# Multipartite entangled states in particle mixing

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- Flavor mixing and entanglement;
- Entanglement in neutrino oscillations:
  - Flavor entanglement;
  - Decoherence;
- Neutrino oscillations as a resource for quantum information.
- Particle mixing and entanglement in Quantum Field Theory.

## Motivations

- Evidence of neutrino mixing and oscillations;

- Importance of entanglement both at a fundamental level and for quantum information;

- Necessity for a treatment of entanglement in the context of Quantum Field Theory*;

- Entanglement in particle physics: entanglement, decoherence, Bell inequalities for the $K^0\bar{K}^0$ (or $B^0\bar{B}^0$) system†;

- Entanglement in neutrino oscillations: entanglement between neutrino and recoil particle‡.

*M.O.Terra Cunha, J.A.Dunningham and V.Vedral, Proc. Royal Soc. A (2007)

†R.A.Bertlmann, Lect. Notes Phys. (2006).

‡B.Kayser et al. Phys Rev. D (2010)

## Entanglement in particle mixing

– Flavor mixing (neutrinos)

$$|\nu_e\rangle = \quad \cos\theta\,|\nu_1\rangle + \sin\theta\,|\nu_2\rangle$$

$$|\nu_\mu\rangle = -\sin\theta\,|\nu_1\rangle + \cos\theta\,|\nu_2\rangle$$

- Correspondence with two-qubit states:

$$|\nu_1\rangle \equiv |1\rangle_1|0\rangle_2 \equiv |10\rangle, \qquad |\nu_2\rangle \equiv |0\rangle_1|1\rangle_2 \equiv |01\rangle,$$

where $|\rangle_i$ denotes states in the Hilbert space for neutrinos with mass $m_i$.

$\Rightarrow$ flavor states are entangled superpositions of the mass eigenstates:

$$|\nu_e\rangle = \quad \cos\theta\,|10\rangle + \sin\theta\,|01\rangle.$$

## Single-particle entanglement*

– A state like $|\psi\rangle_{A,B} = |0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B$ is entangled;

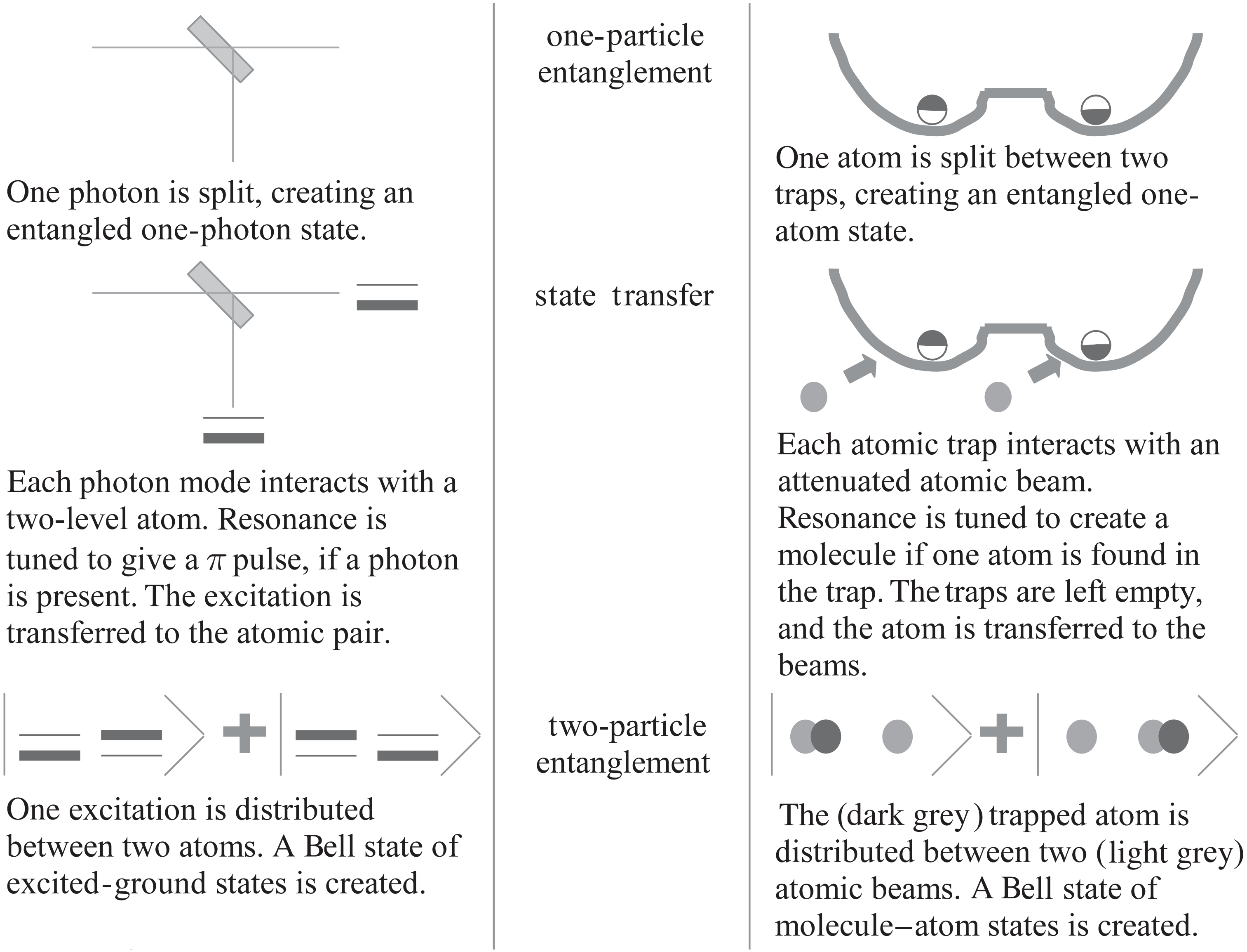
– entanglement among field modes, rather than particles;

– entanglement is a property of composite systems, rather than of many-particle systems;

– entanglement and non-locality are not synonyms;

– single-particle entanglement is as good as two-particle entanglement for applications (quantum cryptography, teleportation, violation of Bell inequalities, etc..).

*J.van Enk, Phys. Rev. A (2005), (2006);
M.O.Terra Cunha, J.A.Dunningham and V.Vedral, Proc. Royal Soc. A (2007);
J.A.Dunningham and V.Vedral, Phys. Rev. Lett. (2007).
S.B.Papp et al. Science (2009)
D.Salart et al. Phys. Rev. Lett (2010)
G.Björk, A.Laghaout, U.L.Andersen Phys. Rev. A (2012)




Protocols for extraction of single-particle entanglement
(from M.O.Terra Cunha, J.A.Dunningham and V.Vedral, Proc. Royal Soc. A (2007))

# Multipartite entanglement

– Characterization of entanglement for multipartite systems is a non-trivial task. Several approaches have been developed: global entanglement, tangle, geometric measures*, etc...

In the 3-qubit case, the two fundamental classes† of states are those of the $GHZ$ state and of the $W$ state:

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)\,,$$

$$|W\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)\,.$$

*T.C.Wei and P.M.Goldbart Phys. Rev. A (2003);
M.Blasone, F.Dell'Anno, S.De Siena and F.Illuminati, Phys. Rev. A (2008).

†W.Dür, G.Vidal, and J.I.Cirac, Phys. Rev. A (2000)

## Multipartite entanglement measures: pure states *

Let $\rho = |\psi\rangle\langle\psi|$ be the density operator corresponding to a pure state $|\psi\rangle$, describing the system $S$ partitioned into $N$ parties.

Bipartition of the $N$-partite system $S = \{S_1, S_2, \ldots, S_N\}$ in two subsystems:

$$S_{A_n} = \{S_{i_1}, S_{i_2}, \ldots, S_{i_n}\}, \qquad 1 \leq i_1 < i_2 < \ldots < i_n \leq N; \quad (1 \leq n < N)$$

and

$$S_{B_{N-n}} = \{S_{j_1}, S_{j_2}, \ldots, S_{j_{N-n}}\}, \qquad 1 \leq j_1 < j_2 < \ldots < j_{N-n} \leq N; \quad i_q \neq j_p$$

– Reduced density matrix of $S_{A_n}$ after tracing over $S_{B_{N-n}}$:

$$\rho_{A_n} \equiv \rho_{i_1, i_2, \ldots, i_n} = Tr_{B_{N-n}}[\rho] = Tr_{j_1, j_2, \ldots, j_{N-n}}[\rho]$$

*D.A.Meyer and N.R.Wallach, J. Math. Phys. (2002);
T.R. de Oliveira, G.Rigolin, and M.C. de Oliveira, Phys. Rev. A (2006);
P.Facchi, G.Florio, and S.Pascazio, Phys. Rev. A (2006).

- von Neumann entropy associated with the above bipartition:

$$E_{vN}^{(A_n;B_{N-n})} = -Tr_{A_n}[\rho_{A_n} \log_2 \rho_{A_n}]\,.$$

- Average von Neumann entropy (global entanglement)

$$\langle E_{vN}^{(n:N-n)} \rangle = \left( \begin{array}{c} N \\ n \end{array} \right)^{-1} \sum_{A_n} E_{vN}^{(A_n;B_{N-n})}\,,$$

where the sum is intended over all the possible bipartitions of the system in two subsystems each with $n$ and $N-n$ elements $(1 \leq n < N)$.

## Multipartite entanglement measures: mixed states

– Entropic measures cannot be used to quantify the entanglement of mixed states $\Rightarrow$ logarithmic negativity.

We denote by

$$\tilde{\rho}_{A_n} \equiv \rho^{PT\, B_{N-n}} = \rho^{PT\, j_1, j_2, \ldots, j_{N-n}}$$

the *bona fide* density matrix, obtained by the partial transposition of $\rho$ with respect to the parties belonging to the subsystem $S_{B_{N-n}}$.

- Logarithmic negativity associated with the above bipartition

$$E_{\mathcal{N}}^{(A_n;B_{N-n})} = \log_2 \| \tilde{\rho}_{A_n} \|_1 \ .$$

- Average logarithmic negativity (global entanglement)

$$\langle E_{\mathcal{N}}^{(n:N-n)} \rangle = \begin{pmatrix} N \\ n \end{pmatrix}^{-1} \sum_{A_n} E_{\mathcal{N}}^{(A_n;B_{N-n})} \, ,$$

where the sum is intended over all the possible bipartitions of the system.

## Multipartite entanglement in neutrino mixing*

– Neutrino mixing (three flavors):

$$|\underline{\nu}_f\rangle \;=\; U(\tilde{\theta},\delta)\,|\underline{\nu}_m\rangle$$

with $|\underline{\nu}_f\rangle = (|\nu_e\rangle, |\nu_\mu\rangle, |\nu_\tau\rangle)^T$ and $|\underline{\nu}_m\rangle = (|\nu_1\rangle, |\nu_2\rangle, |\nu_3\rangle)^T$.

– Mixing matrix (MNSP)

$$U(\tilde{\theta},\delta) \;=\; \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix},$$

where $(\tilde{\theta},\delta) \equiv (\theta_{12},\theta_{13},\theta_{23};\delta)$, $c_{ij} \equiv \cos\theta_{ij}$ and $s_{ij} \equiv \sin\theta_{ij}$.

- Correspondence with three-qubit states:

$$|\nu_1\rangle \equiv |1\rangle_1|0\rangle_2|0\rangle_3 \equiv |100\rangle, \qquad |\nu_2\rangle \equiv |0\rangle_1|1\rangle_2|0\rangle_3 \equiv |010\rangle,$$

$$|\nu_3\rangle \equiv |0\rangle_1|0\rangle_2|1\rangle_3 \equiv |001\rangle$$

*M.Blasone, F.Dell'Anno, S.De Siena, M.Di Mauro and F.Illuminati, Phys. Rev. D (2008).

## Flavor states as generalized W states

– Define the generalized class of three-qubit $W$ states as

$$|\underline{W}^{(3)}(\tilde{\theta};\delta)\rangle \;\equiv\; U^{(3f)}(\tilde{\theta},\delta)\,|\underline{\nu}^{(3)}\rangle$$

$$U^{(3f)}(\tilde{\theta},\delta) \;=\; U(\tilde{\theta},\delta)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta} \end{pmatrix},$$

where $|\underline{W}^{(3)}(\tilde{\theta};\delta)\rangle = \Big(|W_e^{(3)}(\tilde{\theta},\delta)\rangle, |W_\mu^{(3)}(\tilde{\theta},\delta)\rangle, |W_\tau^{(3)}(\tilde{\theta},\delta)\rangle\Big)^T$ and $|\underline{\nu}^{(3)}\rangle = \Big(|\nu_1^{(3)}\rangle, |\nu_2^{(3)}\rangle, |\nu_3^{(3)}\rangle\Big)^T$.

– The entanglement properties of the states associated with matrices $U(\tilde{\theta},\delta)$ and $U^{(3f)}(\tilde{\theta},\delta)$ are identical.

$\Leftrightarrow$ we regard flavor neutrino states as generalized W states.

## Entanglement properties of states with maximal mixing

– Flavor mixing is maximal for

$$\theta_{12}^{max} = \frac{\pi}{4}; \qquad \theta_{23}^{max} = \frac{\pi}{4}; \qquad \theta_{13}^{max} = \arccos\sqrt{\frac{2}{3}}; \qquad \delta^{max} = \frac{\pi}{2}.$$

$\Rightarrow$ all elements of CKM matrix have modulus $= 1/3$:

$$U_{max}^{(3f)} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ iy & iy^2 & i \\ iy^2 & iy & i \end{pmatrix} \qquad \textit{with} \quad y = \exp{(2i\pi/3)}.$$

In this case, all the $|W_{\alpha}^{(3)}(\tilde{\theta}, \delta)\rangle$ states have the same entanglement of $|W^{(3)}\rangle$:

$$E_{vN}^{(A_2;B_1)}(|\underline{W}^{(3)}(\tilde{\theta}^{max};\delta^{max})\rangle) = \langle E_{vN}^{(2:1)}(|\underline{W}^{(3)}(\tilde{\theta}^{max};\delta^{max})\rangle)\rangle = E_{21}^{(3)}.$$

## Entanglement properties of states $|W_\alpha^{(3)}(\tilde{\theta}^{max};\delta)\rangle$ $(\alpha = e,\mu,\tau)$

– We study the dependence of entanglement on the phase $\delta$, with the rotation angles set at their maximal values $\theta_{ij}^{max}$.

The matrix $U^{(3f)}$ becomes

$$U^{(3f)}(\delta) \;=\; \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ -\frac{1}{2}(\sqrt{3}+e^{i\delta}) & \frac{1}{2}(\sqrt{3}-e^{i\delta}) & e^{i\delta} \\ \frac{1}{2}(\sqrt{3}-e^{i\delta}) & -\frac{1}{2}(\sqrt{3}+e^{i\delta}) & e^{i\delta} \end{pmatrix}.$$

We get:

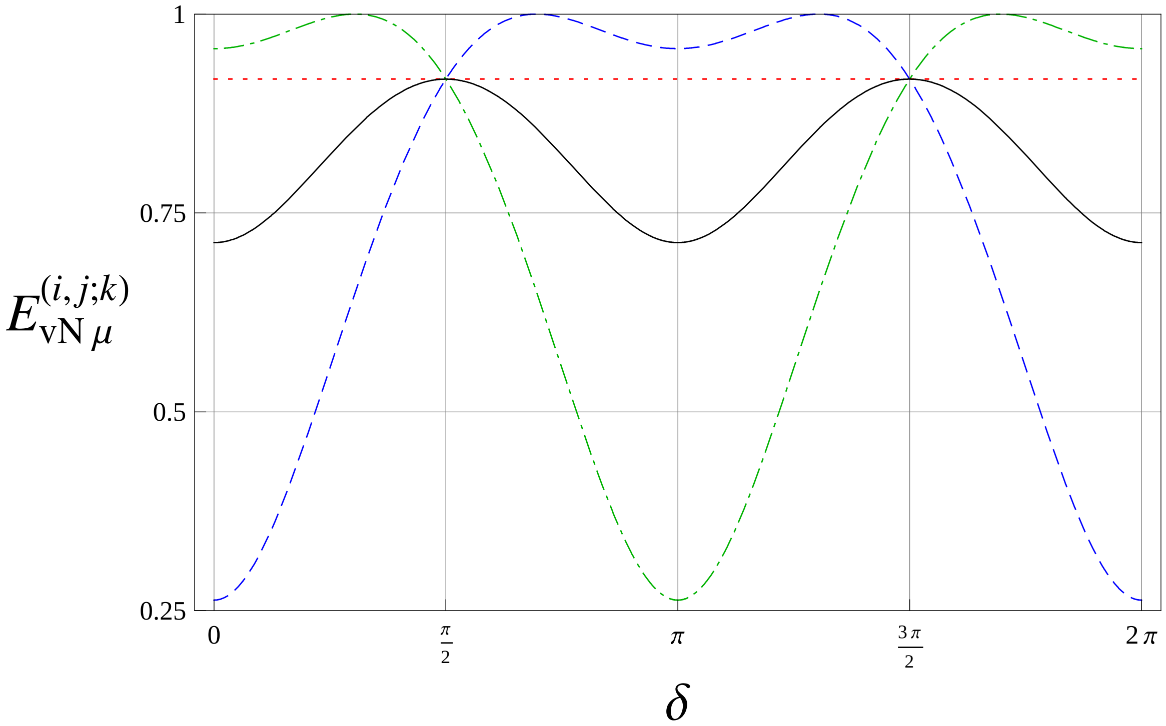
$$E_{vN\,e}^{(1,2;3)} = E_{vN\,e}^{(1,3;2)} = E_{vN\,e}^{(2,3;1)} = E_{vN\,\mu}^{(1,2;3)} = E_{vN\,\tau}^{(1,2;3)} = \log_2 3 - \frac{2}{3},$$

$$E_{vN\,\mu}^{(1,3;2)} = E_{vN\,\tau}^{(2,3;1)} = -\left(\frac{1}{3}-\frac{\cos\delta}{2\sqrt{3}}\right)\log_2\left[\frac{1}{3}-\frac{\cos\delta}{2\sqrt{3}}\right]-\left(\frac{2}{3}+\frac{\cos\delta}{2\sqrt{3}}\right)\log_2\left[\frac{2}{3}+\frac{\cos\delta}{2\sqrt{3}}\right],$$

$$E_{vN\,\mu}^{(2,3;1)} = E_{vN\,\tau}^{(1,3;2)} = -\left(\frac{1}{3}+\frac{\cos\delta}{2\sqrt{3}}\right)\log_2\left[\frac{1}{3}+\frac{\cos\delta}{2\sqrt{3}}\right]-\left(\frac{2}{3}-\frac{\cos\delta}{2\sqrt{3}}\right)\log_2\left[\frac{2}{3}-\frac{\cos\delta}{2\sqrt{3}}\right],$$

where $E_{vN\,\alpha}^{(i,j;k)} \equiv E_{vN}^{(i,j;k)}(|W_\alpha^{(3)}(\delta)\rangle)$.

# Entanglement of the state $|W_\mu^{(3)}(\tilde{\theta}^{max};\delta)\rangle$




von Neumann entropy $E_{vN\,\mu}^{(i,j;k)}$ and average von Neumann entropy $\langle E_{vN\,\mu}^{(2:1)}\rangle$ (full line) as functions of the $CP$-violating phase $\delta$.

$E_{vN\,\mu}^{(i,j;k)}$ is plotted for the bipartitions $i=1, j=2, k=3$ (dotted line); $i=1, j=3, k=2$ (dashed line); $i=2, j=3, k=1$ (dot-dashed line).

$E_{vN\,\mu}^{(1,2;3)}$ is constant and takes the reference value $E_{21}^{(3)} = 0.918296$.

• *"Squeezing"* of entanglement: $E_{vN\,\mu}^{(1,3;2)}$ and $E_{vN\,\mu}^{(2,3;1)}$ vary with $\delta$, attaining the absolute maximum 1 at the points $\delta_1 = \pm\arccos\left(-\frac{1}{\sqrt{3}}\right) \pm 2p\pi$ and $\delta_2 = \pm\arccos\left(\frac{1}{\sqrt{3}}\right) \pm 2p\pi$ (with $p$ integer), respectively, and exceeding the reference value $E_{21}^{(3)}$.

The average von Neumann entropy $\langle E_{vN\,\mu}^{(2:1)} \rangle$ stays below the reference value $E_{21}^{(3)}$, attaining it at the points $\delta = \frac{\pi}{2} \pm p\pi$.

• The free parameter $\delta$ can be used to concentrate and squeeze the entanglement in a specific bipartition, allowing a sharply peaked distribution of entanglement, at the expense of the average von Neumann entropy.

## (Flavor) Entanglement in neutrino oscillations*

– Two-flavor neutrino states

$$|\underline{\nu}^{(f)}\rangle \;=\; \mathbf{U}(\tilde{\theta}, \delta)\, |\underline{\nu}^{(m)}\rangle$$

where $|\underline{\nu}^{(f)}\rangle \;=\; (|\nu_e\rangle, |\nu_\mu\rangle)^T$ and $|\underline{\nu}^{(m)}\rangle \;=\; (|\nu_1\rangle, |\nu_2\rangle)^T$ and $\mathbf{U}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ .

– Flavor states at time t:

$$|\underline{\nu}^{(f)}(t)\rangle \;=\; \mathbf{U}(\tilde{\theta}, \delta)\, \mathbf{U}_0(t)\, \mathbf{U}(\tilde{\theta}, \delta)^{-1}\, |\underline{\nu}^{(f)}\rangle \equiv \widetilde{\mathbf{U}}(t)|\underline{\nu}^{(f)}\rangle \,,$$

with $\mathbf{U}_0(t) = \begin{pmatrix} e^{-iE_1 t} & 0 \\ 0 & e^{-iE_2 t} \end{pmatrix}$.

*M.Blasone, F.Dell'Anno, S.De Siena and F.Illuminati, EPL (2009).

– Transition probability for $\nu_\alpha \to \nu_\beta$

$$P_{\nu_\alpha \to \nu_\beta}(t) = |\langle \nu_\beta | \nu_\alpha(t) \rangle|^2 = |\widetilde{\mathbf{U}}_{\alpha\beta}(t)|^2 \,.$$

• We now take the flavor states at initial time as our qubits:

$$|\nu_e\rangle \equiv |1\rangle_e |0\rangle_\mu \equiv |10\rangle_f, \qquad |\nu_\mu\rangle \equiv |0\rangle_e |1\rangle_\mu \equiv |01\rangle_f,$$

– Starting from $|10\rangle_f$ or $|01\rangle_f$, time evolution generates the (entangled) Bell-like states:

$$|\nu_\alpha(t)\rangle = \widetilde{\mathbf{U}}_{\alpha e}(t)|1\rangle_e |0\rangle_\mu + \widetilde{\mathbf{U}}_{\alpha\mu}(t)|0\rangle_e |1\rangle_\mu, \qquad \alpha = e, \mu.$$

# Entanglement measure

– Let $\rho = |\psi\rangle\langle\psi|$ be the density operator for a pure state $|\psi\rangle$

Bipartition of the $N$-partite system $S = \{S_1, S_2, \ldots, S_N\}$ in two subsystems:

$$S_{A_n} = \{S_{i_1}, S_{i_2}, \ldots, S_{i_n}\}, \quad 1 \leq i_1 < i_2 < \ldots < i_n \leq N; (1 \leq n < N)$$

and

$$S_{B_{N-n}} = \{S_{j_1}, S_{j_2}, \ldots, S_{j_{N-n}}\} \quad , 1 \leq j_1 < j_2 < \ldots < j_{N-n} \leq N; i_q \neq j_p$$

– Reduced density matrix of $S_{A_n}$ after tracing over $S_{B_{N-n}}$:

$$\rho_{A_n} \equiv \rho_{i_1,i_2,\ldots,i_n} = Tr_{B_{N-n}}[\rho] = Tr_{j_1,j_2,\ldots,j_{N-n}}[\rho]$$

- Linear entropy associated to such a bipartition:

$$S_L^{(A_n;B_{N-n})}(\rho) = \frac{d}{d-1}(1 - Tr_{A_n}[\rho_{A_n}^2]),$$

$d$ is the Hilbert-space dimension: $d = \min\{\dim S_{A_n}, \dim S_{B_{N-n}}\} = \min\{2^n, 2^{N-n}\}$.

- Average linear entropy (global entanglement):

$$\langle S_L^{(n:N-n)}(\rho)\rangle = \left(\begin{array}{c} N \\ n \end{array}\right)^{-1} \sum_{A_n} S_L^{(A_n;B_{N-n})}(\rho),$$

sum over all the possible bi-partitions of the system in two subsystems, respectively with $n$ and $N-n$ elements ($1 \leq n < N$).

## Entanglement in neutrino oscillations: two-flavors

Consider the density matrix for the electron neutrino state $\rho^{(e)} = |\nu_e(t)\rangle\langle\nu_e(t)|$, and trace over mode $\mu$ $\Rightarrow$ $\rho_e^{(e)}$.

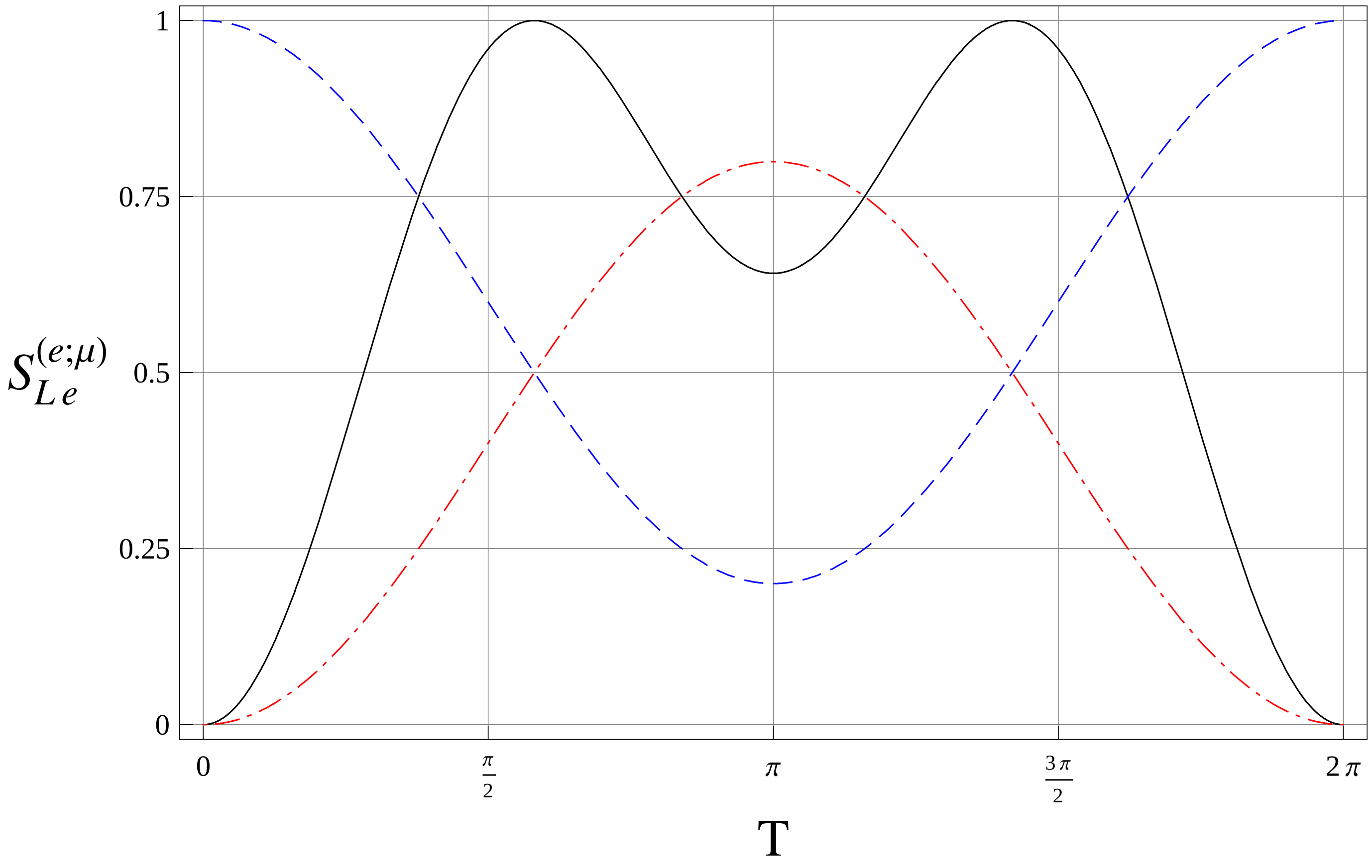
– The associated linear entropy is :

$$S_L^{(e;\mu)}(\rho^{(e)}) = 4\,|\widetilde{\mathbf{U}}_{e\mu}(t)|^2\,|\widetilde{\mathbf{U}}_{ee}(t)|^2 = 4\,P_{\nu_e\to\nu_e}(t)\,P_{\nu_e\to\nu_\mu}(t)$$

– The linear entropy for the state $\rho^{(\alpha)}$ is:

$$\begin{aligned} S_{L\alpha}^{(e;\mu)} = S_{L\alpha}^{(\mu;e)} = \langle S_{L\alpha}^{(1:1)}\rangle &= 4\,|\widetilde{\mathbf{U}}_{\alpha\mu}(t)|^2\,|\widetilde{\mathbf{U}}_{\alpha e}(t)|^2 \\ &= 4|\widetilde{\mathbf{U}}_{\alpha e}(t)|^2\,(1 - |\widetilde{\mathbf{U}}_{\alpha e}(t)|^2) \\ &= 4|\widetilde{\mathbf{U}}_{\alpha\mu}(t)|^2\,(1 - |\widetilde{\mathbf{U}}_{\alpha\mu}(t)|^2)\,. \end{aligned}$$

- Linear entropy given by product of transition probabilities !




Linear entropy $S_{Le}^{(e;\mu)}$ (full) as a function of the scaled time $T = \frac{2Et}{\Delta m_{12}^2}$, with $\sin^2\theta = 0.314$. Transition probabilities $P_{\nu_e \to \nu_e}$ (dashed) and $P_{\nu_e \to \nu_\mu}$ (dot-dashed) are reported for comparison.

## (Flavor) Entanglement in neutrino oscillations: three flavors

### Three-flavor neutrino states

$$|\underline{\nu}^{(f)}\rangle \;=\; \mathbf{U}(\tilde{\theta},\delta)\,|\underline{\nu}^{(m)}\rangle$$

where $|\underline{\nu}^{(f)}\rangle \;=\; (|\nu_e\rangle, |\nu_\mu\rangle, |\nu_\tau\rangle)^T$ and $|\underline{\nu}^{(m)}\rangle \;=\; (|\nu_1\rangle, |\nu_2\rangle, |\nu_3\rangle)^T$

$$\mathbf{U}(\tilde{\theta},\delta) \;=\; \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \;,$$

where $(\tilde{\theta},\delta) \equiv (\theta_{12},\theta_{13},\theta_{23};\delta)$, $c_{ij} \equiv \cos\theta_{ij}$ and $s_{ij} \equiv \sin\theta_{ij}$.

– Flavor states at time t:

$$|\underline{\nu}^{(f)}(t)\rangle \;=\; \mathbf{U}(\tilde\theta,\delta)\,\mathbf{U}_0(t)\,\mathbf{U}(\tilde\theta,\delta)^{-1}\,|\underline{\nu}^{(f)}\rangle \equiv \widetilde{\mathbf{U}}(t)|\underline{\nu}^{(f)}\rangle\,,$$

with $|\underline{\nu}^{(f)}\rangle$ flavor states at $t=0$, $\mathbf{U}_0(t) = diag(e^{-iE_1t}, e^{-iE_2t}, e^{-iE_3t})$,

and $\widetilde{\mathbf{U}}(t) = \mathbf{U}(\tilde\theta,\delta)\,\mathbf{U}_0(t)\,\mathbf{U}(\tilde\theta,\delta)^{-1}$, with $\widetilde{\mathbf{U}}(t=0) = 1\!\mathrm{I}$.

– Transition probability for $\nu_\alpha \to \nu_\beta$

$$P_{\nu_\alpha\to\nu_\beta}(t) \;=\; |\langle\nu_\beta|\nu_\alpha(t)\rangle|^2 \;=\; |\widetilde{\mathbf{U}}_{\alpha\beta}(t)|^2\,,$$

- Take the flavor states at time $t=0$ as the qubits:

$$|\nu_e\rangle \equiv |1\rangle_e|0\rangle_\mu|0\rangle_\tau \equiv |100\rangle_f, \qquad |\nu_\mu\rangle \equiv |0\rangle_e|1\rangle_\mu|0\rangle_\tau \equiv |010\rangle_f,$$

$$|\nu_\tau\rangle \equiv |0\rangle_e|0\rangle_\mu|1\rangle_\tau \equiv |001\rangle_f$$

## Entanglement in neutrino oscillations: three-flavors

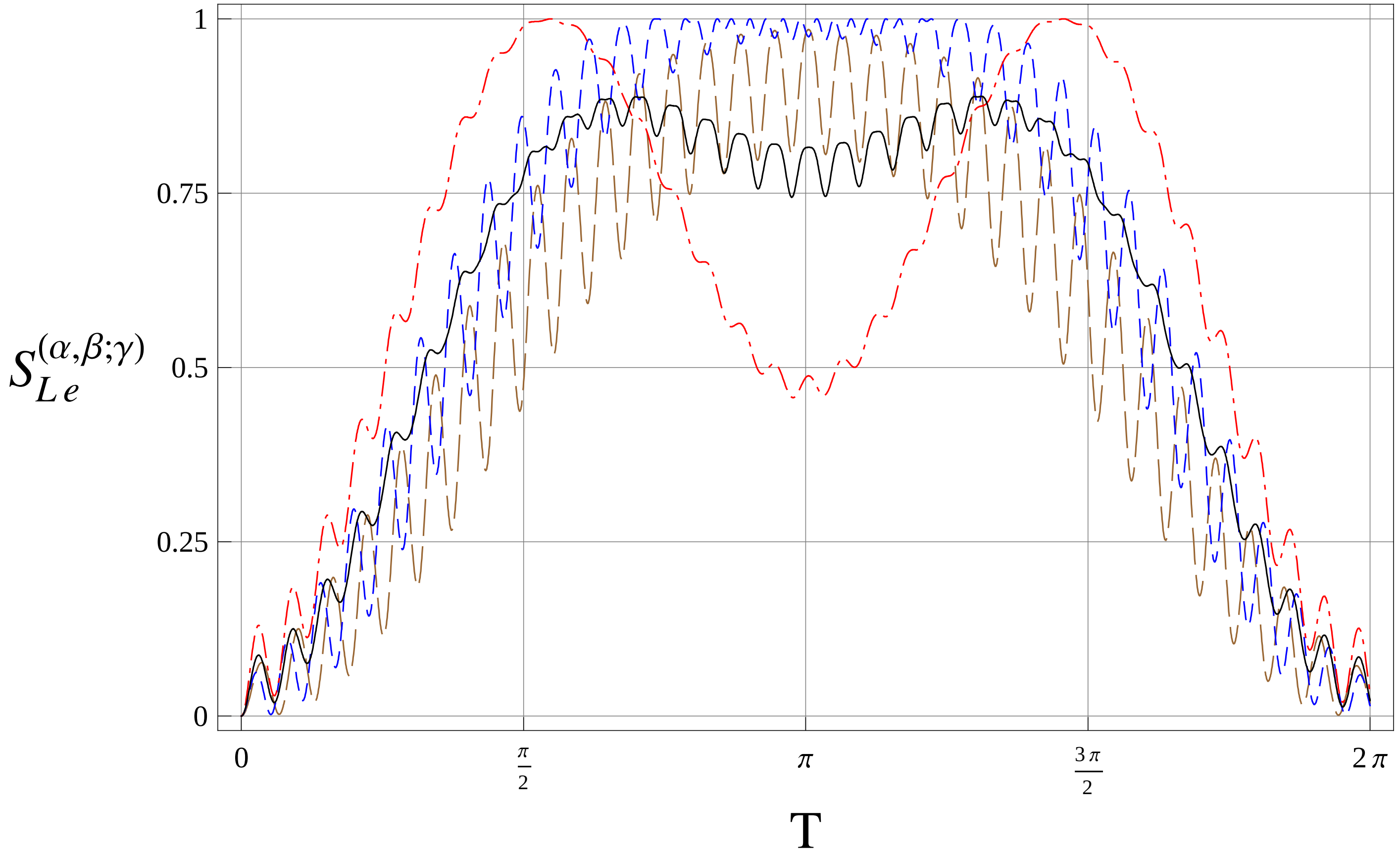
– In the three-flavor case, we obtain

$$\begin{aligned} S_{L\alpha}^{(e,\mu;\tau)} &= 4|\widetilde{\mathbf{U}}_{\alpha\tau}(t)|^2\,(|\widetilde{\mathbf{U}}_{\alpha e}(t)|^2 + |\widetilde{\mathbf{U}}_{\alpha\mu}(t)|^2) \\ &= 4|\widetilde{\mathbf{U}}_{\alpha\tau}(t)|^2\,(1 - |\widetilde{\mathbf{U}}_{\alpha\tau}(t)|^2)\,. \end{aligned}$$

The linear entropies for the two remaining bi-partitions are easily obtained by permuting the indexes $e, \mu, \tau$.

– The average linear entropy is

$$\langle S_{L\alpha}^{(2:1)} \rangle = \frac{8}{3}\,(|\widetilde{\mathbf{U}}_{\alpha e}(t)|^2|\widetilde{\mathbf{U}}_{\alpha\mu}(t)|^2 + |\widetilde{\mathbf{U}}_{\alpha e}(t)|^2|\widetilde{\mathbf{U}}_{\alpha\tau}(t)|^2 + |\widetilde{\mathbf{U}}_{\alpha\mu}(t)|^2|\widetilde{\mathbf{U}}_{\alpha\tau}(t)|^2).$$




Linear entropies $S_{Le}^{(\alpha,\beta;\gamma)}$ and $\langle S_{Le}^{(2;1)} \rangle$ as functions of the scaled time $T = \frac{2Et}{\Delta m_{12}^2}$. Curves correspond to the partial linear entropies $S_{Le}^{(e,\mu;\tau)}$ (long-dashed), $S_{Le}^{(e,\tau;\mu)}$ (dashed), $S_{Le}^{(\mu,\tau;e)}$ (dot-dashed), and to the average linear entropy $\langle S_{Le}^{(2;1)} \rangle$ (full).

Parameters are fixed at central experimental values: $\sin^2\theta_{12} = 0.314$, $\sin^2\theta_{23} = 0.45$, $\sin^2\theta_{12} = 0.008$, $\Delta m_{12}^2 = 7.92 \times 10^{-5} eV^2$, $\Delta m_{23}^2 = 2.6 \times 10^{-3} eV^2$.

Because of $CPT$ invariance, the $CP$ asymmetry $\Delta_{CP}^{\alpha,\beta}$ is equal to the asymmetry under time reversal:

$$\begin{aligned}\Delta_{CP}^{\alpha,\beta} = \Delta_{T}^{\alpha,\beta} &= P_{\nu_\alpha\to\nu_\beta}(t) - P_{\nu_\beta\to\nu_\alpha}(t) \\ &= P_{\nu_\alpha\to\nu_\beta}(t) - P_{\nu_\alpha\to\nu_\beta}(-t)\,.\end{aligned}$$

$\Delta_{CP}^{\alpha,\beta} \neq 0$ for $\delta \neq 0$. Note that $\sum_\beta \Delta_{CP}^{\alpha\beta} = 0$ with $\alpha, \beta = e, \mu, \tau$.

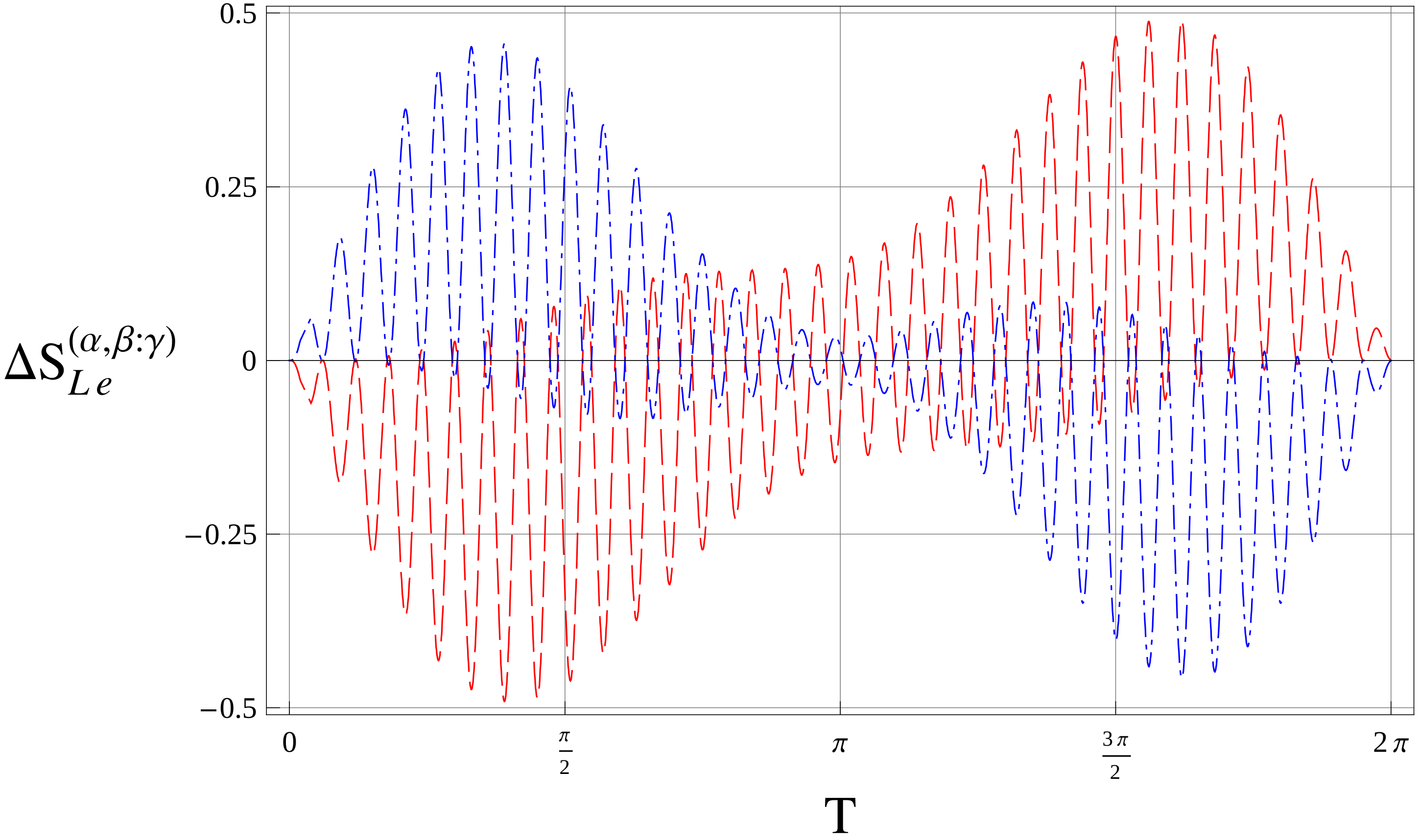
– Define the *"imbalances"*, i.e. the difference between the linear entropies and their time-reversed expressions:

$$\Delta S_{L\lambda}^{(\alpha,\beta;\gamma)} = S_{L\lambda}^{(\alpha,\beta;\gamma)}(t) - S_{L\lambda}^{(\alpha,\beta;\gamma)}(-t)\,,$$

– We have for example:

$$\Delta S_{Le}^{(e,\mu;\tau)} = 4\,\Delta_{CP}^{e,\mu}\,(|\widetilde{\mathbf{U}}_{e\tau}(t)|^2 + |\widetilde{\mathbf{U}}_{\tau e}(t)|^2 - 1)\,,$$

where the last factor is $CP$-even.




The imbalances $\Delta S_{Le}^{(\alpha,\beta;\gamma)}$ as functions of the scaled time $T$. Curves correspond to $\Delta S_{Le}^{(e,\mu;\tau)}$ (long-dashed) and $\Delta S_{Le}^{(e,\tau;\mu)}$ (dot-dashed). The quantity $\Delta S_{Le}^{(\mu,\tau;e)}$ is vanishing.

The $CP$-violating phase is set at the value $\delta = \pi/2$.

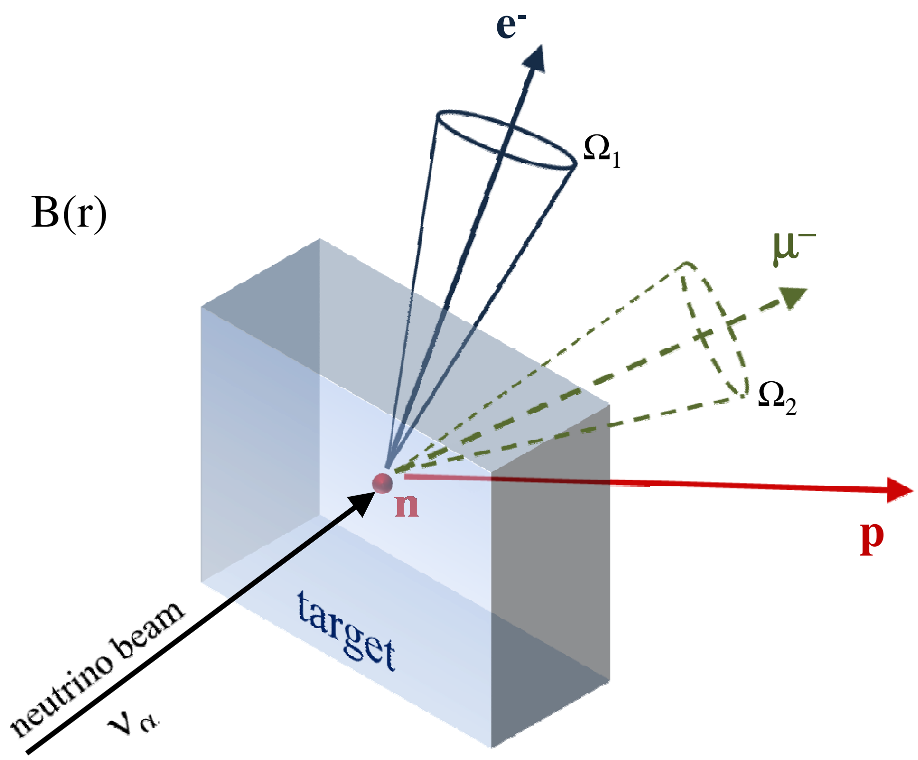
## Neutrino oscillations as a resource for quantum information

• Single-particle entanglement encoded in flavor states $|\underline{\nu}^{(f)}(t)\rangle$ is a real physical resource that can be used, at least in principle, for protocols of quantum information.

– Experimental scheme for the transfer of the flavor entanglement of a neutrino beam into a single-particle system with *spatially separated modes*.

Charged-current interaction between a neutrino $\nu_\alpha$ with flavor $\alpha$ and a nucleon $N$ gives a lepton $\alpha^-$ and a baryon $X$:

$$\nu_\alpha + N \longrightarrow \alpha^- + X \, .$$




## Generation of a single-particle entangled lepton state (two flavors):

In the target the charged-current interaction occurs: $\nu_\alpha + n \longrightarrow \alpha^- + p$ with $\alpha = e\,, \mu$.

A spatially nonuniform magnetic field $\mathbf{B}(\mathbf{r})$ constraints the momentum of the outgoing lepton within a solid angle $\Omega_i$, and ensures spatial separation between lepton paths.

The reaction produces a superposition of electronic and muonic spatially separated states.

• Given the initial Bell-like superposition $|\nu_\alpha(t)\rangle$ the unitary process associated with the weak interaction leads to the superposition

$$|\alpha(t)\rangle \,=\, \Lambda_e|1\rangle_e|0\rangle_\mu + \Lambda_\mu|0\rangle_e|1\rangle_\mu\,,$$

where $|\Lambda_e|^2 + |\Lambda_\mu|^2 = 1$, and $|k\rangle_\alpha$, with $k = 0, 1$, represents the lepton qubit.

The coefficients $\Lambda_\alpha$ are proportional to $\widetilde{\mathbf{U}}_{\alpha\beta}(t)$ and to the cross sections associated with the creation of an electron or a muon.

• Analogy with single-photon system: quantum uncertainty on the so-called "*which path*" of the photon at the output of an unbalanced beam splitter $\Leftrightarrow$ uncertainty on the "*which flavor*" of the produced lepton.

The coefficients $\Lambda_\alpha$ plays the role of the transmissivity and of the reflectivity of the beam splitter.

## Entanglement for mixed particles in QFT*

– Extension of the above analysis to QFT

– Non-trivial nature of mixing transformations in QFT

– Dynamical symmetry approach to entanglement

– Entropic measures in QFT

*M.Blasone, F.Dell'Anno, F.Illuminati and S.De Siena, work in progress.

## Entanglement in relativistic systems

- Necessity* for a treatment of entanglement in the context of Quantum Field Theory.

- Lorentz invariance of entanglement†;

*M.O.Terra Cunha, J.A.Dunningham and V.Vedral, Proc. Royal Soc. A (2007)

†N.Friis, R.A.Bertlmann, M.Huber and B.C.Hiesmayr Phys. Rev. A (2010); M.Huber, N.Friis, A.Gabriel, C.Spengler and B.C.Hiesmayr, EPL (2011)

# Neutrino mixing in QFT

- Mixing relations for two Dirac fields

$$\begin{aligned}\nu_e(x) &= \nu_1(x)\cos\theta + \nu_2(x)\sin\theta \\ \nu_\mu(x) &= -\nu_1(x)\sin\theta + \nu_2(x)\cos\theta\end{aligned}$$

can be written as*

$$\begin{aligned}\nu_e^\alpha(x) &= G_\theta^{-1}(t)\,\nu_1^\alpha(x)\,G_\theta(t) \\ \nu_\mu^\alpha(x) &= G_\theta^{-1}(t)\,\nu_2^\alpha(x)\,G_\theta(t)\end{aligned}$$

– Mixing generator:

$$G_\theta(t) = \exp\left[\theta\int d^3\mathbf{x}\,\left(\nu_1^\dagger(x)\nu_2(x) - \nu_2^\dagger(x)\nu_1(x)\right)\right]$$

*M.Blasone and G.Vitiello, Annals Phys. (1995)

• The vacuum $|0\rangle_{1,2}$ is not invariant under the action of the generator $G_\theta(t)$:

$$|0(t)\rangle_{e,\mu} \equiv G_\theta^{-1}(t)\;|0\rangle_{1,2} = e^{-\theta\big(S_+(t)-S_-(t)\big)}\,|0\rangle_{1,2}$$

• Relation between $|0\rangle_{1,2}$ and $|0(t)\rangle_{e,\mu}$: orthogonality! (for $V\to\infty$)

$$\lim_{V\to\infty}\; {}_{1,2}\langle 0|0(t)\rangle_{e,\mu} = \lim_{V\to\infty}\; e^{V\int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\ln\left(1-\sin^2\theta\,|V_\mathbf{k}|^2\right)^2} = 0$$

with

$$|V_\mathbf{k}|^2 \;\equiv\; \sum_{r,s}|\,v^{r\dagger}_{-\mathbf{k},1}u^{s}_{\mathbf{k},2}\,|^2$$

.

• The "flavor vacuum" $|0(t)\rangle_{e,\mu}$ is a $SU(2)$ generalized coherent state*:

$$|0\rangle_{e,\mu} = \prod_{\mathbf{k},r}\Big[(1-\sin^2\theta\,|V_\mathbf{k}|^2) - \epsilon^r\sin\theta\,\cos\theta\,|V_\mathbf{k}|\,(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2}+\alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1})$$

$$+\,\epsilon^r\sin^2\theta\,|V_\mathbf{k}||U_\mathbf{k}|\,(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},1}-\alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},2}) + \sin^2\theta\,|V_\mathbf{k}|^2\,\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2}\alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}\Big]\,|0\rangle_{1,2}$$

*A. Perelomov, *Generalized Coherent States and Their Applications*, (Springer V., 1986)

• Structure of the annihilation operators for $|0(t)\rangle_{e,\mu}$:

$$\alpha^r_{\mathbf{k},e}(t) = \cos\theta\,\alpha^r_{\mathbf{k},1} + \sin\theta\left(U^*_{\mathbf{k}}(t)\,\alpha^r_{\mathbf{k},2} + \epsilon^r V_{\mathbf{k}}(t)\,\beta^{r\dagger}_{-\mathbf{k},2}\right)$$

$$\alpha^r_{\mathbf{k},\mu}(t) = \cos\theta\,\alpha^r_{\mathbf{k},2} - \sin\theta\left(U_{\mathbf{k}}(t)\,\alpha^r_{\mathbf{k},1} - \epsilon^r V_{\mathbf{k}}(t)\,\beta^{r\dagger}_{-\mathbf{k},1}\right)$$

$$\beta^r_{-\mathbf{k},e}(t) = \cos\theta\,\beta^r_{-\mathbf{k},1} + \sin\theta\left(U^*_{\mathbf{k}}(t)\,\beta^r_{-\mathbf{k},2} - \epsilon^r V_{\mathbf{k}}(t)\,\alpha^{r\dagger}_{\mathbf{k},2}\right)$$

$$\beta^r_{-\mathbf{k},\mu}(t) = \cos\theta\,\beta^r_{-\mathbf{k},2} - \sin\theta\left(U_{\mathbf{k}}(t)\,\beta^r_{-\mathbf{k},1} + \epsilon^r V_{\mathbf{k}}(t)\,\alpha^{r\dagger}_{\mathbf{k},1}\right)$$

• Mixing transformation = Rotation + Bogoliubov transformation .

– Bogoliubov coefficients:

$$U_{\mathbf{k}}(t) \;=\; u^{r\dagger}_{\mathbf{k},2} u^r_{\mathbf{k},1}\, e^{i(\omega_{k,2}-\omega_{k,1})t} \quad ; \qquad V_{\mathbf{k}}(t) = \epsilon^r\, u^{r\dagger}_{\mathbf{k},1} v^r_{-\mathbf{k},2}\, e^{i(\omega_{k,2}+\omega_{k,1})t}$$

$$|U_{\mathbf{k}}|^2 + |V_{\mathbf{k}}|^2 = 1$$

## Currents and charges for mixed fermions *

- Lagrangian in the mass basis:

$$\mathcal{L} = \bar{\Psi}_m \left(i \not\partial - M_d\right) \Psi_m$$

where $\Psi_m^T = (\nu_1, \nu_2)$ and $M_d = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$.

– Lagrangian in the flavor basis:

$$\mathcal{L} = \bar{\Psi}_f \left(i \not\partial - M\right) \Psi_f$$

where $\Psi_f^T = (\nu_e, \nu_\mu)$ and $M = \begin{pmatrix} m_e & m_{e\mu} \\ m_{e\mu} & m_\mu \end{pmatrix}$.

*M.Blasone, P.Jizba and G.Vitiello, Phys. Lett. B (2001)

– Two sets of charges:

$$\begin{aligned} Q_{\nu_i} &= \int d^3\mathbf{x}\, \nu_i^\dagger(x)\, \nu_i(x)\ ; \qquad\qquad i = 1, 2 \\ Q_{\nu_\sigma}(t) &= \int d^3\mathbf{x}\, \nu_\sigma^\dagger(x)\, \nu_\sigma(x)\ ; \qquad\qquad \sigma = e, \mu \end{aligned}$$

with

$$\begin{aligned} Q_{\nu_e}(t) &= \cos^2\theta\, Q_{\nu_1} + \sin^2\theta\, Q_{\nu_2} + \sin\theta\cos\theta \int d^3\mathbf{x} \left[\nu_1^\dagger \nu_2 + \nu_2^\dagger \nu_1\right] \\ Q_{\nu_\mu}(t) &= \sin^2\theta\, Q_{\nu_1} + \cos^2\theta\, Q_{\nu_2} - \sin\theta\cos\theta \int d^3\mathbf{x} \left[\nu_1^\dagger \nu_2 + \nu_2^\dagger \nu_1\right] . \end{aligned}$$

- Problem: find the eigenstates of the above charges.

- The flavor charge operators are diagonal in the flavor ladder operators:

$$
\begin{aligned}
:: Q_{\nu_\sigma}(t) :: \ &\equiv \int d^3\mathbf{x} \ :: \nu_\sigma^\dagger(x)\, \nu_\sigma(x) :: \\
&= \sum_r \int d^3\mathbf{k} \left( \alpha^{r\dagger}_{\mathbf{k},\sigma}(t)\alpha^r_{\mathbf{k},\sigma}(t) - \beta^{r\dagger}_{-\mathbf{k},\sigma}(t)\beta^r_{-\mathbf{k},\sigma}(t) \right), \quad \sigma = e, \mu.
\end{aligned}
$$

Here :: ... :: denotes normal ordering with respect to the flavor vacuum:

$$
:: A :: \equiv A - {}_{e,\mu}\langle 0|A|0\rangle_{e,\mu}
$$

- Define flavor neutrino states with definite momentum and helicity:

$$
|\nu^r_{\mathbf{k},\sigma}\rangle \equiv \alpha^{r\dagger}_{\mathbf{k},\sigma}(0)\, |0\rangle_{e,\mu}
$$

– Such states are eigenstates of the flavor charges (at t=0):

$$
:: Q_{\nu_\sigma} :: |\nu^r_{\mathbf{k},\sigma}\rangle = |\nu^r_{\mathbf{k},\sigma}\rangle
$$

# Lepton charge violation for Pontecorvo states*

– Pontecorvo states:

$$|\nu^r_{\mathbf{k},e}\rangle_P \;=\; \cos\theta\,|\nu^r_{\mathbf{k},1}\rangle \;+\; \sin\theta\,|\nu^r_{\mathbf{k},2}\rangle$$

$$|\nu^r_{\mathbf{k},\mu}\rangle_P \;=\; -\sin\theta\,|\nu^r_{\mathbf{k},1}\rangle \;+\; \cos\theta\,|\nu^r_{\mathbf{k},2}\rangle\,,$$

are *not* eigenstates of the flavor charges.

⇒ *violation of lepton charge conservation* in the production/detection vertices, at tree level:

$$_P\langle\nu^r_{\mathbf{k},e}|\,:Q_e(0):\,|\nu^r_{\mathbf{k},e}\rangle_P \;=\; \cos^4\theta+\sin^4\theta+2|U_{\mathbf{k}}|\,\sin^2\theta\cos^2\theta \;<\; 1,$$

for any $\theta \neq 0$, $\mathbf{k} \neq 0$ and for $m_1 \neq m_2$.

*M. Blasone, A. Capolupo, F. Terranova and G. Vitiello, Phys. Rev. **D** (2005)
C. C. Nishi, Phys. Rev. **D** (2008);
M. Blasone, A. Capolupo, C.-R. Ji and G. Vitiello, Int.J.Mod.Phys.**A** (2010).

– We have, for an electron neutrino state:

$$\begin{aligned}\mathcal{Q}_{\mathbf{k},\nu_\sigma}(t) &\equiv \langle \nu^r_{\mathbf{k},e}| :: Q_{\nu_\sigma}(t) :: |\nu^r_{\mathbf{k},e}\rangle \\ &= \left|\left\{\alpha^r_{\mathbf{k},\sigma}(t), \alpha^{r\dagger}_{\mathbf{k},e}(0)\right\}\right|^2 + \left|\left\{\beta^{r\dagger}_{-\mathbf{k},\sigma}(t), \alpha^{r\dagger}_{\mathbf{k},e}(0)\right\}\right|^2\end{aligned}$$

• Neutrino oscillation formula (exact result)*:

$$\mathcal{Q}_{\mathbf{k},\nu_e}(t) = 1 - |U_{\mathbf{k}}|^2 \sin^2(2\theta) \sin^2\left(\frac{\omega_{k,2}-\omega_{k,1}}{2}t\right) - |V_{\mathbf{k}}|^2 \sin^2(2\theta) \sin^2\left(\frac{\omega_{k,2}+\omega_{k,1}}{2}t\right)$$

$$\mathcal{Q}_{\mathbf{k},\nu_\mu}(t) = |U_{\mathbf{k}}|^2 \sin^2(2\theta) \sin^2\left(\frac{\omega_{k,2}-\omega_{k,1}}{2}t\right) + |V_{\mathbf{k}}|^2 \sin^2(2\theta) \sin^2\left(\frac{\omega_{k,2}+\omega_{k,1}}{2}t\right)$$

- For $k \gg \sqrt{m_1 m_2}$, $|U_{\mathbf{k}}|^2 \to 1$ and $|V_{\mathbf{k}}|^2 \to 0$.

*M.Blasone, P.Henning and G.Vitiello, Phys. Lett. **B** (1999).

## Dynamical symmetry approach to entanglement

• Entanglement can be characterized by total variance of the operators generating the dynamical algebra*.

– Consider the observables $X_i$ elements of the basis of a Lie algebra $\mathcal{L}$ such that the Lie group $G = \exp(i\mathcal{L})$ defines the dynamic symmetry of the system.

• Entanglement of a state $|\psi\rangle$ is given by the total amount of uncertainty:

$$\Delta(\psi) \;=\; \sum_i \left(\langle\psi|X_i^2|\psi\rangle - \langle\psi|X_i|\psi\rangle^2\right)$$

*A. A. Klyachko, B. Öztop, and A. S. Shumovsky, *Phys. Rev.* **A** (2007);

– Consistency with above QM results:

Define flavor annihilation operators:

$$\begin{aligned}\alpha_e(t) &= \cos\theta\,\alpha_1(t) + \sin\theta\,\alpha_2(t)\\ \alpha_\mu(t) &= -\sin\theta\,\alpha_2(t) + \cos\theta\,\alpha_1(t)\end{aligned}$$

– Flavor states:

$$|\nu_\sigma(t)\rangle \equiv \alpha_\sigma^\dagger(t)|0\rangle_m, \qquad \sigma = e, \mu.$$

• Variance of the number operators $N_i = \alpha_i^\dagger \alpha_i$ and $N_\sigma(t) = \alpha_\sigma^\dagger(t)\alpha_\sigma(t)$:

– Variance of $N_i$ $\Rightarrow$ static entanglement:

$$\Delta N_i(\nu_e) = \cos^2\theta\sin^2\theta$$

– Variance of $N_\sigma(t)$ $\Rightarrow$ flavor entanglement:

$$\Delta N_\sigma(\nu_e)(t) = P_{\nu_e\to\nu_e}(t)\,P_{\nu_e\to\nu_\mu}(t)$$

## Entanglement for flavor neutrino states in QFT I

– Entanglement for flavor neutrino states in QFT can be expressed by means of the variances of the neutrino charges:* $Q_{\nu_i}$, $Q_{\nu_\sigma}(t)$

– Variance of $Q_{\nu_i} \rightarrow$ static entanglement:

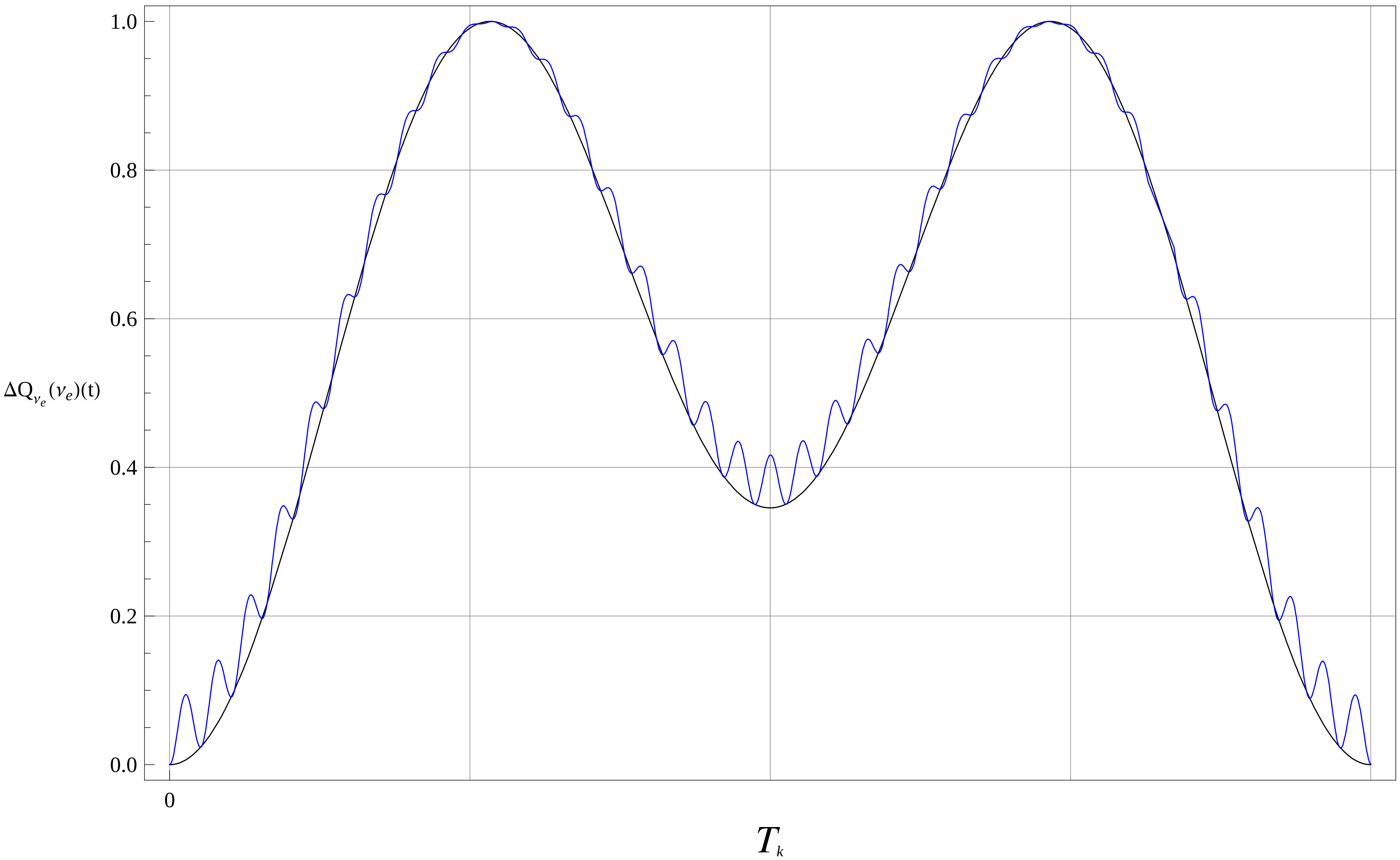
$$\begin{aligned} \Delta Q_{\nu_i}(\nu_e) &= \langle \nu^r_{\mathbf{k},e} | Q^2_{\nu_i}(t) | \nu^r_{\mathbf{k},e} \rangle - \langle \nu^r_{\mathbf{k},e} | Q_{\nu_i} | \nu^r_{\mathbf{k},e} \rangle^2 \\ &= \cos^2\theta \sin^2\theta \end{aligned}$$

– Variance of $Q_{\nu_\sigma} \rightarrow$ flavor entanglement:

$$\begin{aligned} \Delta Q_{\nu_\sigma}(\nu_e)(t) &= \langle \nu^r_{\mathbf{k},e} | Q^2_{\nu_\sigma}(t) | \nu^r_{\mathbf{k},e} \rangle - \langle \nu^r_{\mathbf{k},e} | Q_{\nu_\sigma} | \nu^r_{\mathbf{k},e} \rangle^2 \\ &= \mathcal{Q}^{\mathbf{k}}_{\nu_e \rightarrow \nu_e}(t)\, \mathcal{Q}^{\mathbf{k}}_{\nu_e \rightarrow \nu_\mu}(t) \end{aligned}$$

in formal agreement with results obtained in QM.

*M.Blasone, F. Dell'Anno, S.De Siena and F. Illuminati, J.Phys:Conf.Ser. (2010)




QM vs. QFT flavor entanglement for $|\nu_e(t)\rangle$.

## Entanglement for flavor states in QFT II *

Rewrite QFT flavor neutrino state as:

$$|\nu_e(t)\rangle = [U_{ee}(t)\,\alpha_e^\dagger + U_{e\mu}(t)\,\alpha_\mu^\dagger + U_{e\mu}^{e\bar{e}}(t)\,\alpha_e^\dagger\alpha_\mu^\dagger\beta_e^\dagger + U_{ee}^{\mu\bar{\mu}}(t)\,\alpha_e^\dagger\alpha_\mu^\dagger\beta_\mu^\dagger]\,|0\rangle_{e,\mu}$$

with

$$U_{ee}(t) = e^{-i\omega_1 t}[\cos^2\theta + \sin^2\theta(e^{-i(\omega_2-\omega_1)t}|U|^2 + e^{-i(\omega_2+\omega_1)t}|V|^2)]$$

$$U_{e\mu}(t) = e^{-i\omega_1 t}U\cos\theta\sin\theta(e^{-i(\omega_2-\omega_1)t} - 1)$$

$$U_{e\mu}^{e\bar{e}}(t) = e^{-i\omega_1 t}V\cos\theta\sin\theta(1 - e^{-i(\omega_2+\omega_1)t})$$

$$U_{ee}^{\mu\bar{\mu}}(t) = e^{-i\omega_1 t}UV\sin^2\theta(e^{-i(\omega_2+\omega_1)t} - e^{-i(\omega_2-\omega_1)t}),$$

$$|U_{ee}(t)|^2 + |U_{e\mu}(t)|^2 + |U_{e\mu}^{e\bar{e}}(t)|^2 + |U_{ee}^{\mu\bar{\mu}}(t)|^2 = 1.$$

- In QFT, flavor neutrino states exhibit multiparticle components:

$$|\nu_e\rangle = U_{ee}|1000\rangle + U_{e\mu}|0100\rangle + U_{e\mu}^{e\bar{e}}|1110\rangle + U_{ee}^{\mu\bar{\mu}}|1101\rangle.$$

*M.Blasone, F. Dell'Anno, S.De Siena and F. Illuminati, work in progress

The linear entropies $S_L^{(\alpha;\beta,\gamma,\delta)}$ associated with $|\nu_e\rangle$ are:

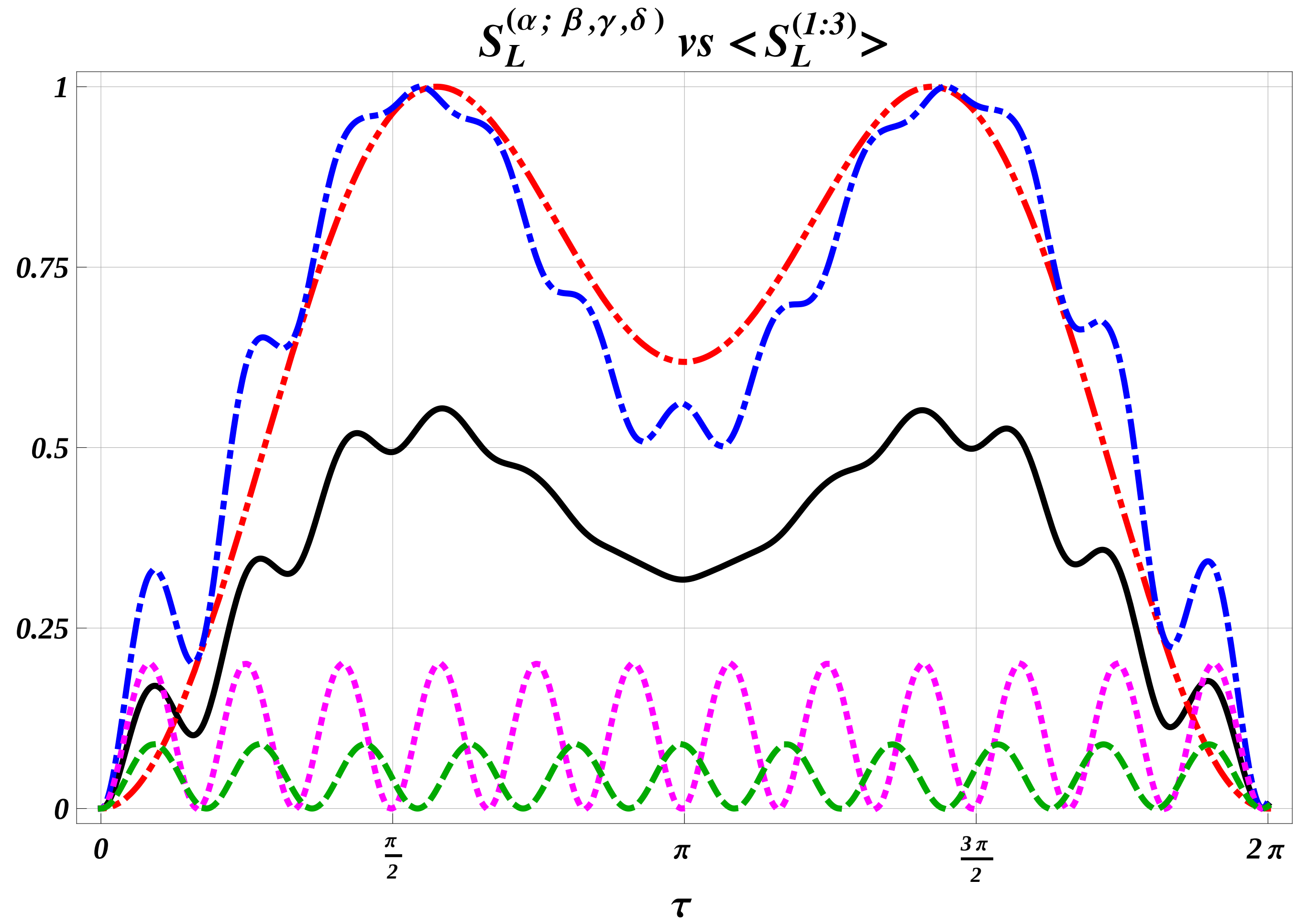
$$S_L^{(\nu_e;\nu_\mu,\bar{\nu}_e,\bar{\nu}_\mu)} = 4|U_{e\mu}|^2(1-|U_{e\mu}|^2)\,, \tag{1}$$

$$S_L^{(\nu_\mu;\nu_e,\bar{\nu}_e,\bar{\nu}_\mu)} = 4|U_{ee}|^2(1-|U_{ee}|^2)\,, \tag{2}$$

$$S_L^{(\bar{\nu}_e;\nu_e,\nu_\mu,\bar{\nu}_\mu)} = 4|U_{e\mu}^{e\bar{e}}|^2(1-|U_{e\mu}^{e\bar{e}}|^2)\,, \tag{3}$$

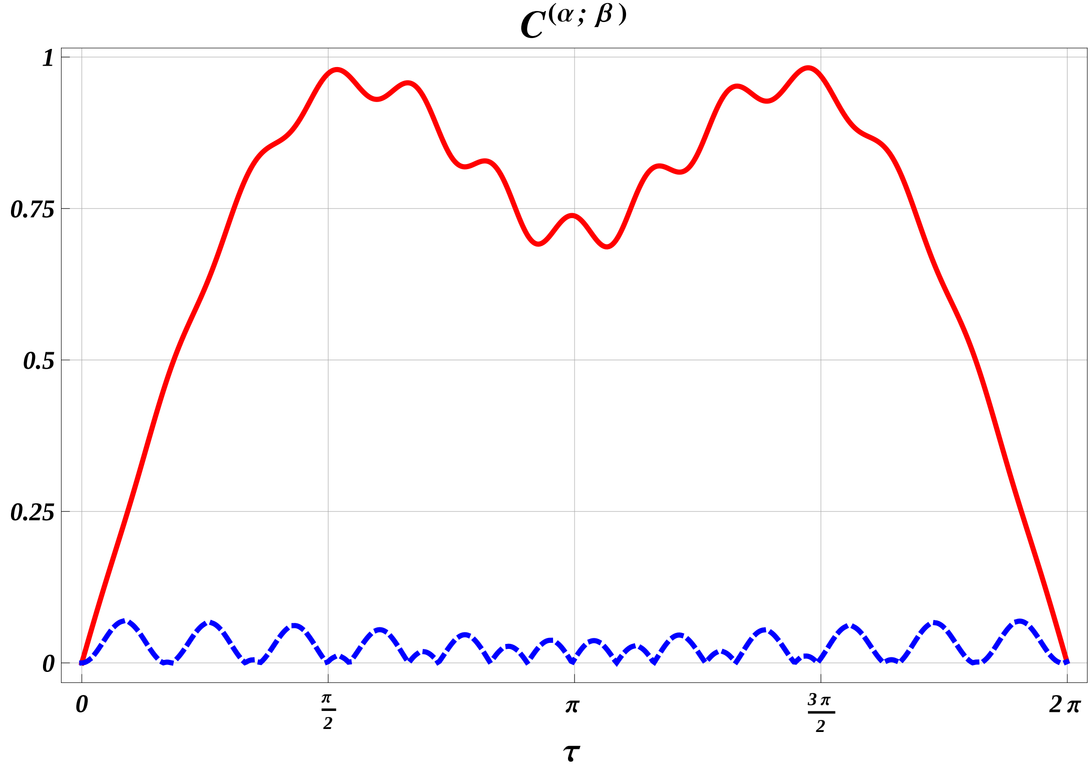
$$S_L^{(\bar{\nu}_\mu;\nu_e,\nu_\mu,\bar{\nu}_e)} = 4|U_{ee}^{\mu\bar{\mu}}|^2(1-|U_{ee}^{\mu\bar{\mu}}|^2)\,. \tag{4}$$

In the quantum mechanical limit, Eqs. (1) and (2) reduce to the Pontecorvo analogs, while Eqs. (3) and (4) go to zero.




Linear entropies: $S_L^{(\nu_e;\nu_\mu,\bar{\nu}_e,\bar{\nu}_\mu)}$ (double-dot-dashed line), $S_L^{(\nu_\mu;\nu_e,\bar{\nu}_e,\bar{\nu}_\mu)}$ (dot-dashed line), $S_L^{(\bar{\nu}_e;\nu_e,\nu_\mu,\bar{\nu}_\mu)}$ (dotted line), $S_L^{(\bar{\nu}_\mu;\nu_e,\nu_\mu,\bar{\nu}_e)}$ (dashed line), and the average linear entropy $\langle S_L^{(1:3)}\rangle$ (full line) as functions of the scaled time $\tau = (\omega_2 - \omega_1)t$.

• Alternatively, we can quantify entanglement between two single parties, by tracing over other degrees of freedom. The resulting state is a mixed state for which we calculate the concurrence.



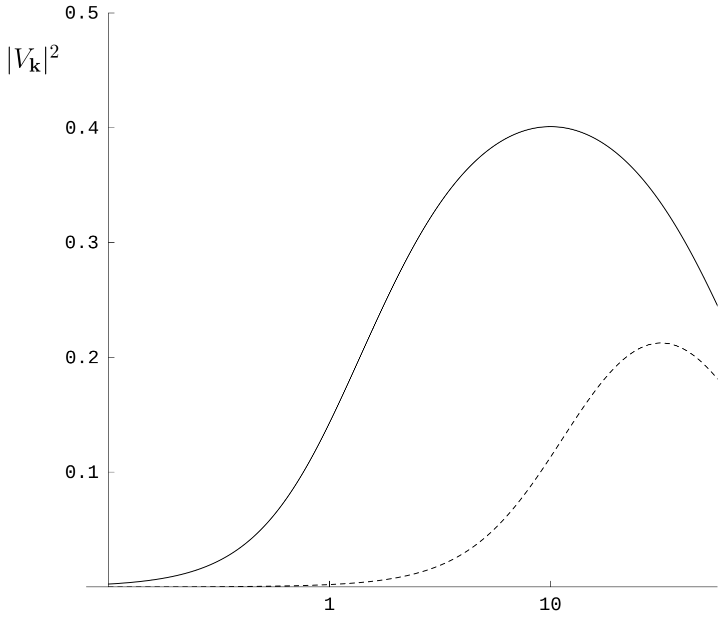

The concurrences $C^{(\alpha;\beta)}$ associated with the reduced mixed states $\rho_{\nu_e,\nu_\mu}$, ($C^{(\nu_e;\nu_\mu)}$, full line), and $\rho_{\bar{\nu}_e,\bar{\nu}_\mu}$ ($C^{(\bar{\nu}_e;\bar{\nu}_\mu)}$, dashed line), as functions of the scaled time $\tau = (\omega_2 - \omega_1)t$.
The concurrences $C^{(\nu_e;\bar{\nu}_e)}$, $C^{(\nu_e;\bar{\nu}_\mu)}$, $C^{(\nu_\mu;\bar{\nu}_e)}$, and $C^{(\nu_\mu;\bar{\nu}_\mu)}$ vanish for every $\tau$.

– Connection among the above QFT results...

## Conclusions

- Elementary particles are produced as entangled states in the SM;
- Quantification of multipartite entanglement for neutrinos and quarks;
- Neutrino oscillations as a resource for quantum information;
- Extension to QFT: entropic measures vs variances.

# Condensation density for mixed fermions




Solid line: $m_1 = 1$, $m_2 = 100$; Dashed line: $m_1 = 10$, $m_2 = 100$.

- $V_{\mathbf{k}} = 0$ when $m_1 = m_2$ and/or $\theta = 0$.

- Max. at $k = \sqrt{m_1 m_2}$ with $V_{max} \to \frac{1}{2}$ for $\frac{(m_2 - m_1)^2}{m_1 m_2} \to \infty$.

- $|V_{\mathbf{k}}|^2 \simeq \frac{(m_2 - m_1)^2}{4k^2}$ for $k \gg \sqrt{m_1 m_2}$.

# Neutrino oscillations (wave packets)*

– Consider, in one dimension, a neutrino with definite flavor, propagating along the $x$ direction:

$$|\nu_\alpha(x,t)\rangle = \sum_j U_{\alpha,j}\,\psi_j(x,t)\,|\nu_j\rangle\,,$$

where $U_{\alpha,j}$ is an element of the mixing matrix, $|\nu_j\rangle$ the mass eigenstate with mass $m_j$, and $\psi_j(x,t)$ its wave function.

– Assume Gaussian distribution $\psi_j(p)$ for the momentum of the massive neutrino $|\nu_j\rangle$:

$$\psi_j(x,t) = \frac{1}{\sqrt{2\pi}}\int dp\,\psi_j(p)\,e^{ipx-iE_j(p)t}\,, \qquad \psi_j(p) = \frac{1}{(2\pi\sigma_p^2)^{1/4}}\,e^{-\frac{1}{4\sigma_p^2}(p-p_j)^2}\,,$$

where $E_j(p) = \sqrt{p^2+m_j^2}$.

– The associated density matrix writes:

$$\rho_\alpha(x,t) = |\nu_\alpha(x,t)\rangle\langle\nu_\alpha(x,t)|\,.$$

If $\sigma_p \ll E_j^2(p_j)/m_j$, one can write $E_j(p) \simeq E_j + v_j(p-p_j)$, with $E_j \equiv \sqrt{p_j^2+m_j^2}$, and $v_j \equiv \frac{\partial E_j(p)}{\partial p}\big|_{p=p_j} = \frac{p_j}{E_j}$ is the group velocity of the wave packet for $\nu_j$.

*C. Giunti, C. W. Kim, Phys. Rev. **D** (1998); *Fundamentals of Neutrino Physics and Astrophysics*, Oxford Univ. Pr. (2007)

– In this case, a Gaussian integration yields:

$$\rho_\alpha(x,t) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \sum_{j,k} U_{\alpha j} U^*_{\alpha k} e^{-i(E_j-E_k)t+i(p_j-p_k)x-\frac{1}{4\sigma_x^2}[(x-v_jt)^2+(x-v_kt)^2]} |\nu_j\rangle\langle\nu_k| \,,$$

where $\sigma_x = (2\sigma_p)^{-1}$. For extremely relativistic neutrinos, one has

$$E_j \simeq E + \xi \frac{m_j^2}{2E} \,, \qquad p_j \simeq E - (1-\xi)\frac{m_j^2}{2E} \,, \qquad v_j \simeq 1 - \frac{m_j^2}{2E_j^2}$$

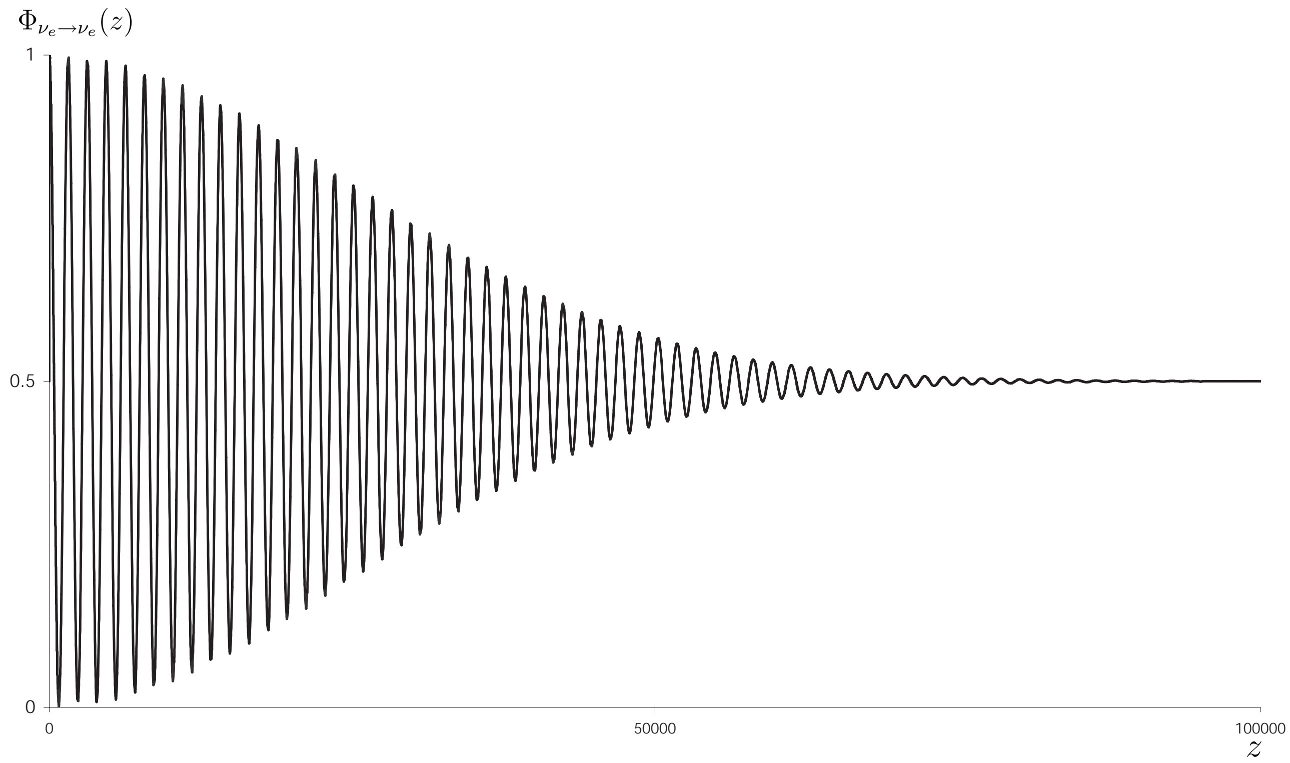
where $E$ is the neutrino energy in the limit of zero mass, and $\xi$ a dimensionless constant depending on the characteristic of the production process.

– The density matrix $\rho_\alpha(x,t)$ provides a space-time description of neutrino dynamics.

– In realistic situations, it is convenient to consider the time-independent density matrix $\rho_\alpha(x)$ obtained by the time average of $\rho_\alpha(x,t)$:

$$\rho_\alpha(x) = \sum_{j,k} U_{\alpha j} U^*_{\alpha k} \exp\left[-i\frac{\Delta m_{jk}^2 x}{2E} - \left(\frac{\Delta m_{jk}^2 x}{4\sqrt{2}E^2\sigma_x}\right)^2 - \left(\xi\frac{\Delta m_{jk}^2}{4\sqrt{2}E\sigma_p}\right)^2\right] |\nu_j\rangle\langle\nu_k| \,,$$

with $\Delta m_{jk}^2 = m_j^2 - m_k^2$.




Flavor oscillations in space.

$$P_{\nu_e\to\nu_e}(x) \;\simeq\; 1-\frac{1}{2}\sin^2(2\theta)\left\{1-\cos\left(2\pi\frac{x}{L^{osc}}\right)\,\exp\left[-\left(\frac{x}{L^{coh}}\right)^2-2\pi^2\left(\frac{\sigma_x}{L^{osc}}\right)^2\right]\right\}$$

- Oscillation length: $L^{osc}=4\pi p/\triangle m^2$

- Coherence length: $L^{coh}=(L^{osc}p)/(\sqrt{2}\pi\sigma_p)$.

## Decoherence in neutrino oscillations

- We analyze the coherence of the quantum superposition of the neutrino mass eigenstates, by looking at the spatial behavior of the multipartite entanglement of the above state*.

By means of the identification $|\nu_i\rangle = |\delta_{i,1}\rangle_1 |\delta_{i,2}\rangle_2 |\delta_{i,3}\rangle_3 \equiv |\delta_{i,1}\delta_{i,2}\delta_{i,3}\rangle$, with $i = 1, 2, 3$, we construct the matrix with elements

$$\langle lmn|\rho_\alpha(x)|ijk\rangle, \qquad \textit{where} \quad i,j,k,l,m,n = 0,1$$

– We analytically compute logarithmic negativities $E_{\mathcal{N}\,\alpha}^{(i,j;k)}$,for $i, j, k = 1, 2, 3$ and $i \neq j \neq k$, and average logarithmic negativity $\langle E_{\mathcal{N}\,\alpha}^{(2:1)}\rangle$, for the neutrino states with flavor $\alpha = e, \mu, \tau$.

*M.Blasone, F.Dell'Anno, S.De Siena, M.Di Mauro and F.Illuminati, Phys. Rev. D (2008).

We assume for the mixing angles the experimental values

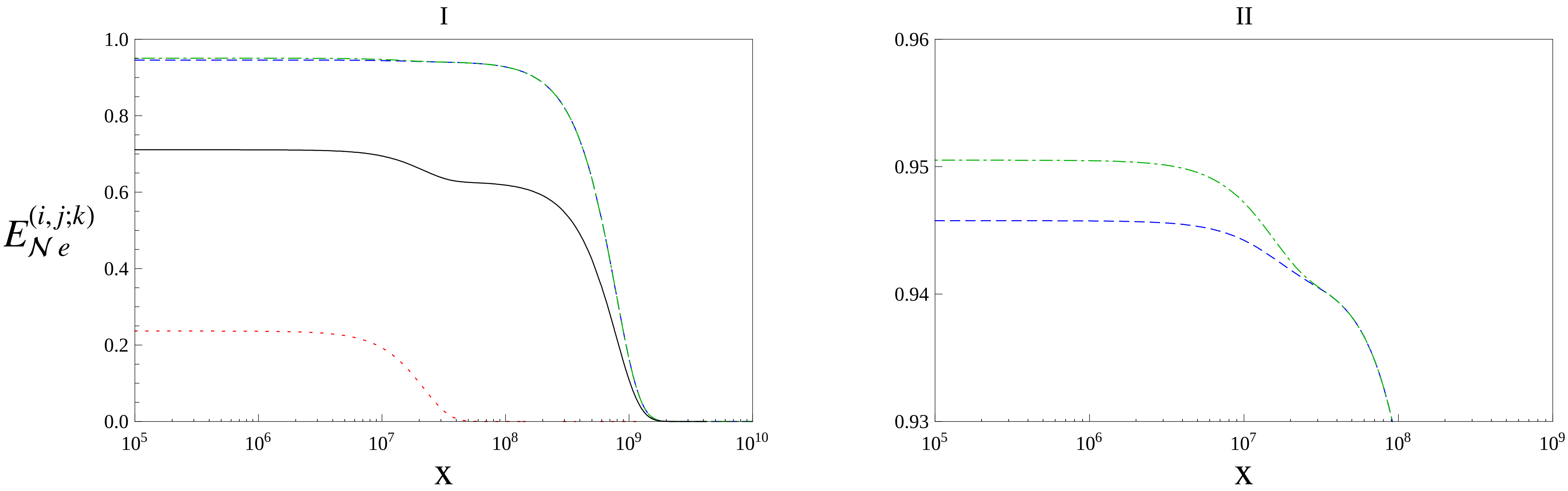
$$\sin^2\theta_{12}^{MNSP} = 0.314(1 \begin{array}{c} +0.18 \\ -0.15 \end{array}), \quad \sin^2\theta_{13}^{MNSP} = (0.8 \begin{array}{c} +2.3 \\ -0.8 \end{array}) \times 10^{-2}, \quad \sin^2\theta_{23}^{MNSP} = 0.45(1 \begin{array}{c} +0.35 \\ -0.20 \end{array})$$

The squared mass differences are fixed at the experimental values*

$$\Delta m_{21}^2 = \delta m^2, \qquad \Delta m_{31}^2 = \Delta m^2 + \frac{\delta m^2}{2}, \qquad \Delta m_{32}^2 = \Delta m^2 - \frac{\delta m^2}{2},$$
$$\delta m^2 = 7.92 \times 10^{-5}\, eV^2, \qquad \delta m^2 = 2.6 \times 10^{-3}\, eV^2.$$

We take $E = 10\, GeV$ and $\sigma_p = 1\, GeV$. The parameter $\xi$ is put to zero for simplicity.

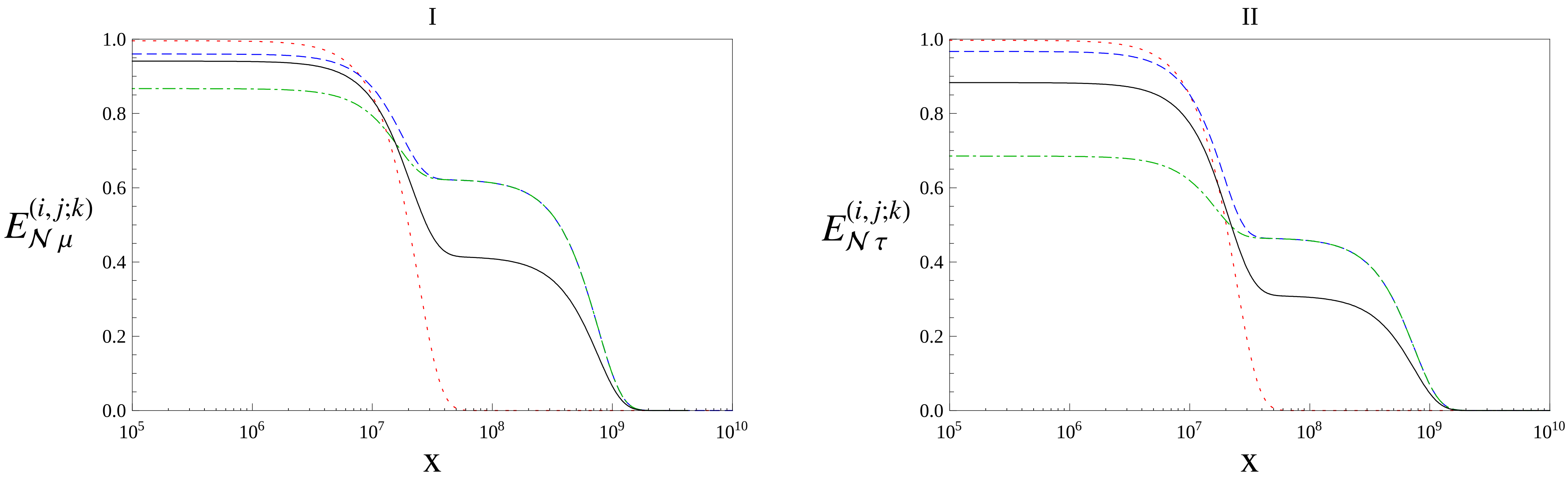
*G. L. Fogli, E. Lisi, A. Marrone, A. Melchiorri, A. Palazzo, P. Serra, J. Silk, and A. Slosar, Phys. Rev. D (2007).




Logarithmic negativities $E_{\mathcal{N}\,e}^{(i,j;k)}$ for all possible bipartitions and average logarithmic negativity $\langle E_{\mathcal{N}\,e}^{(2:1)} \rangle$ (solid line) as functions of the distance $x$ (meters).

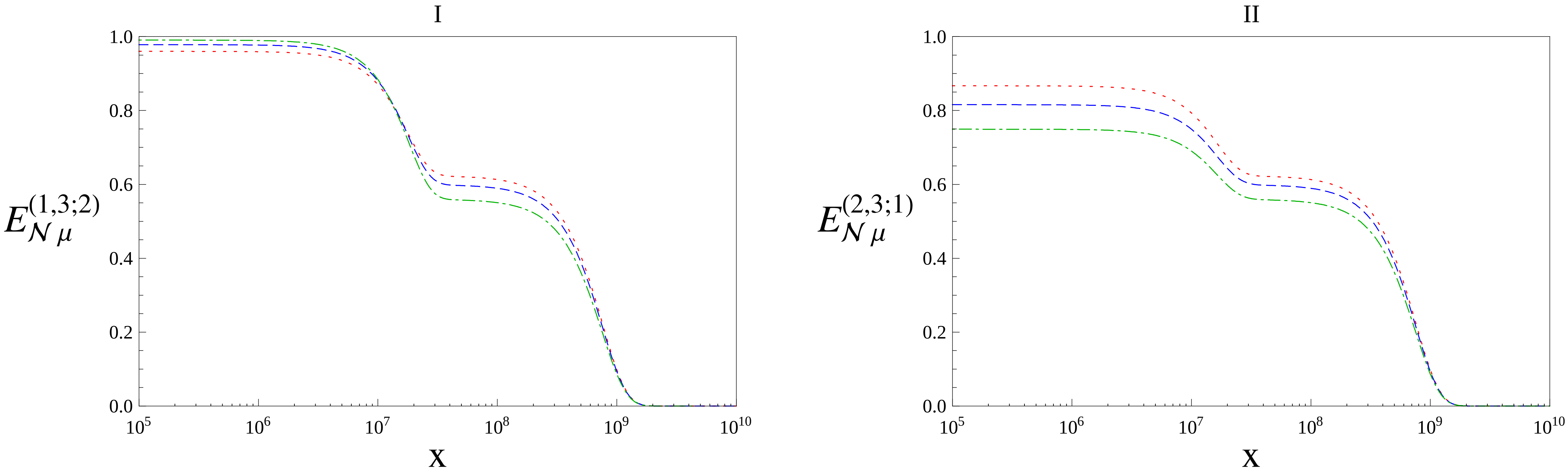
In panel II we plot a zoom of $E_{\mathcal{N}\,e}^{(1,3;2)}$ and $E_{\mathcal{N}\,e}^{(2,3;1)}$

All plotted quantities are independent of the CP-violating phase $\delta$.




Logarithmic negativities $E_{\mathcal{N}\,\alpha}^{(i,j;k)}$ for all possible bipartitions and average logarithmic negativity $\langle E_{\mathcal{N}\,\alpha}^{(2:1)} \rangle$ (solid line), with $\alpha = \mu, \tau$, as functions of the distance $x$ (meters).

The CP-violating phase $\delta$ is put to zero. The $x$ axis is in logarithmic scale, and the dimensions are meters.




Logarithmic negativities $E_{\mathcal{N}\mu}^{(1,3;2)}$ (panel I) and $E_{\mathcal{N}\mu}^{(2,3;1)}$ (panel II) as functions of the distance $x$ (meters) for different choices of the CP-violating phase $\delta$: (a) $\delta = 0$ (dotted line); (b) $\delta = \frac{\pi}{2}$ (dashed line); (b) $\delta = \pi$ (dot-dashed line). $E_{\mathcal{N}\mu}^{(1,2;3)}$ is independent of $\delta$.