal{B}|| = \begin{pmatrix} \frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda & \Sigma^+ & p \\ \Sigma^- & -\frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda & n \\ \Xi^- & \Xi^0 & -\frac{2}{\sqrt{6}}\Lambda \end{pmatrix} . \qquad (3.103)$$

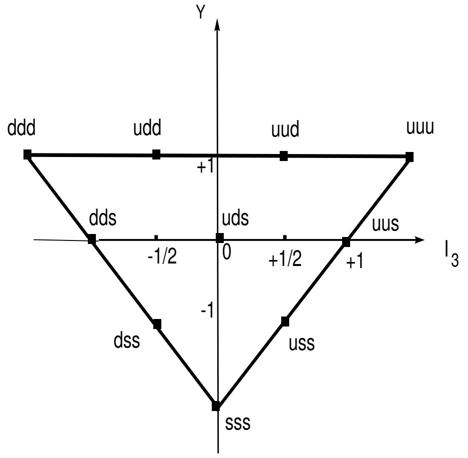


Figure 3.12: The weight diagrams for the baryonic decuplet.

Next we consider the weight diagram for the decuplet, depicted in fig. 3.12. Experimentally these states have spin 3/2 and can be found, together with their masses in Table 3.2. The quark content of these states is as follows (also in this case we have grouped the states in isospin multiplets):

$$\begin{aligned}
I &= 3/2: \\
&\quad |\Delta^{++}>=|uuu>, \ |\Delta^{+}>=|uud>, \\
&\quad |\Delta^{0}>=|udd>, \quad |\Delta^{-}>=|ddd> \\
I &= 1: \\
&\quad |\Sigma^{*+}>=|uus>, \quad |\Sigma^{*-}>=|dds>, \\
&\quad |\Sigma^{*0}>=|uds>, \\
I &= 1/2: |\Xi^{*0}>=|uss>, \quad |\Xi^{*-}>=|dss> \\
I &= 0: \quad |\Omega^{-}>=|sss> \ .
\end{aligned} \tag{3.104}$$

3.6 Mass formulae

Let us write the Dirac lagrangian for the quark fields :

$$\mathcal{L} = \sum_i \bar{\psi}_i (i\gamma^\mu \partial_\mu - m_i)\psi_i \tag{3.105}$$

where the sum runs over the three flavors $i = u, d, s$. The remaining part of the lagrangian containing interaction does not depend on the flavor. Therefore $SU(3)_f$ would be a symmetry if m_i might be supposed equal. Since this is not true, let us write the mass term in matrix form

$$H = \bar{\psi} M \psi \tag{3.106}$$

with

$$\begin{aligned} M &= \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix} = \\ &= \frac{m_u - m_d}{2} F_3 + \frac{m_u + m_d - 2m_s}{\sqrt{3}} F_3 + \frac{m_u + m_d + m_s}{\sqrt{6}} F_0 \end{aligned} \tag{3.107}$$

where $F_a = \lambda_a/2$ and $F_0 = \frac{1}{\sqrt{6}} diag(1,1,1)$. We shall now assume that $SU(3)_f$ but not isospin is violated by mass terms, i.e. $m_s \neq m_u$ but $m_u = m_d$. Let us consider the effect on the mass splitting within baryonic multiplets. One has

$$\begin{aligned} &< B|H|B' >= \frac{m_u - m_d}{2} < B|F_3|B' > + \\ &+ \frac{m_u + m_d - 2m_s}{\sqrt{3}} < B|F_8|B' > + \frac{m_u + m_d + m_s}{\sqrt{6}} < B|F_0|B' >= \\ &= \frac{m_u + m_d - 2m_s}{\sqrt{3}} \left\{ a_8 \mathrm{Tr}\left(B' B^\dagger F_8 \right) + a_8' \mathrm{Tr}\left(B^\dagger B' F_8 \right) \right\} + \\ &+ a_0 \frac{m_u + m_d + m_s}{\sqrt{6}} \mathrm{Tr}(B^\dagger B' F_0) \ . \end{aligned} \tag{3.108}$$

In conclusion we have, for the baryon $B = B'$ with mass m_B:

$$m_B = m_0 + \sqrt{2}\alpha \mathrm{Tr}\left(B^\dagger B F_8 \right) + \sqrt{2}\beta \mathrm{Tr}\left(B B^\dagger F_8 \right) \ . \tag{3.109}$$

Using (3.103) one can express the four different masses of the octet in terms of three parameters. Therefore one gets a constraint:

$$\frac{m_\Xi + m_N}{2} = \frac{m_\Sigma + 3m_\Lambda}{4} \ , \tag{3.110}$$

known as the Gell-Mann Okubo formula for baryons.

Problem. Derive the Gell-Mann Okubo formula. Test its validity using hadron masses as given in Table 3.2 and $m_N = (m_p + m_n)/2$.

For the pseudoscalar mesons we have instead of (3.109):

$$m_M^2 = m_0^2 + \sqrt{2}\alpha \mathrm{Tr}\left(M^\dagger M F_8 \right) + \sqrt{2}\beta \mathrm{Tr}\left(M M^\dagger F_8 \right) \ . \tag{3.111}$$

A reason why in the formula for mesons one should better use quadratic masses might be the presence of quadratic mass term in the scalar lagrangian. Using (3.95) and working as before we get the Gell-Mann Okubo formula for pseudoscalar mesons

$$m_K^2 = \frac{m_\pi + 3m_{\eta_8}^2}{4} \ . \tag{3.112}$$

If we use for m_{η_8} the value reported in table 3.2 we get agreement within 7%. The agreement improves if we admit mixing, as given by eq. (3.93). With the mixing angle given in (3.94) the disagreement decreases to less than 1%. Finally we consider vector mesons. Let us first assume that the matrix for the vector meson octet is, analogously to (3.95),

$$||\mathcal{V}^{(8)}|| = \begin{pmatrix} \frac{\rho^0}{\sqrt{2}} + \frac{\omega_8}{\sqrt{6}} & \rho^+ & K^{*+} \\ \rho^- & -\frac{\rho^0}{\sqrt{2}} + \frac{\omega_8}{\sqrt{6}} & K^{*0} \\ K^{*-} & \bar{K}^{*0} & -2\,\frac{\omega_8}{\sqrt{6}} \end{pmatrix} \ . \tag{3.113}$$

Then the Gell-Mann Okubo formula would give, in this case,

$$3m_{\omega_8}^2 = 4m_{K^*}^2 - m_\rho^2 \ , \tag{3.114}$$

i.e.

$$m_{\omega_8} = 926.5 \text{ MeV} \ , \tag{3.115}$$

which differs significantly from the two experimental masses m_ω and m_ϕ, see Table 3.2. However we expect mixing between the states ω_8 and the singlet ω_0 in other words the quadratic mass term $\omega M^2 \omega$ has the form

$$\omega M^2 \omega = \begin{pmatrix} \omega_8 & \omega_0 \end{pmatrix} \begin{pmatrix} m_{\omega_8}^2 & m_{\omega_8 0}^2 \\ m_{\omega_8 0}^2 & m_{\omega_0}^2 \end{pmatrix} \begin{pmatrix} \omega_8 \\ \omega_0 \end{pmatrix} \ . \tag{3.116}$$

We diagonalize M^2 by the rotation

$$R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{3.117}$$

and we get the eigenvectors

$$\begin{aligned} \omega &= \cos\theta\omega_8 + \sin\theta\omega_0 \\ \phi &= -\sin\theta\omega_8 + \cos\theta\omega_0 \end{aligned} \tag{3.118}$$

together with the eigenvalues m_ω^2 and m_Φ^2, given by the relations

$$m_{\omega_8}^2 = \cos^2\theta\, m_\omega^2 + \sin^2\theta\, m_\phi^2 \tag{3.119}$$
$$m_{\omega_0}^2 = \sin^2\theta\, m_\omega^2 + \cos^2\theta\, m_\phi^2 \tag{3.120}$$
$$m_{\omega_{8\,0}}^2 = \sin 2\theta\, \frac{m_\phi^2 - m_\omega^2}{2}\ . \tag{3.121}$$

Using (3.115), (3.119) and the values in Table 3.2 one gets $\sin\theta = +0.76$. Using the approximate value

$$\sin\theta = \sqrt{\frac{2}{3}}\ , \tag{3.122}$$

together with eq. (3.113) and the analogous of (3.96) for ω_0, we get the ideal mixing results of matrix (3.99).

Problem. Fill in the steps leading to eqns.(3.119)-(3.122)

3.7 Color and heavy quarks

In the constituent quark model the particle Δ^{++} is formed by three identical spin 1/2 u quarks. The three spin 1/2 add to form a spin 3/2 system, therefore the spin wavefunction is symmetric; moreover the orbital angular momentum is zero, therefore the orbital wavefunction is symmetric as well. However the total wavefunction of the uuu system must be antisymmetric since the three identical constituents are fermions. It follows that the total wavefunction must contain an extra factor antisymmetric under fermion exchange. If we attribute an extra degree of freedom $\alpha = 1, 2, 3$ to quarks then the missing piece has the form $\epsilon^{\alpha\beta\gamma}$ and the antisymmetry is guaranteed. This extra degree of freedom is called *color*. It is assumed that color is a *exact* symmetry of strong interactions and that the symmetry group is $SU(3)$, which we denote $SU(3)_c$ to make distinction with $SU(3)_f$. Quarks transforms as the **3** of $SU(3)_c$. It is assumed that all the observed hadrons are color singlets i.e. they are in the IRR **1** of $SU(3)_c$, with irreducible tensor $\delta^{\alpha\beta}$ (mesons) or $\epsilon^{\alpha\beta\gamma}$ (baryons). The internal symmetry $SU(3)_c$ differs from $SU(3)_f$ not only by being exact and not approximate, but also because it is a *local* and not a global gauge symmetry. This means that the strong interaction lagrangian is assumed to be invariant under transformations

$$q^\alpha \to U_{\alpha\beta} q^\beta\ . \tag{3.123}$$

$U \in SU(3)$ is given by an equation similar to (3.29):

$$\exp(i \sum_{a=1}^{8} \phi_a(x)\lambda_a/2) \tag{3.124}$$

with parameters depending on space and time: $x^\mu = (t, \vec{r})$. The local gauge theory based on the color symmetry group $SU(3)_c$ is called Quantum Chromodynamics; its formulation is based on the Yang Mills construction of section 2.1.2. The QCD Lagrangian can be therefore written as follos

$$\mathcal{L} \ = \ -\frac{1}{4} G^a_{\mu\nu} G^{a\,\mu\nu} + \sum_f \bar{q}_f (i\gamma \cdot D - m_f) q_f \tag{3.125}$$

where f denotes the flavor (for $f = u, d, s ...$), the sum over a is from 1 to 8, $D_\mu = \partial_\mu + igT_a G^a_\mu$. T_a are the $SU(3)$ generators in the representation **3** and $G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu - g f_{abc} G^b_\mu G^c_\nu$. The constant g is the strong interaction coupling constant. The particles corresponding to the fields G^a_μ are called gluons. These massless spin one particles belong to the **b**, therefore, similarly to the quarks, are not color-neutral.

Let us conclude with a remark on the number of quarks. We have discussed so far the so-called light quarks u, d, s. Since their masses are much smaller than typical hadronic masses [4], they can be assumed equal to a good extent and this is the basis for the success of the approximate flavor symmetry $SU(3)_f$. If they can be neglected strong interactions have an extra symmetry, the chiral symmetry, to be discussed in chapter 5.

Other three quarks have been discovered in the last 30 years. They are the the *charm* c, the *beauty* b and the *top* t.

3.8 Problems

1. Which would be the quark content of hadrons in the IRR **2**7?
2. Prove (3.73). You might consult the textbook by K. Huang.
3. Derive the result (3.94) using the same procedure adopted to derive the ideal mixing in the vector meson nonet.

[4] The masses of the quarks are as follows. Light quarks: $m_u = 1.5 - 5$ MeV, $m_d = 3 - 9$ MeV, $m_s \approx 120$ MeV. Heavy Quarks: $m_c = 1.1 - 1.4$ GeV, $m_b = 4.1 - 4.4$ GeV, $m_t = 173.8 \pm 5.2$ GeV.

References

For the content of this chapter see K. Huang, *Quark Leptons & Gauge Fields*, World Sci. Singapore, 1982, and H. Georgi cit.

Chapter 4

Leptons

4.1 Leptons

Leptons are particles having no strong interactions. Six of them are known from experiment. Three are neutrinos, basically massless[1]: ν_e, ν_μ, ν_τ. The other three are the electron e (mass 0.51 MeV), the muon μ (mass 106 MeV), the τ (mass 1777 MeV). All these particles are spin 1/2 fermions. We denote the corresponding fields as ψ_e, ψ_{ν_e}, ..., or simply e, ν_e, etc. Together with these particles we have the antiparticles: e^+, $\bar{\nu}_e$, ... It is useful to introduce left handed and right handed components for each field ψ as follows

$$\begin{aligned}
\psi_L &= P_-\psi, \qquad\qquad \psi_R = P_-\psi \\
P_\pm &= \frac{1 \pm \gamma_5}{2}\,, \quad P_\pm^2 = P_\pm\,, \quad P_+P_- = P_-P_+ = 0\ . \qquad (4.1)
\end{aligned}$$

ψ_R and ψ_R are helicity eigenstates .

Problem. Prove this property.

Note the properties:

$$\begin{aligned}
\bar{\psi}_L &= \psi_L^\dagger\gamma_0 = \bar{\psi}P_+\,, \qquad \bar{\psi}_R = \psi_R^\dagger\gamma_0 = \bar{\psi}P_-\,, \\
\psi &= \psi_L + \psi_R\,, \qquad \bar{\psi} = \bar{\psi}_L + \bar{\psi}_R\,, \\
& \bar{\psi}\gamma^\mu\psi = \bar{\psi}_L\gamma^\mu\psi_L + \bar{\psi}_R\gamma^\mu\psi_R\,, \\
\bar{\psi}\psi &= \bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L\,, \quad \bar{\psi}_R\psi_R = \bar{\psi}_L\psi_L = 0\ . \qquad (4.2)
\end{aligned}$$

Only the left-handed neutrinos are considered. We can therefore organize the leptonic fields (also called *matter fields*) into three sets or

[1]Recent experimental data show that neutrinos have a tiny mass. We will neglect this effect here.

families as follows:

$$\{\psi_L^{(j)},\, \psi_R^{(j)}\} \qquad (j=1,2,3 \text{ for 1st, 2nd and 3rd family})\ . \tag{4.3}$$

Here

$$\begin{aligned} \psi_L^{(1)} &= \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L , \qquad && \psi_R^{(1)} = e_R \\ \psi_L^{(2)} &= \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L , \qquad && \psi_R^{(2)} = \mu_R \\ \psi_L^{(3)} &= \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L , \qquad && \psi_R^{(3)} = \tau_R \end{aligned} \tag{4.4}$$

To begin with we assume that these particles are massless; their lagrangian is therefore, for each family,

$$\mathcal{L} = \bar{\psi}_L i\gamma\cdot\partial\psi_L \,+\, \bar{\psi}_R i\gamma\cdot\partial\psi_R\ . \tag{4.5}$$

We assume that under transformations belonging to the symmetry group

$$SU(2)_L \times U(1) \tag{4.6}$$

the fields transform as follows:

$$\begin{aligned} U &\in SU(2)_L : \qquad && \psi_L \to U\psi_L\,, \qquad && \psi_R \to \mathbf{1}\psi_R \\ V &\in U(1) : \qquad && \psi_L \to V\psi_L\,, \qquad && \psi_R \to V\psi_R\ . \end{aligned} \tag{4.7}$$

As a consequence, the lagrangian (4.5) is invariant under (4.6). Next we make a stronger assumption, i.e. that the Dirac Lagrangian for the massless leptons is invariant under local transformation of the group (4.6). We know already that to transform a global internal symmetry into a local gauge symmetry one has to introduce gauge bosons, which play a role analogous to the photon in QED. In our case the substitution is

$$\partial^\mu \to D^\mu = \partial^\mu + ig\mathbf{W}^\mu + ig'X^\mu S \qquad (\mathbf{W}^\mu = W_a^\mu T_a) \tag{4.8}$$

Let us comment on this equation. We have two coupling constants g, g' because of the two factors in (4.6). T_a $(a = 1,2,3)$ and S are generators of $SU(2)_L$ and $U(1)$ respectively, with

$$[T_a, T_b] = i\epsilon_{abc}T_c\,, \qquad [T_a, S] = 0\ . \tag{4.9}$$

W_a^μ and X^μ are four vector bosons. Since ψ_L is a doublet and ψ_R a singlet under $SU(2)_L$, we have

$$T_a\psi_L = \frac{\tau_a}{2}\psi_L\,, \qquad T_a\psi_R = 0\,. \tag{4.10}$$

To define the action of S on the matter fields we define it as follows:

$$S = -T_3 + Q = \frac{Y}{2} \ , \tag{4.11}$$

where Y is called weak hypercharge. Therefore, since

$$Q\psi_L = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \psi_L \, , \qquad Q\psi_R = -\psi_R \, , \tag{4.12}$$

we have

$$S\psi_L = -\frac{1}{2}\psi_L \, , \qquad S\psi_R = -\psi_R \, . \tag{4.13}$$

Due to (4.8) the lagrangian assumes the form for each family

$$\mathcal{L}_1 = \mathcal{L}_L + \mathcal{L}_R \ , \tag{4.14}$$

where, e.g. for the first family

$$\begin{aligned} \mathcal{L}_L &= \bar{\psi}_L i\gamma \cdot D\psi_L = i \begin{pmatrix} \bar{\nu}_{L\,e} & \bar{e}_L \end{pmatrix} \gamma \cdot (\partial + ig\mathbf{W} + ig'XS) \begin{pmatrix} \nu_{L\,e} \\ e_L \end{pmatrix} \\ \mathcal{L}_R &= \bar{\psi}_R i\gamma \cdot D\psi_R = i\, \bar{e}_R \gamma \cdot (\partial + ig'XS) e_R \ . \end{aligned} \tag{4.15}$$

Let us consider the term containing interactions between matter fields and gauge fields in $\mathcal{L}$:

$$\begin{aligned} &- \begin{pmatrix} \bar{\nu}_{L\,e} & \bar{e}_L \end{pmatrix} \gamma \cdot (g\mathbf{W} + g'XS) \begin{pmatrix} \nu_{L\,e} \\ e_L \end{pmatrix} - \bar{e}_R \gamma \cdot (g'XS) e_R = \\ &= -\frac{g}{2\sqrt{2}} \left(\bar{\nu}_e \gamma_\mu (1 - \gamma_5) e W^\mu_\pm + h.c. \right) - \bar{\psi}_L \, \gamma_\mu \left(g W^\mu_3 T_3 + g' X^\mu S \right) \psi_L \\ &- \bar{\psi}_R \, \gamma_\mu \left(g W^\mu_3 T_3 + g' X^\mu S \right) \psi_R \end{aligned} \tag{4.16}$$

where we have used (4.10) and defined

$$W^{\pm\mu} = \frac{W^\mu_1 \pm i W^\mu_2}{\sqrt{2}} \ . \tag{4.17}$$

The first term on the r.h.s of (4.16) describes the interaction of a charged vector boson W with a charged lepton and its neutrino; it produces Feynman diagrams such as the one depicted in fig. 4.1. The charged vector bosons were discovered at CERN in the decade 1980.

Let us consider the other two terms in (4.16). Since we have two neutral vector bosons X and W_3 we may hope to describe both the photon γ of field A^μ and the neutral vector boson Z also discovered at

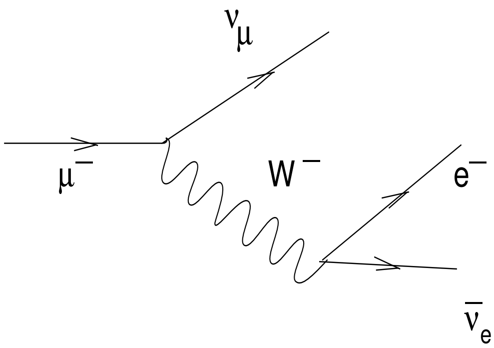


Figure 4.1: The decay $\mu^- \to \nu_\mu e^- \bar{\nu}_e$.

CERN in the 1980's. However we may expect mixing between X and W_3 as they have the same quantum numbers. Therefore we define

$$\begin{aligned} A^\mu &= \sin\theta\, W_3^\mu + \cos\theta\, X^\mu \ , \\ Z^\mu &= \cos\theta\, W_3^\mu - \sin\theta\, X^\mu \ . \end{aligned} \tag{4.18}$$

In the last two terms of (4.16) we have

$$\begin{aligned} & gW_3^\mu T_3 + g' X^\mu S = \\ & = A^\mu \left(g \sin\theta\, T_3 + g' \cos\theta\, S\right) + Z^\mu \left(g\cos\theta\, T_3 - g' \sin\theta\, S\right) \ . \end{aligned} \tag{4.19}$$

We now identify A^μ with the photon field and Z^μ with the field of the Z^0 spin 1 boson.

The coefficient of the photon field must be equal to eQ therefore

$$g \sin\theta\, T_3 + g' \cos\theta\, S = e(T_3 + S) \ , \tag{4.20}$$

which implies the fundamental relations

$$g = \frac{e}{\sin\theta} \qquad g' = \frac{e}{\cos\theta} \ . \tag{4.21}$$

Therefore the coupling to the photon is simply

$$-\bar{\psi}_L\, \gamma_\mu\, (eQA^\mu)\, \psi_L \ - \ \bar{\psi}_R\, \gamma_\mu\, (eQA^\mu)\, \psi_R = eA_\mu \bar{e}\gamma^\mu\, e \quad . \tag{4.22}$$

We can also derive the couplings of Z^0 to leptons:

$$\begin{aligned} \mathcal{L}_{Z\psi\psi} = \quad & -Z^\mu \bar{\psi}_L\, \gamma_\mu\, (g\cos\theta\, T_3 - g' \sin\theta\, S)\, \psi_L \\ & -Z^\mu\, \bar{\psi}_R\, \gamma_\mu\, (g\cos\theta\, T_3 - g' \sin\theta\, S)\, \psi_R = \end{aligned}$$

$$= -\frac{e(-1+2\sin^2\theta)}{2\sin\theta\cos\theta} Z^\mu \bar{e}_L \gamma_\mu e_L - \frac{e\sin\theta}{\cos\theta} Z^\mu \bar{e}_R \gamma_\mu e_R$$
$$-\frac{e}{2\sin\theta\cos\theta} Z^\mu \bar{\nu}_L \gamma_\mu \nu_L \quad . \tag{4.23}$$

Problem. Using the Z^0 couplings determine the width $\Gamma(Z^0 \to \mu^+\mu^-)$.

4.1.1 Gauge fields

Besides the term $\mathcal{L}_1$ describing the kinetic term of the fermions and their couplings to the gauge fields, eq. (4.14), the lagrangian contains the pure gauge fields term, containing the kinetic terms of the gauge bosons and their self-couplings. It has the form discussed in subsection 2.1.2:

$$\mathcal{L}_2 = -\frac{1}{2} Tr \mathbf{W}^{\mu\nu}\mathbf{W}^{\mu\nu} - \frac{1}{4} X_{\mu\nu} X^{\mu\nu} \tag{4.24}$$

When expressed in terms of orthogonal combinations it gives rise to the kinetic terms of the four gauge bosons $W^\pm$, Z, γ and to self interactions, i.e. $W^+W^+\gamma$ or W^+W^-Z couplings.

4.2 Effective theory of weak interactions

4.2.1 Fermi theory of weak interactions

So far we have not yet considered the W mass; we know from experiment that the W boson has indeed a mass $m_W \simeq 80$ GeV. Its propagator would be

$$iD_{\lambda\sigma}(q) = i\, \frac{g_{\lambda\sigma} - q_\lambda q_\sigma / m_W^2}{q^2 - m_W^2} \tag{4.25}$$

Let us compute the amplitude for the process depicted in fig.4.1. We get

$$\begin{aligned}
\mathcal{M} &= i < \nu_\mu(k_1) e^-\bar{(}k_2) \nu_e(k_3) | \mathcal{L} | \mu^-(p) >= \\
&= \left(\frac{-ig}{2\sqrt{2}}\right)^2 \bar{u}(k_1)\gamma^\lambda(1-\gamma_5)u(p) \times \bar{u}(k_2)\gamma^\sigma(1-\gamma_5)v(k_3) iD_{\lambda\sigma}(q) = \\
&\approx i < \nu_\mu(k_1) e^-\bar{(}k_2) \nu_e(k_3) | \frac{G_F}{\sqrt{2}} j^\lambda j_\lambda^\dagger | \mu^-(p) >
\end{aligned} \tag{4.26}$$

with

$$G_F = \frac{g^2}{4\sqrt{2} m_W^2} \tag{4.27}$$

Here we have used the fact that $m_W \gg |q|$. This matrix element is therefore approximately equivalent to the one produced by an effective lagrangian

$$\mathcal{L}_{eff} = \frac{G_F}{\sqrt{2}}\, j^\lambda j^\dagger_\lambda \tag{4.28}$$

where j^λ is the leptonic current is given by

$$j^\lambda = \sum_{\ell=e,\mu,\tau} \bar{\nu}_\ell \gamma^\lambda (1-\gamma_5)\ell \tag{4.29}$$

j^λ changes by 1 the lepton charge and it is therefore called *charged current*. Note the structure $V - A$ of the current and the common coupling to all the families, which is named *universality of the weak current*. The constant G_F is called Fermi constant:

$$G_F = 1.166 \times 10^{-5}\,\mathrm{G}eV^{-2}\ . \tag{4.30}$$

4.2.2 Neutral currents

For small momentum transfer the Fermi theory can be extended to include neutral currents. They relate lepton fields having the same electric charge and can be derived from the couplings (4.23) in the approximation $|q| \ll m_Z$.

Problem. Show that the effective lagrangian for neutral currents can be written as

$$\mathcal{L}_{n.c.} = \frac{G_F}{\sqrt{2}} \left(\frac{m_W}{m_Z}\right)^2 j^\lambda_{n.c.} j_{n.c.\,\lambda} \tag{4.31}$$

Show that the leptonic neutral current has the form

$$j^\lambda_{n.c.} = \sum_{\ell=e,\mu\tau} \Big(\bar{\ell}\gamma^\lambda(a+b\gamma_5)\ell + c\bar{\nu}_\ell\gamma^\lambda(1-\gamma_5)\nu_\ell\Big) \tag{4.32}$$

and determine the coefficients a, b, c.

Neutral currents were discovered at CERN in 1973, 10 years before the discovery of the gauge bosons W, Z^0. The process leading to their discovery was

$$\bar{\nu}_\mu e^- \to \bar{\nu}_\mu e^- \tag{4.33}$$

that cannot take place by charged current. Their discovery was a fundamental success for the unified model of electroweak interactions.

4.3 Glashow-Weinberg-Salam Model

4.3.1 Renormalizability

Let us consider the Fermi lagrangian (4.28) having a coupling constant of dimension $[M]^{-2}$, see (4.30). A theory having a coupling constant with a negative dimension is not renormalizable. Let us discuss this point in some detail.

Since we use $\hbar = c = 1$, then $[M] = [E] = [L]^{-1} = [T]^{-1}$. Action S is dimensionless and therefore the lagrangian density $\mathcal{L}$ has dimension 4: $[\mathcal{L}] = [M]^4$.

Problem. Determine the canonical dimension of spin 0, 1/2, 1 fields.

Now let us consider for example the process $\nu_\mu e^- \to \nu_\mu e^-$. Besides the tree level contribution, which is proportional to G_F and has no loops, we can consider a perturbative correction $\propto G_F^2$ arising from a Feynman diagram with one loop, see Fig. 4.2. The extra factor of G_F in

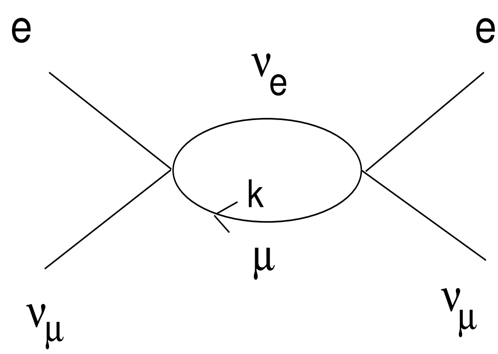


Figure 4.2: The scattering $\nu_\mu e \to \nu_\mu e$ at one loop in the Fermi theory.

the loop diagram, as compared to the tree diagram, is compensated by two extra powers of mass that are provided by two powers of the integration variable k. The diagram is therefore ultraviolet (UV) divergent [2], due to the bad behavior when $|k| \to \infty$. The procedure to eliminate UV divergences consists of: i) regularize the divergent integrals, for

[2]We considered a similar diagram in fig. 1.7 for the $\lambda\phi^4$ theory. Its expression in eq. (1.144) shows a logarithmic divergence $\mathcal{O}(\Lambda^2/s)$ when Λ, the upper limit on the momentum integration (also called UV cutoff) goes to infinity. We omit here the discussion on infrared divergences that can arise from the behavior at small momenta, for example because of vanishing masses in the propagators.

example by cutting the loop integral with some momentum cut-off Λ. This introduces an unphysical dependence on Λ in the amplitude. To get rid of it one adds an *ad hoc* term $\delta\mathcal{L}(\Lambda)$ in the lagrangian (called a *counterterm*) such that in the limit $\Lambda \to \infty$, when one removes the cut-off, the amplitude remains finite. This is the usual procedure, that clearly works only if the number of counterterms is finite. Theories whose UV divergences can be cured in this way are called renormalizable. Fermi theory is not renormalizable, because adding new vertices in new loops implies introducing extra factors of G_F; therefore the corresponding Feynman diagram would have a cut-off dependence $\mathcal{O}(\Lambda)^n$ with larger and larger exponent n. As a consequence an infinite number of counterterms would be necessary to get rid of UV divergences at all orders in the perturbative expansion.

Renormalizable theories are preferred because they allow in principle accurate calculations by the inclusion of terms of higher order in the perturbative series. Let us therefore control if the exact theory, which includes the gauge bosons W and Z represents an improvement. Apparently it does, because the coupling constants g and g' are dimensionless. However a problem still remains and arises from gauge boson masses. Let us consider the Z gauge boson whose propagator is as follows:

$$iD_{\mu\nu}(k) = i\,\frac{-g_{\mu\nu} - k_\mu k_\nu/m_Z^2}{k^2 - m_Z^2 + i\epsilon}\ . \tag{4.34}$$

The term $k_\mu k_\nu/(k^2 - m_Z^2 + i\epsilon)$ behaves as a constant for $|k| \to \infty$. As a consequence the $SU(2) \times U(1)$ model with massive gauge bosons is not renormalizable.

The way to overcome this difficulty was discovered at the end of the decade '60 of last century by Abdus Salam and Steven Weinberg who modified the $SU(2) \times U(1)$ model originally introduced by Glashow. They noticed that the presence of the mass term actually is not allowed by the gauge symmetry. The non-renormalizability of the massive gauge boson model is therefore related to the explicit breaking of the symmetry. There is however a different way by which gauge bosons can acquire mass. It is possible when a gauge theory is spontaneously broken. While in general SSB implies the existence of massless Nambu-Goldstone scalar fields, in presence of gauge fields the NGBs are absent and these missing degrees of freedom appear as longitudinal components of the gauge bosons. In other words the gauge bosons acquire masses. This effect is known as the Higgs-Anderson mechanism. Since the theory is only spontaneously broken, even though the physical states are not symmetric the lagrangian does maintain its symmetry properties. As a consequence one can prove that the spontaneously broken $SU(2) \times U(1)$ model is renormalizable.

4.3.2 Higgs mechanism in the GWS model

Besides the gauge boson mass, another problem of the $SU(2) \times U(1)$ model is the absence of lepton masses. We now show that, if the symmetry $SU(2) \times U(1)$ is spontaneously broken, then not only the gauge bosons, but also the fermions acquire masses.

It is useful to start with one family, when the electron mass will appear in the lagrangian through a term

$$\sim \bar{e}_R e_L \ . \tag{4.35}$$

It will appear by SSB, i.e. through the non vanishing vacuum expectation value of a field. It is natural to assume the existence of a scalar field ϕ and a lagrangian term

$$\mathcal{L}_4 = -f \bar{e}_R \phi^\dagger \psi_L + h.c. \tag{4.36}$$

It is obvious that we need at least a doublet ϕ of complex scalar fields to construct a singlet under the gauge group. In other words

$$\vec{T}\phi = \frac{\vec{\tau}}{2}\,\phi \ . \tag{4.37}$$

On the other hand

$$S\phi = \frac{1}{2}\phi \tag{4.38}$$

and, introducing the components of ϕ we have:

$$\phi = \begin{pmatrix} \varphi^+ \\ \varphi^0 \end{pmatrix} , \tag{4.39}$$

where φ^+ has charge $+1$ while φ^0 is neutral.

Problem. Prove that the assumption (4.38) ensures invariance of (4.36) under transformation of $U(1)$. Prove also that the structure of (4.39) follows from $Q = T_3 + S$.

If now $< 0|\varphi^0|0 >= v/\sqrt{2} \neq 0$, then the electron will have a mass. This point will be discussed below, see eq. (4.62). The scalar field will have a kinetic term with a gauge covariant derivative and a self coupling potential V implementing the hypothesis of SSB, see for comparison (2.49):

$$\mathcal{L}_3 = (D\phi)^\dagger (D\phi) - V(\phi) \tag{4.40}$$

$$V(\phi) = \frac{\lambda}{2}\left(\phi^\dagger\phi - \frac{v^2}{2}\right)^2 \tag{4.41}$$

The vev v is defined by

$$< 0|\phi|0 > = \begin{pmatrix} < \varphi^+ >_0 \\ < \varphi^0 >_0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} , \tag{4.42}$$

Similarly to the choice in subsection 2.2.1 we write the fields in polar components

$$\phi = \begin{pmatrix} \varphi^+ \\ \varphi^0 \end{pmatrix} = e^{-i\vec{\tau}\cdot\vec{\theta}(x)/v} \begin{pmatrix} 0 \\ \frac{v+\eta(x)}{\sqrt{2}} \end{pmatrix} . \tag{4.43}$$

The four real scalar $\vec{\theta}$ and η have zero vev:

$$< \theta_k >_0 = < \eta >_0 = 0 \tag{4.44}$$

and are linear combination of the four components of the complex fields $\varphi^{+,0}$. Let us perform a gauge transformation with

$$U = e^{+i\vec{\tau}\cdot\vec{\theta}(x)/v} . \tag{4.45}$$

Then

$$\phi \rightarrow \phi' = \begin{pmatrix} 0 \\ \frac{v+\eta(x)}{\sqrt{2}} \end{pmatrix} , \tag{4.46}$$

$$\psi_L \rightarrow \psi'_L = U\psi_L , \quad \psi_R \rightarrow \psi'_R = \psi_R , \tag{4.47}$$

$$\frac{\vec{\tau}\cdot\vec{W}^\mu}{2} \rightarrow \frac{\vec{\tau}\cdot\vec{W}'^\mu}{2} = U\,\frac{\vec{\tau}\cdot\vec{W}^\mu}{2}\,U^{-1} + \frac{i}{g}(\partial^\mu U)U^{-1} , \tag{4.48}$$

$$X^\mu \rightarrow X'^\mu = X^\mu . \tag{4.49}$$

Expressing the scalar field lagrangian $\mathcal{L}_3$ in terms of ϕ' produces a lagrangian where the degree of freedoms $\vec{\theta}$ have disappeared and only the scalar field $\eta(x)$ appears:

$$\mathcal{L}_3(\phi') = (D\phi')^\dagger(D\phi') - V(\phi') \tag{4.50}$$

where

$$D_\mu\phi' = \left(\partial_\mu + i\,g\frac{\vec{\tau}\vec{W}'_\mu}{2} + i\,\frac{g'}{2}X_\mu\right)\begin{pmatrix} 0 \\ 1 \end{pmatrix}\frac{v+\eta}{\sqrt{2}} \tag{4.51}$$

$$V(\eta) = \frac{\lambda\,v^2\eta^2}{2} + \frac{\lambda\eta^4}{8} + \frac{\lambda\,v\eta^3}{2} . \tag{4.52}$$

The η field describes a massive neutral particle, the Higgs particle H. It has the a mass

$$m_H^2 = \lambda\, v^2 \tag{4.53}$$

and cubic and quartic self-interactions, as shown by (4.52). The other particles $\vec{\theta}$ are called would-be Goldstone bosons. They are missing in the lagrangian but their role is played by gauge bosons. As a matter of fact let us consider first term in (4.50):

$$(D\phi')^\dagger (D\phi') \tag{4.54}$$

besides the kinetic term of the η field it contains the term

$$\begin{aligned}
&\frac{v^2}{2} \begin{pmatrix} 0 & 1 \end{pmatrix} \left(g\frac{\vec{\tau}\vec{W}'_\mu}{2} + \frac{g'}{2} X_\mu \right) \left(g\frac{\vec{\tau}\vec{W}'^{\mu}}{2} + \frac{g'}{2} X^\mu \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \\
&= \frac{M_W^2}{2} \left(W^{+\mu} W^+_\mu + W^{-\mu} W^-_\mu \right) + \frac{M_Z^2}{2}\, Z^\mu Z_\mu
\end{aligned} \tag{4.55}$$

where

$$M_W^2 = \frac{g^2 v^2}{4}\,, \tag{4.56}$$

and

$$M_Z^2 = \frac{g^2 + g'^2}{4}\, v^2\,. \tag{4.57}$$

It follows that the gauge bosons $W^\pm$ and Z are massive while the photon is massless.

Problem Prove (4.55-4.57).

Problem Using $\tan\theta = g'/g$ prove that

$$M_Z^2 = \frac{e^2\, v^2}{4 \sin^2\theta \cos^2\theta}\,. \tag{4.58}$$

Since the photon is massless, a subgroup $U(1)$, not identical with the $U(1)$ appearing in $SU(2) \times U(1)$, remains unbroken. Therefore, of the four generators of the symmetry group, only three are broken and to them would correspond three Nambu-Goldstone bosons, the $\vec{\theta}$ particles. However they are only would-be NGBs, because they have been eaten by the gauge bosons $W^\pm$ and Z; their degrees of freedom appear as the longitudinal degrees of freedom of $W^\pm$ and Z. Finally the Higgs particle is massive because the subgroup $U(1)_{em}$ is not spontaneously broken.

Let us finally note the relation, valid at the tree level

$$\rho = \frac{M_W^2}{M_Z^2 \cos^2\theta} = 1 \tag{4.59}$$

This gives the so called on-shell definition of the mixing angle θ, called the *Weinberg angle*. Using $m_W = 80.419 \pm 0.056$ GeV and $m_Z = 91.1882 \pm 0.0022$ GeV, the relation (4.59) gives

$$\sin^2\theta \approx 0.22 \ . \tag{4.60}$$

These relations get small corrections from higher order terms in the perturbative expansion; in particular from these corrections an estimate of around 100 GeV, with large uncertainty, is obtained for the mass of the yet undiscovered Higgs particle.

Let us finally determine the electron mass. Starting from (4.36) one gets

$$\begin{aligned}
\mathcal{L}_4 &= -f\bar{e}_R\phi^\dagger\psi_L + h.c. = -f\bar{e}'_R\phi'^\dagger\psi'_L + h.c. = \\
&= -f\bar{e}'_R \begin{pmatrix} 0 & \frac{v+\eta}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} + h.c. \\
&= -m_e\bar{e}e - \frac{f}{\sqrt{2}}\eta\,\bar{e}\,e
\end{aligned} \tag{4.61}$$

with

$$m_e = f\frac{v}{\sqrt{2}} \ . \tag{4.62}$$

The same term giving the electron a mass also gives a Yukawa coupling between the Higgs particle η and the electron, the μ, and the τ lepton.

4.4 Problems

1. Compute the total cross section for $\bar{\nu}_\mu e^- \to \bar{\nu}_\mu e^-$.

2. Determine the cross section for $e^+e^- \to e^+e^-$ for $\sqrt{s} \approx 90$ GeV including electromagnetic and weak contributions. Show that for $\sqrt{s} \approx m_Z$ the latter dominates.

3. Compute the cross section for the process

   $$e^-e^- \to e^-e^-$$

   at the tree level including weak and electromagnetic contributions.

4. Compute $\Gamma(Z^0 \to \nu_e\bar{\nu}_e)$.

5. Compute the cross section for $\bar{\nu}_e e^- \to \bar{\nu}_e e^-$ using the effective *current* $\times$ *current* theory of section 4.2.

References

For the content of this chapter any textbook on Quantum Field Theory with High Energy Physics orientation; for example H. Georgi, *cit.* or O. Nachtmann *Elementary particle Physics*, Springer, Berlin, 1989. For numerical data see Particle Data Group, Review of Particle Properties, http://pdg.lbl.gov/

Chapter 5

Quarks and the Glashow-Weinberg-Salam Model

5.1 Quarks and the GWS Model

Analogously to leptons, see eqns.(4.3) and (4.4), quarks are present in the GWS lagrangian both as doublets ψ_L and singlets ψ_R of the symmetry group organized into three families:

$$\psi_L = \begin{pmatrix} u_L \\ d'_L \end{pmatrix} , \begin{pmatrix} c_L \\ s'_L \end{pmatrix} , \begin{pmatrix} t_L \\ b'_L \end{pmatrix} \tag{5.1}$$

$$\psi_R = u_R\,, \;\; c_R\,, \;\; t_R\,, \;\; d'_R\,, \;\; s'_R\,, \;\; b'_R\,. \tag{5.2}$$

Differently from the leptonic sector we have 6 right fermion fields since all quarks have righ-handed components. The fields $d\,, s\,, b$ have different flavors, but they can mix (analogously to the mixing W^3, X) because weak interactions can change flavor. Therefore the fields having definite group properties , i.e. transforming as the second component of a doublet, are in general linear combinations of the quarks $d\,, s\,, b$ and we denote them as $d'\,, s'\,, b'$. The values of the quark weak hypercharge

$$Y = 2(Q - T_3)\ , \tag{5.3}$$

see (4.11), is in the following table 5.1, where we have also reported the values for leptons and scalar field.

leptons, scalar	Y	quarks	Y
ψ_L	-1	ψ_L	1/3
ψ_R	-2	$u_R,\, c_R,\, t_R$	4/3
ϕ	+1	$d'_R,\, s'_R,\, b'_R$	-2/3

Table 5.1: Weak hypercharge.

Quark fields appear in the lagrangian with terms completely analogous to (4.15) and (4.15). For example the coupling to $W^{\pm}$ is

$$-\frac{g}{\sqrt{2}}\left(J^{\lambda}\,W^{+}_{\lambda}\,+\,h.c.\right) \tag{5.4}$$

where the hadronic current is as

$$J_{\lambda}=\begin{pmatrix}\bar{u}_L & \bar{c}_L & \bar{t}_L\end{pmatrix}\gamma_{\lambda}\begin{pmatrix}d'_L\\ s'_L\\ b'_L\end{pmatrix} \tag{5.5}$$

Similarly one derives the coupling to electromagnetism and Z.

We have to relate the fields d', s', b' to d, s, b. To do that let us consider the Yukawa coupling that, as we know, gives masses to fermions, *via* the spontaneous breaking of the group symmetry. There are in general two couplings proportional to two different 3×3 complex matrices C and C':

$$\begin{aligned}\mathcal{L}_{yuk} \;=\; & -\begin{pmatrix}\bar{d}'_R & \bar{s}'_R & \bar{b}'_R\end{pmatrix}C'\begin{pmatrix}\phi^{\dagger}\begin{pmatrix}u_L\\ d'_L\end{pmatrix}\\ \phi^{\dagger}\begin{pmatrix}c_L\\ s'_L\end{pmatrix}\\ \phi^{\dagger}\begin{pmatrix}t_L\\ b'_L\end{pmatrix}\end{pmatrix}\\ & +\begin{pmatrix}\bar{u}_R & \bar{c}_R & \bar{t}_R\end{pmatrix}C\begin{pmatrix}\phi^{T}\epsilon\begin{pmatrix}u_L\\ d'_L\end{pmatrix}\\ \phi^{T}\epsilon\begin{pmatrix}c_L\\ s'_L\end{pmatrix}\\ \phi^{T}\epsilon\begin{pmatrix}t_L\\ b'_L\end{pmatrix}\end{pmatrix}\,.\end{aligned} \tag{5.6}$$

A term $\bar{\psi}_R\phi^{\dagger}\psi_L$ is manifestly invariant under transformations of $SU(2)_L$ since ϕ is a doublet of this subgroup. But also $\bar{\psi}_R\phi^{T}\epsilon\psi_L$, with $\epsilon=i\sigma_2$

is an invariant. In fact one has $\epsilon\sigma_i + \sigma_i^T\epsilon = 0$ and therefore, under an infinitesimal transformation,

$$\bar{\psi}_R\phi^T\epsilon\psi_L \to \bar{\psi}_R\phi^T(1+\alpha_i\sigma_i^T)\epsilon(1+i\sigma_k\alpha_k)\psi_L = \bar{\psi}_R\phi^T\epsilon\psi_L\ . \quad (5.7)$$

Let us now perform the following thansformations, with U_1, U_2, V rank 3 unitary matrices:

$$\begin{pmatrix} u_R \\ c_R \\ t_R \end{pmatrix} \to U_1 \begin{pmatrix} u_R \\ c_R \\ t_R \end{pmatrix}, \qquad \begin{pmatrix} \bar{d}'_R \\ \bar{s}'_R \\ \bar{b}'_R \end{pmatrix} \to U_2 \begin{pmatrix} \bar{d}'_R \\ \bar{s}'_R \\ \bar{b}'_R \end{pmatrix},$$
$$\begin{pmatrix} \begin{pmatrix} u_L \\ d'_L \end{pmatrix} \\ \begin{pmatrix} c_L \\ s'_L \end{pmatrix} \\ \begin{pmatrix} t_L \\ b'_L \end{pmatrix} \end{pmatrix} \to V \begin{pmatrix} \begin{pmatrix} u_L \\ d'_L \end{pmatrix} \\ \begin{pmatrix} c_L \\ s'_L \end{pmatrix} \\ \begin{pmatrix} t_L \\ b'_L \end{pmatrix} \end{pmatrix} \quad (5.8)$$

These field redefinitions correspond to the following transformations on C, C':

$$\begin{array}{rcl} C & \to & \tilde{C} = U_1^\dagger C V \\ C' & \to & \tilde{C}' = U_2^\dagger C' V\ . \end{array} \quad (5.9)$$

We choose U_1, V such that

$$\tilde{C} = diag(c_u,\, c_c,\, c_t)\ . \quad (5.10)$$

In fact, since

$$CC^\dagger \to U_1^\dagger CC^\dagger U_1\ . \quad (5.11)$$

we can choose U_1 requiring that $U_1^\dagger CC^\dagger U_1$ is diagonal:[1]

$$U_1^\dagger CC^\dagger U_1 = diag(c_u^2,\, c_c^2,\, c_t^2)\ . \quad (5.12)$$

Because of (5.12) the matrix $U_1^\dagger C$ has the form

$$U_1^\dagger C = \begin{pmatrix} c_u & 0 & 0 \\ 0 & c_c & 0 \\ 0 & 0 & c_t \end{pmatrix} W \quad (5.13)$$

[1]The eigenvalues of $CC^\dagger$ must be positive.

with[2] $c_i = c_u,\, c_c,\, c_t \geq 0$. Choosing

$$V = W^\dagger \tag{5.14}$$

one gets

$$\tilde{C} = U_1^\dagger C V = \begin{pmatrix} c_u & 0 & 0 \\ 0 & c_c & 0 \\ 0 & 0 & c_t \end{pmatrix} . \tag{5.15}$$

We can now repeat the procedure with C' to get, instead of (5.13)

$$U_2^\dagger C' = \begin{pmatrix} c_d & 0 & 0 \\ 0 & c_s & 0 \\ 0 & 0 & c_b \end{pmatrix} V_{ckm}^\dagger , \tag{5.16}$$

We cannot render $\tilde{C}'$ diagonal since we have already fixed V in (5.14). We can however apply a new transformation U_2 with

$$U_2 = V_{ckm}^\dagger . \tag{5.17}$$

In conclusion

$$\tilde{C}' = V_{ckm} \begin{pmatrix} c_d & 0 & 0 \\ 0 & c_s & 0 \\ 0 & 0 & c_b \end{pmatrix} V_{ckm}^\dagger . \tag{5.18}$$

5.2 Cabibbo-Kobayashi-Maskawa matrix

The unitary matrix V_{ckm} in (5.18) is called Cabibbo-Kobayashi-Maskawa matrix and is written down as follows:

$$V_{ckm} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} . \tag{5.19}$$

Let us determine the consequences of its presence in the lagrangian. We know that the Yukawa term *via* SSB generates quark masses. In fact working in the unitary gauge (4.45) the scalar field transforms as (4.46) and the Yukawa term trasforms as follows

$$\mathcal{L}_{yuk} = -\frac{v + \eta(x)}{\sqrt{2}} \times$$

[2]If c_i are not positive we can multiply them by appropriate phases that make them positive and reabsorb the phases in W.

$$\times \quad \left\{ \left(\begin{array}{ccc} \bar{d}'_R & \bar{s}'_R & \bar{b}'_R \end{array} \right) V_{ckm} \left(\begin{array}{ccc} c_d & 0 & 0 \\ 0 & c_s & 0 \\ 0 & 0 & c_b \end{array} \right) V^\dagger_{ckm} \left(\begin{array}{c} d'_L \\ s'_L \\ b'_L \end{array} \right) + \right.$$

$$+ \quad \left. \left(\begin{array}{ccc} \bar{u}_R & \bar{c}_R & \bar{t}_R \end{array} \right) \left(\begin{array}{ccc} c_u & 0 & 0 \\ 0 & c_c & 0 \\ 0 & 0 & c_t \end{array} \right) \left(\begin{array}{c} u_L \\ c_L \\ t_L \end{array} \right) + h.c. \right\} . \quad (5.20)$$

The term proportional to v is the quark mass term. Taking into account that

$$\bar{u}_R u_L + h.c. = \bar{u}\frac{1-\gamma_5}{2}u + \bar{u}\frac{1+\gamma_5}{2}u = \bar{u}u \quad (5.21)$$

one gets

$$\mathcal{L}_{mass} \quad = \quad -\left\{ \left(\begin{array}{ccc} \bar{d}' & \bar{s}' & \bar{b}' \end{array} \right) V_{ckm} \left(\begin{array}{ccc} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{array} \right) V^\dagger_{ckm} \left(\begin{array}{c} d' \\ s' \\ b' \end{array} \right) + \right.$$

$$+ \quad \left. \left(\begin{array}{ccc} \bar{u} & \bar{c} & \bar{t} \end{array} \right) \left(\begin{array}{ccc} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{array} \right) \left(\begin{array}{c} u \\ c \\ t \end{array} \right) \right\} , \quad (5.22)$$

where we have put

$$m_j = \frac{c_j\, v}{\sqrt{2}} \quad (5.23)$$

Therefore m_u, m_c, m_t are the masses of u, c and t quarks and the corresponding fermion fields are mass eigenstates. Clearly d', s' and b' are not mass eigenstates: the mass eigenstates are in fact

$$\left(\begin{array}{c} d \\ s \\ b \end{array} \right) = V^\dagger_{ckm} \left(\begin{array}{c} d' \\ s' \\ b' \end{array} \right) . \quad (5.24)$$

The CKM matrix does not play a role any longer in the mass term. However, if we re-express hadronic currents in terms of the mass eigenstates its elements give the weight V_{ij} for the contribution of the $\bar{q}_i q_j$ pair to the current. As a matter of fact we get from (5.5):

$$J_\lambda = \left(\begin{array}{ccc} \bar{u}_L & \bar{c}_L & \bar{t}_L \end{array} \right) \gamma_\lambda V_{ckm} \left(\begin{array}{c} d_L \\ s_L \\ b_L \end{array} \right) . \quad (5.25)$$

We notice that V_{ckm} only appears in the charged currents and not in the neutral currents giving the coupling of quarks to the photon and

Z. To a very good numerical approximation the matrix V_{ij} $(i,j=1,2)$ extracted from V_{ckm} can be approximated as follows

$$V_{ij} = \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix} \tag{5.26}$$

with

$$\sin\theta_c \approx 0.22 \tag{5.27}$$

called Cabibbo's angle. The absolute values of the other matrix elements are as follows: $|V_{ub}| \approx 3.7\times 10^{-3}$, $|V_{cb}| \approx 4.1\times 10^{-2}$, $|V_{td}| \approx 0.9\times 10^{-2}$,$|V_{ts}| \approx 4\times 10^{-3}$,$|V_{tb}| \approx 1.0$. It must be noted that the CKM matrix is complex and unitary and therefore can be parameterized in general by three real parameters and one phase.

5.3 Flavor Changing Neutral Currents and the Glashow-Iliopoulos-Maiani merchanism

The coupling of Z to leptons we wrote in (4.23) holds for all the fermions of the electroweak model and therefore also for quarks; since $S = Q - T_3$ we have:

$$\begin{aligned} \mathcal{L}_{Z\psi\psi} &= -\left[Z^\mu \bar\psi_L\,\gamma_\mu\,(g\cos\theta\,T_3 - g'\sin\theta\,S)\,\psi_L\ + L\to R\right] = \\ &- \frac{e}{\sin\theta\cos\theta}\left[Z^\mu\bar\psi_L\,\gamma_\mu\,(T_3 - \sin\theta^2\,Q)\,\psi_L\ + L\to R\right] \end{aligned} \tag{5.28}$$

Let us consider for example the first doublet

$$\psi_L = \begin{pmatrix} u \\ d' \end{pmatrix}_L \approx \begin{pmatrix} u \\ \cos\theta_c d + \sin\theta_c s \end{pmatrix}_L . \tag{5.29}$$

In particular there exists the Zd_Ls_L coupling:

$$\frac{e}{\sin\theta\cos\theta}\,\sin\theta_c\cos\theta_c\left(\frac{-1}{2} + \frac{\sin^2\theta}{3}\right)\bar d_L\gamma\cdot Z s_L\ . \tag{5.30}$$

If we compare it to the charged current coupling

$$\psi_L\tau_+\gamma\cdot W^+\psi_L \tag{5.31}$$

where we find

$$g\sin\theta_c\bar u_L\gamma\cdot W^+ s_L \tag{5.32}$$

we see that the couplings are of the same numerical order. The coupling (5.30) would be responsible for the decay $K^0 \to \mu^+\mu^-$ while the coupling (5.32) is responsible for the decay $K^+ \to \mu^+\nu_\mu$. Experimentally one has

$$\frac{\Gamma(K_L^0 \to \mu^+\mu^-)}{\Gamma(K^+ \to \mu^+\nu_\mu)} \sim \mathcal{O}\left(10^{-8}\right) \ . \tag{5.33}$$

The explanation for this suppression is in another coupling arising from the second quark doublet:

$$\Psi'_L = \begin{pmatrix} c \\ s' \end{pmatrix}_L \approx \begin{pmatrix} c \\ \cos\theta_c s - \sin\theta_c d \end{pmatrix}_L \ . \tag{5.34}$$

The Zd_Ls_L coupling arising from (5.34) exactly cancel the previous contribution to $K_L^0 \to \mu^+\mu^-$. In conclusion at the tree (no loop) level there are no Flavor Changing Neutral Currents (FCNC). Also at the one-loop level, if m_c is not too high, there is a strong cancellation. This mechanism for suppressing FCNC is called GIM and was invented before the discovery of the charm quark. The discovery in 1973 of the J/Ψ particle, a $\bar{c}c$ vector meson state represented one of the major successes of the high energy theoretical physics of the second half of XX century.

5.4 Hadronic currents and semileptonic decays

If we include quarks the Fermi effective lagrangian becomes

$$\mathcal{L}_{eff} = \frac{G_F}{\sqrt{2}} \left(j^\lambda + J^\lambda\right)\left(j_\lambda^\dagger + J_\lambda^\dagger\right) \tag{5.35}$$

where j_λ is the leptonic current defined in (4.29) and J_λ is the hadronic current given in eq. (5.25). The term $\propto j^\lambda j_\lambda^\dagger$ is relevant in purely leptonic processes (no hadrons). The term $\propto J^\lambda J_\lambda^\dagger$ is active in weak non leptonic processes. The term $\propto j^\lambda J_\lambda^\dagger + h.c.$ can describe the so called semileptonic processes (both hadrons and leptons present). Examples: $\mu^- \to e^-\nu_\mu\bar{\nu}_e$ (leptonic); $K^+ \to \pi^+\pi^0$ (non leptonic); $n \to pe^-\bar{\nu}_e$ (semileptonic). In this section we shall consider some examples of semileptonic decays.

The main problem is the evaluation of the hadronic current J^μ between hadronic states. Let us consider a few cases.

a): $< 0|J^\mu|M(p) >$ (M a pseudoscalar meson of momentum p).

Only the axial current acts; as a matter of fact, the matrix element must be proportional to p^μ, because it is a 4-vector and the only available 4-vector is p^μ. In the meson rest frame only the $\mu = 0$ component is non vanishing. On the r.h.s. we have a scalar; since M is a pseudoscalar meson, J^0 must be a pseudoscalar, because parity is conserved in strong interactions and the matrix element is only determined by strong interactions[3]. This means that, out of the two currents in $J^\mu = V^\mu - A^\mu$, only the axial current is active. It follows that we can write

$$< 0|A^\mu|M(p) >= i\, f_M\, p^\mu \ . \tag{5.36}$$

In particular we have

$$< 0|\bar{u}\gamma^\mu\gamma_5 d|\pi^-(p) >= i\, f_\pi\, p^\mu \ . \tag{5.37}$$

The numerical value of f_π can be extracted from $\Gamma(\pi^- \to \mu^-\bar{\nu}_\mu)$; one finds $f_\pi \approx 130$ MeV.

If M belongs to a $SU(3)_f$ multiplet, we can make use of the approximate flavor symmetry to relate the leptonic constans f_M of different members of the multiplet, because (5.36) must be a singlet. For example if M belongs to the pseudoscalar octet we write

$$< 0|\bar{q}_i T_{ij}\gamma^\mu\gamma_5 q_j|M(p) >= i\, f_M\, p^\mu\, \mathrm{Tr}(TM) \tag{5.38}$$

where M_{ij} is the 3×3 matrix describing in $SU(3)_f$ the pseudoscalar meson. In this way we can relate the different leptonic constants to a unique constant f_M. The reason for the Clebsch-Gordan coefficient $\mathrm{Tr}(TM)$ is as follows. Since strong interactions are $SU(3)_f$ symmetric, the matrix element must be invariant under group operations. If $U \in SU(3)_f$ then $T \to UTU^\dagger$, $M \to UMU^\dagger$ because both T and M are in adjunct representation. The only invariant is $\mathrm{Tr}(TM) \to \mathrm{Tr}(UTU^\dagger UMU^\dagger) = \mathrm{Tr}(TM)$.

b): $< M'(p')|J^\mu|M(p) >$ (M and M' pseudoscalar mesons of momenta p and p').

Let us consider the crossed m.e. $< |J^\mu|M(p)\,\bar{M}'(-p') >$. It is a 4-vector; therefore J^μ must be a vector by parity arguments (in

[3]In fact one has $< 0|V^0|M > = < 0|P^{-1}PV^0P^{-1}P|M >= - < 0|PV^0P^{-1}|M > = - < 0|V^0|M >$ which implies $< 0|V^0|M >= 0$. Notice that we used the transformation law under parity $V^\mu \to PV^\mu P^{-1} = V_\mu$.

the meson pair rest frame, for $\mu = 0$ we have a scalar and also the meson pair is a scalar under parity, therefore J^0 must be a scalar and this excludes the axial currents. Therefore we have

$$< M'(p')|J^\mu|M(p) >= f^{(+)}_{MM'}(p + p') + f^{(-)}_{MM'}(p - p') \qquad (5.39)$$

The factors $f^{(\pm)}$ are scalar functions and therefore can depend only on scalar quantities. The only non trivial scalar that can be constructed using the two momenta p and p' is $q^2 = (p' - p)^2$, therefore $f^{(\pm)} = f^{(\pm)}(q^2)$. The functions $f^{(\pm)}(q^2)$ are called form factors.

Also in this case we can relate different form factors using symmetry arguments. For example if M and M' are in the same $SU(3)_f$ octet we have

$$\begin{aligned} & < M'(p')|\bar{q}_i T_{ij}\gamma^\mu q_j|M(p) >= \\ = \; & \mathrm{Tr}([M, M'^\dagger]T) \left(f^{(+)}(q^2)(p + p') + f^{(-)}(q^2)(p - p') \right) \end{aligned}$$

Notice that if q_j in (5.40) are light quarks, then in the $SU(3)_f$ symmetry limit V^μ is a conserved current (it is the Noether current of the gauge symmetry $SU(3)_f$). This property is known as CVC (conserved vector current).

Since $\partial_\mu V^\mu(x)$ can be expressed *via* a commutator as shown in (1.209):

$$\partial_\mu V^\mu(x) = i[\hat{P}^\mu, V_\mu(x)] \qquad (5.40)$$

we have

$$\begin{aligned} 0 &= < M'(p')|\partial_\mu V^\mu(0)|M(p) >= \\ &= i(p' - p)^\mu < M'(p')|V^\mu(0)|M(p) >= \\ &= i f^{(+)}(q^2)(m_M^2 - m_{M'}^2) - i q^2 f^{(-)}(q^2) \; . \qquad (5.41) \end{aligned}$$

Therefore, in the symmetry limit $m_M^2 = m_{M'}^2$ and $f^{(-)} = 0$.

Since also the electromagnetic current is constructed in the same way, with T_{ij} given in this case by

$$Q = \begin{pmatrix} 2/3 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & -1/3 \end{pmatrix} , \qquad (5.42)$$

then one can relate the matrix element of the $K \to \pi \ell \nu_\ell$ decay to m.e. of the electromagnetic current between pion states. In particular we note that

$$< \pi^\pm(p')|\bar{q} Q \gamma^\mu q|\pi^\pm(p) >= \pm f_+(q^2)(p + p') \; , \qquad (5.43)$$

with $f_+(0) = 1$, whose proof is left to the reader as an exercise.

c): $< B'(p', s')|J^\mu|B(p, s) >$**:** B and B' are spin 1/2 baryons with momenta p and p' and spin components s and s'. Since the hadrons depend on both momentum and spin we can have both vector and axial currents. The Lorentz structure is as follows ($q = p' - p$):

$$
\begin{aligned}
&< B'(p', s')|V^\mu|B(p, s) >= \\
&\bar{u}(p', s')(f_1(q^2)\gamma^\mu - i f_2(q^2)\sigma^{\mu\nu} q_\nu)u(p, s) \; ; \qquad (5.44)
\end{aligned}
$$

$$
\begin{aligned}
&< B'(p', s')|A^\mu|B(p, s) >= \\
&\bar{u}(p', s') \left(g_1(q^2)\gamma^\mu\gamma_5 + g_3(q^2)q^\mu\gamma_5\right) u(p, s) \, , \qquad (5.45)
\end{aligned}
$$

where $q = p - p'$ and $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$ and we have assumed symmetry limit to get rid of other possible Lorentz structures. We discuss the $SU(3)_f$ structure of the form factors in section 5.6.

5.5 Chiral Symmetry

Let us suppose the light quarks massless: $m_u = m_d = m_s = 0$. As we know this should not be a bad approximation, given the numerical values of the quark masses, see the footnote in section 3.7. With vanishing quark masses the strong interaction lagrangian in the light flavor sector has the global gauge symmetry $SU(3)_L \times SU(3)_R$ (chiral symmetry) generated by transformations

$$
\begin{aligned}
q_L = \frac{1-\gamma_5}{2} q \quad &\rightarrow \quad U_L q_L = \exp(i\vec{\alpha}_L \cdot \vec{T}) q_L \\
q_R \quad &\rightarrow \quad U_R q_R = \frac{1+\gamma_5}{2} q = \exp(i\vec{\alpha}_R \cdot \vec{T}) q_R \, . \qquad (5.46)
\end{aligned}
$$

In fact the the only term that is not left invariant is (M =mass matrix)

$$
\begin{aligned}
\bar{q} M q \quad &= \quad (\bar{q}_L + \bar{q}_R) M (q_L + q_R) = \bar{q}_L M q_R + \bar{q}_R M q_L \rightarrow \\
&\rightarrow \bar{q}_L U_L^\dagger M U_R q_R + \bar{q}_R U_R^\dagger M U_R q_L \qquad (5.47)
\end{aligned}
$$

This is different from $\bar{q}Mq$. However if $M = 0$ the lagrangian is chiral symmetric[4]. Going back to chapter 3 we see that the hadron spectra do

[4]If M is a multiple of the unity matrix: $M = m \cdot \mathbf{1}$, and $m \neq 0$, then there is only a subgroup of the chiral group, the vector $SU(3)_V$ that is a symmetry group. This is $SU(3)_f$.

not show a duplication of multiplets, as one would expect in presence of chiral symmetry. For example there is an octet of $J^P = 1^-$ mesons, but there is not another meson multiplet $J^P = 1^+$ degenerate in mass. We conclude that chiral symmetry is spontaneously broken: it is a symmetry of the lagrangian but not of the physical states. According to the Goldstone's theorem there must be as many Goldstone bosons as are the broken generators of the symmetry. We expect only 8 broken generators and indeed we find an octet of light spin 0 mesons, i.e. the 0^- pseudoscalar octet. The fact they have indeed a mass is a consequence of the fact that quark masses are small but, strictly speaking, $m_q \neq 0$. In particular $m_u, m_d \ll m_s$ and the symmetry $SU(2)_L \times SU(2)_R$ should represent a better approximation. This is confirmed by the smallness of the pion mass.

We can a confirmation of this conjecture by the following argument. Let us use (5.37) to compute the divergence of the axial current:

$$< 0|\partial_\mu A^\mu(0)|\pi^-(p) >= f_\pi m_\pi^2 \tag{5.48}$$

on the other hand, from the field equations $i\gamma \cdot \partial\psi = m\psi +$ "colored terms":

$$\partial_\mu A^\mu = \partial_\mu \bar{u}\gamma^\mu\gamma_5 d = i(m_u + m_d)\bar{u}\gamma_5 d \tag{5.49}$$

therefore

$$i(m_u + m_d) < 0|\bar{u}\gamma_5 d|\pi^-(p) >= f_\pi m_\pi^2 \tag{5.50}$$

and $m_u = m_d = 0$ implies $m_\pi^2 = 0$. Note that since in nature m_π is small but not zero, the axial current is not exactly conserved ($\partial_\mu A^\mu \neq 0$). One refers to this circumstance as PCAC (partial conservation of the axial current).

Let us now consider (5.45) with $B = n$ and $B' = p$, and take the derivative of the axial current. Assuming PCAC, $\partial \cdot A \sim 0$ and $\partial \cdot A = 0$ if $m_\pi = 0$. In the chiral limit ($m_q \to 0$) one has

$$2m_N\, g_1(q^2) = q^2 g_3(q^2)\ , \tag{5.51}$$

i.e.

$$g_1(0) = \frac{1}{2m_N}\, q^2 g_3(q^2)\Big|_{q^2\to 0}\ . \tag{5.52}$$

From neutron beta decay one knows that $g_1(0) \sim 1.24$; this implies that $g_3(q^2)$ must have a pole at $q^2 = 0$. The origin of this pole can be understood looking at the Feynman diagram in fig.5.1. Computing the r.h.s. one gets

$$\sqrt{2} g_{\pi pn} \bar{\psi}_p(p')\gamma_5\psi_n(p)\frac{i}{q^2 - m_\pi^2} i f_\pi (p - p')_\mu \tag{5.53}$$

where the matrix element

$$i\mathcal{M}(n \to p\pi^-) = \sqrt{2} g_{\pi pn} \bar{\psi}_p(p')\gamma_5\psi_n(p) \tag{5.54}$$

is obtained from the effective interaction lagrangian

$$\mathcal{L}_{NN\pi} = -ig_{\pi pn}\bar{N}\vec{\tau} \cdot \vec{\phi}_\pi \gamma_5 N \ . \tag{5.55}$$

From experiment $g^2_{\pi pn}/4\pi \simeq 14$. It is clear that the diagram (5.1) only contributes to the form factor g_3. Therefore near the pole mass

$$\bar{\psi}_p(p')g_3(q^2)\, q^\mu\gamma_5\psi_n(p) = \sqrt{2} g_{\pi pn}\bar{\psi}_p(p')\gamma_5\psi_n(p)\frac{i}{q^2 - m_\pi^2} i f_\pi(-q_\mu) \tag{5.56}$$

i.e., taking into account that we assume $m_\pi = 0$:

$$g_3(q^2) = \frac{\sqrt{2} g_{\pi pn}\, f_\pi}{q^2} \tag{5.57}$$

As expected the form factor g_3 has a pole corresponding to a massless from scalar.

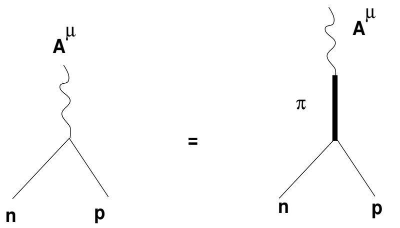


Figure 5.1: Pion pole dominance for $q^2 \to 0$.

From (5.52) and (5.57) we get the celebrate Goldberger-Treiman relation connecting f_π, the axial vector coupling $g_1(0)$, the pion-nucleon coupling constant $g_{\pi pn}$ and the nucleon mass:

$$2m_N g_1(0) = \sqrt{2} g_{\pi pn}\, f_\pi \ . \tag{5.58}$$

Its violation is only of $\sim 5\%$.

5.6 Problems

1. Compute $\Gamma(\pi \to \mu\nu_\mu)$ and compare your result with the experimental value.

2. Compute $\Gamma = \Gamma(K \to \mu\nu_\mu)$. Using the experimental value for Γ determine the ratio f_K/f_π.

3. Show that $< 0|J^\mu|V(p,\epsilon) >$, where V is a vector meson of momentum p and polarization vector ϵ can be written using Lorentz invariance and $SU(3)_f$ symmetry as follows:

$$< 0|J^\mu|V(p,\epsilon) >=< 0|\bar{q}_i T_{ij}\gamma^\mu q_j|V(p,\epsilon) >= f_V \epsilon^\mu \,\mathrm{Tr}[TV] \quad (5.59)$$

where V_{ij} is the 3×3 matrix describing in $SU(3)_f$ the vector meson V.

4. Compute $\Gamma(K^+ \to \pi^0 e^+ \nu_e)$ in terms of the CKM matrix elements and G_F. Use $SU(3)_f$ symmetry as in the text to evaluate the hadronic matrix element.

5. Consider eqns (5.44) and (5.45) without the term f_2, which is a good approximation in semileptonic decays, when the momentum q is small. For the vector form factor we write

$$f_1 = d\mathrm{Tr}(\{B, B'^\dagger\}V)\mathrm{Tr} + f([B, B'^\dagger]V) \quad (5.60)$$

where $\{,\}$ and $[,]$ mean respectively anticommutator and commutator. Using the electromagnetic current prove that $f = 1$ and $d = 0$. Experimentally $f_1(q^2) = f_1(0)/(1 - q^2/m_V^2)^2$ with $m_V^2 \approx 0.7$ GeV^2. For the analogous axial form factor a similar formula holds:

$$g_1(0) = D\mathrm{Tr}(\{B, B'^\dagger\}A)\mathrm{Tr} + D([B, B'^\dagger]A) \quad (5.61)$$

Experimentally one has $D = 0.76$, $F = 0.48$ and $g_1(q^2) = g_1(0)/(1-q^2/m_A^2)^2$ with $m_A^2 \approx 1.0$ GeV^2. Prove that

$$\begin{aligned} g_1(0)\Big|_{n\to p} &= D + F \\ g_1(0)\Big|_{\Lambda\to p} &= -\frac{D+3F}{\sqrt{6}} \,. \end{aligned} \quad (5.62)$$

6. Compute $\Gamma(\Lambda \to p e^- \bar{\nu}_e)$ in terms of the CKM matrix elements and G_F.

References

For the content of this chapter any textbook on Quantum Field Theory with High energy Physics orientation; for example H. Georgi cit. or O. Nachtmann, cit.

Chapter 6

Quark Parton Model

6.1 Partons

Deep inelastic processes are inelastic reactions with probes of high virtuality. Examples are

$$e^+e^- \to X \tag{6.1}$$
$$e\,N \to X \tag{6.2}$$
$$N\,N \to \mu^+\,\mu^-\,X \tag{6.3}$$
$$p\,\bar{p} \to \text{jets (high}\,p_T) + X \tag{6.4}$$

Since from the first pioneering experiments at SLAC it was realized (Feynman) that at the very short distances tested by these processes the hadron participating in the scattering appears constituted by a huge number of elementary puntiform particles, comoving with the hadron. On this basis a model was suggested (*quark parton model*) based on the following assumptions

1. Hadrons are formed by pointlike massless particles, called partons. They can have spin 1/2, in which case they are identified with the quarks. The parton momenta have the same direction of the hadron momentum.

2. Quark-partons have electric charge and therefore interact with electromagnetic and weak currents. A part from that they are basically free. However there can be other subcomponents of the hadrons, with no electric or weak charges and therefore not directly interacting with the probes. They provide the necessary

glue that binds partons in the hadrons and are therefore called gluons[1].

3. To compute hadronic cross sections one has to compute the analogous process off free partons, then multiply by the appropriate probabilities to find partons in the hadron and finally perform an incoherent sum over all the partons.

We shall discuss the reactions (6.1), (6.2) (deep inelastic scattering) leaving some comments on Drell-Yan (6.3) and jet production process (6.4) to section 6.2 at the end of the chapter.

6.1.1 $e^+e^+ \to$ hadrons

Let us consider the high energy inclusive cross section

$$e^+e^+ \to \text{ hadrons } . \tag{6.5}$$

In the parton model it is computed according to the steps depicted schematically in fig. 6.1. Now the last factor in the equation repre-

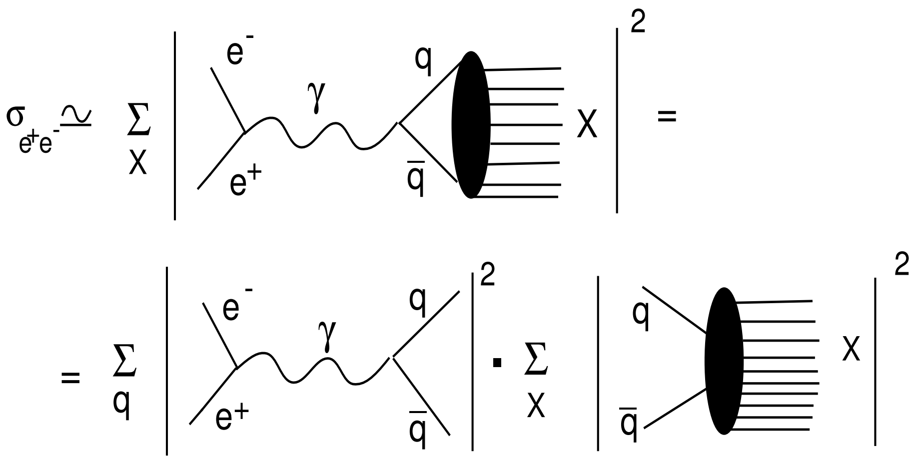


Figure 6.1: $e^+e^+ \to$ hadrons.

sented in this picture amounts to unity, as it represents the probability that the $q\bar{q}$ pair produces any possible hadronic final state. Therefore

[1]We have seen that in QCD gluons are the 8 spin 1 vector bosons responsible of strong interactions between quarks.

in the parton model

$$\sigma(e^+e^- \to \ \text{hadrons}) = \sum_q \sigma(e^+e^- \to \ q\bar{q}) \ . \tag{6.6}$$

It is customary to compare the cross section (6.6) to $\sigma(e^+e^- \to \ \mu^+\mu^-)$ by computing the ratio

$$R_{e^+e^-} = \frac{\sigma(e^+e^- \to \ \text{hadrons})}{\sigma(e^+e^- \to \ \mu^+\mu^-)} \ . \tag{6.7}$$

Now

$$\sigma(e^+e^- \to \ \mu^+\mu^-) = \frac{4\pi\alpha^2}{3s} \ , \tag{6.8}$$

where $s = (p_{e^+} + p_{e^-})^2$.

Problem. Prove that in Quantum Electrodynamics, at the lowest order in the perturbative expansion, and for $\sqrt{s} \gg {}_e$, m_μ one has (6.8).

Therefore

$$R_{e^+e^-} = 3\sum_q e_q^2 \ . \tag{6.9}$$

The factor 3 is a color factor due to the fact that, for each flavor q, three color states are possible for quarks. Eq. (6.9) is confirmed to a good extent from the data; it presents several important features:

1. It holds for energies large enough that low mass effects can be neglected. For example at $\sqrt{s} \simeq 770$ MeV the process is dominated by the ρ^0 resonance through $e^+e^- \to \gamma \to \rho^0 \to \pi^+\pi^-$. Therefore one observes a peak for $s = m_\rho^2$, due to the Breit-Wigner form of the ρ^0 propagator ($\Gamma_\rho \approx 150\text{MeV} \ll m_\rho$):

$$\frac{1}{s - m_\rho^2 + i m_\rho \Gamma_\rho} \ . \tag{6.10}$$

2. It confirms the hypothesis of the quark-partons as spin 1/2 particles.

3. It confirms the hypothesis of the color quantum number for quarks.

4. It shows that quarks, once produced by the electromagnetic probe, behave as free particles. This reflects in the absence of energy dependence in eq. (6.9). This property is called scaling [2]. It should

[2] A function $f(s)$ has the scaling property if it is homogeneous, i.e. if, under a scale transformation $s \to \lambda s$, one has $f(s) \to \lambda^k f(s)$. Here $R(s)$ is homogeneous with $k = 0$ degree.

be mentioned that the first experimental data were obtained at the end of the decade 1960, subsequent more precise data exhibited small deviations from scaling and showed that quarks at short distances, besides electromagnetic interactions, have also strong interactions, although very feeble indeed. This feature is embodied in Quantum-Chromo-Dynamics (QCD), the theory of quarks and gluons since this theory exhibits the property of *asymptotic freedom*; more precisely due to loop effects the physical coupling constant of strong interactions $\alpha_s = g_s^2/4\pi$ depends on the energy, i.e. $s = Q^2$: $\alpha_s(Q^2)$. For large Q^2 it decreases as

$$\alpha_s(Q^2) \sim \frac{\text{const.}}{\ln \frac{Q^2}{\Lambda_{QCD}^2}} \tag{6.11}$$

where $\Lambda_{QCD} \approx 200$ MeV, as determined by experiment. For $Q^2 \gg \Lambda_{QCD}^2$, $\alpha_s(Q^2)$ can be very small and the quarks can be treated as almost free.

5. The sum in (6.9) is extended to all flavors that can be excited for a given energy. For example for $\sqrt{s} < 3$ GeV only the quarks u, d, s can be excited and $R_{e^+e^-} = 2$; for $3 <$ GeV $\sqrt{s} < 9.5$ GeV also the charm quark can be produced and $R_{e^+e^-} = 10/3$, while, for $\sqrt{s} > 9.5$ GeV, the threshold for $b\bar{b}$ is opened and $R_{e^+e^-} = 11/3$.

6.1.2 Deep Inelastic Scattering

The kinematics of the Deep Inelastic Scattering (see fig. 6.2) is as follows

$$\begin{aligned}
k^\mu &= (E, 0, 0, E) \\
k'^\mu &= (E', E' \sin\theta\cos\phi, E' \sin\theta\sin\phi, E'\cos\theta) \\
q'^\mu &= k^\mu - k'^\mu\,, \qquad Q^2 = -q^2 = 4EE'\sin^2\frac{\theta}{2} > 0 \\
p^\mu &= (M, 0, 0, 0)\;.
\end{aligned} \tag{6.12}$$

We define the auxiliary variables

$$\nu = \frac{p\cdot q}{M} = E - E' > 0 \qquad \text{(energy loss)} \tag{6.13}$$

$$y = \frac{\nu}{E} = \frac{p\cdot q}{p\cdot k} = 1 - \frac{E'}{E} \qquad \text{(fractional energy loss)} \tag{6.14}$$

$$x = \frac{Q^2}{2\nu M} = \frac{Q^2}{2MEy} = \frac{1}{\omega} \qquad \text{(Bjorken variable)}\;. \tag{6.15}$$

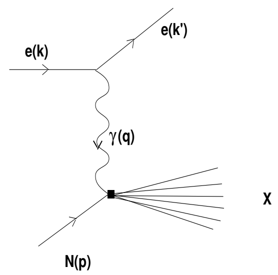


Figure 6.2: Deep inelastic scattering of a virtual photon off a nucleon.

The invariant mass of the hadronic final state is

$$W^2 = m_X^2 = (p+q)^2 = M^2 + q^2 + 2p \cdot q \ . \tag{6.16}$$

Therefore, since $0 \leq m_X^2 - M^2 = 2p \cdot q \ - \ Q^2$, it follows that

$$0 \leq x \leq 1 \ . \tag{6.17}$$

The first DIS experiments were performed at SLAC (1968) then at FNAL and CERN, with a momentum transfer Q^2 as large as 10 GeV^2. The amplitude for this process can be written as follows

$$i\mathcal{M} = (-ie)^2 < e(k')|J^{\mu}_{em}|e(k) >< X|J^{\nu}_{em}|N(p) > \frac{-ig_{\mu\nu}}{q^2} \tag{6.18}$$

and the cross section is as follows

$$d\sigma = \frac{d^3k'}{(2\pi)^3 2E'} \frac{1}{4\sqrt{(p\cdot k)^2 - m^2M^2}} \sum_X (2\pi)^4 \delta^4(p+k-p_X-k')|i\mathcal{M}|^2 \tag{6.19}$$

where $m \sim 0$ is the electron mass. Now

$$\begin{aligned} |i\mathcal{M}|^2 &= \frac{e^4}{Q^4}\overline{\sum}_{spin} < e(k')|J^{\nu}_{em}|e(k) >< e(k)|J^{\mu}_{em}|e(k') > \\ &\times \ < X|J_{em,\,\nu}|N(p) >< N(p)|J_{em,\,\mu}|X > \end{aligned} \tag{6.20}$$

where $\overline{\sum}_{spin}$ means a sum over the final spins and an average over the initial spins. Moreover, with $(k\cdot\gamma) = k^{\mu}\gamma_{\mu}$ and neglecting the electron mass,

$$\overline{\sum}_{spin} < e(k')|J^{\nu}_{em}|e(k) >< e(k)|J^{\mu}_{em}|e(k') >=$$

$$
\begin{aligned}
&= \frac{1}{2}\text{Tr}(k\cdot\gamma)\gamma^\mu(k'\cdot\gamma)\gamma^\nu = 2\left(k^\mu k'^\nu + k^\nu k'^\mu - k\cdot k' g^{\mu\nu}\right)\\
&\equiv 2\,\ell^{\mu\nu}\ . \qquad (6.21)
\end{aligned}
$$

Let us introduce the hadronic tensor

$$
\begin{aligned}
W^{\mu\nu}(p,q) &= \sum_X \frac{(2\pi)^3}{2}\delta^4(p+q-p_X)\\
&\times \overline{\sum}_{spin} < X|J^\nu_{em}|N(p)>< N(p)|J^\mu_{em}|X> \qquad (6.22)
\end{aligned}
$$

One has therefore

$$
\frac{1}{4ME}\sum_X (2\pi)^4\delta^4(p+k-p_X-k')|i\mathcal{M}|^2 = \frac{2\pi}{ME}\ell^{\mu\nu}W_{\mu\nu} \qquad (6.23)
$$

Since $d^3k' = E'^2 dE' d\Omega$ we finally have

$$
\frac{d\sigma}{dE'd\Omega} = \frac{2\alpha^2}{Q^4}\frac{E'}{ME}\ell^{\mu\nu}W_{\mu\nu} \qquad (6.24)
$$

where $\alpha = e^2/4\pi$ is the fine structure constant.

The hadronic tensor satisfies

$$
\begin{aligned}
W^{\mu\nu}(p,q) &= W^{*\,\nu\mu}(p,q)\\
q_\mu W^{\mu\nu}(p,q) &= q_\nu W^{\mu\nu}(p,q) = 0\ . \qquad (6.25)
\end{aligned}
$$

Problem. Prove eqns.(6.25).

The rank 2 tensor $W^{\mu\nu}$ should be constructed using only $q^\mu, p^\mu,\ g^{\mu\nu}$, $\epsilon^{\mu\nu\lambda\sigma}$ and satisfy parity invariance and eqns. (6.25). Therefore it can be expressed in terms of only two invariant real functions W_1, W_2 (called *structure functions*) that, being scalars, can depend only on scalar kinematical variables:

$$
\begin{aligned}
W^{\mu\nu}(p,q) &= \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) W_1(\nu,\,q^2)\\
&+ \left(p^\nu - \frac{p\cdot q q^\nu}{q^2}\right)\left(p^\mu - \frac{p\cdot q q^\mu}{q^2}\right)\frac{W_2(\nu,\,q^2)}{M^2}\ . \qquad (6.26)
\end{aligned}
$$

Problem. Prove eqn.(6.26). Show that a term as follows

$$
i\epsilon^{\mu\nu\lambda\sigma} q_\lambda p_\sigma W_3(\nu,\,q^2) \qquad (6.27)
$$

violates parity. Therefore this term can be present only if parity is not conserved (i.e. in deep inelastic scattering *via* weak interactions).

Using previous expressions, noting that $q \cdot k' = k \cdot k' = -q \cdot k = -q^2/2$, one finally gets

$$\frac{d\sigma}{dE'd\Omega} = \frac{4E'^2\alpha^2}{MQ^4}\left(2W_1 \sin^2\frac{\theta}{2} + W_2 \cos^2\frac{\theta}{2}\right) . \qquad (6.28)$$

Using the y and x variables defined in (6.14), (6.15) and introducing new dimensionless structure functions:

$$F_1(Q^2,\nu) = W_1(Q^2,\nu)\,, \quad F_2(Q^2,\nu) = \frac{\nu}{M}W_2(Q^2,\nu) \qquad (6.29)$$

we can express (6.28) as follows:

$$\frac{d\sigma}{dxdy} = \frac{4\pi\alpha^2 s}{Q^4}\left[xy^2\,F_1(Q^2,\nu) + \left(1 - y - \frac{M^2}{Q^2}\left(\frac{x}{1-x}\right)^2\right)F_2(Q^2,\nu\right], \qquad (6.30)$$

where $s = (p+q)^2 = 2ME = (1-x)Q^2/x$.

Problem. Prove eqns. (6.28) and (6.30).

Our ignorance about strong interactions is hidden in the hadronic tensor, in particular in the structure constants F_j. To compute them let us now apply the hypotheses of the parton model, by which the scattering of the virtual photon is off a parton q_i of momentum ξp, see fig. 6.3 Clearly to get the hadronic tensor $W^{\mu\nu}$ we must first multiply

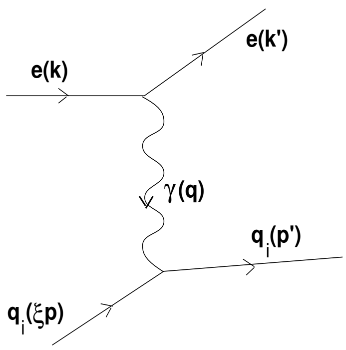


Figure 6.3: Deep inelastic scattering of a virtual photon off a nucleon.

the corresponding tensor for the parton q_i, i.e. $W_i^{\mu\nu}$ by e_i^2, which gives the relative electromagnetic coupling with respect to the proton. Then we multiply the result by the probability of finding a quark of flavor i and longitudinal fractional momentum ξp in the nucleon, that we denote as $q_i(\xi)$; to it we add $\bar{q}_i(\xi)$ the analogous probability for antiquarks. Finally we sum over all flavors and integrate over ξ:

$$W^{\mu\nu} = \sum_i e_i^2 \int d\xi \,[q_i(\xi) + \bar{q}_i(\xi)]\, W_i^{\mu\nu} \tag{6.31}$$

Let us compute the hadronic tensor $W_i^{\mu\nu}$ adapting to the present case eq. (6.22). We have

$$\begin{aligned} W_i^{\mu\nu}(\xi p, q) &= \frac{(2\pi)^3}{2\xi} \int \frac{d^3p'}{(2\pi)^3 2E'} \delta^4(\xi p + q - p') \\ &\times \overline{\sum}_{spin} < q_i(p')|J^\nu_{em}|q_i(\xi p) >< q_i(\xi p)|J^\mu_{em}|q_i(p') > \end{aligned} \tag{6.32}$$

The only relevant difference, is that, since partons are massless, normalization is to the energy, not the mass and we substitute $M \to \xi E$ for the parton energy and analogously $M \to E$ for the nucleon. When expressed by invariant quantities as in (6.30) the factor E vanishes and we remain with the factor $1/\xi$. Now we can compute the matrix element of the current:

$$< q_i(\xi p)|J^\mu_{em}|q_i(p') >= \bar{u}(\xi p)\gamma^\mu u(p') \tag{6.33}$$

so that

$$\begin{aligned} & \overline{\sum}_{spin} < q_i(p')|J^\nu_{em}|q_i(\xi p) >< q_i(\xi p)|J^\mu_{em}|q_i(p') >= \\ = \;& 2\xi \left(p'^\mu p^\nu + p'^\nu p^\mu - p' \cdot p\, g^{\mu\nu}\right) \end{aligned} \tag{6.34}$$

Moreover

$$\int \frac{d^3p'}{2E'} \delta^4(\xi p + q - p') = \int d^4p'\, \delta^4(\xi p + q - p')\delta(p'^2) = \frac{1}{2p\cdot q}\delta(\xi - x)\ . \tag{6.35}$$

Therefore, using $p' = \xi p + q$ and neglecting terms proportional to q^μ that are harmless since $q^\mu \ell_{\mu\nu} = 0$, one has

$$W_i^{\mu\nu}(\xi p, q) = \frac{\delta(\xi - x)}{2p} \frac{1}{p \cdot q} \left(2\xi p^\mu p^\nu - p\cdot q\, g^{\mu\nu}\right)\ , \tag{6.36}$$

and using the decomposition (6.26) and the definitions (6.29) one gets

$$F_1^i \;=\; \frac{1}{2}\delta(\xi - x)$$

$$F_2^i = \xi\delta(\xi - x) \tag{6.37}$$

In conclusion

$$F_1(x) = \frac{1}{2}\sum_i e_i^2\,[q_i(x) + \bar{q}_i(x)] \tag{6.38}$$

$$F_2(x) = 2x\,F_1(x)\ . \tag{6.39}$$

Let us comment on these results. First of all we observe that F_1 and F_2 do not depend on Q^2 and ν but only on the dimensionless ratio $x = Q^2/2M\nu$, i.e. on the Bjorken variable. This means that, analogously to $R_{e^+e^-}$, also these function have the scaling property. A dependence on both Q^2 and x would indicate a scaling violation. This is excluded in the parton model. In fact here the scattering is off partons, i.e. particles with no intrinsic scale (they are massless and point-like). Therefore the dimensionless structure functions can depend by the only non trivial dimensionless variable, i.e. the Bjorken variable x. In QCD, however, the renormalization procedure introduces a further scale, Λ_{QCD} and small scaling violations are admitted [3], so that $F_{1,2} = F_{1,2}(x, Q^2)$. Second we observe a simple relation between F_1 and F_2, i.e. eq.(6.39), called Callan-Gross relation. It is well verified experimentally, a part again from small scaling violations.

Let us define

$$u(x) = \text{probability to find a quark } up \text{ in the proton} \tag{6.40}$$

and analogously $d(x), s(x)$, etc. for the quarks d, s etc. Then

$$F_1^p = \frac{1}{2}\left(\frac{4}{9}[u(x) + \bar{u}(x)] + \frac{1}{9}[d(x) + \bar{d}(x)] + \frac{1}{9}[s(x) + \bar{s}(x)]\right) \tag{6.41}$$

Using isospin invariance the probabilities to find quarks in the neutron can be related to those of the proton, e.g.

$$\begin{aligned} u &\equiv u^p = d^n \\ d &\equiv d^p = u^n \\ s &\equiv s^p = s^n\ , \end{aligned} \tag{6.42}$$

which implies

$$F_1^n = \frac{1}{2}\left(\frac{4}{9}[d(x) + \bar{d}(x)] + \frac{1}{9}[u(x) + \bar{u}(x)] + \frac{1}{9}[s(x) + \bar{s}(x)]\right) \tag{6.43}$$

[3] They were indeed discovered in experiments performed about 10 years after the discovery of the scaling in DIS.

A number of relations, called *sum rules* can be derived in the parton model calculation of DIS. We list them

$$\int_0^1 dx\,[u(x)-\bar{u}(x)] \;=\; 2 \tag{6.44}$$

$$\int_0^1 dx\,[d(x)-\bar{d}(x)] \;=\; 1 \tag{6.45}$$

$$\int_0^1 dx\,[s(x)-\bar{s}(x)] \;=\; 0\;. \tag{6.46}$$

Another sum rule follows from the very definition of x as fraction of the nucleon momentum carried by the quark:

$$\int_0^1 dx\,x\,[u(x)+\bar{u}(x)+d(x)+\bar{d}(x)+s(x)+\bar{s}(x)] = 1-\epsilon\;. \tag{6.47}$$

Experimentally $\epsilon \approx 0.5$ and is interpreted as the total momentum carried by gluons.

It is used to separate in the quark distribution functions the contribution of the *valence* quarks from the contribution of the *sea* quarks. Valence quarks are the constituent quarks; they are uud for protons, udd for neutrons. Sea quarks are the pairs $\bar{q}q$ formed by virtual gluons surrounding the hadrons. Therefore we write

$$q(x) = q_v(x) + q_s(x)\;. \tag{6.48}$$

Since we expect that

$$\bar{u} = \bar{d} = s = \bar{s} = q_s \tag{6.49}$$

we get from previous equations

$$\int_0^1 dx\,u_v(x) \;=\; 2 \tag{6.50}$$

$$\int_0^1 dx\,d_v(x) \;=\; 1 \tag{6.51}$$

and the l.h.s. is interpreted as the number of valence quarks of type u and d in the proton. Also (6.46) becomes obvious due to (6.49), but it should be remembered that $s(x) = \bar{s}(x)$ is stronger than (6.46) and depends on the assumption of the quark model, whereas (6.46) is more general and can be derived using general properties of the hadronic tensor $W_{\mu\nu}$ in the Bj-limit.

6.2 Problems

1. Show that the angular distribution for the process $e^+e^- \to q\,\bar{q}$ is of the type
$$1 + \cos^2\theta\ . \tag{6.52}$$
This is the angular distribution of two-jet events found in e^+e^- annihilation at high energy. These events are characterized by a final state where most of the particle momenta are in two cone regions. The regions are opposite to each other so that $\vec{P}_{tot} = 0$ in the center of mass frame and are distributed according the angular law (6.52). Jet production is interpreted as the effect of a final state interaction (*hadronization*) producing hadrons out of the quark pair.

2. Determine the angular distribution of the jets in the hypothesis of spin 0 quarks. Comment on the difference with (6.52). The experimental evidence for the law (6.52) is one of the best reasons why we believe that quarks have spin 1/2.

3. Show that the Callan Gross relation only holds for spin 1/2 quarks.

4. Derive the momentum sum rule.

5. For the neutrino induced $\nu N \to \mu^- X$ deep inelastic scattering prove that the most general form of the hadronic tensor contains an extra term
$$F_3\epsilon^{\mu\nu\alpha\beta}\ . \tag{6.53}$$
Derive an expression for the cross section.

6. Derive the Gross-Llewellyn Smith sum rule (6.3).

Chapter 7

Problems for the final exam

1. Prove eq. (1.109):

$$D(x) = -i \int \frac{d^3k}{(2\omega_k\, 2\pi)^3} \left\{ e^{-i(\omega_k x_0 - \vec{k}\cdot\vec{x})}\, \theta(x_0) + e^{+i(\omega_k x_0 - \vec{k}\cdot\vec{x})}\, \theta(x_0) \right\} . \tag{7.1}$$

2. Prove

$$i\, D(x-y) = \frac{1}{Z[0]} \int [D\phi]\phi(x)\phi(y) \exp\left\{ i \int d^dx \left[-\frac{1}{2}\phi(\Box + m^2 - i\epsilon)\phi \right] \right\} \tag{7.2}$$

and comment on the relation between this equation and the result

$$i\, D(x-y) =< 0|T(\phi(x)\phi(y))|0 > \tag{7.3}$$

obtained for the propagator in the operatorial approach.

3. Prove that in QED in the massless limit, for $e^+e^- \to \mu^+\mu^-$

$$\frac{1}{4}\overline{\sum_{spin}}\Big|\mathcal{M}\Big|^2 = \frac{2e^4(t^2+u^2)}{s^2} \tag{7.4}$$

and for $e^-\mu^- \to e^-\mu^-$

$$\frac{1}{4}\overline{\sum_{spin}}\Big|\mathcal{M}\Big|^2 = \frac{2e^4(s^2+u^2)}{t^2} . \tag{7.5}$$

Comment on the crossing symmetry.

4. Prove (2.34)

$$\delta \mathbf{F}_{\mu\nu} = i[\alpha, \mathbf{F}_{\mu\nu}] \tag{7.6}$$

5. Prove that if U is a local gauge transformation, $D^\mu \to U D^\mu U^\dagger$, from which prove that the gauge fields can appear in the lagrangian only through gauge covariant derivatives.

6. In $SU(3)$ draw the Young tables and compute the dimensions of the IRRs defined by: $(m,n) = (0,1)$, $(0,2)$, $(2,0)$, $(1,1)$, $(1,2)$, $(2,1)$, $(3,0)$, $(0,3)$, $(0,4)$, $(4,0)$, $(1,3)$, $(3,1)$, $(2,2)$.

7. Prove the Gell-Mann Okubo formula for baryons:

$$\frac{m_\Xi + m_N}{2} = \frac{m_\Sigma + 3m_\Lambda}{4} \ . \tag{7.7}$$

8. Compute the decay rate relative to the process: $Z \to \ell^+ \ell^-$, where ℓ is a charged lepton. Put $m_\ell = 0$.

9. Compute the decay rate relative to the process: $Z \to \nu_\ell \bar{\nu}_\ell$.

10. Compute the decay rate relative to the process: $H \to W^+ W^-$.

11. Compute the decay rate relative to the process: $H \to Z^0 Z^0$.

12. Compute the decay rate relative to the process: $H \to f\bar{f}$, where f is a fermion; put $m_f \neq 0$.

13. Compute $\Gamma(\mu^-(p_1) \to e^-(p_2)\bar{\nu}_e(k_2)\nu_\mu(k_1))$. Use the Fermi approximation.

14. Compute the cross section relative to the scattering process: $\nu_\mu(k) d(p) \to \mu^-(k') u(p')$. Use the Fermi approximation.

15. Compute $\sigma(\bar{\nu}_\mu(k) u(p) \to \mu^+(k') d(p'))$. Use the Fermi approximation.

16. Compute $\Gamma(K^+ \to \pi^0 e^+ \nu_e)$ in terms of the CKM matrix elements and G_F. Use $SU(3)_f$ symmetry as in the text to evaluate the hadronic matrix element.

17. Consider eqns (5.44) and (5.45)

$$\begin{aligned} &< B'(p', s')|V^\mu|B(p,s) >= \\ &\bar{u}(p', s')(f_1(q^2)\gamma^\mu - i f_2(q^2)\sigma^{\mu\nu} q_\nu) u(p,s) \ ; \end{aligned} \tag{7.8}$$

$$< B'(p',s')|A^\mu|B(p,s) >= \\ \bar{u}(p',s')\left(g_1(q^2)\gamma^\mu\gamma_5 + g_3(q^2)q^\mu\gamma_5\right)u(p,s)\,, \tag{7.9}$$

where $q = p - p'$ and $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu,\gamma^\nu]$. Justify the neglect of the term f_2 in semileptonic decays. Using $SU(3)_f$ symmetry justify

$$f_1 = d\mathrm{Tr}(\{B, B'^\dagger\}V)\mathrm{Tr} + f([B, B'^\dagger]V) \tag{7.10}$$

where $\{,\}$ and $[,]$ mean respectively anticommutator and commutator, with $f = 1$ and $d = 0$. For the analogous axial form factor prove:

$$g_1(0) = D\mathrm{Tr}(\{B, B'^\dagger\}A)\mathrm{Tr} + D([B, B'^\dagger]A) \tag{7.11}$$

Prove that

$$\begin{aligned} g_1(0)\Big|_{n\to p} &= D + F \\ g_1(0)\Big|_{\Lambda\to p} &= -\frac{D+3F}{\sqrt{6}}\,. \end{aligned} \tag{7.12}$$

18. Compute $\Gamma(\Lambda \to pe^-\bar{\nu}_e)$ in terms of the CKM matrix elements and G_F.

19. Prove the equation

$$\frac{d\sigma}{dE'd\Omega} = \frac{4E'^2\alpha^2}{MQ^4}\left(2W_1\sin^2\frac{\theta}{2} + W_2\cos^2\frac{\theta}{2}\right)\,. \tag{7.13}$$

Using the y and x variables defined in (6.14), (6.15) and introducing the dimensionless structure functions:$F_1(Q^2,\nu) = W_1(Q^2,\nu)$, $F_2(Q^2,\nu) = \frac{\nu}{M}W_2(Q^2,\nu)$ derive:

$$\frac{d\sigma}{dxdy} = \frac{4\pi\alpha^2 s}{Q^4}\left[xy^2\,F_1(Q^2,\nu) + \left(1 - y - \frac{M^2}{Q^2}\left(\frac{x}{1-x}\right)^2\right)F_2(Q^2,\nu\right] \tag{7.14}$$

where $s = (p+q)^2 = 2ME = (1-x)Q^2/x$.

20. Derive the cross section for the neutrino induced $\nu N \to \mu^- X$ deep inelastic scattering in terms of the structure functions F_1, F_2, F_3.

21. Derive the Gross-Llewellyn Smith sum rule.