Invalid approximations in the oscillation formula

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Many derivations of the standard mass oscillation formula are based upon invalid approximations or worse. We discuss some of these and correct them, obtaining different oscillation formulas according to the kinematic assumptions made.

1. A CORRECT DERIVATION OF THE STANDARD FORMULA

The standard oscillation formula applied to both neutral kaons and neutrinos reads:

$$P(f \to f') = \sin^2(2\theta)\sin^2\left[L\Delta m^2/4E\right] \qquad (1)$$

where an initial flavour f (neutrino, kaon, ...) oscillates into flavour f'. In the above formula L indicates the distance traveled from source to sink (assumed the same for both mass eigenstates), E is the energy of each mass-eigenstates (m-e) (equal energy hypothesis) or more generally the average m-e energy ($\overline{E}$), $\Delta m^2 = m_1^2 - m_2^2$, and θ is the mixing angle defined in the standard way:

$$\Psi_f = \cos(\theta)\Psi_1 + \sin(\theta)\Psi_2 \qquad (2)$$

$$\Psi_{f'} = -\sin(\theta)\Psi_1 + \cos(\theta)\Psi_2 \qquad (3)$$

A classical derivation of the above standard formula assumes a *common rest frame* for the m-e. This implies a common proper time and consequently equal velocities for the m-e in *any* Lorentz frame. We calculate in the rest frame: so $E = m$ and $\mathbf{p} = 0$. Note that this is also a common momentum scenario in this frame, but a very particular one, since there is no $\mathbf{x}$ dependence when $\mathbf{p} = 0$.

$$\begin{aligned}\Psi(\tau,\mathbf{x}) &= \cos(\theta)e^{(-im_1\tau)}\Psi_1 + \sin(\theta)e^{(-im_2\tau)}\Psi_2 \\ &= e^{(-im_1\tau)}[\cos^2(\theta)\Psi_f - \cos(\theta)\sin(\theta)\Psi_{f'} \\ &\quad + e^{(-i\tau\Delta m)}\{\sin^2(\theta)\Psi_f + \cos(\theta)\sin(\theta)\Psi_{f'}\}]\end{aligned}$$

Hence the amplitude for $f \to f'$ is given by:

$$\begin{aligned}A(f \to f') &= e^{(-im_1\tau)}\{\cos(\theta)\sin(\theta) \\ &\qquad \times[e^{(-i\tau\Delta m)} - 1]\} \\ &= e^{(-im_1\tau)}\sin(2\theta)\sin(\tau\Delta m/2) \qquad (4)\end{aligned}$$

Note the oscillation argument $\xi = \tau\Delta m/2$ is just one half of the phase differences of the m-e. We shall use this in what follows to abbreviate all the demonstrations.

In an arbitrary inertial frame we must make the substitutions:

$$t = \gamma\tau \ ; \ \overline{E} = \gamma\overline{m} \ ; 2\overline{m}\Delta m = \Delta m^2 \ . \qquad (5)$$

This yields:

$$\begin{aligned}A(f \to f') &= \sin\left(\frac{t}{\gamma}\frac{\Delta m^2}{2\overline{m}}\frac{1}{2}\right) \\ &= \sin(t\Delta m^2/4\overline{E}) \qquad (6)\end{aligned}$$

For ultra relativistic particles we can substitute, at this point, t with L and if *furthermore* $\Delta m \ll m$ the standard formula is found with the oscillating phase:

$$\xi \simeq \frac{\Delta m^2 L}{4\overline{E}} = 1.27L[km]\Delta m^2[eV]/\overline{E}[GeV] \qquad (7)$$

The above is conventionally referred to as the constant velocity derivation.

2. AN INVALID DERIVATION OF THE STANDARD FORMULA

We now summarize an authoritative, but incorrect, derivation of the standard formula, which appears in the Particle Data Group 1998 [1].

The plane wave phase in an arbitrary frame is $\phi = Et - \mathbf{p}\cdot\mathbf{x}$. Simplifying to one dimension with $\mathbf{p}$ along the x axis ϕ becomes $\phi = Et - px$. Now let us make the *approximation* $t = x$ and use: $E - p = m^2/(E+p)$ which tends in the ultra relativistic limit to $m^2/2E \approx m^2/2p$. Thus the phase factor ϕ for each m-e becomes: $(E-p)t \approx m^2t/2p$ or $m^2t/2E$.

In the *common momentum* scenario the first form yields:

$$\xi = \frac{1}{2}(\phi_1 - \phi_2) = \frac{1}{2}\frac{t\Delta m^2}{2p} \approx \frac{L\Delta m^2}{4p} \approx \frac{L\Delta m^2}{4\overline{E}} \quad (8)$$

The above derivation is **incorrect**. The approximation $t = x$ must *not* be made before subtracting the phase differences. Firstly, it is always unwise to make approximations in large numbers which are then subtracted to yields a very small result. Secondly, it is easy to see that the approximation $t = x$ produces a significant effect. Indeed, it corresponds to assuming that the velocity of the particle is that of light $v = 1$. For consistency this implies $E = p$. Now, if *both* of these approximations are applied simultaneously then each phase $\phi = Et - px = 0$ and hence $\xi = 0$.

The correct calculation delays making any approximation until after subtracting the phases. We use in the following $v = p/E = x/t$, whence $Et - px = px[(E/p)(t/x) - 1] = px[1/v^2 - 1]$. Substituting for x the value L we find:

$$\phi = pL[1/v^2 - 1]\,. \quad (9)$$

In the common momentum scenario and with a unique L only the velocity in the above formula is mass dependent. Now,

$$\Delta(1/v^2) = \frac{E_1^2}{p^2} - \frac{E_2^2}{p^2} = \frac{\Delta E^2}{p^2} = \frac{\Delta m^2}{p^2} \quad (10)$$

Thus, $\phi_1 - \phi_2 = pL(\Delta m^2)/p^2 = L\Delta m^2/p$ and consequently,

$$\xi = \frac{1}{2}\frac{L\Delta m^2}{p} \approx 2[L\Delta m^2/4\overline{E}] \quad (11)$$

This is exactly twice the standard result because both the approximations $t = x$ and $E = p$ made *before* subtracting the phases looses half of the full argument. Making simultaneously both, as we have pointed out, looses *all* phases.

It easy to see that a similar result is obtained for the *common energy scenario*, with E in the place of the above average energy. With a unique t and now average L,

$$\phi_1 - \phi_2 = \overline{L}\Delta p^2 = \frac{\overline{L}}{\overline{p}}\Delta p^2 = -\frac{\overline{L}}{\overline{p}}\Delta m^2 \quad (12)$$

then

$$\xi = \frac{1}{2}\frac{\overline{L}}{\overline{p}}\Delta m^2 \approx \frac{\overline{L}\Delta m^2}{2E} \quad (13)$$

Again in the ultra relativistic limit this is approximately twice the standard result.

3. VARIATIONS UPON A THEME

3.1. The ket derivation

Kets depend only upon t, for an energy eigenstate we have:

$$|\nu(t) >= e^{-iEt}|\nu(0)> \quad (14)$$

Hence the phase that appears is only Et (e.g. Perkins [2]). Thus, $\xi = \frac{1}{2}(\Delta E)t$. If we now assume, as does Perkins, that we are in an equal momentum frame (if it exists for the system) $p_1 = p_2$ then $\Delta E = \Delta m^2/2\overline{E}$ and we obtain anew the standard result.

This derivation is clearly not Lorentz invariant. This means that the result changes if we change Lorentz frame. Indeed if we adopt Lipkin's thesis [3] that equal energies must be used, then the above derivation yields zero. The error is due to the fact that kets depend *explicitly* only upon t, by *implicitly* also upon $\mathbf{x}$. Indeed $< \mathbf{x}|\nu >= \nu(\mathbf{x}, t)$, the wave function, and the space dependence here is not a property of the bra which is a space-eigenstate, but of the ket. This text book derivation is therefore also invalid.

3.2. A variation upon the previous case

For common momentum, the space dependence is the *common* $exp(i\mathbf{p}\cdot\mathbf{x})$. Now a common phase factor in the m-e phases can be factored out and thus does not contribute to the oscillation argument. We are tempted to conclude that, in this frame, we may ignore the $\mathbf{x}$ dependence of the phase !

This argument, which would lead to the same standard result as above, is also incorrect, because $\mathbf{x}$ represents the distances traveled by the particles. The assumption of the same momentum and hence different energies, for different masses, means that we are dealing with different velocities and therefore, normally, different distances. If one imposes a common distance L, one must allow for different time intervals and this also invalidates the previous derivation.

We wish to add here a few words about the appearance of different time intervals. Contrary to convention this situation is physically highly probable. Only in the case of "instantaneous" wave function creation is there an obligatory unique time interval. Instantaneous events over a finite space region in one frame are, of course, not instantaneous in any other frame (except for space rotations). Hence different time intervals are to be considered the norm.

3.3. The wave packet derivation

This is a variation upon the Kayser derivation outlined in the previous section. The calculation which again yields the standard result is based upon a saddle point method. It is formally more correct than the plane wave argument. Unfortunately even here [4] Kayser drops a "small" term in the phase proportional to $(t - x)$ before estimating the integral involved. *Without dropping anything* repeating the Kayser calculation yields exactly twice his oscillation result, in agreement with our plane wave derivation in the equal momentum or energy scenarios.

4. Conclusions

1. The standard oscillation formula (1) is obtained correctly **only** in the *common velocity* scenario. Okun et al. [5] claims that this scenario is unphysical because it implies $E_1/E_2 = m_1/m_2$, which may be a very large/small number for neutrinos. However it is not incompatible with the data if, as for kaons, $\Delta m \ll \overline{m}$.

2. The *common energy*, or *common momentum* scenario, leads to twice the standard oscillation phase (in the ultra relativistic limit) unless invalid assumptions, or improper approximations, are made. It is also to be noted that the existence of such frames is far from being demonstrated.

3. A more detailed discussions of some of the results presented here can be found in ref. [6] where formulas valid in the non relativistic region are also given.

REFERENCES

1. Particle Data Group, Eur. Phys. Jour. C 3 (1998) 307.
2. D. H. Perkins, Introduction to High Energy Physics, Addison-Wesley (1986)
3. H. J. Lipkin, Phys. Lett. B 348 (1995) 604.
4. B. Kayser, Phys. Rev. D24 (1981) 110.
5. L. B. Okun and E. S. Tsukerman, hep-ph/0007262.
6. S. De Leo, G. Ducati, P. Rotelli, hep-ph/9906460, to be published in Mod. Phys, Lett. A .